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On mod p Riemann-Roch formulae for mapping class groups

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Abstract.

We provide affirmative evidences for conjectural mod p Riemann-Roch formulae for mapping class groups by considering Steenrod operations on mod p Morita-Mumford classes.

§1. Introduction

Let Σ_g be a closed oriented surface of genus $g \geq 2$ and let Γ_g be its mapping class group. Namely, it is the group consisting of path components of $\mathrm{Diff}_+\Sigma_g$, which is the group of orientation preserving diffeomorphisms of Σ_g. Any cohomology class of Γ_g can be considered as a characteristic class of oriented surface bundles. Indeed, by a theorem of Earle and Eells [4], the classifying space $B\mathrm{Diff}_+\Sigma_g$ of oriented Σ_g-bundles is an Eilenberg-MacLane space $K(\Gamma_g, 1)$ so that we have a natural isomorphism

$$H^*(B\mathrm{Diff}_+\Sigma_g; \mathbb{Z}) \cong H^*(\Gamma_g; \mathbb{Z}).$$

Morita [11] and Mumford [12] independently introduced a series of cohomology classes $e_k \in H^{2k}(\Gamma_g; \mathbb{Z})$ of Γ_g which are called Morita-Mumford classes (or Mumford-Morita-Miller classes in the literature). Over the rationals, the natural homomorphism

$$\mathbb{Q}[e_1, e_2, e_3, \dots] \to H^*(\Gamma_g; \mathbb{Q})$$

is an isomorphism in dimensions less than $2g/3$ by the proof of Mumford conjecture [9]. On the contrary, less is known about integral or mod p Morita-Mumford classes of Γ_g. In [1], the author proposed a conjecture

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concerning of integral Morita-Mumford classes. To be precise, let B_{2k} be the $2k$-th Bernoulli number, and define N_{2k}, D_{2k} to be coprime integers satisfying $B_{2k}/2k = N_{2k}/D_{2k}$. Let $s_k \in H^{2k}(\Gamma_g;\mathbb{Z})$ be the k-th Newton class of Γ_g which will be defined in §3.

Conjecture 1 (integral Riemann-Roch formulae for Γ_g).

$$N_{2k}e_{2k-1} = D_{2k}s_{2k-1} \in H^*(\Gamma_g;\mathbb{Z})$$

holds for all $k \geq 1$ and $g \geq 2$.

With rational coefficients, it is deduced from the Grothendieck-Riemann-Roch theorem. See [11, 12]. The conjecture is affirmative for $k = 1$ (i.e. $e_1 = 12s_1 \in H^2(\Gamma_g;\mathbb{Z})$ for all $g \geq 2$), since $H^2(\Gamma_g,\mathbb{Z}) \cong \mathbb{Z}$ for $g \geq 3$ as was proved by Harer [6] (see [1] for the case $g = 2$). The author and Kawazumi [2] showed that the conjecture is affirmative for any cyclic subgroup of Γ_g. Kawazumi [8] showed that a slightly weaker version of the conjecture holds for hyperelliptic mapping class groups. In addition, a result of Galatius, Madsen and Tillmann [5, Theorem 1.2] can be regarded as an affirmative evidence of the conjecture for the stable mapping class group (see [1, Section 7] for the detail). Now let p be a prime and $\mathbb{F}_p$ the field consisting of p elements. Passing to the mod p cohomology, Conjecture 1 leads to the following conjecture:

Conjecture 2 (mod p Riemann-Roch formulae for Γ_g).

$$N_{2k}e_{2k-1} = D_{2k}s_{2k-1} \in H^*(\Gamma_g;\mathbb{F}_p)$$

holds for all $k \geq 1$ and $g \geq 2$.

The purpose of this paper is to provide affirmative evidences of Conjecture 2. Our main result is the following:

Theorem 1.1. *Let p be an odd prime and $g \geq 2$. If*

$$N_{2k}e_{2k-1} = D_{2k}s_{2k-1} \in H^*(\Gamma_g;\mathbb{F}_p)$$

holds for some $k \geq 1$, and if $\binom{2k-1}{i}$ is prime to p, then

$$N_{2k+i(p-1)}e_{2k-1+i(p-1)} = D_{2k+i(p-1)}s_{2k-1+i(p-1)} \in H^*(\Gamma_g;\mathbb{F}_p).$$

In other words, the affirmative solution of Conjecture 2 for some k implies that for $k + i(p-1)/2$, provided $\binom{2k-1}{i}$ is prime to p. In particular, since Conjecture 1 and hence Conjecture 2 are affirmative for $k = 1$ as was mentioned earlier, one has the following result:

Corollary 1.2. *Let p be an odd prime. Then*

$$N_{p^n+1} e_{p^n} = D_{p^n+1} s_{p^n} \in H^*(\Gamma_g; \mathbb{F}_p)$$

for all $n \geq 0$ and $g \geq 2$.

Theorem 1.1 is proved by considering reduced power operations on mod p Morita-Mumford and Newton classes of Γ_g, together with Kummer's congruences on Bernoulli numbers. Similar considerations are possible for $p = 2$ by using squaring operations in place of reduced power operations.

The rest of the paper is organized as follows. In §2, we will recall the definition of Morita-Mumford classes, and compute the action of Steenrod operations on them. In §3, we will recall the definition of Newton classes. The proof of Theorem 1.1 will be given in §4.

Notation. There are some conflicting notations of Bernoulli numbers. We define B_{2k} $(k \geq 1)$ by a power series expansion

$$\frac{z}{e^z - 1} + \frac{z}{2} = 1 + \sum_{k=1}^{\infty} \frac{B_{2k}}{(2k)!} z^{2k}.$$

Thus $B_2 = 1/6, B_4 = -1/30, B_6 = 1/42, B_8 = -1/30, B_{10} = 5/66$, and so on. Our notation is consistent with [2, 7] but differs from [1, 8].

§2. Morita-Mumford classes

2.1. Definition

Let $\pi : E \to B$ be an oriented Σ_g-bundle, $T_{E/B}$ the tangent bundle along the fiber of π, and $e \in H^2(E; \mathbb{Z})$ the Euler class of $T_{E/B}$. Then the k-th Morita-Mumford class $e_k \in H^{2k}(B; \mathbb{Z})$ of π is defined by

$$e_k := \pi_!(e^{k+1})$$

where $\pi_! : H^*(E; \mathbb{Z}) \to H^{*-2}(B; \mathbb{Z})$ is the Gysin homomorphism (or the integration along the fiber). The structure group of oriented Σ_g-bundles is $\mathrm{Diff}_+\Sigma_g$. Hence passing to the universal Σ_g-bundle

$$E\mathrm{Diff}_+\Sigma_g \times_{\mathrm{Diff}_+\Sigma_g} \Sigma_g \to B\mathrm{Diff}_+\Sigma_g$$

we obtain the cohomology classes $e_k \in H^*(B\mathrm{Diff}_+\Sigma_g; \mathbb{Z})$. As was mentioned in Introduction, the classifying space $B\mathrm{Diff}_+\Sigma_g$ is an Eilenberg-MacLane space $K(\Gamma_g, 1)$ so that we obtain the *k-th Morita-Mumford class $e_k \in H^{2k}(\Gamma_g; \mathbb{Z})$ of Γ_g.*

2.2. Steenrod operations

For an odd prime p, let

$$\mathrm{P}^i : H^k(-;\mathbb{F}_p) \to H^{k+2i(p-1)}(-;\mathbb{F}_p)$$

be the i-th reduced power operation. We will compute the action of P^i's on mod p Morita-Mumford classes. To this end let us return to an oriented smooth Σ_g-bundle $\pi : E \to B$ and choose a smooth embedding $E \to \mathbb{R}^n$ of E in some Euclidean space $\mathbb{R}^n$. The normal bundle $N^f E$ of the resulting embedding $f : E \to B \times \mathbb{R}^n$ is called the *normal bundle along the fiber*:

$$T_{E/B} \oplus N^f E \cong E \times \mathbb{R}^n \quad \text{(product bundle)}.$$

Let $q_\bullet(N^f E) \in H^*(E, \mathbb{F}_p)$ be the total Wu class of $N^f E$ defined by $q_\bullet(N^f E) = \phi^{-1} \circ \mathrm{P} \circ \phi(1)$ where ϕ is the Thom isomorphism for $N^f E$ (see [10, p.228] for instance). Applying the generalized Riemann-Roch theorem [3, p.65 Theorem 9] to the total reduced power operation $\mathrm{P} = \sum_i \mathrm{P}^i$, one has

$$\mathrm{P}(\pi_!(u)) = \pi_!(\mathrm{P}(u) \cdot q_\bullet(N^f E)) \tag{1}$$

for every $u \in H^*(E;\mathbb{F}_p)$. For the mod p Euler class $e \in H^2(E;\mathbb{F}_p)$ of $T_{E/B}$, one has $\mathrm{P}(e) = \mathrm{P}^0(e) + \mathrm{P}^1(e) = e + e^p$ and hence

$$\mathrm{P}(e^{k+1}) = \mathrm{P}(e)^{k+1} = (e + e^p)^{k+1} = e^{k+1}(1 + e^{p-1})^{k+1}$$

by Cartan formula. On the other hand, one has

$$q_\bullet(T_{E/B}) \cdot q_\bullet(N^f E) = q_\bullet(T_{E/B} \oplus N^f E) = q_\bullet(E \times \mathbb{R}^n) = 1$$
$$q_\bullet(T_{E/B}) = 1 + e^{p-1} \quad \text{(see [10, p.228])}.$$

Applying $u = e^{k+1}$ to (1), one has

$$\begin{aligned}\mathrm{P}(e_k) = \mathrm{P}(\pi_!(e^{k+1})) &= \pi_!(e^{k+1}(1 + e^{p-1})^{k+1} \cdot q_\bullet(N^f E)) \\ &= \pi_!(e^{k+1}(1 + e^{p-1})^{k} \cdot q_\bullet(T_{E/B}) \cdot q_\bullet(N^f E)) \\ &= \pi_!(e^{k+1}(1 + e^{p-1})^{k})\end{aligned}$$

and hence

$$\mathrm{P}(e_k) = \pi_!\left(\sum_{i=0}^{k} \binom{k}{i} e^{k+i(p-1)+1}\right) = \sum_{i=0}^{k} \binom{k}{i} e_{k+i(p-1)}. \tag{2}$$

Since reduced power operations are natural with respect to bundle maps, the last equality (2) is valid in $H^*(\Gamma_g;\mathbb{F}_p)$. Thus we have proved the following proposition:

Proposition 2.1. *For* $e_k \in H^*(\Gamma_g; \mathbb{F}_p)$, *one has*

$$\mathrm{P}^i(e_k) = \binom{k}{i} e_{k+i(p-1)} \in H^*(\Gamma_g; \mathbb{F}_p).$$

Now let $\mathrm{Sq}^i : H^k(-, \mathbb{F}_2) \to H^{k+i}(-, \mathbb{F}_2)$ be the i-th squaring operation. Applying the generalized Riemann-Roch theorem to the total squaring operation $\mathrm{Sq} = \sum_i \mathrm{Sq}^i$, one has

$$\mathrm{Sq}(\pi_!(u)) = \pi_!(\mathrm{Sq}(u) \cdot \omega_\bullet(N^f E)) \tag{3}$$

for every $u \in H^*(E; \mathbb{F}_2)$, where $\omega_\bullet(N^f E) \in H^*(E; \mathbb{F}_2)$ is the total Stiefel-Whitney class of $N^f E$. With the equation (3) in mind, the proof of the following proposition is similar to that of Proposition 2.1, and is left to the reader:

Proposition 2.2. *For* $e_k \in H^*(\Gamma_g; \mathbb{F}_2)$, *one has*

$$\mathrm{Sq}^{2i}(e_k) = \binom{k}{i} e_{k+i}, \ \mathrm{Sq}^{2i+1}(e_k) = 0 \in H^*(\Gamma_g; \mathbb{F}_2).$$

§3. Newton classes

Let $U(n)$ be the n-dimensional unitary group. The k-th Newton class $s_k \in H^*(BU(n); \mathbb{Z})$ is the characteristic class associated to the formal sum $\sum_{l=1}^n x_l^k$. Steenrod operations on mod p Newton classes $s_k \in H^*(BU(n); \mathbb{F}_p)$ can be computed quite easily, as in the proposition below.

Proposition 3.1. *For an odd prime* p, *one has*

$$\mathrm{P}^i(s_k) = \binom{k}{i} s_{k+i(p-1)} \in H^*(BU(n); \mathbb{F}_p).$$

For $p = 2$, *one has*

$$\mathrm{Sq}^{2i}(s_k) = \binom{k}{i} s_{k+i}, \ \mathrm{Sq}^{2i+1}(s_k) = 0 \in H^*(BU(n); \mathbb{F}_2).$$

Proof. By the definition of the Newton class s_k, it suffices to prove the case $n = 1$. Since $s_k = c_1^k \in H^*(BU(1); \mathbb{Z})$ where c_1 is the first Chern class, for an odd prime p, one has

$$\mathrm{P}^i(s_k) = \mathrm{P}^i(c_1^k) = \binom{k}{i} c_1^{k+i(p-1)} = \binom{k}{i} s_{k+i(p-1)}$$

as desired. The case $p = 2$ is similar. Q.E.D.

Now we recall the definition of Newton classes of Γ_g. The natural action of Γ_g on the first real homology $H_1(\Sigma_g;\mathbb{R})$ induces a homomorphism $\Gamma_g \to Sp(2g,\mathbb{R})$. The homomorphism yields a continuous map

$$\eta : K(\Gamma_g, 1) \to BU(g),$$

for the maximal compact subgroup of $Sp(2g,\mathbb{R})$ is isomorphic to $U(g)$. The *k-th Newton class* $s_k \in H^{2k}(\Gamma_g;\mathbb{Z})$ *of* Γ_g is defined to be the pullback of $s_k \in H^*(BU(g);\mathbb{Z})$ by η (we use the same symbol).

§4. Proof of the main results

The proof of Theorem 1.1 is based on the following facts concerning of number theoretic properties of Bernoulli numbers:

Theorem 4.1. *Let p be a prime number.*

(1) $p|D_{2k}$ *if and only if* $p-1|2k$.

(2) *If* $p-1 \not| \, 2k$ *and* $k \equiv h \pmod{p-1}$ *then*

$$\frac{B_{2k}}{2k} \equiv \frac{B_{2h}}{2h} \pmod{p}.$$

The last congruence is considered in $\mathbb{Z}_{(p)} = \{m/n \in \mathbb{Q} \mid (n,p) = 1\}$. Namely, for $r, s \in \mathbb{Z}_{(p)}$, we write $r \equiv s \pmod{p}$ if $r-s = m/n$, $(n,p) = 1$, and $p|m$. The first statement (1) is called von Staudt's theorem, while the second statement (2) is called the Kummer's congruence. See [7] for the proof of Theorem 4.1.

Proof of Theorem 1.1. Applying the i-th reduced power operation to the both sides of equality $N_{2k}e_{2k-1} = D_{2k}s_{2k-1}$, one has

$$N_{2k}e_{2k-1+i(p-1)} = D_{2k}s_{2k-1+i(p-1)} \in H^*(\Gamma_g;\mathbb{F}_p) \tag{4}$$

by Proposition 2.1 and 3.1.

Suppose first $p-1$ divides $2k$. It follows from von Staudt's theorem that $D_{2k} \equiv D_{2k+i(p-1)} \equiv 0 \pmod{p}$. Consequently, the equality $N_{2k}e_{2k-1} = D_{2k}s_{2k-1}$ implies the condition $e_{2k-1} = 0$, and the equality (4) implies $e_{2k-1+i(p-1)} = 0$. Since

$$N_{2k+i(p-1)}e_{2k-1+i(p-1)} = D_{2k+i(p-1)}s_{2k-1+i(p-1)}$$

is equivalent to $e_{2k-1+i(p-1)} = 0$, theorem follows.

Now suppose $p-1$ does not divide $2k$. By von Staudt's theorem, D_{2k} and $D_{2k+i(p-1)}$ are prime to p. Choose integers I_{2k} and $I_{2k+i(p-1)}$

satisfying $I_{2k}D_{2k} \equiv 1$ and $I_{2k+i(p-1)}D_{2k+i(p-1)} \equiv 1 \pmod p$. Then the equality (4) is equivalent to

$$N_{2k}I_{2k} \cdot e_{2k-1+i(p-1)} = s_{2k-1+i(p-1)}. \tag{5}$$

Now the Kummer's congruence implies

$$N_{2k}I_{2k} \equiv N_{2k+i(p-1)}I_{2k+i(p-1)} \pmod p.$$

Consequently, one has

$$N_{2k+i(p-1)}I_{2k+i(p-1)} \cdot e_{2k-1+i(p-1)} = s_{2k-1+i(p-1)}$$

and hence

$$N_{2k+i(p-1)}e_{2k-1+i(p-1)} = D_{2k+i(p-1)}s_{2k-1+i(p-1)}$$

as desired. Q.E.D.

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