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L^2-torsion invariants and the Magnus representation of the mapping class group

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Abstract.

In this paper, we study a series of L^2-torsion invariants from the viewpoint of the mapping class group of a surface. We establish some vanishing theorems for them. Moreover we explicitly calculate the first two invariants and compare them with hyperbolic volumes.

§1. Magnus representation

Let $\Sigma_{g,1}$ be a compact oriented smooth surface of genus g with a boundary $\partial\Sigma_{g,1} \cong S^1$. In this paper, we always assume that $g \geq 1$. We take and fix a base point $* \in \partial\Sigma_{g,1}$ of $\Sigma_{g,1}$. Let $\mathcal{M}_{g,1}$ be the mapping class group of $\Sigma_{g,1}$, namely, the group of all isotopy classes of orientation preserving diffeomorphisms of $\Sigma_{g,1}$ relative to the boundary. We denote $\pi_1(\Sigma_{g,1}, *)$ by Γ, which is a free group of rank $2g$, and fix a generating system $\Gamma = \langle x_1, \dots, x_{2g}\rangle$. Let $\mathbb{Z}\Gamma$ be the group ring of Γ over $\mathbb{Z}$. We write $\varphi_* \in \mathrm{Aut}(\Gamma)$ to the automorphism induced from $\varphi \in \mathcal{M}_{g,1}$. The following result, usually called the Dehn-Nielsen-Baer theorem, is classical and fundamental to study the mapping class group $\mathcal{M}_{g,1}$ by using combinatorial group theories (see [9] Section 2.9).

Proposition 1.1 (Zieschang [27]). *The above induced homomorphism $\mathcal{M}_{g,1} \ni \varphi \mapsto \varphi_* \in \mathrm{Aut}(\Gamma)$ is injective.*

As a corollary, we see that φ can be determined by the words $\varphi_*(x_1), \dots, \varphi_*(x_{2g}) \in \Gamma$. Since the fundamental formula $\gamma = 1 +$

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$\sum_{i=1}^{2g}(\partial\gamma/\partial x_i)(x_i - 1)$ holds in $\mathbb{Z}\Gamma$ for any $\gamma \in \Gamma$, the word $\varphi_*(x_j)$ is determined by $\{\partial\varphi_*(x_j)/\partial x_i\}$. Here $\partial/\partial x_i : \mathbb{Z}\Gamma \to \mathbb{Z}\Gamma$ denotes Fox's free differential. See [1] Section 3.1 for a systematic treatment of the subject. The Magnus representation of the mapping class group is defined as follows.

Definition 1.2. The Magnus representation of $\mathcal{M}_{g,1}$ is defined by the assignment

$$r : \mathcal{M}_{g,1} \ni \varphi \mapsto \left(\overline{\frac{\partial\varphi_*(x_j)}{\partial x_i}}\right)_{ij} \in GL(2g, \mathbb{Z}\Gamma),$$

where $\overline{\sum_g \lambda_g g} = \sum_g \lambda_g g^{-1}$ for any element $\sum_g \lambda_g g \in \mathbb{Z}\Gamma$.

Remark 1.3. By the expression $\gamma = 1 + \sum_i(\partial\gamma/\partial x_i)(x_i - 1)$, it follows that r is injective. However, it is not a group homomorphism, just a crossed homomorphism. According to the practice, we call it simply the Magnus representation of $\mathcal{M}_{g,1}$.

Now for a matrix $B \in M(n, \mathbb{C})$, let us recall that its characteristic polynomial

$$\det(tI - B)$$

is one of the fundamental tools in the linear algebra. Here I denotes the identity matrix of degree n. If we can define a characteristic polynomial of $r(\varphi)$, it may be useful tool to study the mapping class group. In order to define it for a Magnus matrix $r(\varphi)$, we need to clarify the following two points.

(1) What is the determinant over a non-commutative group ring?
(2) What is the meaning of a variable "t" in the group?

As an answer to these problems, we can formulate that

- the variable t lives in the fundamental group of the mapping torus of φ,
- a characteristic polynomial "det"$(tI - r(\varphi))$ with respect to the Fuglede-Kadison determinant.

In the later sections, we explain that the characteristic polynomial of $r(\varphi)$ is defined as a real number and it essentially gives the L^2-torsion and the hyperbolic volume of the mapping torus of φ. Moreover taking the lower central series of the surface group Γ, we obtain a family of Magnus representations, so that we can introduce a sequence of L^2-torsion invariants as an approximate sequence of the hyperbolic volume.

This paper is organized as follows. In the next section, we briefly recall the definition of the Fuglede-Kadison determinant. In Section 3, we summarize some properties of the L^2-torsion of 3-manifolds and explain a relation to the Magnus representation. We introduce a sequence of L^2-torsion invariants for a surface bundle over the circle in Section 4 and give some formulas for them in Section 5. In the last section, we discuss some vanishing theorems for L^2-torsion invariants.

§2. Fuglede-Kadison determinant

In this section, we review the combinatorial definition of the Fuglede-Kadison determinant over a non-commutative group ring and its basic properties (see [19] for details).

The idea to define a determinant over a group ring comes from the following observation. That is, for a matrix $B \in GL(n, \mathbb{C})$ with the (non-zero) eigenvalues $\lambda_1, \dots, \lambda_n$, we can formally calculate

$$\begin{aligned}\log|\det B|^2 &= \log\prod_{i=1}^{n}\lambda_i\overline{\lambda_i} = \sum_{i=1}^{n}\log\lambda_i\overline{\lambda_i} \\ &= \sum_{i=1}^{n}\left(\sum_{p=1}^{\infty}\frac{(-1)^{p+1}}{p}\left(\lambda_i\overline{\lambda_i}-1\right)^p\right) \\ &= -\sum_{p=1}^{\infty}\left(\sum_{i=1}^{n}\frac{1}{p}\left(1-\lambda_i\overline{\lambda_i}\right)^p\right) \\ &= -\sum_{p=1}^{\infty}\frac{1}{p}\mathrm{tr}\left((I-BB^*)^p\right)\end{aligned}$$

by the power series expansion of log, where B^* is the adjoint matrix of B. More precisely, if we take a sufficiently large constant $K > 0$, we obtain

$$|\det B| = K^n \exp\left(-\frac{1}{2}\sum_{p=1}^{\infty}\frac{1}{p}\mathrm{tr}\left(\left(I-K^{-2}BB^*\right)^p\right)\right) \in \mathbb{R}_{>0}.$$

Thus if we can define a certain "trace" over a group ring, we get a determinant by using this formula.

Let π be a discrete group and $\mathbb{C}\pi$ denote its group ring over $\mathbb{C}$. For an element $\sum_{g\in\pi}\lambda_g g \in \mathbb{C}\pi$, we define the $\mathbb{C}\pi$-trace $\mathrm{tr}_{\mathbb{C}\pi} : \mathbb{C}\pi \to \mathbb{C}$ by

$$\mathrm{tr}_{\mathbb{C}\pi}\left(\sum_{g\in\pi}\lambda_g g\right) = \lambda_e \in \mathbb{C},$$

where e is the unit element in π. For an $n \times n$-matrix $B = (b_{ij}) \in M(n, \mathbb{C}\pi)$, we extend the definition of $\mathbb{C}\pi$-trace by means of

$$\mathrm{tr}_{\mathbb{C}\pi}(B) = \sum_{i=1}^{n} \mathrm{tr}_{\mathbb{C}\pi}(b_{ii}).$$

Next let us recall the definition of the L^2-Betti number of an $n \times m$-matrix $B \in M(n, m, \mathbb{C}\pi)$. We consider the bounded π-equivariant operator

$$R_B : \oplus_{i=1}^{n} l^2(\pi) \to \oplus_{i=1}^{m} l^2(\pi)$$

defined by the natural right action of B. Here $l^2(\pi)$ is the complex Hilbert space of the formal sums $\sum_{g\in\pi} \lambda_g g$ which are square summable. We fix a positive real number K so that $K \geq ||R_B||_\infty$ holds, where $||R_B||_\infty$ is the operator norm of R_B.

Definition 2.1. The L^2-Betti number of a matrix $B \in M(n, m, \mathbb{C}\pi)$ is defined by

$$b(B) = \lim_{p\to\infty} \mathrm{tr}_{\mathbb{C}\pi} \left((I - K^{-2}BB^*)^p \right) \in \mathbb{R}_{\geq 0},$$

where $B^* = (\bar{b}_{ji})$ and $\overline{\sum \lambda_g g} = \sum \bar{\lambda}_g g^{-1}$ for each entry.

Roughly speaking, the L^2-Betti number $b(B)$ measures the size of the kernel of a matrix B. Hereafter we assume $b(B) = 0$. Then, for a matrix with coefficients in a non-commutative group ring, we can introduce the desired determinant as follows.

Definition 2.2. The Fuglede-Kadison determinant of a matrix $B \in M(n, m, \mathbb{C}\pi)$ is defined by

$$\det_{\mathbb{C}\pi}(B) = K^n \exp\left(-\frac{1}{2} \sum_{p=1}^{\infty} \frac{1}{p} \mathrm{tr}_{\mathbb{C}\pi} \left((I - K^{-2}BB^*)^p \right) \right) \in \mathbb{R}_{>0},$$

if the infinite sum of non-negative real numbers in the above exponential converges to a real number.

Remark 2.3. It is shown that the L^2-Betti number $b(B)$ and the Fuglede-Kadison determinant $\det_{\mathbb{C}\pi}(B)$ are independent of the choice of the constant K (see [16] for example).

Here we consider the condition of the convergence. For any matrix $B \in M(n, \mathbb{C})$, the condition

$$\lim_{p\to\infty} \mathrm{tr} \left((I - K^{-2}BB^*)^p \right) = 0$$

implies that B has no zero eigenvalues, and then $|\det B|$ converges. In the case of group rings, if $\det_{\mathbb{C}\pi}(B)$ converges, then $b(B) = 0$. But it is not a sufficient condition, so that we need additional one. It is a problem to decide when $\det_{\mathbb{C}\pi}(B)$ converges. Under the assumption that $b(B) = 0$, such a sufficient condition is given by the positivity of the Novikov-Shubin invariant $\alpha(B)$. Then the convergence of the infinite sum in the Fuglede-Kadison determinant is guaranteed. The Novikov-Shubin invariant of an operator R_B measures how concentrated the spectrum of $R_B^* R_B$ is. However, in general, it is hard to check the positivity of the Novikov-Shubin invariant.

To avoid the difficulty, we need to consider the determinant class condition for groups (see [19], [24] for details). A group π is of det ≥ 1-class if for any $B \in M(n, m, \mathbb{Z}\pi)$ the Fuglede-Kadison determinant of B satisfies $\det_{\mathbb{C}\pi}(B) \geq 1$. There are no known examples of groups which are not of det ≥ 1-class. Further recently it was proved that there is a certain large class $\mathcal{G}$ of groups for which they are of det ≥ 1-class. It includes amenable groups and countable residually finite groups. If we can see that π belongs to $\mathcal{G}$, namely it is of det ≥ 1-class, the convergence of the Fuglede-Kadison determinant is guaranteed when the L^2-Betti number is vanishing. See [18], [19], [24] for definitions and properties of these subjects.

§3. L^2-torsion of 3-manifolds

In this section, we quickly recall the definition of the L^2-torsion of 3-manifolds. It is an L^2-analogue of the Reidemeister and the Ray-Singer torsion and essentially gives Gromov's simplicial volume under certain general conditions [2], [3], [4], [8], [14], [15], [20], [21], [22]. See [19] and its references for historical background, related works and so on.

Let M be a compact connected orientable 3-manifold. We fix a CW-complex structure on M. We may assume that the action of $\pi_1 M$ on the universal covering $\widetilde{M}$ is cellular (if necessary, we have only to take a subdivision of the original structure). We consider the $\mathbb{C}\pi_1 M$-chain complex

$$0 \longrightarrow C_3(\widetilde{M}, \mathbb{C}) \xrightarrow{\partial_3} C_2(\widetilde{M}, \mathbb{C}) \xrightarrow{\partial_2} C_1(\widetilde{M}, \mathbb{C}) \xrightarrow{\partial_1} C_0(\widetilde{M}, \mathbb{C}) \longrightarrow 0$$

of $\widetilde{M}$. Since the boundary operator ∂_i is a matrix with coefficients in $\mathbb{C}\pi_1 M$, if we take the adjoint operator $\partial_i^* : C_{i-1}(\widetilde{M}, \mathbb{C}) \to C_i(\widetilde{M}, \mathbb{C})$ as in the previous section, we can define the ith (combinatorial) Laplace operator $\Delta_i : C_i(\widetilde{M}, \mathbb{C}) \to C_i(\widetilde{M}, \mathbb{C})$ by

$$\Delta_i = \partial_{i+1} \circ \partial_{i+1}^* + \partial_i^* \circ \partial_i.$$

Let us suppose that all the L^2-Betti numbers $b(\Delta_i)$ vanish and the fundamental group $\pi_1 M$ is of det ≥ 1-class. Thereby as a generalization of the classical Reidemeister torsion, the L^2-torsion $\tau(M)$ is defined by

Definition 3.1.

$$\tau(M) = \prod_{i=0}^{3} \det_{\mathbb{C}\pi_1 M}(\Delta_i)^{(-1)^{i+1} i} \in \mathbb{R}_{>0}.$$

As for the positivity of Novikov-Shubin invariants $\alpha(\Delta_i)$ for the Laplace operator Δ_i, it is known that $\alpha(\Delta_i) > 0$ holds under some general assumptions (see [15]). For example, if a compact connected orientable 3-manifold M satisfies

(1) $\pi_1 M$ is infinite,
(2) M is an irreducible 3-manifold or $S^1 \times S^2$ or $\mathbb{R}P^3 \sharp \mathbb{R}P^3$,
(3) if $\partial M \neq \phi$, it consists of tori,
(4) if $\partial M = \phi$, M is finitely covered by a 3-manifold which is a hyperbolic, Seifert or Haken 3-manifold,

then $b(\Delta_i) = 0$ and $\alpha(\Delta_i) > 0$ for each i. Therefore, we see that the L^2-torsion $\tau(M)$ is also well-defined in view of these conditions.

Remark 3.2. The above condition (4) is automatically satisfied by Perelman's proof of Thurston's Geometrization Conjecture.

As a notable property of the L^2-torsion, it is known that $\log \tau(M)$ can be interpreted as Gromov's simplicial volume $||M||$ and hyperbolic volume $\mathrm{vol}(M)$ (see [7]) of M. See [21] for the heart of the proof.

Theorem 3.3. *Let M be a compact connected orientable irreducible 3-manifold with an infinite fundamental group such that ∂M is empty or a disjoint union of incompressible tori. Then it holds that*

$$\log \tau(M) = C||M||,$$

where C is the universal constant not depending on M. In particular, if M is a hyperbolic 3-manifold, we obtain

$$\log \tau(M) = -\frac{1}{3\pi}\mathrm{vol}(M).$$

Next we review Lück's formula for the L^2-torsion of 3-manifolds ([16] Theorem 2.4). From this formula, we see that $\log \tau$ is a characteristic polynomial of the Magnus representation of the mapping class group.

Theorem 3.4. *Let M be as in the above theorem. We suppose that ∂M is non-empty and $\pi_1 M$ has a deficiency one presentation*

$$\langle s_1, \ldots, s_{n+1} \mid r_1, \ldots, r_n \rangle .$$

Put A to be the $n \times n$-matrix with entries in $\mathbb{Z}\pi_1 M$ obtained from the matrix $(\partial r_i / \partial s_j)$ by deleting one of the columns. Then the logarithm of the L^2-torsion of M is given by

$$\begin{aligned} \log \tau(M) &= -2 \log \det_{\mathbb{C}\pi_1 M}(A) \\ &= -2n \log K + \sum_{p=1}^{\infty} \frac{1}{p} \mathrm{tr}_{\mathbb{C}\pi_1 M} \left((I - K^{-2} A A^*)^p \right), \end{aligned}$$

where K is a constant satisfying $K \geq ||R_A||_\infty$.

To see a relation between the Magnus representation and the L^2-torsion, we describe the above Lück's formula for a surface bundle over the circle.

For an orientation preserving diffeomorphism φ of $\Sigma_{g,1}$, we form the mapping torus M_φ by taking the product $\Sigma_{g,1} \times [0,1]$ and gluing $\Sigma_{g,1} \times \{0\}$ and $\Sigma_{g,1} \times \{1\}$ via φ. This gives a surface bundle over S^1. Its diffeomorphism type is determined by the monodromy map φ, and conversely the monodromy map φ is determined by a given surface bundle up to conjugacy and isotopy. Here an isotopy fixes setwisely the points on the boundary $\partial\Sigma_{g,1}$. We take a deficiency one presentation of the fundamental group $\pi = \pi_1(M_\varphi, *)$,

$$\pi = \left\langle x_1, \ldots, x_{2g}, t \mid r_i : t x_i t^{-1} = \varphi_*(x_i),\ 1 \leq i \leq 2g \right\rangle,$$

where the base point $*$ of π and $\Gamma = \pi_1(\Sigma_{g,1}, *)$ is the same one on the fiber $\Sigma_{g,1} \times \{0\} \subset M_\varphi$ and $\varphi_* : \Gamma \to \Gamma$ is the automorphism induced by $\varphi : \Sigma_{g,1} \to \Sigma_{g,1}$. It should be noted that π is isomorphic to the semi-direct product of Γ and $\pi_1 S^1 \cong \mathbb{Z} = \langle t \rangle$.

Applying the free differential calculus to the relations r_i $(1 \leq i \leq 2g)$, we obtain the Alexander matrix

$$A = \left(\frac{\partial r_i}{\partial x_j} \right) \in M(2g, \mathbb{Z}\pi).$$

Then Lück's formula for a surface bundle over the circle is given by

$$\begin{aligned} \log \tau(M_\varphi) &= -2 \log \det_{\mathbb{C}\pi}(A) \\ &= -4g \log K + \sum_{p=1}^{\infty} \frac{1}{p} \mathrm{tr}_{\mathbb{C}\pi} \left((I - K^{-2} A A^*)^p \right), \end{aligned}$$

where K is a constant satisfying $K \geq ||R_A||_\infty$.

This formula enables us to interpret the L^2-torsion $\log \tau$ of a surface bundle over the circle as the characteristic polynomial of the Magnus representation $r(\varphi)$. In fact, an easy calculation shows that

$$A = \left(\frac{\partial r_i}{\partial x_j}\right) = tI - {}^t\overline{r(\varphi)}.$$

Then if we take the Fuglede-Kadison determinant in $M(2g, \mathbb{C}\pi)$, we have

$$\begin{aligned}\det{}_{\mathbb{C}\pi}\left(tI - {}^t\overline{r(\varphi)}\right) &= \det{}_{\mathbb{C}\pi}\left(tI - {}^t\overline{r(\varphi)}\right)^* \\ &= \det{}_{\mathbb{C}\pi}\left(t^{-1}I - r(\varphi)\right)\end{aligned}$$

because $\mathrm{tr}_{\mathbb{C}\pi}(BB^*) = \mathrm{tr}_{\mathbb{C}\pi}(B^*B)$ holds. Therefore the L^2-torsion is interpreted as the characteristic polynomial of $r(\varphi)$.

§4. Definition of L^2-torsion invariants

As was seen in Section 3, Lück's formula gives a way to calculate the simplicial volume from a presentation of the fundamental group. However, in general, it seems to be difficult to evaluate the exact values from the formula. In this section, we introduce a sequence of L^2-torsion invariants which approximates the original one for a surface bundle over the circle. See [12] for details.

In order to construct such a sequence of L^2-torsion invariants, we consider the lower central series of Γ. Namely, we take the descending infinite sequence

$$\Gamma_1 = \Gamma \supset \Gamma_2 \supset \cdots \supset \Gamma_k \supset \cdots,$$

where $\Gamma_k = [\Gamma_{k-1}, \Gamma_1]$ for $k \geq 2$. Let N_k be the kth nilpotent quotient $N_k = \Gamma/\Gamma_k$ and $p_k : \Gamma \to N_k$ be the natural projection.

In the previous section, we considered a chain complex $C_*(\widetilde{M_\varphi}, \mathbb{C})$ of $\mathbb{C}\pi$-modules. Instead of this complex, we can use the chain complex

$$C_*(M_\varphi, l^2(\pi)) = l^2(\pi) \otimes_{\mathbb{C}\pi} C_*(\widetilde{M_\varphi}, \mathbb{C})$$

to define the same L^2-torsion $\tau(M_\varphi)$. This point of view allows us to introduce a sequence of the L^2-torsion invariants.

The group Γ_k is a normal subgroup of π, so that we can take the quotient group $\pi(k) = \pi/\Gamma_k$. It should be noted that $\pi(k)$ is isomorphic to the semi-direct product $N_k \rtimes \mathbb{Z}$. We denote the induced projection

$\pi \to \pi(k)$ by the same letter p_k. Thereby we can consider the chain complex

$$C_*\left(M_\varphi, l^2(\pi(k))\right) = l^2(\pi(k)) \otimes_{\mathbb{C}\pi} C_*(\widetilde{M}_\varphi, \mathbb{C})$$

through the projection p_k. By using the Laplace operator

$$\Delta_i^{(k)} : C_i\left(M_\varphi, l^2(\pi(k))\right) \to C_i\left(M_\varphi, l^2(\pi(k))\right)$$

on this complex, we can formally define the kth L^2-torsion invariant $\tau_k(M_\varphi)$ as follows.

Definition 4.1.

$$\tau_k(M_\varphi) = \prod_{i=0}^{3} \det{}_{\mathbb{C}\pi(k)}(\Delta_i^{(k)})^{(-1)^{i+1}i}.$$

Of course, this definition is well-defined if every L^2-Betti number $b(\Delta_i^{(k)})$ vanishes and every $\pi(k)$ is of det ≥ 1-class. The next lemma is easily proved (see [12], [17]).

Lemma 4.2. *The L^2-Betti numbers of $\Delta_i^{(k)}$ are all zero.*

Recall the class $\mathcal{G}$ of groups. It is the smallest class of groups which contains the trivial group and is closed under the following processes: (i) amenable quotients, (ii) colimits, (iii) inverse limits, (iv) subgroups and (v) quotients with finite kernel (see [19], [24]). It is known that $\mathcal{G}$ contains all amenable groups. By definition, $N_k = \Gamma/\Gamma_k$ is a nilpotent group and in particular an amenable group. Hence every N_k belongs to $\mathcal{G}$. Further for any automorphism $\varphi_* : N_k \to N_k$, its mapping torus extension (*HNN*-extension) $N_k \rtimes \mathbb{Z}$ also belongs to $\mathcal{G}$. Therefore we have

Lemma 4.3. *The group $\pi(k)$ belongs to $\mathcal{G}$.*

As a result, we can conclude that our L^2-torsion invariants τ_k can be defined for any $k \geq 1$ and they are all homotopy invariants (see [19], [24]).

Now let us describe a formula of the kth L^2-torsion invariant $\tau_k(M_\varphi)$ and establish a relation to the Magnus representation of the mapping class group. Let $p_{k*} : \mathbb{C}\pi \to \mathbb{C}\pi(k)$ be an induced homomorphism over the group rings. For $k \geq 1$, we put

$$A_k = \left(p_{k*}\left(\frac{\partial r_i}{\partial x_j}\right)\right) \in M(2g, \mathbb{C}\pi(k)).$$

Moreover we fix a constant K_k satisfying $K_k \geq ||R_{A_k}||_\infty$. Then we have

$$\begin{aligned}\log \tau_k(M_\varphi) &= -2\log \det_{\mathbb{C}\pi(k)}(R_{A_k}) \\ &= -4g\log K_k + \sum_{p=1}^{\infty}\frac{1}{p}\mathrm{tr}_{\mathbb{C}\pi(k)}\left(\left(I - K_k^{-2}A_k{A_k}^*\right)^p\right),\end{aligned}$$

by virtue of the same argument as Theorem 3.4.

For the kth invariant τ_k, we have taken the lower central series $\{\Gamma_k\}$ of Γ and the nilpotent quotients $\{N_k\}$. These quotients induce a sequence of representations (more precisely, crossed homomorphisms)

$$r_k : \mathcal{M}_{g,1} \to GL(2g, \mathbb{Z}N_k)$$

for $k \geq 1$ (see [23]). They naively approximate the original Magnus representation $r : \mathcal{M}_{g,1} \to GL(2g, \mathbb{Z}\Gamma)$. By the similar observation as before, the kth invariant $\log \tau_k(M_\varphi)$ can be regarded as the characteristic polynomial of $r_k(\varphi)$ with respect to the Fuglede-Kadison determinant in $M(2g, \mathbb{C}\pi(k))$.

From the viewpoint of the Magnus representation of the mapping class group, it seems natural to raise the following problem.

Problem 4.4. *Show that the sequence* $\{\tau_k(M_\varphi)\}$ *converges to* $\tau(M_\varphi)$ *when we take the limit on* k.

In general, such an approximation problem for the L^2-torsion seems to be difficult. However, similar convergence results are known for the L^2-Betti numbers. In fact, Lück shows in [18] a theorem relating L^2-Betti numbers to ordinary Betti numbers of finite coverings. This result is generalized to more general settings by Schick in [24].

As for the Fuglede-Kadison determinant, Lück proves in [19] the following. Let $f : \mathbb{Q}[\mathbb{Z}] \to \mathbb{Q}[\mathbb{Z}]$ be the $\mathbb{Q}[\mathbb{Z}]$-map given by multiplication with $p(t) \in \mathbb{Q}[\mathbb{Z}]$ and $f_{(2)} : l^2(\mathbb{Z}) \to l^2(\mathbb{Z})$ be the linear operator obtained from f by tensoring with $l^2(\mathbb{Z})$ over $\mathbb{Q}[\mathbb{Z}]$. Further let $f_{[n]} : \mathbb{C}[\mathbb{Z}/n] \to \mathbb{C}[\mathbb{Z}/n]$ be the linear operator obtained from f by taking the tensor product with $\mathbb{C}[\mathbb{Z}/n]$ over $\mathbb{Q}[\mathbb{Z}]$. We then get an approximation result:

$$\log \det_{\mathbb{C}[\mathbb{Z}]}\left(f_{(2)}\right) = \lim_{n\to\infty}\frac{\log \det_{\mathbb{C}[\mathbb{Z}/n]}\left(f_{[n]}\right)}{n}$$

(see [11] for a similar statement). In [19] Lück also points out that there exists a purely algebraic example where Fuglede-Kadison determinants do not converge.

On the other hand, in general, we have at least an inequality for the Fuglede-Kadison determinant in the limit statement (see [24]). That is, for the operator R_{A_k} we see that

$$\log \det_{\mathbb{C}\pi}(R_A) \geq \limsup_k \log \det_{\mathbb{C}\pi(k)}(R_{A_k})$$

holds. In the last section, we shall discuss Problem 4.4 again and give an affirmative answer under certain conditions.

§5. Formulas of τ_1 and τ_2

In this section, we give explicit formulas of the first two invariants of a sequence of our L^2-torsion invariants. They are really computable formulas, so that we can make a systematic calculation for low genus cases. In particular, we compare them with hyperbolic volumes. The results discussed here are a summary of our previous paper [12] (see also [10], [11]).

First we consider the Magnus representation

$$r_1 : \mathcal{M}_{g,1} \to GL(2g, \mathbb{Z}N_1).$$

Here $N_1 = \Gamma/\Gamma_1$ is the trivial group and then the above representation is the same as the usual homological action of $\mathcal{M}_{g,1}$ on $H_1(\Sigma_{g,1}, \mathbb{Z})$. Namely we have the representation

$$r_1 : \mathcal{M}_{g,1} \to \mathrm{Aut}\,(H_1(\Sigma_{g,1}, \mathbb{Z}), \langle\ ,\ \rangle) \cong \mathrm{Sp}(2g, \mathbb{Z}),$$

where $\langle\ ,\ \rangle$ denotes the intersection form on the first homology group. Further $\pi(1) = \pi/\Gamma_1 \cong \mathbb{Z} = \langle t \rangle$ and its group ring $\mathbb{C}\langle t \rangle$ is a commutative Laurent polynomial ring $\mathbb{C}[t, t^{-1}]$. Then the matrix A_1 is nothing but the usual characteristic matrix of ${}^t r_1(\varphi)$. In this case, it is described by the usual determinant for a matrix with commutative entries.

In order to state the theorem, we recall a definition from number theory (see [6] and its references). For a Laurent polynomial $F(\mathbf{t}) \in \mathbb{C}[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$, the Mahler measure of F is defined by

$$m(F) = \int_0^1 \cdots \int_0^1 \log \left| F(e^{2\pi\sqrt{-1}\theta_1}, \dots, e^{2\pi\sqrt{-1}\theta_n}) \right| d\theta_1 \cdots d\theta_n,$$

where we assume that undefined terms are omitted. Namely we define the integrand to be zero whenever we hit a zero of F.

Theorem 5.1 ([12]). *The logarithm of the first invariant τ_1 is given by*

$$\log \tau_1(M_\varphi) = -2m\left(\Delta_{r_1(\varphi)}\right),$$

where $\Delta_{r_1(\varphi)}(t) = \det A_1 = \det(tI - r_1(\varphi))$. *Moreover if* $\Delta_{r_1(\varphi)}(t)$ *has a factorization* $\Delta_{r_1(\varphi)}(t) = \prod_{i=1}^{2g}(t - \alpha_i)$ $(\alpha_i \in \mathbb{C})$, *then we have*

$$\log \tau_1(M_\varphi) = -2 \sum_{i=1}^{2g} \log \max\{1, |\alpha_i|\}.$$

Remark 5.2. In other words, $\log \tau_1(M_\varphi)$ is given by the integral of the Alexander polynomial of M_φ over the circle S^1 (see [16], for the exterior of a knot K in the 3-sphere S^3). Further, $\log \tau_1(M_\varphi)$ can be described by the asymptotic behavior of the order of the first homology group of a cyclic covering (see [11]).

The point of the proof is to identify the Hilbert space $l^2(\mathbb{Z})$ with $L^2(\mathbb{R}/\mathbb{Z})$ in terms of the Fourier transforms. Then the $\mathbb{C}\langle t\rangle$-trace $\mathrm{tr}_{\mathbb{C}\langle t\rangle} : l^2(\mathbb{Z}) \to \mathbb{C}$ can be realized as the integration

$$L^2(\mathbb{R}/\mathbb{Z}) \ni f(\theta) \mapsto \int_0^1 f(\theta)d\theta \in \mathbb{C}$$

(see [12] for details). From this description and Kronecker's theorem ([6] Theorem 2), we obtain a certain vanishing theorem of the first invariant.

Corollary 5.3. *The logarithm of* $\tau_1(M_\varphi)$ *vanishes if and only if every eigenvalue of* $r_1(\varphi) \in \mathrm{Sp}(2g, \mathbb{Z})$ *is a root of unity.*

This corollary seems to be interesting from the viewpoint of Problem 4.4. Because in some case, we can say that the first invariant τ_1 already approximates the simplicial volume. In particular, Corollary 5.3 implies that a torus bundle M_φ $(g = 1)$ with a hyperbolic structure (namely, $|\mathrm{tr}\,(r_1(\varphi))| \geq 3$) has always non-trivial L^2-torsion invariant $\tau_1(M_\varphi)$. Summing up, we have

Corollary 5.4. *For any* $\varphi \in \mathcal{M}_{1,1}$, *its mapping torus* M_φ *admits a hyperbolic structure if and only if* M_φ *has a non-trivial* L^2*-torsion invariant* $\tau_1(M_\varphi)$.

Therefore, the first invariant τ_1 already approximates the simplicial volume in genus one case.

Remark 5.5. It is known that if the characteristic polynomial of $r_1(\varphi) \in \mathrm{Sp}(2g, \mathbb{Z})$ is irreducible over $\mathbb{Z}$, has no roots of unity as eigenvalues and is not a polynomial in t^n for any $n > 1$, then φ is pseudo-Anosov (see Casson-Bleiler [5]). In this case, $\mathrm{vol}(M_\varphi) \neq 0$ and further $\log \tau_1(M_\varphi) \neq 0$ by Corollary 5.3.

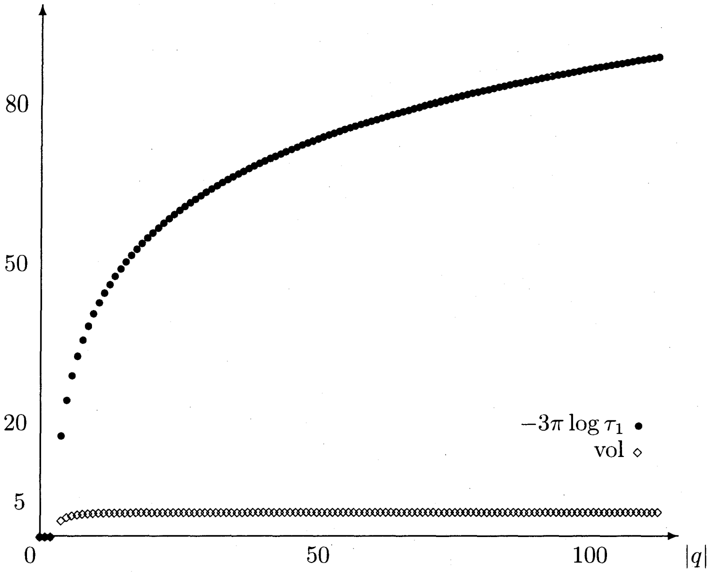

Fig. 1. $-3\pi \log \tau_1(M_\varphi)$ and $\mathrm{vol}(M_\varphi)$ vs. $|q|$

Example 5.6. It is well-known that the mapping class group of the two dimensional torus $T^2 = \mathbb{R}^2/\mathbb{Z}^2$ is isomorphic to $SL(2, \mathbb{Z})$. Taking a matrix $\begin{pmatrix} q & 1 \\ -1 & 0 \end{pmatrix} \in SL(2, \mathbb{Z})$, it gives a diffeomorphism φ on T^2. We may assume that it is the identity on some embedded disk by an isotopic deformation and it gives an element of $\mathcal{M}_{1,1}$. We use the same symbol φ for this mapping class. An easy calculation shows that

$$r_1(\varphi) = \begin{pmatrix} q & 1 \\ -1 & 0 \end{pmatrix}$$

and

$$\Delta_{r_1(\varphi)}(t) = \det(tI - r_1(\varphi)) = t^2 - qt + 1.$$

We put $\xi_\pm = (q \pm \sqrt{q^2 - 4})/2$ (the eigenvalues of the matrix $r_1(\varphi)$). If $|q| \leq 2$, then $|\xi_\pm| = 1$. Hence $\log \tau_1(M_\varphi) = 0$ in these cases. On the other hand, either $|\xi_+|$ or $|\xi_-|$ is greater than one when $|q| \geq 3$, so that M_φ has a non-trivial L^2-torsion invariant τ_1 in these cases. In fact, the logarithm of the first invariant is given by

$$\log \tau_1(M_\varphi) = -2 \log \max \{|\xi_+|, |\xi_-|\}.$$

The values of $\log \tau_1$ for the traces q and $-q$ are the same, so that it is a function of $|\mathrm{tr}(r_1(\varphi))|$. We put a graph of the L^2-torsion invariant

$-3\pi \log \tau_1(M_\varphi)$ and the hyperbolic volume $\mathrm{vol}(M_\varphi)$ as a function of $|q|$ in Fig 1.

Example 5.7. Next we consider the genus two case. Let $t_1, \ldots, t_5$ be the Lickorish-Humphries generators of $\mathcal{M}_{2,1}$. We take the element $\varphi = t_1 t_3 t_5{}^2 t_2^{-1} t_4^{-1} \in \mathcal{M}_{2,1}$. As was shown in [5], the characteristic polynomial of $r(\varphi)$ is

$$\begin{aligned}\Delta_{r_1(\varphi)}(t) &= \det(tI - r_1(\varphi)) \\ &= t^4 - 9t^3 + 21t^2 - 9t + 1\end{aligned}$$

and irreducible over $\mathbb{Z}$. Moreover it has no roots of unity as zeros. Hence, φ is pseudo-Anosov and M_φ has a non-trivial L^2-torsion invariant $\tau_1(M_\varphi)$. In fact, we have

$$-3\pi \log \tau_1(M_\varphi) = 52.954.... \quad \text{and} \quad \mathrm{vol}(M_\varphi) = 11.466....$$

Remark 5.8. In the above two examples, we used SnapPea [26] to compute the hyperbolic volumes.

Now in the following, we consider the second invariant τ_2. In the case of genus one, we can prove the vanishing of $\log \tau_2(M_\varphi)$.

Theorem 5.9 ([11]). $\log \tau_2(M_\varphi) = 0$ *for any* $\varphi \in \mathcal{M}_{1,1}$.

This follows from the fact that the group $\pi(2)$ is isomorphic to the fundamental group of a closed torus bundle over the circle. Such a 3-manifold admits no hyperbolic structures, so that the original L^2-torsion is trivial and we obtain the assertion.

On the other hand, in the case of $g \geq 2$, it is difficult to describe $\log \tau_2$ explicitly on the full mapping class group $\mathcal{M}_{g,1}$. However, we can do it on the Torelli group. Let φ be an element of the Torelli group $\mathcal{I}_{g,1}$, namely φ acts trivially on the first homology group $H_1(\Sigma_{g,1}, \mathbb{Z})$. Then we notice that $\log \tau_1(M_\varphi) = 0$ holds for any $\varphi \in \mathcal{I}_{g,1}$ (see Corollary 5.3). To give an explicit formula of $\log \tau_2$, we consider the Magnus representation

$$r_2 : \mathcal{M}_{g,1} \to GL(2g, \mathbb{Z}N_2),$$

where $N_2 = \Gamma/[\Gamma, \Gamma] \cong H_1(\Sigma_{g,1}, \mathbb{Z})$. If we restrict r_2 to the Torelli group $\mathcal{I}_{g,1}$, this is really a homomorphism (see [23] Corollary 5.4). Then our formula for the second L^2-torsion invariant is the following. The proof is similar to one for Theorem 5.1.

Theorem 5.10 ([12]). *For any mapping class* $\varphi \in \mathcal{I}_{g,1}$, *the logarithm of the second* L^2*-torsion invariant* $\tau_2(M_\varphi)$ *is given by*

$$\log \tau_2(M_\varphi) = -2m\left(\Delta_{r_2(\varphi)}\right),$$

where $\Delta_{r_2(\varphi)}(y_1,\dots,y_{2g},t) = \det A_2 = \det(tI - \overline{r_2(\varphi)})$ *and* y_i *denotes the homology class corresponding to* x_i.

Now we suppose $F(\mathbf{t}) \in \mathbb{Z}[t_1^{\pm 1},\dots,t_n^{\pm 1}]$ is primitive. We define F to be a generalized cyclotomic polynomial if it is a monomial times a product of one-variable cyclotomic polynomials evaluated at monomials.

The next corollary immediately follows from the theorem of Boyd, Lawton and Smyth (see [6] Theorem 4).

Corollary 5.11. *For any mapping class* $\varphi \in \mathcal{I}_{g,1}$, $\log \tau_2(M_\varphi) = 0$ *if and only if* $\Delta_{r_2(\varphi)}$ *is a generalized cyclotomic polynomial.*

As a typical element of the Torelli group $\mathcal{I}_{g,1}$, we first consider a BSCC-map φ_h $(1 \le h \le g)$ of genus h. That is, a Dehn twist along a bounding simple closed curve on $\Sigma_{g,1}$ which separates $\Sigma_{g,1}$ into $\Sigma_{h,1}$ and genus $g-h$ surface with two boundaries. We then see from [25] Corollary 4.3 that $\Delta_{r_2(\varphi_h)} = (t-1)^{2g}$. This is clearly a generalized cyclotomic polynomial, so that $\log \tau_2(M_{\varphi_h}) = 0$.

Second we consider a BP-map $\psi_h = D_c D_{c'}^{-1}$ of genus h $(1 \le h \le g-1)$, where c and c' are disjoint homologous simple closed curves on $\Sigma_{g,1}$ and D_c denotes the Dehn twist along c. Since

$$\Delta_{r_2(\psi_h)} = (t-1)^{2g-2h}(t - y_{g+h+1})^{2h}$$

holds (see [25]), where y_{g+h+1} denotes the homology class corresponding to the $(h+1)$th meridian of $\Sigma_{g,1}$, we also have $\log \tau_2(M_{\psi_h}) = 0$.

The next example shows the non-triviality of the second L^2-torsion invariant $\log \tau_2$.

Example 5.12. Let $\varphi = t_3 \varphi_1 t_3^{-1} \varphi_1 \in \mathcal{I}_{2,1}$. Then we see from a computation in [25] that

$$\Delta_{r_2(\varphi)} = (t-1)^4 + t(t-1)^2(y_1 - 2 + {y_1}^{-1})(y_2 - 2 + {y_2}^{-1}).$$

This is not a generalized cyclotomic polynomial, so that the mapping torus M_φ has a non-trivial L^2-torsion invariant $\tau_2(M_\varphi)$. In fact we can numerically compute it by means of Lawton's result (see [13]). More precisely we have

$$\begin{aligned} -3\pi \log \tau_2(M_\varphi) &= 6\pi m\left(\Delta_{r_2(\varphi)}\right) \\ &= 6\pi \lim_{r\to\infty} m\left(\Delta_{r_2(\varphi)}(u,u,u^r)\right) \\ &= 19.28.... \end{aligned}$$

§6. Vanishing of $\log \tau_k$ for reducible mapping classes

From the Nielsen-Thurston theory (see [5]), the mapping classes of a surface are classified into the following three types: (i) periodic, (ii) reducible and (iii) pseudo-Anosov. In our point of view, the most interesting object is a pseudo-Anosov map φ. Because the corresponding mapping torus M_φ has non-trivial hyperbolic volume.

In this final section, we show two vanishing theorems for $\log \tau_k$. We introduced an infinite sequence $\{\tau_k\}$ as an approximation of the hyperbolic volume. Thus if it behaves well with the index k, we ought to prove

$$\lim_{k\to\infty} \log \tau_k = 0$$

for non-hyperbolic 3-manifolds (see Problem 4.4). As a first step of this observation, we obtain the following.

Theorem 6.1. *If $\varphi \in \mathcal{M}_{g,1}$ is the product of Dehn twists along any disjoint non-separating simple closed curves on $\Sigma_{g,1}$ which are mutually non-homologous, then $\log \tau_k(M_\varphi) = 0$ for any $k \geq 1$.*

Remark 6.2. The mapping torus M_φ for $\varphi \in \mathcal{M}_{g,1}$ as above admits no hyperbolic structures, so that $\mathrm{vol}(M_\varphi) = 0$ holds.

Proof. At first, we prove the theorem for the genus one case. After that we give the outline of the proof in the higher genus case.

Let D_c be a Dehn twist along a non-separating simple closed curve c on $\Sigma_{1,1}$. Taking a conjugation, we can assume that the curve c is one of the standard generators of $\pi_1(\Sigma_{1,1})$. We then see that $\varphi = D_c{}^q$ is represented by a matrix $\begin{pmatrix} 1 & q \\ 0 & 1 \end{pmatrix} \in SL(2,\mathbb{Z})$. Thus we can choose a deficiency one presentation

$$\langle x, y, t \mid txt^{-1} = x,\ tyt^{-1} = x^q y \rangle$$

of $\pi_1(M_\varphi)$. Applying the free differential calculus to the relators $txt^{-1}x^{-1}$ and $tyt^{-1}(x^q y)^{-1}$, we obtain the Alexander matrix

$$A = \begin{pmatrix} t-1 & 0 \\ -\partial(x^q)/\partial x & t - x^q \end{pmatrix}.$$

Here we remark that the generators t and x can be commuted by the relation $txt^{-1} = x$. Hence in this case, the kth Alexander matrix A_k coincides with the original matrix A. In particular, t and x always commute. As we saw in Section 5, the L^2-torsion invariant $\tau_k(M_\varphi)$ ($k \geq$

1) can be computed by using the usual determinant and the Mahler measure in such a situation. Since

$$\det A = (t-1)(t-x^q)$$

is a generalized cyclotomic polynomial, we obtain $\log \tau_k(M_\varphi) = 0$ as desired (see Corollary 5.11).

In the higher genus case, we can assume that the mapping class φ is given by

$$\begin{aligned}
&\varphi_*(x_1) = x_1,\ \varphi_*(x_2) = x_1^{q_1} x_2, \ldots, \\
&\varphi_*(x_{2l-1}) = x_{2l-1},\ \varphi_*(x_{2l}) = x_{2l-1}^{q_l} x_{2l}, \\
&\varphi_*(x_{2l+1}) = x_{2l+1}, \ldots, \varphi_*(x_{2g}) = x_{2g}
\end{aligned}$$

by taking a conjugation, where $q_1, \ldots, q_l \in \mathbb{Z}$ and $1 \leq l \leq g-1$. We then obtain the following presentation of $\pi_1(M_\varphi)$:

$$\langle x_1, \ldots, x_{2g}, t \mid t x_i t^{-1} = \varphi_*(x_i),\ 1 \leq i \leq 2g \rangle.$$

Since the Alexander matrix A is the direct sum of the 2×2-matrix in the genus one case, we obtain $\log \tau_k(M_\varphi) = 0$ by the similar arguments. Q.E.D.

As another affirmative answer to Problem 4.4, we can show the vanishing of $\log \tau_k$ for the following mapping classes (see [12]). That is, we consider the case where there exists an integer n such that M_{φ^n} is topologically the product of $\Sigma_{g,1}$ and S^1. Here its bundle structure is non-trivial in general. Namely the nth power φ^n of a given monodromy φ is not trivial. A typical example is the Dehn twist along the simple closed curve on $\Sigma_{g,1}$ parallel to the boundary. The difference between an isotopy fixing the boundary pointwisely and such one setwisely, it gives birth to the difference between a bundle structure and a topological type. We then obtain

Theorem 6.3 ([12]). $\log \tau_k(M_\varphi) = 0$ *for any* $k \geq 1$.

It is easy to see that such a 3-manifold does not admit a hyperbolic structure. Hence it has trivial simplicial volume.

The above two examples are both non-hyperbolic cases, so that we conclude the present paper with the following problem.

Problem 6.4. *Show*

$$\lim_{k \to \infty} \log \tau_k(M_\varphi) = \log \tau(M_\varphi)$$

for a pseudo-Anosov diffeomorphism φ.

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References

[1] J. Birman, Braids, links, and mapping class groups, Ann. of Math. Stud., **82**, Princeton Univ. Press, Princeton, NJ, 1974.

[2] D. Burghelea, L. Friedlander, T. Kappeler and P. McDonald, Analytic and Reidemeister torsion for representations in finite type Hilbert modules, Geom. Funct. Anal., **6** (1996), 751–859.

[3] A. Carey, M. Farber and V. Mathai, Determinant lines, von Neumann algebras and L^2 torsion, J. Reine. Angew. Math., **484** (1997), 153–181.

[4] A. Carey and V. Mathai, L^2-torsion invariants, J. Funct. Anal., **110** (1992), 377–409.

[5] A. Casson and S. Bleiler, Automorphisms of surfaces after Nielsen and Thurston, Cambridge Univ. Press, Cambridge, 1988.

[6] G. Everest, Measuring the height of a polynomial, Math. Intelligencer, **20** (1998), 9–16.

[7] M. Gromov, Volume and bounded cohomology, Inst. Hautes Études Sci. Publ. Math., **56** (1982), 5–99.

[8] E. Hess and T. Schick, L^2-torsion of hyperbolic manifolds, Manuscripta Math., **97** (1998), 329–334.

[9] N. V. Ivanov, Mapping class groups, Handbook of geometric topology, North-Holland, Amsterdam, 2002, pp. 523–633.

[10] T. Kitano, T. Morifuji and M. Takasawa, Numerical calculation of L^2-torsion invariants, Interdiscip. Inform. Sci., **9** (2003), 35–42.

[11] T. Kitano, T. Morifuji and M. Takasawa, L^2-torsion invariants and homology growth of a torus bundle over S^1, Proc. Japan Acad. Ser. A Math. Sci., **79** (2003), 76–79.

[12] T. Kitano, T. Morifuji and M. Takasawa, L^2-torsion invariants of a surface bundle over S^1, J. Math. Soc. Japan, **56** (2004), 503–518.

[13] W. M. Lawton, A problem of Boyd concerning geometric means of polynomials, J. Number Theory, **16** (1983), 356–362.

[14] J. Lott, Heat kernels on covering spaces and topological invariants, J. Differential Geom., **35** (1992), 471–510.

[15] J. Lott and W. Lück, L^2-topological invariants of 3-manifolds, Invent. Math., **120** (1995), 15–60.

[16] W. Lück, L^2-torsion and 3-manifolds, Low-dimensional topology, Knoxville, TN, 1992, Conf. Proc. Lecture Notes Geom. Topology, III, Int. Press, Cambridge, MA, 1994, pp. 75–107.

[17] W. Lück, L^2-Betti numbers of mapping tori and groups, Topology, **33** (1994), 203–214.

[18] W. Lück, Approximating L^2-invariants by their finite-dimensional analogues, Geom. Funct. Anal., **4** (1994), 455–481.

[19] W. Lück, L^2-invariants: theory and applications to geometry and K-theory, Ergeb. Math. Grenzgeb. (3), **44**, Springer-Verlag, Berlin, 2002.

[20] W. Lück and M. Rothenberg, Reidemeister torsion and the K-theory of von Neumann algebras, K-Theory, **5** (1991), 213–264.

[21] W. Lück and T. Schick, L^2-torsion of hyperbolic manifolds of finite volume, Geom. Funct. Anal., **9** (1999), 518–567.

[22] V. Mathai, L^2-analytic torsion, J. Funct. Anal., **107** (1992), 369–386.

[23] S. Morita, Abelian quotients of subgroups of the mapping class group of surfaces, Duke Math. J., **70** (1993), 699–726.

[24] T. Schick, L^2-determinant class and approximation of L^2-Betti numbers, Trans. Amer. Math. Soc., **353** (2001), 3247–3265 (electronic).

[25] M. Suzuki, A class function on the Torelli group, Kodai Math. J., **26** (2003), 304–316.

[26] J. Weeks, SnapPea, `http://www.geometrygames.org/SnapPea` .

[27] H. Zieschang, E. Vogt and H-D. Coldewey, Surfaces and planar discontinuous groups, Translated from the German by John Stillwell, Lecture Notes in Math., **835**, Springer, Berlin, 1980.

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