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A generalization of Chakiris' fibrations

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Abstract.

Chakiris [5] constructed examples of holomorphic Lefschetz fibrations of genus 2 with separating singular fibers and proved a classification theorem for such fibrations in the late 1970's. We generalize some parts of his construction topologically to give new examples of hyperelliptic Lefschetz fibrations of arbitrary genus with separating singular fibers which include homeomorphic but non-diffeomorphic 4-manifolds.

§1. Introduction

According to remarkable works of Donaldson [6] and Gompf [14] (see also [1]), there is a Lefschetz fibration over the 2-sphere with prescribed fundamental group. The classification of all Lefschetz fibrations over the 2-sphere is not possible in nature. Many examples of Lefschetz fibrations are given in terms of positive relations in mapping class groups (see [14], [35], [4], [21], [15], [16], and [8]).

Hyperelliptic Lefschetz fibrations, which are (relative minimalizations of) double branched coverings of simple 4-manifolds (see [32], [12]), include all Lefschetz fibrations of genus 1 and 2 and many important examples. It would be rather hopeful to classify hyperelliptic Lefschetz fibrations over the 2-sphere. Siebert and Tian [32] conjectured that every hyperelliptic Lefschetz fibration over the 2-sphere without separating singular fibers is holomorphic. They solved it affirmatively in genus 2 case under assumption of monodromy transitivity [33]. Their conjecture is closely related to a smooth analogue of an earlier theorem of Chakiris [5] which asserts that every holomorphic fibration of genus 2 without virtual reducible singular fibers is a fiber sum of three typical fibrations. On the other hand, it does not seem to be known how many hyperelliptic

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Lefschetz fibrations with separating singular fibers exist. Matsumoto's genus 2 Lefschetz fibration [25] and its generalization in arbitrary even genus due to Cadavid [4] and Korkmaz [21] are well-known examples of such fibrations. Chakiris [5] also showed a mysterious classification theorem of Lefschetz fibrations of genus 2 with separating singular fibers, which we would like to call the '1/19-theorem'.

In this paper we generalize some parts of Chakiris' construction topologically to give new examples of hyperelliptic Lefschetz fibrations of arbitrary genus with separating singular fibers. In Section 2 we review definitions and basic properties of Lefschetz fibrations and relations in mapping class groups. In Section 3 we construct new positive relations in hyperelliptic mapping class groups and in Section 4 we investigate various properties of the corresponding hyperelliptic Lefschetz fibrations over the 2-sphere. In particular, we exhibit infinitely many pairs of homeomorphic but non-diffeomorphic hyperelliptic Lefschetz fibrations with separating singular fibers.

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§2. Lefschetz fibrations and positive relations

In this section we briefly review Lefschetz fibrations and relations in mapping class groups.

2.1. Lefschetz fibrations and their monodromies

We first review the definition and basic properties of Lefschetz fibrations. More details can be found in Matsumoto [25] and Gompf and Stipsicz [14].

Let Σ_g be a closed oriented surface of genus g.

Definition 2.1. Let M be a closed oriented smooth 4-manifold. A smooth map $f : M \to S^2$ is called a *Lefschetz fibration* of genus g if it satisfies the following conditions:

(i) f has finitely many critical values $b_1, \dots, b_n \in S^2$ and f is a smooth fiber bundle over $S^2 - \{b_1, \dots, b_n\}$ with fiber Σ_g;

(ii) for each i $(i = 1, \dots, n)$, there exists a unique critical point p_i in the *singular fiber* $F_i := f^{-1}(b_i)$ such that f is locally written as $f(z_1, z_2) = z_1^2 + z_2^2$ with respect to some local complex coordinates around p_i and b_i which are compatible with orientations of M and S^2;

(iii) no fiber contains a (-1)-sphere.

Let $\mathcal{M}_g$ be the mapping class group of Σ_g, namely the group of all isotopy classes of orientation-preserving diffeomorphisms of Σ_g. We follow the functional notation: for $\varphi, \psi \in \mathcal{M}_g$, the symbol $\psi\varphi$ means that we apply φ first and then ψ. We denote by $\mathcal{F}$ the free group generated by all isotopy classes $\mathcal{S}$ of simple closed curves on Σ_g. There is a natural epimorphism $\varpi : \mathcal{F} \to \mathcal{M}_g$ which sends (the isotopy class of) a simple closed curve a on Σ_g to the right-handed Dehn twist t_a along a. We often denote the image $\varpi(W)$ of a word W in the generators $\mathcal{S}$ by $\overline{W}$. We set $\mathcal{R} := \text{Ker } \varpi$ and call each element of $\mathcal{R}$ a *relator* in the generators $\mathcal{S}$ of $\mathcal{M}_g$. We put ${}_W(c) := t_{a_r}^{\varepsilon_r} \cdots t_{a_1}^{\varepsilon_1}(c) \in \mathcal{S}$ for $c \in \mathcal{S}$ and $W = a_r^{\varepsilon_r} \cdots a_1^{\varepsilon_1} \in \mathcal{F}$ $(a_1, \dots, a_r \in \mathcal{S}, \varepsilon_1, \dots, \varepsilon_r \in \{\pm 1\})$ and ${}_W V := {}_W(c_1) \cdots {}_W(c_s) \in \mathcal{F}$ for $V = c_1 \cdots c_s \in \mathcal{F}$ $(c_1, \dots c_s \in \mathcal{S})$.

Let $f : M \to S^2$ be a Lefschetz fibration of genus g as in the definition above. Since f restricted over $S^2 - \{b_1, \dots, b_n\}$ is a smooth fiber bundle with fiber Σ_g, we consider the homomorphism

$$\chi : \pi_1(S^2 - \{b_1, \dots, b_n\}) \to \pi_1(\text{BDiff}_+\Sigma_g) \cong \pi_0(\text{Diff}_+\Sigma_g) = \mathcal{M}_g$$

induced by the classifying map $S^2 - \{b_1, \dots, b_n\} \to \text{BDiff}_+\Sigma_g$, which is called the *holonomy homomorphism* (cf. Morita [28]) or the *monodromy representation* of f. Let γ_i $(i = 1, \dots, n)$ be the loop consisting of the boundary circle of a small disk neighborhood of b_i oriented clockwise and a path connecting a point on the circle to the base point $b_0 \in S^2 - \{b_1, \dots, b_n\}$. We choose these loops $\gamma_1, \dots, \gamma_n$ so that the composition $\gamma_1 \cdots \gamma_n$ is null-homotopic on $S^2 - \{b_1, \dots, b_n\}$ and any two of them intersect only at b_0. Thus we obtain a presentation

$$\pi_1(S^2 - \{b_1, \dots, b_n\}, b_0) = \langle \gamma_1, \dots, \gamma_n | \gamma_1 \cdots \gamma_n = 1 \rangle.$$

For each i $(i = 1, \dots, n)$, $\chi(\gamma_i)$ is known to be a right-handed Dehn twist t_{c_i} along some essential simple closed curve c_i on Σ_g. Hence we have a *positive* relation

$$t_{c_1} \cdots t_{c_n} = \chi(\gamma_1 \cdots \gamma_n) = 1 \in \mathcal{M}_g$$

or a *positive* relator $c_1 \cdots c_n \in \mathcal{R}$ associated to the Lefschetz fibration $f : M \to S^2$. Each c_i is called the *vanishing cycle* of the singular fiber F_i. F_i is called *non-separating* (or *irreducible*, *type* I) if c_i does not separate Σ_g into two connected components and *separating* (or *reducible*, *type* II_h) if c_i separates Σ_g into subsurfaces of genus h and $g - h$.

Suppose that $g \geq 2$. Kas [19] and Matsumoto [25] proved that there exists a one-to-one correspondence between the isomorphism classes of Lefschetz fibrations $f : M \to S^2$ and the conjugacy classes of homomorphisms χ which sends each loop going around b_i to a right-handed

Dehn twist along an essential simple closed curve on Σ_g. Although the positive relator $c_1 \cdots c_n \in \mathcal{R}$ actually depends on a choice of a loop system $(\gamma_1, \dots, \gamma_n)$ on $S^2 - \{b_1, \dots, b_n\}$, its equivalence class modulo conjugations of all factors $c_1, \dots, c_n$ by a fixed element W of $\mathcal{F}$:

$$c_1 \cdot \cdots \cdot c_n \sim {}_W(c_1) \cdot \cdots \cdot {}_W(c_n),$$

and *elementary transformations* :

$$c_1 \cdot \cdots \cdot c_i \cdot c_{i+1} \cdot \cdots \cdot c_n \sim c_1 \cdot \cdots \cdot c_{i+1} \cdot {}_{c_{i+1}^{-1}}(c_i) \cdot \cdots \cdot c_n,$$
$$c_1 \cdot \cdots \cdot c_i \cdot c_{i+1} \cdot \cdots \cdot c_n \sim c_1 \cdot \cdots \cdot {}_{c_i}(c_{i+1}) \cdot c_i \cdot \cdots \cdot c_n$$

$(i = 1, \dots, n-1)$, is uniquely determined by the isomorphism class of the Lefschetz fibration $f : M \to S^2$. Conversely, any positive relator $\varrho \in \mathcal{R}$ can be realized as a relator associated to some Lefschetz fibration over S^2. We denote (the isomorphism class of) such a Lefschetz fibration by $M_\varrho \to S^2$.

Let $f : M \to S^2$ and $f' : M' \to S^2$ be Lefschetz fibrations of genus g and $\varrho, \varrho' \in \mathcal{R}$ corresponding positive relators. Take regular values $b_0, b_0' \in S^2$ of f, f' and consider the fiber $F := f^{-1}(b_0), F' := f'^{-1}(b_0')$ and their open fibered neighborhoods $\nu F \subset M, \nu F' \subset M'$, respectively. Using a fiber-preserving, orientation-reversing diffeomorphism $\varphi : \partial(M - \nu F) \to \partial(M' - \nu F')$, we can glue $M - \nu F$ and $M' - \nu F'$ together and construct a new manifold $M \#_F M'$ which will admit a Lefschetz fibration $f \# f' : M \#_F M' \to S^2$ of genus g. We call this fibration a *fiber sum* of $f : M \to S^2$ and $f' : M' \to S^2$. The diffeomorphism type of $M \#_F M'$ and the isomorphism type of $f \# f'$ might depend on the choice of the diffeomorphism φ. A positive relator corresponding to the fiber sum $f \# f'$ can be written as $\varrho \cdot {}_W\varrho'$ for some $W \in \mathcal{F}$ which depends on the choice of φ.

Let $\iota : \Sigma_g \to \Sigma_g$ be (the mapping class of) a hyperelliptic involution, an involution on Σ_g with $2g+2$ fixed points, and $\mathcal{H}_g$ the centralizer of ι in $\mathcal{M}_g$, which is called the *hyperelliptic mapping class group.* Note that $\mathcal{H}_1 = \mathcal{M}_1$ and $\mathcal{H}_2 = \mathcal{M}_2$, while $\mathcal{H}_g \neq \mathcal{M}_g$ for $g \geq 3$. We set $\mathcal{S}^H := \{a \in \mathcal{S} \mid t_a \in \mathcal{H}_g\}$. We denote by $\mathcal{F}^H$ the subgroup of $\mathcal{F}$ generated by $\mathcal{S}^H$ and put $\mathcal{R}^H := \mathcal{R} \cap \mathcal{F}^H$. A Lefschetz fibration $f : M \to S^2$ of genus g is said to be *hyperelliptic* if its holonomy homomorphism χ can be chosen in the conjugacy class so that the image $\operatorname{Im}\chi$ is included in $\mathcal{H}_g$. If the canonical projection $\mathcal{H}_g \to \mathcal{M}_{0,2g+2} \to S_{2g+2}$ maps $\operatorname{Im}\chi$ onto a transitive subgroup of S_{2g+2}, we say that the monodromy of f is *transitive*, otherwise *intransitive*, where $\mathcal{M}_{0,2g+2}$ is the mapping class group of the 2-sphere with $2g+2$ marked points and S_{2g+2} is the symmetric group of degree $2g+2$.

2.2. Basic relations in mapping class groups

We next review several relations in the mapping class group $\mathcal{M}_g$ (cf. [42], [21], and [8]).

The relator $A = A(a) := a \in \mathcal{R}$ is called an *identity relator*, where a is a null-homotopic simple closed curve on Σ_g.

For (isotopy classes of) simple closed curves a and b, we denote their geometric intersection number by $i(a,b)$. Let a and b be simple closed curves on Σ_g and c the simple closed curve $t_b(a)$. The relation

$$t_c = t_b t_a t_b^{-1}$$

in $\mathcal{M}_g$ is called the *braid relation.* (It follows from the braid relation $t_{t_b(a)} = t_b t_a t_b^{-1}$ that $\overline{{}_b(a)} = \overline{bab^{-1}}$, $\overline{{}_{b^{-1}}(a)} = \overline{b^{-1}ab}$, and elementary transformations keep positive relators being positive relators.) If $i(a,b) = n$, we put

$$T_n = T(a,b) := bab^{-1}c^{-1} \in \mathcal{R}.$$

If $i(a,b) = 1$, we have another braid relation $t_b = t_a t_c t_a^{-1}$. This relation together with the original relation $t_c = t_b t_a t_b^{-1}$ yields Artin's relation $t_a t_b t_a = t_b t_a t_b$.

An ordered n-tuple $(c_1, \ldots, c_n)$ of simple closed curves on Σ_g is called a *chain* of length n if c_i and c_{i+1} intersect transversely at one point $(i = 1, \ldots, n-1)$ and other c_i and c_j never intersect. When the length n is even (resp. odd), a regular neighbourhood of a chain $(c_1, \ldots, c_n)$ is a subsurface of Σ_g which is of genus $h = n/2$ (resp. $h = (n-1)/2$) and has one boundary component (resp. two boundary components). We denote simple closed curves parallel to the boundary by d (resp. d_1 and d_2). The relation

$$t_d = (t_{c_1} \cdots t_{c_{2h}})^{4h+2} \quad (\text{resp. } t_{d_1} t_{d_2} = (t_{c_1} \cdots t_{c_{2h+1}})^{2h+2})$$

is called the *chain relation* of length n, or the *even* (resp. *odd*) chain relation (see Wajnryb [42]). We put

$$C_{2h} = C(c_1, \ldots, c_{2h}) := (c_1 \cdots c_{2h})^{4h+2} d^{-1} \in \mathcal{R},$$
$$C_{2h+1} = C(c_1, \ldots, c_{2h+1}) := (c_1 \cdots c_{2h+1})^{2h+2} d_1^{-1} d_2^{-1} \in \mathcal{R}.$$

Remark 2.2. The even (resp. odd) chain relation above holds even if we permute the factors of the chain:

$$(c_{\sigma(1)} \cdots c_{\sigma(2h)})^{4h+2} d^{-1} \in \mathcal{R} \; (\text{resp. } (c_{\tau(1)} \cdots c_{\tau(2h+1)})^{2h+2} d_1^{-1} d_2^{-1} \in \mathcal{R}),$$

where $\sigma \in S_{2h}$ (resp. $\tau \in S_{2h+1}$) is an arbitrary permutation (cf. Matsumoto [24]). Hirose told the author an elementary proof of this fact.

We need the following definition and lemma for constructions of positive relators in the next section.

Definition 2.3 (Smith [35], cf. [8]). Let $\varrho = W_1^{-1}W_2 \in \mathcal{R}$ be a relator with W_1 and W_2 positive words in $\mathcal{F}$. Suppose that a positive relator $\varsigma \in \mathcal{R}$ includes W_1 as a subword: $\varsigma = UW_1V$, where U and V are positive words in $\mathcal{F}$. Then we can construct a new positive relator $\varsigma' = \varsigma V^{-1}\varrho V = UW_2V$ in $\mathcal{R}$. This operation $\varsigma \mapsto \varsigma'$ is called a *ϱ-substitution* to ς. If a positive relator $\hat{\varsigma}$ is obtained by applying a sequence of $\varrho^{\pm 1}$-substitutions to ς, we denote it by $\varsigma \equiv \hat{\varsigma} \pmod{\varrho}$.

Lemma 2.4. *Let $(c_1, \ldots, c_n)$ be a chain of length n on Σ_g. The following equivalence holds for $k = 1, \ldots, n-1$ and $i = 1, \ldots, k+1$.*

$$(c_1c_2\cdots c_n)^i \equiv (c_1c_2\cdots c_k)^i \cdot (c_{k+1}c_k\cdots c_{k-i+2}) \cdot (c_{k+2}c_{k+1}\cdots c_{k-i+3}) \\ \cdots\cdot (c_nc_{n-1}\cdots c_{n-i+1}) \pmod{T_0, T_1}$$

Proof. Straightforward from the proof of Lemma 4.6 of [8]. Q.E.D.

§3. Hyperelliptic Chakiris relations

In this section we construct new examples of positive relators in the hyperelliptic mapping class group $\mathcal{H}_g$. We first review basic relations in $\mathcal{H}_g$ (cf. [3] and [8]).

Let $(c_1, \ldots, c_{2g+1})$ be a chain of length $2g+1$ on Σ_g. Suppose that each c_i is invariant under the hyperelliptic involution ι. Hence the right-handed Dehn twists $t_{c_1}, \ldots, t_{c_{2g+1}}$ belong to $\mathcal{H}_g$. The chain relator $C_{2g+1} = (c_1\cdots c_{2g+1})^{2g+2}d_1^{-1}d_2^{-1}$ of length $2g+1$ combined with two identity relators $A(d_1) = d_1$ and $A(d_2) = d_2$ is a positive relator

$$C_{\mathrm{I}} := C_{2g+1}A(d_2)A(d_1) = (c_1\cdots c_{2g+1})^{2g+2} \in \mathcal{R}^H.$$

The chain relator $C_{2g} = (c_1\cdots c_{2g})^{4g+2}d^{-1}$ of length $2g$ combined with an identity relator $A(d) = d$ is a positive relator

$$C_{\mathrm{II}} := C_{2g}A(d) = (c_1\cdots c_{2g})^{4g+2} \in \mathcal{R}^H.$$

It is well-known that the images of

$$I := c_1c_2\cdots c_{2g}c_{2g+1}^2c_{2g}\cdots c_2c_1 \ \in \mathcal{F}^H,$$
$$J := (c_1c_2\cdots c_{2g})^{2g+1} \ \in \mathcal{F}^H$$

under ϖ represent (the mapping class of) the hyperelliptic involution ι (see Birman and Hilden [3], p. 108, Equation (8), and [7], Lemma 4.13).

3.1. Even genus

Suppose that $g \geq 2$ and g is even. We first consider the following three elements of $\mathcal{F}^H$.

$$\begin{aligned}
P_0 &:= ((c_1 c_2 \cdots c_g)^{g+1} \\
&\quad \cdot (c_{g+1} \cdots c_3 c_2) \cdot (c_{g+2} \cdots c_4 c_3) \cdot \cdots \cdot (c_{2g} \cdots c_{g+2} c_{g+1}))^2, \\
Q_0 &:= (c_1 c_2 \cdots c_{2g+1})^{g+1} \cdot (c_g \cdots c_2 c_1)^{2g+2} \\
&\quad \cdot (c_1^{-1} c_2^{-1} \cdots c_g^{-1} c_{g+1} c_g \cdots c_2 c_1) \cdot (c_2^{-1} c_3^{-1} \cdots c_{g+1}^{-1} c_{g+2} c_{g+1} \cdots c_3 c_2) \\
&\quad \cdot \cdots \cdot (c_{g+1}^{-1} c_{g+2}^{-1} \cdots c_{2g}^{-1} c_{2g+1} c_{2g} \cdots c_{g+2} c_{g+1}), \\
R_0 &:= (c_g \cdots c_2 c_1)^{2g+2} \\
&\quad \cdot (c_1^{-1} c_2^{-1} \cdots c_g^{-1} c_{g+1} c_g \cdots c_2 c_1) \cdot (c_2^{-1} c_3^{-1} \cdots c_{g+1}^{-1} c_{g+2} c_{g+1} \cdots c_3 c_2) \\
&\quad \cdot \cdots \cdot (c_{g+1}^{-1} c_{g+2}^{-1} \cdots c_{2g}^{-1} c_{2g+1} c_{2g} \cdots c_{g+2} c_{g+1}) \cdot (c_{2g+1} \cdots c_2 c_1)^{g+1}.
\end{aligned}$$

Lemma 3.1. *The words P_0, Q_0, and R_0 represent $\iota, 1$, and ι in $\mathcal{H}_g$, respectively.*

Proof. Using Lemma 2.4, we have

$$\begin{aligned}
(c_1 c_2 \cdots c_{2g})^{g+1} &\equiv (c_1 c_2 \cdots c_g)^{g+1} \cdot (c_{g+1} \cdots c_2 c_1) \cdot (c_{g+2} \cdots c_3 c_2) \\
&\quad \cdot \cdots \cdot (c_{2g} \cdots c_{g+1} c_g) \pmod{T_0, T_1}.
\end{aligned}$$

We rewrite the right-hand side by using braid relations to obtain

$$\begin{aligned}
(c_1 c_2 \cdots c_{2g})^{g+1} &\equiv (c_1 c_2 \cdots c_g)^{g+1} \cdot (c_{g+1} \cdots c_3 c_2) \cdot (c_{g+2} \cdots c_4 c_3) \\
&\quad \cdot \cdots \cdot (c_{2g} \cdots c_{g+2} c_{g+1}) \cdot c_1 c_2 \cdots c_g \pmod{T_0, T_1}.
\end{aligned}$$

Again from Lemma 2.4, we have

$$\begin{aligned}
(c_1 c_2 \cdots c_{2g})^g &\equiv (c_1 c_2 \cdots c_g)^g \cdot (c_{g+1} \cdots c_3 c_2) \cdot (c_{g+2} \cdots c_4 c_3) \\
&\quad \cdot \cdots \cdot (c_{2g} \cdots c_{g+2} c_{g+1}) \pmod{T_0, T_1}.
\end{aligned}$$

We combine the last two equivalences to obtain

$$\begin{aligned}
J &= (c_1 c_2 \cdots c_{2g})^{2g+1} \\
&\equiv ((c_1 c_2 \cdots c_g)^{g+1} \cdot (c_{g+1} \cdots c_3 c_2) \cdot (c_{g+2} \cdots c_4 c_3) \\
&\quad \cdot \cdots \cdot (c_{2g} \cdots c_{g+2} c_{g+1}))^2 \pmod{T_0, T_1} \\
&= P_0.
\end{aligned}$$

Hence we have $\overline{P_0} = \overline{J} = \iota$.

As is already mentioned, the image of

$$C_{\mathrm{I}} = (c_1 c_2 \cdots c_{2g+1})^{2g+2} \in \mathcal{F}^H$$

under ϖ is equal to 1 in $\mathcal{H}_g$. By virtue of Lemma 2.4 and Corollary A.2, we have

$$\begin{aligned}(c_1 c_2 \cdots c_{2g+1})^{g+1} &\equiv (c_1 c_2 \cdots c_g)^{g+1} \cdot (c_{g+1} \cdots c_2 c_1) \cdot (c_{g+2} \cdots c_3 c_2) \\ &\quad \cdots\cdots (c_{2g+1} \cdots c_{g+2} c_{g+1}) \pmod{T_0, T_1} \\ &\equiv (c_g \cdots c_2 c_1)^{g+1} \cdot (c_{g+1} \cdots c_2 c_1) \cdot (c_{g+2} \cdots c_3 c_2) \\ &\quad \cdots\cdots (c_{2g+1} \cdots c_{g+2} c_{g+1}) \pmod{T_0, T_1}.\end{aligned}$$

It is easy to check by manipulating braid relations as Lemma 2.1 of [21] that the following equivalence holds.

$$\begin{aligned}&(c_1^{-1} c_2^{-1} \cdots c_g^{-1})^{g+1} \cdot (c_{g+1} \cdots c_2 c_1) \\ &\quad \cdot (c_{g+2} \cdots c_3 c_2) \cdots\cdots (c_{2g+1} \cdots c_{g+2} c_{g+1}) \\ &\equiv (c_1^{-1} c_2^{-1} \cdots c_g^{-1} c_{g+1} c_g \cdots c_2 c_1) \cdot (c_2^{-1} c_3^{-1} \cdots c_{g+1}^{-1} c_{g+2} c_{g+1} \cdots c_3 c_2) \\ &\quad \cdots\cdots (c_{g+1}^{-1} c_{g+2}^{-1} \cdots c_{2g}^{-1} c_{2g+1} c_{2g} \cdots c_{g+2} c_{g+1}) \qquad \pmod{T_0, T_1}\end{aligned}$$

Gathering these equivalences, we obtain

$$\begin{aligned}&(c_1 c_2 \cdots c_{2g+1})^{g+1} \\ &\equiv (c_g \cdots c_2 c_1)^{2g+2} \cdot (c_1^{-1} c_2^{-1} \cdots c_g^{-1})^{g+1} (c_{g+1} \cdots c_2 c_1) \cdot (c_{g+2} \cdots c_3 c_2) \\ &\quad \cdots\cdots (c_{2g+1} \cdots c_{g+2} c_{g+1}) \pmod{T_0, T_1} \\ &\equiv (c_g \cdots c_2 c_1)^{2g+2} \cdot (c_1^{-1} c_2^{-1} \cdots c_g^{-1} c_{g+1} c_g \cdots c_2 c_1) \\ &\quad \cdot (c_2^{-1} c_3^{-1} \cdots c_{g+1}^{-1} c_{g+2} c_{g+1} \cdots c_3 c_2) \\ &\quad \cdots\cdots (c_{g+1}^{-1} c_{g+2}^{-1} \cdots c_{2g}^{-1} c_{2g+1} c_{2g} \cdots c_{g+2} c_{g+1}) \pmod{T_0, T_1}.\end{aligned}$$

We multiply both sides of the equivalence by $(c_1 c_2 \cdots c_{2g+1})^{g+1}$ and conclude

$$Q_0 \equiv C_{\mathrm{I}} \pmod{T_0, T_1} \quad \text{and} \quad \overline{Q}_0 = \overline{C}_{\mathrm{I}} = 1 \in \mathcal{H}_g.$$

It follows from the equivalence above that

$$\begin{aligned}R_0 &= (c_1 c_2 \cdots c_{2g+1})^{-(g+1)} \cdot Q_0 \cdot (c_{2g+1} \cdots c_2 c_1)^{g+1} \\ &\equiv (c_1 c_2 \cdots c_{2g+1})^{-(g+1)} \cdot C_{\mathrm{I}} \cdot (c_{2g+1} \cdots c_2 c_1)^{g+1} \quad \pmod{T_0, T_1} \\ &= (c_1 c_2 \cdots c_{2g+1})^{g+1} \cdot (c_{2g+1} \cdots c_2 c_1)^{g+1}.\end{aligned}$$

The element

$$D_i := [I, c_i] = Ic_iI^{-1}c_i^{-1} \quad (i = 1, \ldots, 2g+1)$$

of $\mathcal{F}^H$ is a relator of $\mathcal{H}_g$: $D_i \in \mathcal{R}^H$ (see Birman and Hilden [3], Theorem 8). Applying D_i-substitutions repeatedly, we have

$$\begin{aligned}
&(c_1c_2\cdots c_{2g+1})^{g+1}\cdot(c_{2g+1}\cdots c_2c_1)^{g+1} \\
&\equiv I\cdot(c_1c_2\cdots c_{2g+1})^{g}\cdot(c_{2g+1}\cdots c_2c_1)^{g} \quad (\mathrm{mod}\ D_1,\ldots,D_{2g+1}) \\
&\equiv \cdots\cdots \\
&\equiv I^{g+1} \quad (\mathrm{mod}\ D_1,\ldots,D_{2g+1}).
\end{aligned}$$

We combine these equivalence to obtain

$$R_0 \equiv I^{g+1}\ (\mathrm{mod}\ T_0, T_1, D_1, \ldots, D_{2g+1}) \quad \text{and} \quad \overline{R}_0 = \overline{I}^{g+1} = \iota$$

because g is even. This completes the proof of the lemma. Q.E.D.

Let $d \in \mathcal{S}^H$ be the boundary curve of a regular neighborhood of $c_1 \cup \cdots \cup c_g$. We now define three positive words in the generators $\mathcal{S}^H$:

$$\begin{aligned}
P :=\ & d\cdot {}_W(c_{g+1}\cdots c_3c_2)\cdot\cdots\cdot {}_W(c_{2g}\cdots c_{g+2}c_{g+1}) \\
&\cdot(c_{g+1}\cdots c_3c_2)\cdot\cdots\cdot(c_{2g}\cdots c_{g+2}c_{g+1}) \quad (W := (c_1c_2\cdots c_g)^{-(g+1)}); \\
Q :=\ & (c_1c_2\cdots c_{2g+1})^{g+1}\cdot d\cdot {}_{W_1}(c_{g+1})\cdot {}_{W_2}(c_{g+2})\cdot\cdots\cdot {}_{W_{g+1}}(c_{2g+1}); \\
R :=\ & d\cdot {}_{W_1}(c_{g+1})\cdot {}_{W_2}(c_{g+2})\cdot\cdots\cdot {}_{W_{g+1}}(c_{2g+1})\cdot(c_{2g+1}\cdots c_2c_1)^{g+1} \\
& (W_i := (c_{i+g-1}\cdots c_{i+1}c_i)^{-1} \ (i = 1,\ldots,g+1)).
\end{aligned}$$

Theorem 3.2. *The words P, Q, and R in $\mathcal{F}^H$ are positive words representing $\iota, 1$, and ι in $\mathcal{H}_g$, respectively.*

Proof. We apply C_g^{-1}-substitutions to P_0, Q_0, and R_0 and rewrite them as follows.

$$\begin{aligned}
P_0 &= (c_1c_2\cdots c_g)^{2g+2}\cdot(c_1c_2\cdots c_g)^{-(g+1)} \\
&\quad \cdot(c_{g+1}\cdots c_3c_2)\cdot\cdots\cdot(c_{2g}\cdots c_{g+2}c_{g+1}) \\
&\quad \cdot(c_1c_2\cdots c_g)^{g+1}\cdot(c_{g+1}\cdots c_3c_2)\cdot\cdots\cdot(c_{2g}\cdots c_{g+2}c_{g+1}) \\
&\equiv (c_1c_2\cdots c_g)^{2g+2}\cdot({}_W(c_{g+1})\cdots {}_W(c_3)\,{}_W(c_2)) \\
&\quad \cdot\cdots\cdot({}_W(c_{2g})\cdots {}_W(c_{g+2})\,{}_W(c_{g+1})) \\
&\quad \cdot(c_{g+1}\cdots c_3c_2)\cdot\cdots\cdot(c_{2g}\cdots c_{g+2}c_{g+1}) \quad (\mathrm{mod}\ T_n) \\
&\equiv P \quad (\mathrm{mod}\ C_g),
\end{aligned}$$

$$\begin{aligned} Q_0 &\equiv (c_1c_2\cdots c_{2g+1})^{g+1}\cdot(c_g\cdots c_2c_1)^{2g+2}\\ &\quad\cdot w_1(c_{g+1})\cdot w_2(c_{g+2})\cdot\cdots\cdot w_{g+1}(c_{2g+1}) \qquad (\mathrm{mod}\ T_n)\\ &\equiv Q \quad (\mathrm{mod}\ C_g),\\ R_0 &\equiv (c_g\cdots c_2c_1)^{2g+2}\cdot w_1(c_{g+1})\cdot w_2(c_{g+2})\cdot\cdots\cdot w_{g+1}(c_{2g+1})\\ &\quad\cdot(c_{2g+1}\cdots c_2c_1)^{g+1} \qquad (\mathrm{mod}\ T_n)\\ &\equiv R \quad (\mathrm{mod}\ C_g). \end{aligned}$$

Thus the proof is completed. Q.E.D.

The next corollary immediately follows from the theorem.

Corollary 3.3. *The words $P^2, Q, R^2, PR, PI, PJ, RI, RJ$ in $\mathcal{F}^H$ are positive relators for $\mathcal{H}_g$.*

3.2. Odd genus

Suppose that $g \geq 3$ and g is odd. We first consider the following two elements of $\mathcal{F}^H$.

$$\begin{aligned} Q_0 &:= (c_1c_2\cdots c_{2g+1})^{g+2}\cdot(c_{g-1}\cdots c_2c_1)^{2g+2}\\ &\quad\cdot(c_1^{-1}c_2^{-1}\cdots c_{g-1}^{-1}c_gc_{g-1}\cdots c_2c_1)\cdot(c_2^{-1}c_3^{-1}\cdots c_g^{-1}c_{g+1}c_g\cdots c_3c_2)\\ &\quad\cdot\cdots\cdot(c_{g+2}^{-1}c_{g+3}^{-1}\cdots c_{2g}^{-1}c_{2g+1}c_{2g}\cdots c_{g+3}c_{g+2}),\\ R_0 &:= c_1c_2\cdots c_{2g+1}\cdot(c_{g-1}\cdots c_2c_1)^{2g+2}\\ &\quad\cdot(c_1^{-1}c_2^{-1}\cdots c_{g-1}^{-1}c_gc_{g-1}\cdots c_2c_1)\cdot(c_2^{-1}c_3^{-1}\cdots c_g^{-1}c_{g+1}c_g\cdots c_3c_2)\\ &\quad\cdot\cdots\cdot(c_{g+2}^{-1}c_{g+3}^{-1}\cdots c_{2g}^{-1}c_{2g+1}c_{2g}\cdots c_{g+3}c_{g+2})\cdot(c_{2g+1}\cdots c_2c_1)^{g+1}. \end{aligned}$$

Lemma 3.4. *Both of the words Q_0 and R_0 are relators for $\mathcal{H}_g$.*

Proof. As is already mentioned, the image of

$$C_{\mathrm{I}} = (c_1c_2\cdots c_{2g+1})^{2g+2} \in \mathcal{F}^H$$

under ϖ is equal to 1 in $\mathcal{H}_g$. By virtue of Lemma 2.4 and Corollary A.2, we have

$$\begin{aligned} (c_1c_2\cdots c_{2g+1})^g &\equiv (c_1c_2\cdots c_{g-1})^g\cdot(c_g\cdots c_2c_1)\cdot(c_{g+1}\cdots c_3c_2)\\ &\quad\cdot\cdots\cdot(c_{2g+1}\cdots c_{g+3}c_{g+2}) \quad (\mathrm{mod}\ T_0, T_1)\\ &\equiv (c_{g-1}\cdots c_2c_1)^g\cdot(c_g\cdots c_2c_1)\cdot(c_{g+1}\cdots c_3c_2)\\ &\quad\cdot\cdots\cdot(c_{2g+1}\cdots c_{g+3}c_{g+2}) \quad (\mathrm{mod}\ T_0, T_1). \end{aligned}$$

It is easy to check by manipulating braid relations as Lemma 2.1 of [21] that the following equivalence holds.

$$\begin{aligned}
&(c_1^{-1}c_2^{-1}\cdots c_{g-1}^{-1})^{g+2}\cdot(c_g\cdots c_2c_1)\\
&\qquad\cdot(c_{g+1}\cdots c_3c_2)\cdot\cdots\cdot(c_{2g+1}\cdots c_{g+3}c_{g+2})\\
&\equiv(c_1^{-1}c_2^{-1}\cdots c_{g-1}^{-1}c_gc_{g-1}\cdots c_2c_1)\cdot(c_2^{-1}c_3^{-1}\cdots c_g^{-1}c_{g+1}c_g\cdots c_3c_2)\\
&\quad\cdot\cdots\cdot(c_{g+2}^{-1}c_{g+3}^{-1}\cdots c_{2g}^{-1}c_{2g+1}c_{2g}\cdots c_{g+3}c_{g+2})\pmod{T_0,T_1}
\end{aligned}$$

Gathering these equivalences, we obtain

$$\begin{aligned}
&(c_1c_2\cdots c_{2g+1})^g\\
&\equiv(c_{g-1}\cdots c_2c_1)^{2g+2}\cdot(c_1^{-1}c_2^{-1}\cdots c_{g-1}^{-1})^{g+2}(c_g\cdots c_2c_1)\cdot(c_{g+1}\cdots c_3c_2)\\
&\quad\cdot\cdots\cdot(c_{2g+1}\cdots c_{g+3}c_{g+2})\pmod{T_0,T_1}\\
&\equiv(c_{g-1}\cdots c_2c_1)^{2g+2}\cdot(c_1^{-1}c_2^{-1}\cdots c_{g-1}^{-1}c_gc_{g-1}\cdots c_2c_1)\\
&\quad\cdot(c_2^{-1}c_3^{-1}\cdots c_g^{-1}c_{g+1}c_g\cdots c_3c_2)\\
&\quad\cdot\cdots\cdot(c_{g+2}^{-1}c_{g+3}^{-1}\cdots c_{2g}^{-1}c_{2g+1}c_{2g}\cdots c_{g+3}c_{g+2})\pmod{T_0,T_1}.
\end{aligned}$$

We multiply both sides of the equivalence by $(c_1c_2\cdots c_{2g+1})^{g+2}$ and conclude

$$Q_0\equiv C_{\mathrm{I}}\pmod{T_0,T_1}\quad\text{and}\quad\overline{Q}_0=\overline{C}_{\mathrm{I}}=1\ \in\mathcal{H}_g.$$

It follows from the equivalence above that

$$\begin{aligned}
R_0&=(c_1c_2\cdots c_{2g+1})^{-(g+1)}\cdot Q_0\cdot(c_{2g+1}\cdots c_2c_1)^{g+1}\\
&\equiv(c_1c_2\cdots c_{2g+1})^{-(g+1)}\cdot C_{\mathrm{I}}\cdot(c_{2g+1}\cdots c_2c_1)^{g+1}\pmod{T_0,T_1}\\
&=(c_1c_2\cdots c_{2g+1})^{g+1}\cdot(c_{2g+1}\cdots c_2c_1)^{g+1}.
\end{aligned}$$

Applying D_i-substitutions repeatedly, we have

$$\begin{aligned}
&(c_1c_2\cdots c_{2g+1})^{g+1}\cdot(c_{2g+1}\cdots c_2c_1)^{g+1}\\
&\equiv I\cdot(c_1c_2\cdots c_{2g+1})^g\cdot(c_{2g+1}\cdots c_2c_1)^g\pmod{D_1,\dots,D_{2g+1}}\\
&\equiv\cdots\cdots\\
&\equiv I^{g+1}\pmod{D_1,\dots,D_{2g+1}}.
\end{aligned}$$

We combine these equivalence to obtain

$$R_0\equiv I^{g+1}\pmod{T_0,T_1,D_1,\dots,D_{2g+1}}\quad\text{and}\quad\overline{R}_0=\overline{I}^{g+1}=1$$

because g is odd. This completes the proof of the lemma. Q.E.D.

Let $d \in \mathcal{S}^H$ be the boundary curve of a regular neighborhood of $c_1 \cup \cdots \cup c_{g-1}$. We now define two positive words in the generators $\mathcal{S}^H$:

$$\begin{aligned} Q &:= (c_1 c_2 \cdots c_{2g+1})^{g+2} \cdot d \cdot (c_{g-1} \cdots c_2 c_1)^2 \\ &\quad \cdot {}_{W_1}(c_g) \cdot {}_{W_2}(c_{g+1}) \cdot \cdots \cdot {}_{W_{g+2}}(c_{2g+1}); \\ R &:= c_1 c_2 \cdots c_{2g+1} \cdot d \cdot (c_{g-1} \cdots c_2 c_1)^2 \\ &\quad \cdot {}_{W_1}(c_g) \cdot {}_{W_2}(c_{g+1}) \cdot \cdots \cdot {}_{W_{g+2}}(c_{2g+1}) \cdot (c_{2g+1} \cdots c_2 c_1)^{g+1} \\ &\quad (W_i := c_i^{-1} c_{i+1}^{-1} \cdots c_{i+g-2}^{-1} \ (i = 1, \dots, g+2)). \end{aligned}$$

Theorem 3.5. *Both of the words Q and R in $\mathcal{F}^H$ are positive relators for $\mathcal{H}_g$.*

Proof. We apply C_g^{-1}-substitutions to Q_0 and R_0 and rewrite them as follows.

$$\begin{aligned} Q_0 &\equiv (c_1 c_2 \cdots c_{2g+1})^{g+2} \cdot (c_{g-1} \cdots c_2 c_1)^{2g} \cdot (c_{g-1} \cdots c_2 c_1)^2 \\ &\quad \cdot {}_{W_1}(c_g) \cdot {}_{W_2}(c_{g+1}) \cdot \cdots \cdot {}_{W_{g+2}}(c_{2g+1}) \pmod{T_n} \\ &\equiv Q \pmod{C_{g-1}}, \\ R_0 &\equiv c_1 c_2 \cdots c_{2g+1} \cdot (c_{g-1} \cdots c_2 c_1)^{2g} \cdot (c_{g-1} \cdots c_2 c_1)^2 \\ &\quad \cdot {}_{W_1}(c_g) \cdot {}_{W_2}(c_{g+1}) \cdot \cdots \cdot {}_{W_{g+2}}(c_{2g+1}) \cdot (c_{2g+1} \cdots c_2 c_1)^{g+1} \pmod{T_n} \\ &\equiv R \pmod{C_{g-1}}. \end{aligned}$$

Thus the proof is completed. Q.E.D.

Let $d_1, d_2 \in \mathcal{S}^H$ be the boundary curves of a regular neighborhood of $c_1 \cup \cdots \cup c_g$.

Remark 3.6. Substituting the same words as P_0, Q_0, and R_0 for even genus by the inverse C_g^{-1} of a chain relator C_g, we have the following positive words for odd genus:

$$\begin{aligned} P' &:= d_1^2 d_2^2 \cdot {}_{W}(c_{g+1} \cdots c_3 c_2) \cdot \cdots \cdot {}_{W}(c_{2g} \cdots c_{g+2} c_{g+1}) \\ &\quad \cdot (c_{g+1} \cdots c_3 c_2) \cdot \cdots \cdot (c_{2g} \cdots c_{g+2} c_{g+1}) \quad (W := (c_1 c_2 \cdots c_g)^{-(g+1)}); \\ Q' &:= (c_1 c_2 \cdots c_{2g+1})^{g+1} \cdot d_1^2 d_2^2 \cdot {}_{W_1}(c_{g+1}) \cdot {}_{W_2}(c_{g+2}) \cdot \cdots \cdot {}_{W_{g+1}}(c_{2g+1}); \\ R' &:= d_1^2 d_2^2 \cdot {}_{W_1}(c_{g+1}) \cdot {}_{W_2}(c_{g+2}) \cdot \cdots \cdot {}_{W_{g+1}}(c_{2g+1}) \cdot (c_{2g+1} \cdots c_2 c_1)^{g+1} \\ &\quad (W_i := (c_{i+g-1} \cdots c_{i+1} c_i)^{-1} \ (i = 1, \dots, g+1)), \end{aligned}$$

which are not in $\mathcal{F}^H$. The words P, Q, and R in $\mathcal{F}$ are positive words representing $\iota, 1$, and ι in $\mathcal{M}_g$, respectively.

§4. Generalized Chakiris fibrations

In this section we study various properties of hyperelliptic Lefschetz fibrations arising from hyperelliptic Chakiris relations given in the previous section.

We denote the signature and the Euler characteristic of a compact oriented smooth 4-manifold M by $\sigma = \mathrm{Sign}(M)$ and e, respectively. It is easily seen that $e = -4(g-1) + n$ for a Lefschetz fibration $f : M \to S^2$ of genus g with n singular fibers. We often denote by n_0 (resp. n_+) the number of non-separating (resp. separating) singular fibers of f: $n = n_0 + n_+$.

Let $(c_1, \ldots, c_{2g+1})$ be a chain of length $2g+1$ on Σ_g. Suppose that each c_i is invariant under the hyperelliptic involution ι. Hence the right-handed Dehn twists $t_{c_1}, \ldots, t_{c_{2g+1}}$ belong to $\mathcal{H}_g$.

Three hyperelliptic Lefschetz fibrations $M_{I^2}, M_{C_{\mathrm{I}}}$, and $M_{C_{\mathrm{II}}} = M_{J^2}$ without separating singular fibers have been studied from various points of view (see [25], [26], [34], [14], [32], [33], [2], [30], [7], and [8]). For example, the number n of singular fibers, and the values of signature σ and the Euler characteristic e for these manifolds are calculated as in the following table.

LF	$n = n_0$	σ	e
M_{I^2}	$4(2g+1)$	$-4(g+1)$	$4(g+2)$
$M_{C_{\mathrm{I}}}$	$2(g+1)(2g+1)$	$-2(g+1)^2$	$2(2g^2+g+3)$
$M_{C_{\mathrm{II}}}$	$4g(2g+1)$	$-4g(g+1)$	$4(2g^2+1)$

$M_{I^2}, M_{C_{\mathrm{I}}}$, and $M_{C_{\mathrm{II}}} = M_{J^2}$ are known to be simply-connected, non-spin, and have a (-1)-section. M_{I^2} and $M_{C_{\mathrm{I}}}$ have transitive monodromy, whereas $M_{C_{\mathrm{II}}}$ has intransitive monodromy.

Lemma 4.1 (cf. Wajnryb [42], Lemma 21, Auroux [2], Lemma 3.4). *Three fiber sums $M_{C_{\mathrm{I}}^2} = \#_F 2M_{C_{\mathrm{I}}}$, $M_{I^{2g+2}} = \#_F (g+1) M_{I^2}$, and $M_{C_{\mathrm{II}}I^2} = M_{C_{\mathrm{II}}} \#_F M_{I^2}$ are isomorphic as Lefschetz fibrations.*

Proof. Using Corollary A.3 and applying elementary transformations repeatedly, we have

$$\begin{aligned}
C_{\mathrm{I}}^2 &\sim (c_1 c_2 \cdots c_{2g+1})^{4g+4} \sim (c_1 c_2 \cdots c_{2g+1})^{2g+2} \cdot (c_{2g+1} \cdots c_2 c_1)^{2g+2} \\
&= (c_1 c_2 \cdots c_{2g+1})^{2g+1} \cdot I \cdot (c_{2g+1} \cdots c_2 c_1)^{2g+1} \\
&\sim (c_1 c_2 \cdots c_{2g+1})^{2g+1} \cdot {}_I(c_{2g+1} \cdots c_2 c_1)^{2g+1} \cdot I \\
&= (c_1 c_2 \cdots c_{2g+1})^{2g+1} \cdot (c_{2g+1} \cdots c_2 c_1)^{2g+1} \cdot I \\
&\sim \cdots\cdots \sim I^{2g+2},
\end{aligned}$$

where we use ${}_I(c_i) = \iota(c_i) = c_i$ $(i = 1, \ldots, 2g+1)$.

We apply elementary transformations and Lemma 2.4 to obtain $JI \sim C_{\mathrm{I}}$:

$$\begin{aligned} & JI \\ = \ & J \cdot c_1 c_2 \cdots c_{2g} c_{2g+1}^2 c_{2g} \cdots c_2 c_1 \sim \ {}_J(c_1 c_2 \cdots c_{2g+1}) \cdot J \cdot c_{2g+1} \cdots c_2 c_1 \\ = \ & c_1 c_2 \cdots c_{2g+1} \cdot J \cdot c_{2g+1} \cdots c_2 c_1 \sim \ c_1 c_2 \cdots c_{2g+1} \cdot (c_1 c_2 \cdots c_{2g+1})^{2g+1} \\ = \ & C_{\mathrm{I}}. \end{aligned}$$

From this equivalence, we have

$$C_{\mathrm{I}}^2 \sim (JI)^2 = JIJI \sim J \cdot {}_I J \cdot I^2 = J^2 I^2 = C_{\mathrm{II}} \cdot I^2$$

as claimed. Q.E.D.

4.1. Even genus

Suppose that $g \geq 2$ and g is even. Let $d \in \mathcal{S}^H$ be the boundary curve of a regular neighborhood of $c_1 \cup \cdots \cup c_g$. We obtain positive relators $P^2, Q, R^2, PR, PI, PJ, RI, RJ \in \mathcal{R}^H$ and corresponding hyperelliptic Lefschetz fibrations

$$M_{P^2},\ M_Q,\ M_{R^2},\ M_{PR},\ M_{PI},\ M_{PJ},\ M_{RI},\ M_{RJ}$$

of genus g over S^2 from Corollary 3.3. These are non-spin 4-manifolds because a component of a separating singular fiber represents a homology class of square -1. Each of M_{P^2}, M_Q, M_{PI}, and M_{PJ}, which does not include the word R in its monodromy, has a smooth (-1)-section which naturally comes from C_{II} or $C_{\mathrm{I}} \sim JI$ (cf. Smith [36], Lemma 2.3). $M_Q, M_{R^2}, M_{PR}, M_{PI}, M_{RI}$, and M_{RJ} have transitive monodromy, whereas M_{P^2} and M_{PJ} have intransitive monodromy.

We first examine the fundamental groups of these manifolds.

Proposition 4.2. *The fundamental group $\pi_1(M_{P^2})$ of M_{P^2} is isomorphic to $\mathbb{Z}_2$, while the manifolds $M_Q, M_{R^2}, M_{PR}, M_{PI}, M_{PJ}, M_{RI}$, M_{RJ} are simply connected.*

Proof. We orient $c_1, c_2, \ldots, c_{2g+1}$ so that $c_i \cdot c_{i+1} = +1$ $(i = 1, \ldots, 2g)$ and take oriented simple closed curves $e_1, e_2, \ldots, e_g$ so that $\{c_2, c_4, \ldots, c_{2g}, e_1, e_2, \ldots, e_g\}$ is a symplectic basis of $H_1(\Sigma_g; \mathbb{Z})$ (i.e. $c_{2i} \cdot e_j = \delta_{ij}$, $c_{2i} \cdot c_{2j} = e_i \cdot e_j = 0$ $(i, j = 1, \ldots, g)$). Connecting these curves to a base point $*$ of Σ_g by appropriate arcs, we consider them also to be elements of $\pi_1(\Sigma_g, *)$ which satisfy $[c_2, e_1][c_4, e_2] \cdots [c_{2g}, e_g] = 1$. Namely,

$$\pi_1(\Sigma_g, *) = \langle c_2, c_4, \ldots, c_{2g}, e_1, e_2, \ldots, e_g \mid [c_2, e_1][c_4, e_2] \cdots [c_{2g}, e_g] \rangle.$$

Let $i : \Sigma_g \hookrightarrow M_\varrho$ be the inclusion map from a general fiber into the total space M_ϱ, where $\varrho = P^2, Q, R^2, PR, PI, PJ, RI, RJ$. The induced homomorphism $i_\# : \pi_1(\Sigma_g) \to \pi_1(M_\varrho)$ is surjective and the kernel of $i_\#$ includes the normal subgroup N of $\pi_1(M_\varrho)$ generated by the vanishing cycles of M_ϱ (cf. Amorós et al. [1], Lemma 3.2).

If $\varrho \neq P^2$, then M_ϱ has vanishing cycles $c_1, c_2, \ldots, c_{2g}$. We can choose arcs connecting $c_1, c_3, \ldots, c_{2g-1}$ to the base point $*$ such that $c_1 = e_1^{-1}$, $c_{2i-1} = e_{i-1}^{-1} c_{2i} e_i c_{2i}^{-1}$ $(i = 2, \ldots, g)$ as elements of $\pi_1(\Sigma_g, *)$. Thus we obtain a presentation

$$\begin{aligned}\pi_1(\Sigma_g, *)/N &= \langle c_2, c_4, \ldots, c_{2g}, e_1, e_2, \ldots, e_g \mid [c_2, e_1][c_4, e_2] \cdots [c_{2g}, e_g], \\ &\qquad c_2, c_4, \ldots, c_{2g}, e_1^{-1}, e_{i-1}^{-1} c_{2i} e_i c_{2i}^{-1} \ (i = 2, \ldots, g), \text{etc...}\rangle \\ &= \{1\}.\end{aligned}$$

Hence we have $\pi_1(M_\varrho) = \{1\}$.

If $\varrho = P^2$, the kernel of $i_\#$ coincides with N and $\pi_1(M_{P^2})$ is isomorphic to $\pi_1(\Sigma_g, *)/N$ because M_{P^2} has a smooth (-1)-section which naturally comes from a chain relation C_{2g} of length $2g$ (cf. Smith [36], Lemma 2.3, Amorós et al. [1], Lemma 3.2). Since M_{P^2} has vanishing cycles $c_2, c_3, \ldots, c_{2g}$, we obtain a presentation

$$\begin{aligned}\pi_1(\Sigma_g, *)/N' &= \langle c_2, c_4, \ldots, c_{2g}, e_1, e_2, \ldots, e_g \mid [c_2, e_1][c_4, e_2] \cdots [c_{2g}, e_g], \\ &\qquad c_2, c_4, \ldots, c_{2g}, e_{i-1}^{-1} c_{2i} e_i c_{2i}^{-1} \ (i = 2, \ldots, g)\rangle \\ &= \langle e_1 \rangle,\end{aligned}$$

where N' is the normal subgroup of $\pi_1(\Sigma_g, *)$ generated by $c_2, c_3, \ldots, c_{2g}$. Thus $\pi_1(M_\varrho)$ is cyclic because there is a natural surjective homomorphism $\pi_1(\Sigma_g, *)/N' \to \pi_1(\Sigma_g, *)/N \cong \pi_1(M_\varrho)$. The first homology group $H_1(M_{P^2}; \mathbb{Z})$ is isomorphic to $H_1(\Sigma_g; \mathbb{Z})/N_0$, where $N_0 := ND/N$ and $D := [\pi_1(\Sigma_g), \pi_1(\Sigma_g)]$. Since $c_{2i-1} = e_i - e_{i-1}\,(i = 2, \ldots, g)$, $w(c_i) = -c_i\,(i = 2, \ldots, g)$, $w(c_i) = c_i\,(i = g+2, \ldots, 2g)$, $w(c_{g+1}) = e_g + e_{g-1}$ in $H_1(\Sigma_g; \mathbb{Z})$, we have

$$H_1(M_{P^2}; \mathbb{Z}) \cong H_1(\Sigma_g; \mathbb{Z})/N_0 = \mathbb{Z}[e_1]/(2e_1) \cong \mathbb{Z}_2.$$

Hence $\pi_1(M_{P^2})$ is isomorphic to $\mathbb{Z}_2$. Q.E.D.

We next mention other invariants for our examples. The numbers n_0, n_+ of singular fibers, and the values of signature σ and the Euler characteristic e for the manifolds above are calculated as in the following table. (By virtue of Proposition 4.2, these manifolds satisfy $b_2^+ = (e + \sigma)/2 - 1$ and $b_2^- = (e - \sigma)/2 - 1$.)

LF	n_0	n_+	σ	e
M_{P^2}	$4g^2$	2	$-2(g^2+1)$	$2(2g^2-2g+3)$
M_Q	$2(g+1)^2$	1	$-(g^2+2g+3)$	$2g^2+7$
M_{R^2}	$4(g+1)^2$	2	$-2(g^2+2g+3)$	$2(2g^2+2g+5)$
M_{PR}	$2(2g^2+2g+1)$	2	$-2(g^2+g+2)$	$4(g^2+2)$
M_{PI}	$2(g+1)^2$	1	$-(g^2+2g+3)$	$2g^2+7$
M_{PJ}	$2g(3g+1)$	1	$-(3g^2+2g+1)$	$6g^2-2g+5$
M_{RI}	$2(g^2+4g+2)$	1	$-(g^2+4g+5)$	$2g^2+4g+9$
M_{RJ}	$2(3g^2+3g+1)$	1	$-(3g^2+4g+3)$	$6g^2+2g+7$

b_2^+ is always odd for these manifolds and it is greater than 1 except in the case of M_{P^2}, M_Q, and M_{PI} for $g=2$.

Remark 4.3. The values of signature of hyperelliptic Lefschetz fibrations in the table above are calculated by using the local signature formula [27], [7] or by computing signature contributions of relators in their monodromies [8]. Ozbagci's signature formula [29] and these methods are suited for explicit computation of signature of Lefschetz fibrations. For example, Hasegawa [17] and Yun [43] independently computed signatures of Gurtas' fibrations [15], [16] by using these formulae.

Remark 4.4. It is likely that M_Q and M_{PI} are isomorphic as Lefschetz fibrations although the author could not relate them by elementary transformations.

We recall a theorem of Chakiris [5].

Theorem 4.5 (Chakiris' 1/19-theorem [5], Theorem 4.9). *Let $M \to \mathbb{CP}^1$ be a holomorphic Lefschetz fibration of genus 2 with $n \geq 19n_+$.*

(1) *If the monodromy is transitive, then M is isomorphic to M_W, where $W = I^pR^qQ^rC_{\mathrm{I}}^s$ $(p,q,r,s \geq 0,\ p \equiv q \pmod 2)$. In particular, M is a fiber sum of copies of $M_{I^2}, M_{C_{\mathrm{I}}}, M_Q, M_{R^2}$, and M_{RI}.*

(2) *If the monodromy is intransitive, then M is isomorphic to M_W, where $W = P^kJ^l$ $(k,l \geq 0,\ k \equiv l \pmod 2)$. In particular, M is a fiber sum of copies of $M_{C_{\mathrm{II}}}, M_{P^2}$, and M_{PJ}.*

Although we do not give any proof of this theorem, we generalize some lemmas in [5] which were used to prove the theorem.

We define an element of $\mathcal{F}^H$ by

$$K := (d \cdot {}_{W_1}(c_{g+1}) \cdot {}_{W_2}(c_{g+2}) \cdot \cdots \cdot {}_{W_{g+1}}(c_{2g+1}))^2.$$

It immediately follows from the proofs of Lemma 3.1 for $Q_0 \in \mathcal{R}^H$ and Theorem 3.2 for $Q \in \mathcal{R}^H$ that K is a positive relator for $\mathcal{H}_g$.

Proposition 4.6 (cf. Chakiris [5], 'the second' Lemma 4.8). *Both of the hyperelliptic Lefschetz fibrations $M_{Q^2} = M_Q \#_F M_Q$ and M_{R^2} are isomorphic to a fiber sum $M_K \#_F M_{C_\mathrm{I}} = M_{KC_\mathrm{I}}$ of M_K and M_{C_I}.*

Proof. We apply elementary transformations to Q^2 as follows.

$$\begin{aligned}
Q^2 &\sim (d \cdot {}_{W_1}(c_{g+1}) \cdot {}_{W_2}(c_{g+2}) \cdot \cdots \cdot {}_{W_{g+1}}(c_{2g+1}) \cdot (c_1 c_2 \cdots c_{2g+1})^{g+1})^2 \\
&\sim (d \cdot {}_{W_1}(c_{g+1}) \cdot {}_{W_2}(c_{g+2}) \cdot \cdots \cdot {}_{W_{g+1}}(c_{2g+1}))^2 \\
&\quad \cdot (c_1 c_2 \cdots c_{2g+1})^{g+1} \cdot ({}_U(c_1) \cdot {}_U(c_2) \cdot \cdots \cdot {}_U(c_{2g+1}))^{g+1} \\
(U &:= (d \cdot {}_{W_1}(c_{g+1}) \cdot {}_{W_2}(c_{g+2}) \cdot \cdots \cdot {}_{W_{g+1}}(c_{2g+1}) \cdot (c_1 c_2 \cdots c_{2g+1})^{g+1})^{-1}) \\
&= (d \cdot {}_{W_1}(c_{g+1}) \cdot {}_{W_2}(c_{g+2}) \cdot \cdots \cdot {}_{W_{g+1}}(c_{2g+1}))^2 \cdot (c_1 c_2 \cdots c_{2g+1})^{2g+2} \\
&= K \cdot C_\mathrm{I},
\end{aligned}$$

where we use ${}_U(c_i) = \overline{U}(c_i) = c_i$ $(i = g+1, \ldots, 2g+1)$ because U is conjugate to Q^{-1} and then $\overline{U} = 1$ in $\mathcal{H}_g$.

We apply similar elementary transformations to R^2 as follows.

$$\begin{aligned}
R^2 &\sim (d \cdot {}_{W_1}(c_{g+1}) \cdot {}_{W_2}(c_{g+2}) \cdot \cdots \cdot {}_{W_{g+1}}(c_{2g+1}))^2 \\
&\quad \cdot (c_{2g+1} \cdots c_2 c_1)^{g+1} \cdot ({}_{R^{-1}}(c_{2g+1}) \cdot \cdots \cdot {}_{R^{-1}}(c_2) \cdot {}_{R^{-1}}(c_1))^{g+1} \\
&= (d \cdot {}_{W_1}(c_{g+1}) \cdot {}_{W_2}(c_{g+2}) \cdot \cdots \cdot {}_{W_{g+1}}(c_{2g+1}))^2 \cdot (c_{2g+1} \cdots c_2 c_1)^{2g+2} \\
&\sim (d \cdot {}_{W_1}(c_{g+1}) \cdot {}_{W_2}(c_{g+2}) \cdot \cdots \cdot {}_{W_{g+1}}(c_{2g+1}))^2 \cdot (c_1 c_2 \cdots c_{2g+1})^{2g+2} \\
&= K \cdot C_\mathrm{I},
\end{aligned}$$

where we use ${}_{R^{-1}}(c_i) = \overline{R}^{-1}(c_i) = \iota(c_i) = c_i$ $(i = g+1, \ldots, 2g+1)$ and Corollary A.3. Q.E.D.

Remark 4.7. It is not difficult to show that M_K is isomorphic to Cadavid-Korkmaz' generalization, which we denote by M_{CK} in [8], of Matsumoto's genus 2 Lefschetz fibration on $S^2 \times T^2 \# 4\overline{\mathbb{CP}}^2$ (see [25], Example B). Hirose told the author a combinatorial proof of this fact. It is known that $n_0 = 2g+2, n_+ = 2, \sigma = -4$, and $e = 8 - 2g$ for M_K (see [4], [21], and [8]).

Both of the hyperelliptic Lefschetz fibrations M_{PJ} and $M_{RI^{g-1}}$ have $6g^2 + 2g$ non-separating singular fibers and one separating singular fiber. From the local signature formula [27], [7], they have the same signature $\sigma = -(3g^2 + 2g + 1)$ and the same Euler characteristic $e = 6g^2 - 2g + 5$. They are simply-connected and non-spin. It follows from Freedman's classification theorem that both M_{PJ} and $M_{RI^{g-1}}$ are homeomorphic to $(3g^2/2 - 2g + 1)\mathbb{CP}^2 \# (9g^2/2 + 2)\overline{\mathbb{CP}}^2$. However they are not isomorphic as Lefschetz fibrations because the monodromy of $M_{RI^{g-1}}$ is transitive while that of M_{PJ} is intransitive.

Theorem 4.8. *If $g \geq 4$ and g is even, then $M_{PJ}, M_{RI^{g-1}}$, and $(3g^2/2 - 2g + 1)\mathbb{CP}^2 \,\#(9g^2/2 + 2)\overline{\mathbb{CP}}^2$ are mutually non-diffeomorphic.*

Proof. We first note that $b_2^+ = 3g^2/2 - 2g + 1 > 1$ and odd for these manifolds. If $g \geq 4$, $M_{RI^{g-1}}$ is isomorphic to a non-trivial fiber sum $M_{RI} \#_F (g/2 - 1) M_{I^2}$. It follows from a theorem of Usher [41] that $M_{RI^{g-1}}$ is a minimal symplectic 4-manifold. Since $b_2^+ > 1$, $M_{RI^{g-1}}$ does not contain any smooth (-1)-sphere as a consequence of Seiberg-Witten theory [39], [40], [22] (cf. [14], Remark 10.2.4(a)). On the other hand, M_{PJ} has a smooth (-1)-section which naturally comes from a chain relation C_{2g} of length $2g$ (cf. Smith [36], Lemma 2.3). Hence M_{PJ} and $M_{RI^{g-1}}$ can not be diffeomorphic.

By Gompf's theorem ([14], Theorem 10.2.18) M_{PJ} (resp. $M_{RI^{g-1}}$) admits a symplectic structure ω_{PJ} (resp. $\omega_{RI^{g-1}}$). It follows from results of Taubes [38] (cf. [14], Theorem 10.1.11) that the classes $\pm c_1(M_{PJ}, \omega_{PJ})$ (resp. $\pm c_1(M_{RI^{g-1}}, \omega_{RI^{g-1}})$) are Seiberg-Witten basic classes. On the other hand, the 4-manifold $(3g^2/2 - 2g + 1)\mathbb{CP}^2 \#(9g^2/2 + 2)\overline{\mathbb{CP}}^2$ has vanishing Seiberg-Witten invariants because it decomposes as the connected sum of $(3g^2/2 - 2g)\mathbb{CP}^2$ and $\mathbb{CP}^2 \#(9g^2/2 + 2)\overline{\mathbb{CP}}^2$ [23] (cf. [14], Theorem 2.4.6). Hence M_{PJ} (resp. $M_{RI^{g-1}}$) is not diffeomorphic to $(3g^2/2 - 2g + 1)\mathbb{CP}^2 \#(9g^2/2 + 2)\overline{\mathbb{CP}}^2$. Q.E.D.

Remark 4.9. Theorem 4.8 is a variant of Fuller's theorem [11], which states that $\#_F g M_{I^2}, M_{C_{\rm II}}$, and $(2g^2 - 2g + 1)\mathbb{CP}^2 \#(6g^2 + 2g + 1)\overline{\mathbb{CP}}^2$ are homeomorphic but mutually non-diffeomorphic for every $g \geq 2$ (see also [25], [13], and [7]). Fuller's theorem can be reproved by the same method as the proof of Theorem 4.8 without using Kirby calculus.

In contrast to Theorem 4.8 we show the following theorem about fiber sums.

Theorem 4.10 (cf. Chakiris [5], 'the first' Lemma 4.8)**.** *The fiber sum $M_{PJI^2} = M_{PJ} \#_F M_{I^2}$ is isomorphic to the fiber sum $M_{RI^{g+1}} = M_{RI^{g-1}} \#_F M_{I^2}$ as Lefschetz fibrations for every even $g \geq 2$. In particular, these manifolds are diffeomorphic to each other.*

We postpone the proof of this theorem to Appendix B.

Remark 4.11. Theorem 4.10 is a variant of Lemma 4.1. Such kinds of stability for hyperelliptic Lefschetz fibrations under taking fiber sums with copies of M_{I^2} were formulated by Auroux [2] and Kharlamov and Kulikov [20].

If $g = 2$, the manifolds M_{PJ}, M_{RI}, and $K3\#\overline{\mathbb{CP}}^2$ are homeomorphic to $3\mathbb{CP}^2\#20\overline{\mathbb{CP}}^2$ by Freedman's classification theorem. The next theorem was suggested by the referee.

Theorem 4.12. *If $g = 2$, then M_{PJ}, $K3\#\overline{\mathbb{CP}}^2$, and $3\mathbb{CP}^2\#20\overline{\mathbb{CP}}^2$ are mutually non-diffeomorphic.*

Proof. $0 \in H^2(K3;\mathbb{Z})$ is the only Seiberg-Witten basic class of $K3$ [10] (cf. [14], Corollary 3.1.15). The blowup formula for Seiberg-Witten invariants [9] (cf. [14], Theorem 2.4.9) tells us that the only basic classes of $K3\#\overline{\mathbb{CP}}^2$ are $\pm E$, where $E \in H^2(K3\#\overline{\mathbb{CP}}^2;\mathbb{Z})$ is the Poincaré dual of the homology class of the exceptional sphere. On the other hand, consider M_{PJ} and its symplectic structure ω_{PJ}. The classical adjunction formula implies that the canonical class $K_{PJ} = -c_1(M_{PJ},\omega_{PJ})$ of the symplectic manifold (M_{PJ},ω_{PJ}) satisfies $K_{PJ} \cdot F = 2$, where $F \in H^2(M_{PJ};\mathbb{Z})$ is the Poincaré dual of the homology class of the fiber. Moreover, as observed in the proof of Theorem 4.8, M_{PJ} has a smooth (-1)-section and we have $S \cdot F = 1$, where $S \in H^2(M_{PJ};\mathbb{Z})$ is the Poincaré dual of the homology class of the section. Taubes' result [38] (cf. [14], Theorem 10.1.11) shows that $\pm K_{PJ}$ are basic classes for M_{PJ}. But the blowup formula shows that there must be other basic classes $\pm(K_{PJ} - 2S)$ which are distinct from $\pm K_{PJ}$ since they pair trivially with F (see [14], Exercise 10.1.20). Therefore M_{PJ} is not diffeomorphic to $K3\#\overline{\mathbb{CP}}^2$.

The manifold $3\mathbb{CP}^2\#20\overline{\mathbb{CP}}^2$ has vanishing Seiberg-Witten invariants for the same reason as the proof of Theorem 4.8. Hence this manifold is diffeomorphic neither to M_{PJ} nor to $K3\#\overline{\mathbb{CP}}^2$. Q.E.D.

We notice that M_{RI} for $g = 2$ is not diffeomorphic to $3\mathbb{CP}^2\#20\overline{\mathbb{CP}}^2$, but we can not distinguish M_{RI} from other two manifolds.

Problem 4.13. *Determine whether M_{PJ} and M_{RI} are diffeomorphic or not when $g = 2$. Is M_{RI} diffeomorphic to $K3\#\overline{\mathbb{CP}}^2$?*

Remark 4.14. Sato [30] listed the pairs (n_0, n_+) of numbers of singular fibers possibly realized by some genus 2 Lefschetz fibration with (-1)-sphere. Hirose [18] has constructed examples of genus 2 Lefschetz fibrations with (-1)-sphere which actually realize the pairs $(16,2), (18,1)$, and $(28,1)$. If $g = 2$, the pairs (n_0, n_+) of numbers of singular fibers of the Lefschetz fibrations $M_{P^2}, M_Q, M_{PR}, M_{PI}, M_{PJ}, M_{RI}$ are $(16,2), (18,1), (26,2), (18,1), (28,1), (28,1)$, respectively. M_{P^2}, M_Q, M_{PI}, M_{PJ} also realize three pairs (n_0, n_+) in Sato's table ([30], Table 1) because they contain (-1)-spheres. M_{RI} for $g = 2$ turns out to be isomorphic to Auroux's fibration X_2 in [2] (see Appendix B). Sato told the author that X_2 admits no (-1)-section but contains a (-1)-sphere as a 'double section'. Hence M_{RI} also realizes the pair $(28, 1)$.

Remark 4.15. Usher's theorem [41], which is used in the proof of Theorem 4.8 and is an affirmative solution to a conjecture of Stipsicz [37], is proved also by Sato [31] when $g = 2$.

4.2. Odd genus

Suppose that $g \geq 3$ and g is odd. Let $d \in \mathcal{S}^H$ be the boundary curve of a regular neighborhood of $c_1 \cup \cdots \cup c_{g-1}$. We obtain positive relators $Q, R \in \mathcal{R}^H$ by Theorem 3.5. We also have positive relators for $\mathcal{H}_g$ defined by

$$\begin{aligned} K_1 :=& (c_1 c_2 \cdots c_{2g+1} \cdot d \cdot (c_{g-1} \cdots c_2 c_1)^2 \\ & \cdot w_1(c_g) \cdot w_2(c_{g+1}) \cdots w_{g+2}(c_{2g+1}))^2, \\ K_2 :=& (c_1 c_2 \cdots c_{2g+1})^2 (d \cdot (c_{g-1} \cdots c_2 c_1)^2 \\ & \cdot w_1(c_g) \cdot w_2(c_{g+1}) \cdots w_{g+2}(c_{2g+1}))^2, \end{aligned}$$

which are constructed in the same way as K for even genus (see the proofs of Lemma 3.4 for $Q_0 \in \mathcal{R}^H$ and Theorem 3.5 for $Q \in \mathcal{R}^H$).

Thus we obtain the corresponding hyperelliptic Lefschetz fibrations

$$M_Q,\ M_R,\ M_{K_1},\ M_{K_2}$$

of genus g over S^2 with transitive monodromy. These are non-spin 4-manifolds because a component of a separating singular fiber represents a homology class of square -1. Each of M_Q, M_{K_1}, and M_{K_2} has a smooth (-1)-section which naturally comes from C_{I} (cf. Smith [36], Lemma 2.3).

Proposition 4.16. *The manifolds M_Q, M_R, M_{K_1}, and M_{K_2} are simply connected.*

Proof. Quite similar to the proof of Proposition 4.2. Q.E.D.

The numbers n_0, n_+ of singular fibers, and the values of signature σ and the Euler characteristic e for the manifolds above are calculated as in the following table. (By virtue of Proposition 4.16, these manifolds satisfy $b_2^+ = (e + \sigma)/2 - 1$ and $b_2^- = (e - \sigma)/2 - 1$.)

LF	n_0	n_+	σ	e
M_Q and M_R	$2(g^2 + 4g + 1)$	1	$-(g + 2)^2$	$2g^2 + 4g + 7$
M_{K_1} and M_{K_2}	$2(5g + 1)$	2	$-2(2g + 3)$	$2(3g + 4)$

The values of signature of hyperelliptic Lefschetz fibrations in the table above are calculated by using formulae in [27], [7], or [8]. b_2^+ is always odd for these manifolds and it is greater than 1.

Proposition 4.17. *The four fiber sums $M_{Q^2} = M_Q \#_F M_Q$, $M_{R^2} = M_R \#_F M_R$, $M_{K_i C_{\mathrm{I}}} = M_{K_i} \#_F M_{C_{\mathrm{I}}}$ $(i = 1, 2)$ of copies of hyperelliptic Lefschetz fibrations M_Q, M_R, $M_{C_{\mathrm{I}}}$, M_{K_i} $(i = 1, 2)$ are isomorphic to each other.*

Proof. We apply elementary transformations to Q^2 as follows.

$$\begin{aligned}
Q^2 &\sim (c_1 c_2 \cdots c_{2g+1} \cdot d \cdot (c_{g-1} \cdots c_2 c_1)^2 \\
&\qquad \cdot {}_{W_1}(c_g) \cdot {}_{W_2}(c_{g+1}) \cdot \cdots \cdot {}_{W_{g+2}}(c_{2g+1}) \cdot (c_1 c_2 \cdots c_{2g+1})^{g+1})^2 \\
&\sim (c_1 c_2 \cdots c_{2g+1} \cdot d \cdot (c_{g-1} \cdots c_2 c_1)^2 \\
&\qquad \cdot {}_{W_1}(c_g) \cdot {}_{W_2}(c_{g+1}) \cdot \cdots \cdot {}_{W_{g+2}}(c_{2g+1}))^2 \\
&\qquad \cdot (c_1 c_2 \cdots c_{2g+1})^{g+1} \cdot ({}_U(c_1) \cdot {}_U(c_2) \cdot \cdots \cdot {}_U(c_{2g+1}))^{g+1} \\
(U &:= (c_1 c_2 \cdots c_{2g+1} \cdot d \cdot (c_{g-1} \cdots c_2 c_1)^2 \\
&\qquad \cdot {}_{W_1}(c_g) \cdot {}_{W_2}(c_{g+1}) \cdot \cdots \cdot {}_{W_{g+2}}(c_{2g+1}) \cdot (c_1 c_2 \cdots c_{2g+1})^{g+1})^{-1}) \\
&= (c_1 c_2 \cdots c_{2g+1} \cdot d \cdot (c_{g-1} \cdots c_2 c_1)^2 \cdot {}_{W_1}(c_g) \cdot {}_{W_2}(c_{g+1}) \cdot \cdots \\
&\qquad \cdot {}_{W_{g+2}}(c_{2g+1}))^2 \cdot (c_1 c_2 \cdots c_{2g+1})^{2g+2} \\
&= K_1 \cdot C_{\mathrm{I}},
\end{aligned}$$

where we use ${}_U(c_i) = \overline{U}(c_i) = c_i$ $(i = 1, \ldots, 2g+1)$ because U is conjugate to Q^{-1} and then $\overline{U} = 1$ in $\mathcal{H}_g$.

$$\begin{aligned}
Q^2 &\sim (d \cdot (c_{g-1} \cdots c_2 c_1)^2 \\
&\qquad \cdot {}_{W_1}(c_g) \cdot {}_{W_2}(c_{g+1}) \cdot \cdots \cdot {}_{W_{g+2}}(c_{2g+1}) \cdot (c_1 c_2 \cdots c_{2g+1})^{g+2})^2 \\
&\sim (d \cdot (c_{g-1} \cdots c_2 c_1)^2 \cdot {}_{W_1}(c_g) \cdot {}_{W_2}(c_{g+1}) \cdot \cdots \cdot {}_{W_{g+2}}(c_{2g+1}))^2 \\
&\qquad \cdot (c_1 c_2 \cdots c_{2g+1})^{g+2} \cdot ({}_V(c_1) \cdot {}_V(c_2) \cdot \cdots \cdot {}_V(c_{2g+1}))^{g+2} \\
(V &:= (d \cdot (c_{g-1} \cdots c_2 c_1)^2 \\
&\qquad \cdot {}_{W_1}(c_g) \cdot {}_{W_2}(c_{g+1}) \cdot \cdots \cdot {}_{W_{g+2}}(c_{2g+1}) \cdot (c_1 c_2 \cdots c_{2g+1})^{g+2})^{-1}) \\
&= (d \cdot (c_{g-1} \cdots c_2 c_1)^2 \cdot {}_{W_1}(c_g) \cdot {}_{W_2}(c_{g+1}) \cdot \cdots \cdot {}_{W_{g+2}}(c_{2g+1}))^2 \\
&\qquad \cdot (c_1 c_2 \cdots c_{2g+1})^{2g+4} \\
&\sim K_2 \cdot C_{\mathrm{I}},
\end{aligned}$$

where we use ${}_V(c_i) = \overline{V}(c_i) = c_i$ $(i = 1, \ldots, 2g+1)$ because V is conjugate to Q^{-1} and then $\overline{V} = 1$ in $\mathcal{H}_g$.

We apply similar elementary transformations to R^2 as follows.

$$\begin{aligned} R^2 = & \ (c_1 c_2 \cdots c_{2g+1} \cdot d \cdot (c_{g-1} \cdots c_2 c_1)^2 \\ & \cdot {}_{W_1}(c_g) \cdot {}_{W_2}(c_{g+1}) \cdot \cdots \cdot {}_{W_{g+2}}(c_{2g+1}) \cdot (c_{2g+1} \cdots c_2 c_1)^{g+1})^2 \\ \sim & \ (c_1 c_2 \cdots c_{2g+1} \cdot d \cdot (c_{g-1} \cdots c_2 c_1)^2 \\ & \cdot {}_{W_1}(c_g) \cdot {}_{W_2}(c_{g+1})) \cdot \cdots \cdot {}_{W_{g+2}}(c_{2g+1}))^2 \\ & \cdot (c_{2g+1} \cdots c_2 c_1)^{g+1} \cdot ({}_{R^{-1}}(c_{2g+1}) \cdot \cdots \cdot {}_{R^{-1}}(c_2) \cdot {}_{R^{-1}}(c_1))^{g+1} \\ = & \ (c_1 c_2 \cdots c_{2g+1} \cdot d \cdot (c_{g-1} \cdots c_2 c_1)^2 \\ & \cdot {}_{W_1}(c_g) \cdot {}_{W_2}(c_{g+1})) \cdot \cdots \cdot {}_{W_{g+2}}(c_{2g+1}))^2 \cdot (c_{2g+1} \cdots c_2 c_1)^{2g+2} \\ = & \ (c_1 c_2 \cdots c_{2g+1} \cdot d \cdot (c_{g-1} \cdots c_2 c_1)^2 \\ & \cdot {}_{W_1}(c_g) \cdot {}_{W_2}(c_{g+1})) \cdot \cdots \cdot {}_{W_{g+2}}(c_{2g+1}))^2 \cdot (c_1 c_2 \cdots c_{2g+1})^{2g+2} \\ = & \ K_1 \cdot C_{\mathrm{I}}, \end{aligned}$$

where we use ${}_{R^{-1}}(c_i) = \overline{R}^{-1}(c_i) = c_i$ $(i = 1, \ldots, 2g+1)$ and Corollary A.3. Q.E.D.

Remark 4.18. The author does not know any explicit examples of hyperelliptic Lefschetz fibrations of odd genus with separating singular fibers other than $M_Q, M_R, M_{K_1}, M_{K_2}$, and fiber sums of their copies. M_Q (resp. M_{K_1}) might not be isomorphic to M_R (resp. M_{K_2}) although the author does not know any invariants which distinguish these fibrations.

Let $d_1, d_2 \in \mathcal{S}^H$ be the boundary curves of a regular neighborhood of $c_1 \cup \cdots \cup c_g$.

Remark 4.19. Substituting the same word as Q_0 for even genus by the inverse C_g^{-1} of a chain relator C_g twice, we have the following positive word for odd genus:

$$K' := (d_1^2 d_2^2 \cdot {}_{W_1}(c_{g+1}) \cdot {}_{W_2}(c_{g+2}) \cdot \cdots \cdot {}_{W_{g+1}}(c_{2g+1}))^2$$

which is not in $\mathcal{F}^H$ (see Remark 3.6). This is a positive relator in $\mathcal{M}_g$ and the corresponding Lefschetz fibration $M_{K'}$ is nothing but Cadavid-Korkmaz' fibration for odd genus, which we denote by M_{CK} in [8]. It is known that $n_0 = 2g+10, n_+ = 0, \sigma = -8$, and $e = 14 - 2g$ for M_K (see [4], [21], and [8]).

§5. Concluding remarks

Hyperelliptic Lefschetz fibrations form a very special and beautiful class of Lefschetz fibrations. But it seems that there is much room to be studied. For example, a famous conjecture of Siebert and Tian [32] for hyperelliptic Lefschetz fibrations without separating singular fibers is only partially solved in genus 2 case. Moreover it is not clear whether all hyperelliptic Lefschetz fibrations over the 2-sphere have been discovered.

On the other hand, hyperelliptic Lefschetz fibrations are rich enough to include many explicit examples with interesting properties. For example, generalizations of Matsumoto's fibrations [25] and Chakiris' fibrations [5] give examples of homeomorphic but non-diffeomorphic 4-manifolds which become diffeomorphic after taking fiber sums with only one copy of M_{I^2} (see Remark 4.9, Remark 4.11, Theorem 4.8, and Theorem 4.10). These examples together with stabilization theorems of Auroux [2] and Kharlamov and Kulikov [20] seem to be 'fiber sum analogues' of 4-manifolds which dissolve after taking connected sums with only one copy of $S^2 \times S^2$ together with Wall's stabilization theorem for connected sums of simply-connected 4-manifolds with copies of $S^2 \times S^2$.

If hyperelliptic Lefschetz fibrations are investigated very well, they would be recognized as new fundamental 4-manifolds and might play interesting roles such as elliptic surfaces in 4-manifold topology.

§Appendix A. A reversing lemma

Let $(c_1, c_2, \ldots, c_n)$ be a chain of length n on Σ_g and W_1, W_2 positive words in the generators $c_1, c_2, \ldots, c_n \in \mathcal{S}$. We write $W_1 \approx W_2$ if W_1 can be transformed into W_2 by replacing $c_i c_{i+1} c_i$ with $c_{i+1} c_i c_{i+1}$, $c_{i+1} c_i c_{i+1}$ with $c_i c_{i+1} c_i$ for $i = 1, \ldots, n-1$, and $c_i c_j$ with $c_j c_i$ for $|i-j| > 1$ repeatedly. This relation is an equivalence relation on the set of positive words in the generators $c_1, c_2, \ldots, c_n$. It is not difficult to verify that Lemma 2.4 is true even if we replace $\equiv \pmod{T_0, T_1}$ with $\approx$.

Lemma A.1 (Chakiris [5], Lemma 3.5). *The following equivalence holds.*

$$(c_1 c_2 \cdots c_n)^{n+1} \approx (c_n \cdots c_2 c_1)^{n+1} \quad (n = 1, \ldots, 2g).$$

Proof. We set $W_1 := (c_1 c_2 \cdots c_n)^{n+1}$ and $W_2 := (c_n \cdots c_2 c_1)^{n+1}$ for $n = 1, \ldots, 2g$. We prove $W_1 \approx W_2$ by induction on n.

If $n = 2$, then W_1 is transformed into W_2 as follows:

$$W_1 = (c_1 c_2)^3 = c_1 c_2 c_1 \cdot c_2 c_1 c_2 \approx c_2 c_1 c_2 \cdot c_1 c_2 c_1 = (c_2 c_1)^3 = W_2.$$

Suppose that $W_1 \approx W_2$ is valid for $n-1$:

$$(c_1 c_2 \cdots c_{n-1})^n \approx (c_{n-1} \cdots c_2 c_1)^n.$$

We consider W_1 and W_2 for n. Applying Lemma 2.4 for $\approx$ and using the assumption, we have

$$\begin{aligned} W_1 = (c_1c_2\cdots c_n)^{n+1} &= (c_1c_2\cdots c_n)^n \cdot c_1c_2\cdots c_n \\ &\approx (c_1c_2\cdots c_{n-1})^n \cdot c_n\cdots c_2c_1 \cdot c_1c_2\cdots c_n \\ &\approx (c_{n-1}\cdots c_2c_1)^n \cdot c_n\cdots c_2c_1 \cdot c_1c_2\cdots c_n. \end{aligned}$$

Manipulating braid relations as Lemma 2.1 of [21], we transform the right-hand side as follows:

$$\begin{aligned} &(c_{n-1}\cdots c_2c_1)^n \cdot c_n\cdots c_2c_1 \cdot c_1c_2\cdots c_n \\ &\approx c_n\cdots c_2c_1 \cdot (c_n\cdots c_3c_2)^n \cdot c_1c_2\cdots c_n \\ &= c_n\cdots c_2c_1 \cdot (c_n\cdots c_3c_2)^{n-1} \cdot c_n\cdots c_2c_1 \cdot c_2\cdots c_n \\ &\approx c_n\cdots c_2c_1 \cdot (c_n\cdots c_3c_2)^{n-2} \cdot (c_n\cdots c_2c_1)^2 \cdot c_3\cdots c_n \\ &\approx \cdots\cdots \\ &\approx c_n\cdots c_2c_1 \cdot c_n\cdots c_3c_2 \cdot (c_n\cdots c_2c_1)^{n-1} \cdot c_n \\ &\approx (c_n\cdots c_2c_1)^{n+1} = W_2. \end{aligned}$$

We have thus shown $W_1 \approx W_2$ as claimed. Q.E.D.

Let $\varrho \in \mathcal{R}$ be a relator including $W_1 = (c_1c_2\cdots c_n)^{n+1}$ as a subword: $\varrho = UW_1V$ $(U, V \in \mathcal{F})$. We put $\varrho' := UW_2V \in \mathcal{F}$, where $W_2 = (c_n\cdots c_2c_1)^{n+1}$.

Corollary A.2. *The word ϱ' is also a relator in $\mathcal{R}$ and $\varrho \equiv \varrho'$* (mod T_0, T_1).

Proof. It immediately follows from the definition of $\approx$ that $W_1 \approx W_2$ implies $\varrho \equiv \varrho'$ (mod T_0, T_1) and then $\varrho' \in \mathcal{R}$. Q.E.D.

We set $W_1 = (c_1c_2\cdots c_n)^{n+1}$ and $W_2 = (c_n\cdots c_2c_1)^{n+1}$ for $n = 1, \dots, 2g$ again. Let $\varrho \in \mathcal{R}$ be a positive relator including W_1 as a subword: $\varrho = UW_1V$ ($U, V \in \mathcal{F}$ and U, V are positive). We put $\varrho' := UW_2V \in \mathcal{F}$.

Corollary A.3. *The word ϱ' is also a positive relator in $\mathcal{R}$ and $\varrho \sim \varrho'$.*

Proof. The word ϱ' is obviously positive. From Corollary A.2, $\varrho \equiv \varrho'$ (mod T_0, T_1) and then $\varrho' \in \mathcal{R}$. We can show that $W_1 \approx W_2$ implies $\varrho \sim \varrho'$ because

$$\begin{aligned} &\cdots\cdot c_i \cdot c_{i+1} \cdot c_i \cdot\cdots \sim \cdots\cdot c_i \cdot {}_{c_{i+1}}(c_i) \cdot c_{i+1} \cdot\cdots \\ \sim &\cdots\cdot {}_{c_1c_{i+1}}(c_i) \cdot c_i \cdot c_{i+1} \cdot\cdots = \cdots\cdot c_{i+1} \cdot c_i \cdot c_{i+1} \cdot\cdots \end{aligned}$$

for $i = 1, \ldots, n-1$ and

$$\cdots \cdot c_i \cdot c_j \cdot \cdots \sim \cdots \cdot {}_{c_j}(c_i) \cdot c_i \cdot \cdots = \cdots \cdot c_j \cdot c_i \cdot \cdots$$

for $|i-j| > 1$ by braid relations. Q.E.D.

§Appendix B. Proof of Theorem 4.10

It is easy to see that the image of

$$H := c_{2g+1}c_{2g} \cdots c_2 c_1^2 c_2 \cdots c_{2g} c_{2g+1} \ \in \mathcal{F}^H$$

under ϖ represents the hyperelliptic involution ι.

We need the following lemma to show that $PJI^2 \sim RI^{g+1}$.

Lemma B.1 (cf. Chakiris [5], Lemma 4.7). *The equivalence*

$$P \cdot {}_{c_{2g+1}^{-1}} J \sim RH^{g-1}$$

holds for every even $g \geq 2$.

Proof. We first notice that $P \cdot {}_{c_{2g+1}^{-1}} J$ and RH^{g-1} belong to $\mathcal{R}^H$ because $\varpi(P \cdot {}_{c_{2g+1}^{-1}} J) = \iota t_{c_{2g+1}}^{-1} \iota t_{c_{2g+1}} = 1$ and $\varpi(RH^{g-1}) = \iota^g = 1$.

We use such elementary transformations as in the proof of Corollary A.3 and cyclic permutations, which are expressed as compositions of elementary transformations, repeatedly in this proof. Using Corollary A.3, Lemma 2.4, and manipulation of braid relations as Lemma 2.1 of [21], we obtain the following long sequence of elementary transformations.

$$\begin{aligned}
&P \cdot {}_{c_{2g+1}^{-1}} J \\
&\sim {}_{c_{2g+1}^{-1}} J \cdot P = {}_{c_{2g+1}^{-1}} (c_1 c_2 \cdots c_{2g})^{2g+1} \cdot P \sim {}_{c_{2g+1}^{-1}} (c_{2g} \cdots c_2 c_1)^{2g+1} \cdot P \\
&\sim {}_{c_{2g+1}^{-1}} ((c_{2g} \cdots c_2 c_1)^g \cdot (c_g c_{g-1} \cdots c_{2g}) \cdot \\
&\qquad\qquad \cdots \cdot (c_1 c_2 \cdots c_{g+1}) \cdot (c_g \cdots c_2 c_1)^{g+1}) \cdot P \\
&= {}_{c_{2g+1}^{-1}} ((c_{2g} \cdots c_2 c_1)^g \cdot (c_g c_{g-1} \cdots c_{2g}) \cdot \\
&\qquad\qquad \cdots \cdot (c_1 c_2 \cdots c_{g+1})) \cdot (c_g \cdots c_2 c_1)^{g+1} \cdot P \\
&\sim {}_{c_{2g+1}^{-1}} ((c_{2g} \cdots c_2 c_1)^g \cdot (c_g c_{g-1} \cdots c_{2g}) \cdot \cdots \cdot (c_1 c_2 \cdots c_{g+1})) \cdot W^{-1} \cdot P \\
&= {}_{c_{2g+1}^{-1}} ((c_{2g} \cdots c_2 c_1)^g \cdot (c_g c_{g-1} \cdots c_{2g}) \cdot \cdots \cdot (c_1 c_2 \cdots c_{g+1})) \cdot W^{-1} \\
&\quad \cdot d \cdot {}_W(c_{g+1} \cdots c_3 c_2) \cdot \cdots \cdot {}_W(c_{2g} \cdots c_{g+2} c_{g+1}) \\
&\quad \cdot (c_{g+1} \cdots c_3 c_2) \cdot \cdots \cdot (c_{2g} \cdots c_{g+2} c_{g+1}) \\
&\sim {}_{c_{2g+1}^{-1}} ((c_{2g} \cdots c_2 c_1)^g \cdot (c_g c_{g-1} \cdots c_{2g}) \cdot \cdots \cdot (c_1 c_2 \cdots c_{g+1})) \cdot W^{-1}
\end{aligned}$$

$$
\begin{aligned}
&\cdot {}_{W}(c_{g+1}\cdots c_3c_2)\cdot\cdots\cdot {}_{W}(c_{2g}\cdots c_{g+2}c_{g+1})\\
&\cdot (c_{g+1}\cdots c_3c_2)\cdot\cdots\cdot(c_{2g}\cdots c_{g+2}c_{g+1})\cdot {}_{(d^{-1}P)^{-1}}(d)\\
\sim\ & {}_{c_{2g+1}^{-1}}((c_{2g}\cdots c_2c_1)^g\cdot(c_gc_{g-1}\cdots c_{2g})\cdot\cdots\cdot(c_1c_2\cdots c_{g+1}))\\
&\cdot (c_{g+1}\cdots c_3c_2)\cdot\cdots\cdot(c_{2g}\cdots c_{g+2}c_{g+1})\cdot W^{-1}\\
&\cdot (c_{g+1}\cdots c_3c_2)\cdot\cdots\cdot(c_{2g}\cdots c_{g+2}c_{g+1})\cdot d\\
\sim\ & ({}_{c_{2g}}c_{2g+1}c_{2g-1}\cdots c_2c_1)^g\\
&\cdot (c_gc_{g-1}\cdots c_{2g-1}{}_{c_{2g}}c_{2g+1})\cdot(c_{g-1}c_{g-2}\cdots c_{2g})\cdot\cdots\cdot(c_1c_2\cdots c_{g+1})\\
&\cdot (c_{g+1}\cdots c_3c_2)\cdot\cdots\cdot(c_{2g}\cdots c_{g+2}c_{g+1})\cdot(c_1c_2\cdots c_g)^{g+1}\\
&\cdot (c_{g+1}\cdots c_3c_2)\cdot\cdots\cdot(c_{2g}\cdots c_{g+2}c_{g+1})\cdot d\\
&\qquad (\text{n.b. } {}_{c_{2g+1}^{-1}}c_{2g} = {}_{c_{2g}}c_{2g+1})\\
\sim\ & ({}_{c_{2g}}c_{2g+1}c_{2g-1}\cdots c_2c_1)^g\\
&\cdot (c_gc_{g-1}\cdots c_{2g-1}{}_{c_{2g}}c_{2g+1})\cdot(c_{g-1}c_{g-2}\cdots c_{2g})\cdot\cdots\cdot(c_1c_2\cdots c_{g+1})\\
&\cdot (c_{g+1}\cdots c_2c_1)\cdot(c_{g+2}\cdots c_2c_1)\cdot\cdots\cdot(c_{2g}\cdots c_2c_1)\\
&\cdot c_g\cdot(c_{g-1}c_g)\cdot(c_{g-2}c_{g-1}c_g)\cdot\cdots\cdot(c_1c_2\cdots c_g)\\
&\cdot (c_{g+1}\cdots c_3c_2)\cdot\cdots\cdot(c_{2g}\cdots c_{g+2}c_{g+1})\cdot d\\
\sim\ & ({}_{c_{2g}}c_{2g+1}c_{2g-1}\cdots c_2c_1)^g\\
&\cdot (c_gc_{g-1}\cdots c_{2g-1}{}_{c_{2g}}c_{2g+1})\cdot(c_{g-1}c_{g-2}\cdots c_{2g})\cdot\cdots\cdot(c_1c_2\cdots c_{g+1})\\
&\cdot (c_{g+1}\cdots c_2c_1)\cdot(c_{g+2}\cdots c_2c_1)\cdot\cdots\cdot(c_{2g}\cdots c_2c_1)\\
&\cdot c_g\cdot(c_{g-1}c_g)\cdot\cdots\cdot(c_2c_3\cdots c_g)\cdot(c_1c_2\cdots c_g)\\
&\cdot (c_{g+1}c_{g+2}\cdots c_{2g})\cdot\cdots\cdot(c_3c_4\cdots c_{g+2})\cdot(c_2c_3\cdots c_{g+1})\cdot d\\
\sim\ & ({}_{c_{2g}}c_{2g+1}c_{2g-1}\cdots c_2c_1)^g\\
&\cdot (c_gc_{g-1}\cdots c_{2g-1}{}_{c_{2g}}c_{2g+1})\cdot(c_{g-1}c_{g-2}\cdots c_{2g})\cdot\cdots\cdot(c_1c_2\cdots c_{g+1})\\
&\cdot (c_{g+1}\cdots c_2c_1)\cdot(c_{g+2}\cdots c_2c_1)\cdot\cdots\cdot(c_{2g}\cdots c_2c_1)\\
&\cdot (c_1c_2\cdots c_{2g})\cdot\cdots\cdot(c_1c_2\cdots c_{g+2})\cdot(c_1c_2\cdots c_{g+1})\cdot d\\
\sim\ & ({}_{c_{2g}}c_{2g+1}c_{2g-1}\cdots c_2c_1)^g\\
&\cdot (c_gc_{g-1}\cdots c_{2g-1}{}_{c_{2g}}c_{2g+1})\cdot(c_{g-1}c_{g-2}\cdots c_{2g})\cdot\cdots\cdot(c_1c_2\cdots c_{g+1})\\
&\cdot (c_{2g}\cdots c_2c_1^2c_2\cdots c_{2g})\cdot\cdots\cdot(c_{g+2}\cdots c_2c_1^2c_1\cdots c_{g+2})\\
&\cdot (c_{g+1}\cdots c_2c_1^2c_1\cdots c_{g+1})\cdot d\\
\sim\ & ({}_{c_{2g}}c_{2g+1}c_{2g-1}\cdots c_2c_1)^g\\
&\cdot (c_gc_{g-1}\cdots c_{2g-1}{}_{c_{2g}}c_{2g+1})\cdot(c_{g-1}c_{g-2}\cdots c_{2g})\cdot\cdots\cdot(c_1c_2\cdots c_{g+2})\\
&\cdot c_{g+1}\cdots c_2c_1^2c_2\cdots c_{2g}\cdot(c_{2g-1}\cdots c_2c_1^2c_2\cdots c_{2g-1})
\end{aligned}
$$

$$\cdots\cdot(c_{g+2}\cdots c_2c_1^2c_1\cdots c_{g+2})\cdot(c_{g+1}\cdots c_2c_1^2c_1\cdots c_{g+1})\cdot d$$

$$\sim ({}_{c_{2g}}c_{2g+1}c_{2g-1}\cdots c_2c_1)^g$$

$$\cdot(c_gc_{g-1}\cdots c_{2g-1\,c_{2g}}c_{2g+1}c_{2g})\cdot(c_{g-1}c_{g-2}\cdots c_{2g})\cdot\cdots\cdot(c_1c_2\cdots c_{g+2})$$

$$\cdot c_g\cdots c_2c_1^2c_2\cdots c_{2g}\cdot(c_{2g-1}\cdots c_2c_1^2c_2\cdots c_{2g-1})$$

$$\cdots\cdot(c_{g+2}\cdots c_2c_1^2c_1\cdots c_{g+2})\cdot(c_{g+1}\cdots c_2c_1^2c_1\cdots c_{g+1})\cdot d$$

$$\sim ({}_{c_{2g}}c_{2g+1}c_{2g-1}\cdots c_2c_1)^g\cdot(c_gc_{g-1}\cdots c_{2g}c_{2g+1})\cdot(c_{g-1}c_{g-2}\cdots c_{2g})$$

$$\cdots\cdot(c_1c_2\cdots c_{g+2})\cdot c_g\cdots c_2c_1^2c_2\cdots c_{2g}\cdot(c_{2g-1}\cdots c_2c_1^2c_2\cdots c_{2g-1})$$

$$\cdots\cdot(c_{g+2}\cdots c_2c_1^2c_1\cdots c_{g+2})\cdot(c_{g+1}\cdots c_2c_1^2c_1\cdots c_{g+1})\cdot d$$

$$\sim ({}_{c_{2g}}c_{2g+1}c_{2g-1}\cdots c_2c_1)^g\cdot c_{2g}\cdots c_{g+2}c_{g+1}\cdot(c_gc_{g-1}\cdots c_{2g+1})$$

$$\cdot(c_{g-1}c_{g-2}\cdots c_{2g})\cdot\cdots\cdot(c_1c_2\cdots c_{g+2})\cdot c_1c_2\cdots c_{2g}$$

$$\cdot(c_{2g-1}\cdots c_2c_1^2c_2\cdots c_{2g-1})\cdot\cdots\cdot(c_{g+1}\cdots c_2c_1^2c_1\cdots c_{g+1})\cdot d$$

$$\sim ({}_{c_{2g}}c_{2g+1}c_{2g-1}\cdots c_2c_1)^g\cdot c_{2g}\cdots c_2c_1\cdot(c_{g+1}c_{g+2}\cdots c_{2g+1})$$

$$\cdot(c_gc_{g-1}\cdots c_{2g})\cdot\cdots\cdot(c_2c_3\cdots c_{g+2})\cdot c_1c_2\cdots c_{2g}$$

$$\cdot(c_{2g-1}\cdots c_2c_1^2c_2\cdots c_{2g-1})\cdot\cdots\cdot(c_{g+1}\cdots c_2c_1^2c_1\cdots c_{g+1})\cdot d$$

$$\sim c_{2g}\cdots c_2c_1\cdot(c_{2g+1}c_{2g}\cdots c_3c_2)^g\cdot(c_{g+1}c_{g+2}\cdots c_{2g+1})$$

$$\cdot(c_gc_{g-1}\cdots c_{2g})\cdot\cdots\cdot(c_2c_3\cdots c_{g+2})\cdot c_1c_2\cdots c_{2g}$$

$$\cdot(c_{2g-1}\cdots c_2c_1^2c_2\cdots c_{2g-1})\cdot\cdots\cdot(c_{g+1}\cdots c_2c_1^2c_1\cdots c_{g+1})\cdot d$$

$$\sim c_{g+1}\cdots c_2c_1\cdot(c_{2g+1}c_{2g}\cdots c_3c_2)^g\cdot(c_{g+1}c_{g+2}\cdots c_{2g+1})$$

$$\cdot(c_gc_{g-1}\cdots c_{2g})\cdot\cdots\cdot(c_2c_3\cdots c_{g+2})\cdot c_1c_2\cdots c_{2g}$$

$$\cdot(c_{2g-1}\cdots c_2c_1^2c_2\cdots c_{2g-1})\cdot\cdots\cdot(c_{g+1}\cdots c_2c_1^2c_1\cdots c_{g+1})$$

$$\cdot c_{2g}\cdot\cdots\cdot c_{g+3}c_{g+2}\cdot d$$

$$\sim c_{g+1}\cdots c_2c_1\cdot(c_{2g+1}c_{2g}\cdots c_3c_2)^g\cdot(c_{g+1}c_{g+2}\cdots c_{2g+1})$$

$$\cdot(c_gc_{g-1}\cdots c_{2g})\cdot\cdots\cdot(c_2c_3\cdots c_{g+2})\cdot(c_1c_2\cdots c_{g+1})$$

$$\cdot(c_{2g}\cdots c_2c_1^2c_2\cdots c_{2g})\cdot\cdots\cdot(c_{g+2}\cdots c_2c_1^2c_1\cdots c_{g+2})\cdot d$$

$$\sim(c_{2g+1}\cdots c_{g+4}c_{g+3})\cdot\cdots\cdot(c_{2g+1}c_{2g})\cdot c_{2g+1}$$

$$\cdot(c_{g+1}\cdots c_2c_1)\cdot(c_{g+2}\cdots c_3c_2)\cdot\cdots\cdot(c_{2g+1}\cdots c_3c_2)$$

$$\cdot(c_{g+1}c_{g+2}\cdots c_{2g+1})\cdot\cdots\cdot(c_2c_3\cdots c_{g+2})\cdot(c_1c_2\cdots c_{g+1})$$

$$\cdot(c_{2g}\cdots c_2c_1^2c_2\cdots c_{2g})\cdot\cdots\cdot(c_{g+2}\cdots c_2c_1^2c_1\cdots c_{g+2})\cdot d$$

$$\sim(c_{g+1}\cdots c_2c_1)\cdot(c_{g+2}\cdots c_3c_2)\cdot\cdots\cdot(c_{2g+1}\cdots c_3c_2)$$

$$\cdot(c_{g+1}c_{g+2}\cdots c_{2g+1})\cdot\cdots\cdot(c_2c_3\cdots c_{g+2})\cdot(c_1c_2\cdots c_{g+1})$$

$$\cdot(c_{2g}\cdots c_2c_1^2c_2\cdots c_{2g})\cdot\cdots\cdot(c_{g+2}\cdots c_2c_1^2c_1\cdots c_{g+2})$$

$$
\begin{aligned}
&\quad \cdot (c_{2g+1} \cdots c_{g+4} c_{g+3}) \cdot \cdots \cdot (c_{2g+1} c_{2g}) \cdot c_{2g+1} \cdot d \\
&\sim (c_{g+1} \cdots c_2 c_1) \cdot (c_{g+2} \cdots c_3 c_2) \cdot \cdots \cdot (c_{2g+1} \cdots c_{g+2} c_{g+1}) \\
&\quad \cdot c_2 \cdot (c_3 c_2) \cdot \cdots \cdot (c_g \cdots c_3 c_2) \\
&\quad \cdot (c_{g+1} c_{g+2} \cdots c_{2g+1}) \cdot \cdots \cdot (c_2 c_3 \cdots c_{g+2}) \cdot (c_1 c_2 \cdots c_{g+1}) \\
&\quad \cdot (c_{2g} \cdots c_2 c_1^2 c_2 \cdots c_{2g}) \cdot \cdots \cdot (c_{g+2} \cdots c_2 c_1^2 c_1 \cdots c_{g+2}) \\
&\quad \cdot (c_{2g+1} \cdots c_{g+4} c_{g+3}) \cdot \cdots \cdot (c_{2g+1} c_{2g}) \cdot c_{2g+1} \cdot d \\
&\sim (c_{g+1} \cdots c_2 c_1) \cdot (c_{g+2} \cdots c_3 c_2) \cdot \cdots \cdot (c_{2g+1} \cdots c_{g+2} c_{g+1}) \\
&\quad \cdot c_g \cdot (c_{g-1} c_g) \cdot \cdots \cdot (c_2 c_3 \cdots c_g) \\
&\quad \cdot (c_{g+1} c_{g+2} \cdots c_{2g+1}) \cdot \cdots \cdot (c_2 c_3 \cdots c_{g+2}) \cdot (c_1 c_2 \cdots c_{g+1}) \\
&\quad \cdot (c_{2g} \cdots c_2 c_1^2 c_2 \cdots c_{2g}) \cdot \cdots \cdot (c_{g+2} \cdots c_2 c_1^2 c_1 \cdots c_{g+2}) \\
&\quad \cdot (c_{2g+1} \cdots c_{g+4} c_{g+3}) \cdot \cdots \cdot (c_{2g+1} c_{2g}) \cdot c_{2g+1} \cdot d \\
&\sim (c_{g+1} \cdots c_2 c_1) \cdot (c_{g+2} \cdots c_3 c_2) \cdot \cdots \cdot (c_{2g+1} \cdots c_{g+2} c_{g+1}) \\
&\quad \cdot c_g \cdot (c_{g-1} c_g) \cdot \cdots \cdot (c_2 c_3 \cdots c_g) \\
&\quad \cdot (c_{g+1} \cdots c_2 c_1) \cdot (c_{g+2} \cdots c_3 c_2) \cdot \cdots \cdot (c_{2g+1} \cdots c_{g+2} c_{g+1}) \\
&\quad \cdot (c_{2g} \cdots c_2 c_1^2 c_2 \cdots c_{2g}) \cdot \cdots \cdot (c_{g+2} \cdots c_2 c_1^2 c_1 \cdots c_{g+2}) \\
&\quad \cdot (c_{2g+1} \cdots c_{g+4} c_{g+3}) \cdot \cdots \cdot (c_{2g+1} c_{2g}) \cdot c_{2g+1} \cdot d \\
&\sim (c_{g+1} \cdots c_2 c_1) \cdot (c_{g+2} \cdots c_3 c_2) \cdot \cdots \cdot (c_{2g+1} \cdots c_{g+2} c_{g+1}) \\
&\quad \cdot (c_{g+1} \cdots c_2 c_1) \cdot (c_{g+2} \cdots c_3 c_2) \cdot \cdots \cdot (c_{2g+1} \cdots c_{g+2} c_{g+1}) \\
&\quad \cdot c_{2g+1} \cdot (c_{2g} c_{2g+1}) \cdot \cdots \cdot (c_{g+3} c_{g+4} \cdots c_{2g+1}) \\
&\quad \cdot (c_{2g} \cdots c_2 c_1^2 c_2 \cdots c_{2g}) \cdot \cdots \cdot (c_{g+2} \cdots c_2 c_1^2 c_1 \cdots c_{g+2}) \\
&\quad \cdot (c_{2g+1} \cdots c_{g+4} c_{g+3}) \cdot \cdots \cdot (c_{2g+1} c_{2g}) \cdot c_{2g+1} \cdot d \\
&\sim (c_{g+1} \cdots c_2 c_1) \cdot (c_{g+2} \cdots c_3 c_2) \cdot \cdots \cdot (c_{2g+1} \cdots c_{g+2} c_{g+1}) \\
&\quad \cdot (c_{g+1} \cdots c_2 c_1) \cdot (c_{g+2} \cdots c_3 c_2) \cdot \cdots \cdot (c_{2g+1} \cdots c_{g+2} c_{g+1}) \\
&\quad \cdot (c_{2g+1} \cdots c_2 c_1^2 c_2 \cdots c_{2g+1})^{g-1} \cdot d \\
&\sim (c_{g+1} \cdots c_2 c_1) \cdot (c_{g+2} \cdots c_3 c_2) \cdot \cdots \cdot (c_{2g+1} \cdots c_{g+2} c_{g+1}) \\
&\quad \cdot (c_{g+1} c_{g+2} \cdots c_{2g+1}) \cdot \cdots \cdot (c_2 c_3 \cdots c_g) \cdot (c_1 c_2 \cdots c_{g+1}) \cdot H^{g-1} \cdot d \\
&\sim (c_g \cdots c_2 c_1)^{g+1} \cdot {}_{W_1}(c_{g+1}) \cdot {}_{W_2}(c_{g+2}) \cdot \cdots \cdot {}_{W_{g+1}}(c_{2g+1}) \\
&\quad \cdot (c_{g+1} c_{g+2} \cdots c_{2g+1}) \cdot \cdots \cdot (c_2 c_3 \cdots c_g) \cdot (c_1 c_2 \cdots c_{g+1}) \cdot H^{g-1} \cdot d \\
&\sim {}_{W_1}(c_{g+1}) \cdot {}_{W_2}(c_{g+2}) \cdot \cdots \cdot {}_{W_{g+1}}(c_{2g+1}) \\
&\quad \cdot (c_{g+1} c_{g+2} \cdots c_{2g+1}) \cdot \cdots \cdot (c_2 c_3 \cdots c_g) \cdot (c_1 c_2 \cdots c_{g+1}) \\
&\quad \cdot (c_g \cdots c_2 c_1)^{g+1} \cdot H^{g-1} \cdot d \quad (\text{n.b. } {}_{H^{g-1}}(c_i) = c_i \ (i = 1, \ldots, g))
\end{aligned}
$$

$$\sim {}_{W_1}(c_{g+1}) \cdot {}_{W_2}(c_{g+2}) \cdot \cdots \cdot {}_{W_{g+1}}(c_{2g+1}) \cdot (c_{2g+1} \cdots c_2c_1)^{g+1} \cdot H^{g-1} \cdot d$$
$$\sim d \cdot {}_{W_1}(c_{g+1}) \cdot {}_{W_2}(c_{g+2}) \cdot \cdots \cdot {}_{W_{g+1}}(c_{2g+1}) \cdot (c_{2g+1} \cdots c_2c_1)^{g+1} \cdot H^{g-1}$$
$$= R \cdot H^{g-1}$$

Thus the proof is completed. Q.E.D.

By virtue of Lemma 4.1, Corollary A.3, and Lemma B.1, we can show the equivalence $PJI^2 \sim RI^{g+1}$ as follows.

$$\begin{aligned}
PJI^2 &= P \cdot JI \cdot I = P \cdot C_{\mathrm{I}} \cdot c_1c_2 \cdots c_{2g}c_{2g+1}^2c_{2g} \cdots c_2c_1 \\
&\sim {}_{PC_{\mathrm{I}}}(c_1c_2 \cdots c_{2g+1}) \cdot PC_{\mathrm{I}} \cdot c_{2g+1} \cdots c_2c_1 \sim H \cdot PC_{\mathrm{I}} \\
&\sim H \cdot P \cdot (c_{2g+1} \cdots c_2c_1)^{2g+2} \\
&\sim c_{2g} \cdots c_2c_1 \cdot HP \cdot c_{2g+1} \cdot (c_{2g} \cdots c_2c_1c_{2g+1})^{2g+1} \\
&\sim HP \cdot {}_{(HP)^{-1}}(c_{2g} \cdots c_2c_1) \cdot c_{2g+1} \cdot (c_{2g} \cdots c_2c_1c_{2g+1})^{2g+1} \\
&\sim HP \cdot (c_{2g}c_{2g+1}c_{2g-1} \cdots c_2c_1)^{2g+2} \\
&\sim HP \cdot (c_{2g+1} \cdot {}_{c_{2g+1}^{-1}}(c_{2g}) \cdot c_{2g-1} \cdots c_2c_1)^{2g+2} \\
&\sim HP \cdot {}_{c_{2g+1}^{-1}}(c_{2g+1} \cdots c_2c_1)^{2g+2} \sim HP \cdot {}_{c_{2g+1}^{-1}}C_{\mathrm{I}} \\
&\sim HP \cdot {}_{c_{2g+1}^{-1}}(JH) \sim HP \cdot {}_{c_{2g+1}^{-1}}J \cdot {}_{c_{2g+1}^{-1}}H \\
&\sim H \cdot RH^{g-1} \cdot {}_{c_{2g+1}^{-1}}H \sim RH^{g-1} \cdot {}_{c_{2g+1}^{-1}}H \cdot H \\
&= RH^{g-1} \cdot c_{2g+1} \cdot {}_{c_{2g+1}^{-1}}(c_{2g}) \cdot c_{2g-1} \cdots c_2c_1 \\
&\qquad \cdot c_1c_2 \cdots c_{2g-1} \cdot {}_{c_{2g+1}^{-1}}(c_{2g}) \cdot c_{2g+1} \cdot H \\
&\sim RH^{g-1} \cdot c_{2g}c_{2g-1} \cdots c_2c_1^2c_2 \cdots c_{2g-1}c_{2g}c_{2g+1}^2 \cdot H \\
&\sim c_{2g+1} \cdot H \cdot RH^{g-1} \cdot c_{2g} \cdots c_2c_1^2c_2 \cdots c_{2g}c_{2g+1} \\
&\sim H \cdot RH^{g-1} \cdot {}_{(HRH^{g-1})^{-1}}(c_{2g+1}) \cdot c_{2g} \cdots c_2c_1^2c_2 \cdots c_{2g}c_{2g+1} \\
&= H \cdot RH^{g-1} \cdot H \sim RH^{g+1} \sim c_1c_2 \cdots c_{2g+1} \cdot RH^g \cdot c_{2g+1} \cdots c_2c_1 \\
&\sim R \cdot {}_{R^{-1}}(c_1c_2 \cdots c_{2g+1}) \cdot H^g \cdot c_{2g+1} \cdots c_2c_1 \\
&= RI^{g+1}
\end{aligned}$$

This completes the proof of Theorem 4.10.

References

[1] J. Amorós, F. Bogomolov, L. Katzarkov and T. Pantev, Symplectic Lefschetz fibrations with arbitrary fundamental groups, J. Diff. Geom., **54** (2000), 489–545.

[2] D. Auroux, Fiber sums of genus 2 Lefschetz fibrations, Turkish J. Math., **25** (2001), 1–10.

[3] J. Birman and H. Hilden, On mapping class groups of closed surfaces as covering spaces, Advances in the Theory of Riemann surfaces, Ann. Math. Stud., **66**, Princeton Univ. Press, 1971, pp. 81–115.

[4] C. Cadavid, A remarkable set of words in the mapping class group, Dissertation, Univ. of Texas, Austin, 1998.

[5] K. N. Chakiris, The monodromy of genus two pencils, Dissertation, Colombia Univ., 1978.

[6] S. K. Donaldson, Lefschetz pencils on symplectic manifolds, J. Differential Geom., **53** (1999), 205–236.

[7] H. Endo, Meyer's signature cocycle and hyperelliptic fibrations (with Appendix written by T. Terasoma), Math. Ann., **316** (2000), 237–257.

[8] H. Endo and S. Nagami, Signature of relations in mapping class groups and non-holomorphic Lefschetz fibrations, Trans. Amer. Math. Soc., **357** (2005), 3179–3199.

[9] R. Fintushel and R. J. Stern, Immersed spheres in 4-manifolds and the immersed Thom conjecture, Turkish J. Math., **19** (1995), 27–40.

[10] R. Fintushel and R. J. Stern, Rational blowdowns of smooth 4-manifolds, J. Differential Geom., **46** (1997), 181–235.

[11] T. Fuller, Diffeomorphism types of genus 2 Lefschetz fibrations, Math. Ann., **311** (1998), 163–176.

[12] T. Fuller, Hyperelliptic Lefschetz fibrations and branched covering spaces, Pacific J. Math., **196** (2000), 369–393.

[13] T. Fuller, Lefschetz fibrations of 4-dimensional manifolds, Cubo A Math. J., **5** (2003), 275–294.

[14] R. E. Gompf and A. I. Stipsicz, 4-manifolds and Kirby calculus, Grad. Stud. Math., **20**, Amer. Math. Soc., 1999.

[15] Y. Z. Gurtas, Positive Dehn twist expressions for some new involutions in mapping class group I, II, preprint, arXiv:math.GT/0404310, math.GT/0404311.

[16] Y. Z. Gurtas, Positive Dehn twist expressions for some elements of finite order in mapping class group, preprint, arXiv:math.GT/0501385.

[17] I. Hasegawa, A combinatorial proof for the signature of the Lefschetz fibration with Gurtas' monodromy, preprint, 2005.

[18] S. Hirose, Examples of non-minimal Lefschetz fibrations of genus 2, preprint, 2006.

[19] A. Kas, On the handlebody decomposition associated to a Lefschetz fibration, Pacific J. Math., **89** (1980), 89–104.

[20] V. M. Kharlamov and V. S. Kulikov, On braid monodromy factorizations, Izv. Math., **67** (2003), 499–534.

[21] M. Korkmaz, Noncomplex smooth 4-manifolds with Lefschetz fibrations, Internat. Math. Res. Notices, 2001, no. 3, 115–128.

[22] D. Kotschick, The Seiberg-Witten invariants of symplectic four-manifolds (after C. H. Taubes), Séminaire Bourbaki, 48ème annèe, 1995-96, n° 812, Astérisque, **241** (1997), 195–220.

[23] D. Kotschick, J. W. Morgan and C. H. Taubes, Four-manifolds without symplectic structures but with nontrivial Seiberg-Witten invariants, Math. Res. Letters, **2** (1995), 119–124.

[24] M. Matsumoto, A presentation of mapping class groups in terms of Artin groups and geometric monodromy of singularities, Math. Ann., **316** (2000), 401–418.

[25] Y. Matsumoto, Lefschetz fibrations of genus two – a topological approach, In: Topology and Teichmüller Spaces, Proceedings of the 37th Taniguchi Symposium, World Scientific, Singapore, 1996, pp. 123–148.

[26] Y. Matsumoto, Splitting of certain singular fibers of genus 2, Bol. Soc. Nat. Mexicana, **10** (2004), 331–355.

[27] T. Morifuji, On Meyer's function of hyperelliptic mapping class groups, J. Math. Soc. Japan, **55** (2003), 117–129.

[28] S. Morita, Characteristic classes of surface bundles, Invent. Math., **90** (1987), 551–577.

[29] B. Ozbagci, Signatures of Lefschetz fibrations, Pacific J. Math., **202** (2002), 99–118.

[30] Y. Sato, 2-spheres of square -1 and the geography of genus-2 Lefschetz fibrations, preprint, 2003.

[31] Y. Sato, A conjecture of Stipsicz – minimality of fiber sums of genus 2 Lefschetz fibrations, preprint (in Japanese), 2006.

[32] B. Siebert and G. Tian, On hyperelliptic C^∞-Lefschetz fibrations of four-manifolds, Commun. Contemp. Math., **1** (1999), 466–488.

[33] B. Siebert and G. Tian, On the holomorphicity of genus two Lefschetz fibrations, Ann. of Math. (2), **161** (2005), 959–1020.

[34] I. Smith, Lefschetz fibrations and the Hodge bundle, Geom. Topol., **3** (1999), 211–233.

[35] I. Smith, Lefschetz pencils and divisors in moduli space, Geom. Topol., **5** (2001), 579–608.

[36] I. Smith, Geometric monodromy and the hyperbolic disc, Quart. J. Math., **52** (2001), 217–228.

[37] A. I. Stipsicz, Indecomposability of certain Lefschetz fibrations, Proc. Amer. Math. Soc., **129** (2000), 1499–1502.

[38] C. H. Taubes, The Seiberg-Witten invariants and symplectic forms, Math. Res. Letters, **1** (1994), 809–822.

[39] C. H. Taubes, Counting pseudo-holomorphic submanifolds in dimension 4, J. Differential Geom., **44** (1996), 818–893.

[40] C. H. Taubes, The Seiberg-Witten and Gromov invariants, Math. Res. Letters, **2** (1995), 221–238.

[41] M. Usher, Minimality and symplectic sums, Internat. Math. Res. Notices, 2006, Article ID 49857, 1–17.
[42] B. Wajnryb, An elementary approach to the mapping class group of a surface, Geom. Topol., **3** (1999), 405–466.
[43] K.-H. Yun, On the signature of a Lefschetz fibration coming from an involution, Topology Appl., **153** (2006), 1994–2012.

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