Advanced Studies in Pure Mathematics 52, 2008
Groups of Diffeomorphisms
pp. 1–20

Asymptotic geometry of foliations and pseudo-Anosov flows — a survey

Sérgio R. Fenley

This expository article describes certain aspects of large scale geometry of foliations and flows in 3-manifolds with Gromov hyperbolic fundamental group. The first part is a survey of ideas and previous results about asymptotic behavior of leaves of foliations and flow lines. It reviews classical ideas such as quasi-isometries, quasigeodesics and ideal boundaries. This part includes a description of the seminal work of Cannon-Thurston on fibrations and suspension pseudo-Anosov flows as well as other important results concerning certain classes of foliations. The second part of the article describes more recent topics: What information can be obtained about the asymptotic structure of the universal cover of a manifold using only the dynamics of a pseudo-Anosov flow in the manifold? We explain how to create a dynamical systems ideal boundary for a certain class of such flows and a corresponding compactification of the universal cover. We then explain how these objects are strongly related to the large scale geometry of the manifolds, the large scale geometry of the flows themselves and also of some classes of foliations. Details and proofs of the results in the second part are found in [Fe8].

Let $\mathcal{F}$ be a Reebless foliation in M^3 with Gromov hyperbolic fundamental group, which is not virtually $\mathbf{Z}$. The lifted foliation to the universal cover $\widetilde{M}$ will be denoted by $\widetilde{\mathcal{F}}$. By Novikov's fundamental theorem, the leaves of $\mathcal{F}$ are incompressible and lift to simply connected leaves in $\widetilde{M}$ [No]. The leaves in $\widetilde{\mathcal{F}}$ cannot be spheres or else $\widetilde{M}$ would

Received July 13, 2007.
Revised October 9, 2007.
2000 *Mathematics Subject Classification.* Primary: 53C23, 57R30, 37C85, 37D20; Secondary: 57M50, 37D50, 58D19.
Research partially supported by NSF grant DMS-0305313.

be $\mathbf{S}^2 \times \mathbf{R}$ and $\pi_1(M)$ would be virtually $\mathbf{Z}$. Hence the leaves of $\widetilde{\mathcal{F}}$ are topological planes and they are properly embedded in $\widetilde{M}$ [No]. As $\mathcal{F}$ is Reebless then M is irreducible [Ros] and the universal cover of M is homeomorphic to $\mathbf{R}^3$ [Pa].

In terms of geometry, the universal cover $\widetilde{M}$ is Gromov hyperbolic and can be compactified with an ideal boundary, the Gromov ideal boundary of $\widetilde{M}$ [Gr]. Since M is irreducible and $\pi_1(M)$ is not virtually $\mathbf{Z}$, then Bestvina and Mess [Be-Me] proved that the ideal boundary is a topological 2-sphere S^2_∞. The *limit set* of a subset B of $\widetilde{M}$ is the set of accumulation points of B in S^2_∞. For such manifolds, the relationship between sets in $\widetilde{M}$ and their respective limit sets in S^2_∞ is of the utmost importance [Mo, Th1, Th2, Th3, Gr]. For background in Gromov hyperbolic groups and spaces see [Gr, Gh-Ha].

Notice that if M is closed, irreducible, atoroidal and with infinite fundamental group (which is not virtually $\mathbf{Z}$), then Perelman's results [Pe1, Pe2, Pe3] imply that M is hyperbolic. Here we do not make use of Perelman's results. In particular one goal of the second part of the article is to analyse which geometric information about M can be obtained solely from dynamical systems constructions.

The leaves of $\widetilde{\mathcal{F}}$ are properly embedded in $\widetilde{M}$ and hence only accumulate in the sphere at infinity. For such foliations there are two very important questions [Ga5]:

Property/Question 1) How do leaves approach their limit sets? Is it in a continuous manner? We explain continuity below;

Property/Question 2) Are leaves geometrically good in $\widetilde{M}$, that is, does distance along the leaves compare well with distance in the manifold?

Let us first explain question 2). Let $k \geq 1, C \geq 0$. A (k, C) *quasi-isometric embedding* between metric spaces (X, d), (X', d') is a map f from X to X' so that for any a, b in X then

$$\frac{1}{k} d(a,b) - C \ \leq \ d'(f(a), f(b)) \ \leq \ k d(a,b) + C.$$

The rough meaning of this is as follows: the constant C (which can be quite big) means that small distances ($\leq C$) do not matter. In particular the map f does not need to be continuous at all. The meaning of k is that for big distances the map f is essentially k-bilipschitz. A *quasi-isometric embedding* is a map which is a (k, C) quasi-isometric embedding for some (k, C). A *quasi-isometry* is a quasi-isometric embedding for which there is some positive C' so that $f(X)$ is C' dense in

X'. Then up to this type of error in distances, the metric spaces X, X' are the same. Quasi-isometries are extremely important in hyperbolic spaces: Mostow used quasi-isometries to prove a very strong rigidity result for hyperbolic manifolds [Mo]. Quasi-isometries were also extremely important in Thurston's proof of the geometrization conjecture in the Haken case [Th1, Th2, Th3, Mor].

For a Reebless foliation in such M^3 then one can also show that the leaves are uniformly Gromov hyperbolic [Pl, Su, Gr]. The meaning of question 2) above for foliations in M^3 with Gromov hyperbolic fundamental group is the following: Consider a leaf F of $\widetilde{\mathcal{F}}$ with its path metric and we ask whether the inclusion $i : F \to \widetilde{M}$ is a quasi-isometric embedding. If this is true then the limit set of F is a Jordan curve in S^2_∞ [Th1, Gr]. If every leaf F of $\widetilde{\mathcal{F}}$ is quasi-isometrically embedded in $\widetilde{M}$, then we say that $\mathcal{F}$ is a *quasi-isometric foliation.* Notice that uniformity of the quasi-isometry constants is <u>not required</u>. See remark below on the existence of these foliations.

We now explain question 1). In addition to the leaves being uniformly Gromov hyperbolic, in fact there is a metric in M so that leaves are hyperbolic [Ca] (meaning sectional curvature is constant $= -1$). Hence the inclusion $i : F \to \widetilde{M}$ of a leaf of $\widetilde{\mathcal{F}}$ can be thought of as a map from the hyperbolic plane $\mathbf{H}^2$ to $\widetilde{M}$. The hyperbolic plane has a canonical compactification to a closed disk with a circle at infinity S^1_∞. So the asymptotic behavior of leaves of $\widetilde{\mathcal{F}}$ or how they approach their limit sets can be phrased in the following way: given the inclusion $i : F \to \widetilde{M}$ does it extend to a continuous map between compactifications $F \cup S^1_\infty \to \widetilde{M} \cup S^2_\infty$? If this is the case for all leaves of $\widetilde{\mathcal{F}}$ then we say that $\mathcal{F}$ has the *continuous extension property.* The restriction to S^1_∞ expresses the limit set of the respective leaf as a continuous image of the circle. It is a classical result that for a given leaf F of $\widetilde{\mathcal{F}}$ property 2) is stronger than property 1) see [Mo, Th2, Gr]. In this way questions 1) and 2) are basic questions about the relationship between the foliation structure and the large scale geometric structure of the universal cover of these manifolds.

The level zero result in this area is that there are no quasi-isometric foliations in M^3 with $\pi_1(M)$ Gromov hyperbolic [Fe1]. In particular for every foliation there is at least one leaf of $\widetilde{\mathcal{F}}$ which is not quasi-isometrically embedded [Fe1]. This is easy to show if the leaf space of $\widetilde{\mathcal{F}}$ is not Hausdorff. In case this leaf space is Hausdorff, then it is homeomorphic to the real numbers $\mathbf{R}$. In that case the foliation is called an $\mathbf{R}$-*covered* foliation. Then the limit set of every leaf of $\widetilde{\mathcal{F}}$

is the whole sphere [Fe1]. It follows that in this case no leaf of $\widetilde{\mathcal{F}}$ is quasi-isometrically embedded. We remark that this particular result holds in higher dimensions: if $\pi_1(M^n)$ is Gromov hyperbolic and $\mathcal{F}$ is a codimension one foliation with Gromov hyperbolic leaves, then $\mathcal{F}$ is not a quasi-isometric foliation. For the remainder of the article we restrict to M being 3-dimensional.

By the the above remarks, question 2) always has a negative answer as stated. However if we consider singular foliations of a certain type, there are some examples of quasi-isometric foliations as we will see below.

As a result of the negative answer to question 2), then for general foliations one considers the continuous extension question. How can one think about this? The first situation that comes to mind is that of a compact leaf S of $\mathcal{F}$. What happens in this case? The compact leaf is incompressible or, in other words, π_1-injective (if S is two sided) [No]. It follows that M is Haken and therefore hyperbolic, by Thurston's proof of geometrization in the Haken case [Th1, Th2, Mor]. Here there is a fundamental dichotomy proved using results of Thurston [Th2] and Bonahon [Bon]: either the geometry of the compact leaf is very well behaved or is very badly distorted. The result is that either $\pi_1(S)$ corresponds to a quasi-Fuchsian group or S is a virtual fiber of a fibration over the circle. One of the equivalent ways to characterize the first situation is that any lift $\widetilde{S}$ to $\widetilde{M}$ is quasi-isometrically embedded in $\widetilde{M}$. In this case the limit set of any such $\widetilde{S}$ is a Jordan curve. Geometrically this is a very well behaved situation. The second possibility implies that the limit set of any such $\widetilde{S}$ is the whole sphere. This is the geometrically wild situation. Thurston wondered what could be said about virtual fibers, in particular whether their lifts to $\widetilde{M}$ extend continuously to the ideal boundary (In fact, before that, Thurston wondered if manifolds fibering over the circle could be hyperbolic. If that were true, then the limit set of the lift of a fiber is the whole sphere and that was completely mysterious at that point.. The double limit theorem was the main new tool to prove geometrization in that case [Th3]).

In a seminal work, Cannon and Thurston [Ca-Th] studied fibrations over the circle and proved the continuous extension property for them. This work was done in the early 1980's but only very recently was published.

How was this done in the fibration case? At this point we should introduce a fundamental tool in the study of foliations in 3-manifolds. In the case of a fibration in M^3 closed, atoroidal, it follows that the monodromy of the fiber is a *pseudo-Anosov* homeomorphism [Th4, Bl-Ca].

In particular it has stable and unstable singular 1-dimensional measured foliations in the fiber which are transverse to each other. These measured foliations are projectively invariant under a representative of the monodromy of the fibration. The singularities are of p-prong type. The suspension flow associated to this map preserves the suspended (2-dimensional) foliations and is an example of a pseudo-Anosov flow, see general definition below. Cannon and Thurston [Ca-Th] introduced a metric in M called the singular solv metric, which is quasi-isometric to the ambient metric when lifted to $\widetilde{M}$ and so that: given any stable/unstable leaf L in the universal cover then there is a distance decreasing retraction of $\widetilde{M}$ into L. This implies that L is quasi-isometrically embedded in the original metric of $\widetilde{M}$. So the leaves of the stable/unstable foliations are quasi-isometrically embedded. We stress that the stable/unstable foliations are singular 2-dimensional foliations. This shows that question 2) can have a positive answer if one considers certain types of singular 2-dim foliations in M^3 with $\pi_1(M)$ Gromov hyperbolic. The singular solv metric is obtained using the structure of pseudo-Anosov homeomorphisms of surfaces.

An important property here is that if L is quasi-isometrically embedded then L is a bounded distance (depending on the constants k, C) from the convex hull of the limit set of L. This also works for manifolds with $\pi_1(M)$ Gromov hyperbolic [Gr, Gh-Ha]. Hence if leaves L are far away from a base point it turns out that their limit sets are very small when seen from the base point in the visual measure. Cannon and Thurston use information on how stable/unstable leaves intersect leaves of $\widetilde{\mathcal{F}}$ to show that the leaves of $\widetilde{\mathcal{F}}$ (for $\mathcal{F}$ a fibration) extend continuously to the circle at infinity.

The main facts used here were the following:

I) A (suspension) pseudo-Anosov flow Φ transverse to $\mathcal{F}$ so that its stable and unstable foliations are quasi-isometric (leaves are quasi-isometrically embedded in $\widetilde{M}$),

II) Detailed information of how stable/unstable leaves in $\widetilde{M}$ intersect leaves of $\widetilde{\mathcal{F}}$. In particular considering the ideal compactification of a leaf F of $\widetilde{\mathcal{F}}$ with an ideal circle S^1_∞ then: intrinsic ideal points of F have neighborhood systems defined by intersections of F with stable/unstable leaves in the universal cover.

Information I) is strong metric information and information II) refers to topology and dynamical systems.

We now define a general pseudo-Anosov flow. Let Φ be a flow in M^3 closed. Then Φ is a *pseudo-Anosov flow* if the following conditions hold [Mo1, Fe8]: Φ has no point orbits and there are two dimensional foliations Λ^s, Λ^u, stable and unstable respectively , which are possibly singular and so that

a) There are at most finitely many "singular" orbits and off of the singular orbits Λ^s, Λ^u are non singular foliations, transverse to each other and intersect exactly along the flow lines of Φ;

b) A singular orbit α is a closed orbit and a leaf of Λ^s (or Λ^u) containing α is homeomorphic to $(P \times I)/f$ where P is a p-prong in the plane ($p \geq 3$) and f is a homeomorphism from $P \times \{1\}$ to $P \times \{0\}$;

c) Finally all orbits in a leaf of Λ^s are forward asymptotic and likewise in a leaf Λ^u the orbits are asymptotic in the backward direction.

Pseudo-Anosov flows are extremely common amongst 3-manifolds: some classes of examples are: 1) Anosov flows. For example geodesic flows in the unit tangent bundle of closed surfaces of negative Gaussian curvature. Also suspensions of Anosov diffeomorphisms of the torus (here there are no singular orbits); 2) Suspension pseudo-Anosov flows; 3) Pseudo-Anosov flows survive most Dehn surgeries – perform Dehn surgery along a closed orbit of the flow (suppose orientation preserving curve), then outside of two lines in Dehn surgery space it follows that the resulting flow is still pseudo-Anosov [Fri]. In particular, using Thurston's Dehn surgery theorem [Th1], it follows that there is an enormous class of pseudo-Anosov flows where the underlying manifold is hyperbolic.

We should remark that foliations come in many different flavors. Some have compact leaves, others have all leaves dense in the manifold, some are **R**-covered, some have one sided branching (see definition below). Some are strongly connected with the topology of the manifold (finite depth foliations – see definition below); others have strong dynamical properties (Anosov foliations). A priori there is not a single method that allows one to study the continuous extension property for all of them at the same time.

We now describe a couple of important classes of foliations for which something is known concerning the continuous extension property. Both of these results were obtained right around the same time in the late 90's.

The first result concerns finite depth foliations. A foliation is *proper* if all its leaves are properly embedded in the manifold (here it is M, not $\widetilde{M}$). In particular they never limit in themselves (in the foliation sense). By minimality there are always compact leaves which are defined to have depth zero. The depth is then defined inductively: a leaf has depth i, if the maximum depth of the leaves where it limits on is $i-1$ (hence a depth one leaf only limits on compact leaves). The foliation $\mathcal{F}$ is said to be *finite depth* if it is proper and there is n positive integer which is an upper bound to the depths of all the leaves. The minimum such n is the depth of the foliation. In a fundamental paper Gabai [Ga1] proved that if M closed, oriented, irreducible, has non finite second homology then it has Reebless, finite depth foliations. These foliations are directly associated with a hierarchy of the manifold [Fe-Mo]: one cuts along leaves and finally arrives at a collection of balls. As such, these foliations have been successfully used by Gabai and others to obtain deep results in 3-dimensional topology [Ga1, Ga2, Ga3].

As in the fibration case one looks for a transverse pseudo-Anosov flow. This was studied by Mosher and Gabai [Mo2], but in general one can only obtain almost transversality: A pseudo-Anosov flow Φ is *almost transverse* to a foliation $\mathcal{F}$ if the following happens [Mo2]: after a possible blow up of some singular orbits of Φ into a collection of annuli (a type of flow blow up), then one obtains a flow which is transverse to the foliation $\mathcal{F}$. When there is a blow up, then the resulting flow is not pseudo-Anosov, but it still has stable/unstable singular foliations (which are not transverse to each other at the blown up annuli – they both contain the blown up annuli). This flow has several important properties. Given $\mathcal{F}$ a Reebless, transversely orientable, finite depth foliation in an atoroidal closed 3-manifold, then Mosher and Gabai [Mo2] constructed a pseudo-Anosov flow Φ which is almost transverse to $\mathcal{F}$. This also has consequences for manifolds obtained by Dehn surgeries on certain orbits of the flow [Mo2]. Concerning the asymptotic behavior of foliations the following happens: in some cases where $\mathcal{F}$ has depth one, we were able to prove that the stable/unstable foliations of Φ are quasi-isometric foliations [Fe4]. These were the first examples of quasi-isometric singular foliations which were not obtained from suspension pseudo-Anosov flows. In some situations we were also able to prove that in the universal cover, the intersections of leaves of $\widetilde{\mathcal{F}}$ with stable/unstable leaves have good properties. This allowed us to prove the continuous extension property for some depth one foliations [Fe4].

Another result concerning the continuous extension property was obtained by Thurston [Th5]. He considered uniform foliations and slitherings. A foliation is *uniform* if for any two leaves in $\widetilde{M}$ they are at a finite Hausdorff distance from each other. Assuming that the foliation is Reebless this implies that $\mathcal{F}$ is $\mathbf{R}$-covered. A related concept is that of slitherings: a *slithering* is a map f from the universal covering $\widetilde{M}$ of M to the circle, so that f is a submersion everywhere and f is group equivariant under the action of $\pi_1(M)$ by covering translations. See details in [Th5]. The slithering map f induces an $\mathbf{R}$-covered foliation in M which is uniform. In [Th5], Thurston proved, amongst other things, that if M is atoroidal, acylindrical with a transversely orientable, uniform foliation then it has a transverse pseudo-Anosov flow Φ. The proof is a very clever extension of the fibration situation. He produces the universal circle for such foliations as defined in [Th6, Th7] to relate the circles at infinity of different leaves of $\widetilde{\mathcal{F}}$. In the atoroidal case he produces 2 laminations transverse to $\mathcal{F}$ and transverse to each other, which blow down to a pseudo-Anosov flow transverse to $\mathcal{F}$ [Th5]. Once that is done, then using the uniform property it is not hard to show that there is a retraction from $\widetilde{M}$ to stable/unstable leaves which does not increase distances too much. The continuous extension follows as in the fibering case.

The following result is a much more recent result about the continuous extension property. In the Cannon-Thurston result they obtained a pseudo-Anosov flow with stable/unstable foliations which are quasi-isometric. There is a weaker property, namely that the flow Φ is quasigeodesic. A *quasigeodesic* in $\widetilde{M}$ is a quasi-isometric embedding $g: I \to \widetilde{M}$, where I is an interval in $\mathbf{R}$ or the integers. Given a flow Φ let $\widetilde{\Phi}$ be the lifted flow to $\widetilde{M}$. A flow Φ is quasigeodesic if all flow lines of $\widetilde{\Phi}$ in $\widetilde{M}$ are quasigeodesics. As for quasi-isometric objects, it is also true that quasigeodesics are extremely important in manifolds with Gromov hyperbolic fundamental group [Mo, Th1, Gr, Gh-Ha]. It is not too hard to prove that if Λ^s (or Λ^u) is a quasi-isometric foliation, then the flow Φ is a quasigeodesic flow [Fe4]. The result below is very recent and shows that the concept of quasigeodesic flows is extremely useful for studying asymptotic behavior of foliations:

Theorem 1. [Fe7] *Let $\mathcal{F}$ be a foliation in M^3 closed with $\pi_1(M)$ Gromov hyperbolic. Suppose that there is a pseudo-Anosov flow Φ which is almost transverse to $\mathcal{F}$ and so that Φ is a quasigeodesic flow. Then $\mathcal{F}$ has the continuous extension property.*

We first remark that if $\mathcal{F}$ is almost transverse to a pseudo-Anosov flow, then it follows that $\mathcal{F}$ is necessarily a Reebless foliation [Fe7]. For the purposes of this survey article we restrict ourselves to the situation when the flow is actually transverse to the foliation. In order to prove theorem 1, one first proves the following: given a leaf F of $\widetilde{\mathcal{F}}$ and a leaf L of $\widetilde{\Lambda}^s$ then the intersection $L \cap F$ is a 1-dimensional set in F, which may have prongs. We show that each ray of $L \cap F$ accumulates only in a single point in the ideal boundary $\partial_\infty F$ of F. This allows us to describe the neighborhood system of a point q in $\partial_\infty F$ using only intersections of F with leaves of $\widetilde{\Lambda}^s$ and $\widetilde{\Lambda}^u$. This gives the property II) which was mentioned in the Cannon-Thurston analysis. This description does not use any geometric hypothesis on M or Φ – it uses only dynamical properties of the flow. The next step is to use the geometry. The fact that flow lines of $\widetilde{\Phi}$ are quasigeodesics is weaker than the foliations Λ^s, Λ^u being quasi-isometric. But quasigeodesic behavior of Φ still has geometric consequences for the stable/unstable leaves of $\widetilde{\Lambda}^s, \widetilde{\Lambda}^u$ in $\widetilde{M}$. A quasigeodesic is a bounded distance (depending on the quasi-isometric constants (k, C)) from a minimal geodesic in $\widetilde{M}$ [Gr, Gh-Ha]. It follows that the convex hulls of the limit sets of the leaves of $\widetilde{\Lambda}^s, \widetilde{\Lambda}^u$ in $\widetilde{M}$ are at a bounded distance from the leaves themselves. This property together with fact II) and an extended analysis of the topological situation allows one to prove the continuous extension property for $\mathcal{F}$ under the hypothesis of theorem 1. See details in [Fe7].

One case where theorem 1 was successfully used is that of finite depth foliations. We previously proved jointly with Mosher that the pseudo-Anosov flows constructed by Mosher and Gabai which are almost transverse to finite depth foliations are quasigeodesic [Fe-Mo]. Theorem 1 then implies the continuous extension property for all Reebless, finite depth foliations.

Notice also that the condition on $\mathcal{F}$ in theorem 1 is an open condition: if it is satisfied by $\mathcal{F}$, then it is satisfied by any $\mathcal{F}'$ sufficiently near $\mathcal{F}$. This allows one to prove the continuous extension property for many other foliations.

The upshot is that in order to analyse the continuous extension property for foliations, one approach is to find an almost transverse pseudo-Anosov flow which is quasigeodesic. Notice that this has happened in all the situations analysed so far. The big question is: how does one find a quasigeodesic flow transverse to $\mathcal{F}$? In the fibration case it is extremely easy to prove that any transverse flow (pseudo-Anosov or not) is a quasigeodesic flow. This is because any such flow is a suspension flow and using geometric properties of the foliation it is easy to see that the flow is quasigeodesic. It is also very easy to prove quasigeodesic behavior for large classes of transverse flows in the uniform case.

How was the quasigeodesic behavior obtained for Φ in the finite depth foliation? Assuming that $\mathcal{F}$ is not a fake fibration over the circle, then it has compact leaves which are quasi-Fuchsian and therefore any lift F of this compact leaf A to $\widetilde{M}$ is quasi-isometrically embedded [Th1, Th2, Bon]. Notice that this is highly non trivial geometric information from the foliation. This implies that flow lines of Φ which keep intersecting compact leaves in M lift to flow lines of $\widetilde{\Phi}$ which keep intersecting these quasi-isometric leaves of $\widetilde{\mathcal{F}}$ in $\widetilde{M}$. Here is the key observation: because the leaf A is compact, these intersections with lifts of A form a discrete subset of $\widetilde{M}$. As the quasi-isometric leaves escape in $\widetilde{M}$ their limit sets shrink to a point u of S^2_∞. This is because lifts of compact leaves to $\widetilde{M}$ are quasi-isometrically embedded. The point u is the only possible limit point for the given flow line of $\widetilde{\Phi}$. This genesis of an argument can be carefully expanded, using the explicit structure of finite depth foliations to show that the flow (almost) transverse to the foliation is a quasigeodesic flow.

We summarize what we have obtained so far: in all of these cases we start with a foliation which has certain strong geometric properties and using these geometric properties, one can show that a transverse pseudo-Anosov flow is quasigeodesic. Then using the geometric properties of the flow, we can prove the continuous extension property for the foliation.

The difficulty with this approach is that we have to start with some geometric property of the foliation. So in the case that $\mathcal{F}$ does not have compact leaves and is not uniform there is no geometric information to start with. Notice that this is the generic situation. For example this occurs if $\mathcal{F}$ has dense leaves and is not $\mathbf{R}$-covered, or if $\mathcal{F}$ is $\mathbf{R}$-covered and not uniform, or if $\mathcal{F}$ is not finite depth. How does one deal with such cases?

We now describe a new, different approach: in this approach one starts with the pseudo-Anosov flow and its dynamic properties and obtains asymptotic information about $\widetilde{M}$ and geometric information about the flow, using only the dynamics of the flow. This can then be used to analyse certain classes of foliations and prove the continuous extension property. In order to apply this approach we will only use topological or dynamical information about the foliation and will not need geometric information. So for the time being we completely forget foliations and concentrate on pseudo-Anosov flows.

The first goal is to analyse what a pseudo-Anosov flow can say about the asymptotic structure of the universal cover of the manifold. We first study the orbit space of the flow $\widetilde{\Phi}$ in $\widetilde{M}$. A very important basic fact is that the lifted flow to the universal cover is topologically a product flow and the orbit space $\mathcal{O}$ of this flow is homeomorphic to an open disk (or the plane) [Fe2, Fe-Mo]. This is just a topological statement – the geometric or homotopic behavior of the flow is not a product at all. The fundamental group of M acts on $\widetilde{M}$ by covering translations which preserve the collection of flow lines in $\widetilde{M}$. Hence $\pi_1(M)$ acts naturally on $\mathcal{O}$. As the flow is a product flow in $\widetilde{M}$, then the lifted stable and unstable foliations to $\widetilde{M}$ project to one dimensional (perhaps singular) foliations in $\mathcal{O}$. These are denoted by $\mathcal{O}^s, \mathcal{O}^u$, the stable and unstable foliations in $\mathcal{O}$. They are transverse to each other and have only p-prong singularities. These are singular 1-dimensional foliations in the plane. The fundamental group also preserves the stable/unstable foliations in $\widetilde{M}$ and therefore acts on $\mathcal{O}^s, \mathcal{O}^u$. With this topological structure in the orbit space $\mathcal{O}$ one can construct a natural ideal boundary to $\mathcal{O}$:

Theorem 2. [Fe8] *Let Φ be a pseudo-Anosov flow in M^3 closed. Let $\widetilde{\Phi}$ be the lifted flow to the universal cover and $\mathcal{O}$ be the orbit space of $\widetilde{\Phi}$. Then there is an ideal circle boundary $\partial\mathcal{O}$ obtained using only the topological and dynamical structure of the stable and unstable foliations in $\mathcal{O}$. There is also a compactfication $\mathcal{D} = \mathcal{O} \cup \partial\mathcal{O}$ which is homeomorphic to a closed disk. The fundamental group acts by homeomorphism on $\mathcal{D}$.*

Notice there is absolutely no restriction on the pseudo-Anosov flow. We stress that so far we have a boundary for the orbit space $\mathcal{O}$, which is 2-dimensional and not a boundary for $\widetilde{M}$ which has dimension 3. The flow boundary of $\widetilde{M}$ will be constructed later.

The prototype here is a suspension pseudo-Anosov flow – then $\mathcal{O}$ is identified with a lift of a fiber of the fibration of M over the circle. The fiber is a hyperbolic surface. The lift of the fiber is identified with the hyperbolic plane $\mathbf{H}^2$. In this case the ideal circle boundary of $\mathcal{O}$ is identified with the circle at infinity S^1_∞ of the lift of the fiber (this is the ideal circle of $\mathbf{H}^2$). But this uses geometry in $\mathcal{O}$ and not just dynamics. Notice however that any point q in the circle boundary S^1_∞ has a neighborhood basis in $\mathbf{H}^2 \cup S^1_\infty$ which is bounded by leaves of $\mathcal{O}^s$ or $\mathcal{O}^u$. The topology in $\mathcal{O} \cup \partial\mathcal{O}$ can also be defined using only these neighborhood systems. In this way we can define the ideal boundary and ideal compactification of $\mathcal{O}$ using only $\mathcal{O}^s, \mathcal{O}^u$. We want to do something similar in the general case.

Before describing the general situation, notice that in general there is no geometry (even coarse geometry) in the space $\mathcal{O}$. It is only a topological space and the fundamental group acts by homeomorphisms on this space. We now explain ideal points of $\mathcal{O}$ in the general case:

For an arbitrary pseudo-Anosov flow, an ideal point of $\mathcal{O}$ is defined by a sequence of chains – a chain is a finite collection of segments alternatively in $\mathcal{O}^s, \mathcal{O}^u$ and 2 rays, forming a properly embedded path in $\mathcal{O}$. One can think of a chain as a properly embedded polygonal arc in $\mathcal{O}$. The sides of the polygonal arc are alternativately in $\mathcal{O}^s, \mathcal{O}^u$.

In the case of fibrations, the chains are very simple: they are just leaves of $\mathcal{O}^s$ or $\mathcal{O}^u$. In that case an ideal point of $\mathcal{O}$ is determined by a sequence of nested leaves of (say) $\mathcal{O}^s$ which escape compact sets in $\mathcal{O}$. This defines a neighborhood system of this ideal point of $\mathcal{O}$. In general it is necessary to use chains in $\mathcal{O}^s, \mathcal{O}^u$ rather than just leaves of $\mathcal{O}^s, \mathcal{O}^u$ because of the existence of perfect fits as defined below. Any ray in a leaf of $\mathcal{O}^s, \mathcal{O}^u$ is properly embedded and defines an ideal point in $\mathcal{O}$, but there are many other points.

Now we define perfect fits. An unstable leaf G of Λ^u makes a *perfect fit* with a stable leaf F of Λ^s if G and F do not intersect but any other unstable leaf sufficiently near G (and in the F side) will intersect F and vice versa. The same happens to projections to the orbit space $\mathcal{O}$. In the orbit space we can think of a perfect fit as a proper embedding of a rectangle minus a corner. Stable leaves are horizontal segments, unstable leaves are vertical. The 2 leaves which limit in the missing corner form a perfect fit. Roughly speaking, perfect fits are associated to freely homotopic closed orbits of the flow Φ: if there are freely homotopic closed orbits of Φ then there are such orbits α, β which are freely homotopic to the inverse of each other and so that [Fe3, Fe4]: If one lifts then coherently to orbits $\widetilde{\alpha}, \widetilde{\beta}$ of $\widetilde{\Phi}$ then $\widetilde{W^s}(\widetilde{\alpha}), \widetilde{W^u}(\widetilde{\beta})$ form a perfect fit. Notice one is stable and the other unstable. If there are perfect fits as above then the collection of unstable leaves intersecting $\widetilde{W^s}(\widetilde{\alpha})$ is nested, but does not escape in $\widetilde{M}$ – they limit on $\widetilde{W^u}(\widetilde{\beta})$ at least. It follows that when projected to the orbit space $\mathcal{O}$, the elements of the corresponding sequence of leaves of $\mathcal{O}^u$ are not "shrinking" to a single ideal point. In fact they limit at least to a leaf l of $\mathcal{O}^u$. This is why one needs to consider the more general object of chains in $\mathcal{O}^s, \mathcal{O}^u$.

The ideal points of $\mathcal{O}$ are defined as equivalence classes of escaping, nested sequences of chains which satisfy a convexity property. The convexity property is a technical property which guarantees for instance that the chains are properly embedded in $\mathcal{O}$. See [Fe8] for detailed proofs that $\mathcal{D}$ has a well defined topology making it homeormorphic to a closed disk. The proof is quite involved.

One main goal in introducing an ideal boundary for $\mathcal{O}$ is that it leads to an understanding of the asymptotic behavior of $\widetilde{M}$ in certain cases. The flow ideal boundary of $\widetilde{M}$ is constructed as a quotient of the ideal boundary of $\mathcal{O}$ as described in the next theorem:

Theorem 3. [Fe8] *Let Φ be a pseudo-Anosov flow without perfect fits and not topologically conjugate to a suspension Anosov flow. Let $\mathcal{O}$ be its orbit space and $\partial\mathcal{O}$ its ideal boundary. Let $\cong$ be the equivalence relation in $\partial\mathcal{O}$ generated by: two points are in the same class if they are the ideal points of the same stable or unstable leaf in $\mathcal{O}$. Let $\mathcal{R}$ be the set of equivalence classes with the quotient topology. Then $\mathcal{R}$ is a 2-sphere. The fundamental group of M acts on $\mathcal{R}$ by homeomorphisms. There is a natural topology in $\widetilde{M} \cup \mathcal{R}$ making it into a compactification of $\widetilde{M}$. The action of $\pi_1(M)$ on $\widetilde{M}$ extends to an action on $\widetilde{M} \cup \mathcal{R}$. The quotient map from $\partial\mathcal{O}$ to $\mathcal{R}$ is a group invariant Peano curve associated to the flow Φ. All of this uses only the dynamics of the flow Φ.*

How is the space $\mathcal{R}$ obtained from the ideal boundary of $\mathcal{O}$? Here is some information: If x, y points in $\partial\mathcal{O}$ are related under $\cong$ and x, y are ideal points of (say) the same stable leaf in $\mathcal{O}^s$, then the condition on perfect fits implies that there is no unstable leaf with ideal point x (or y). Hence if p is the maximum number of prongs in singular leaves of $\mathcal{O}^s$ (or $\mathcal{O}^u$), then any equivalence class of $\cong$ has at most p points. One can then use a theorem of Moore on classical topology and use the topological structure of $\mathcal{O}^s, \mathcal{O}^u$ [Fe4] to show that $\partial\mathcal{O}/\cong$ is a 2-dimensional sphere $\mathcal{R}$ where the group $\pi_1(M)$ acts naturally.

Unless otherwise stated, we assume from now on that Φ has no perfect fits and is not topologically conjugate to a suspension Anosov flow. One example of such a flow is a suspension pseudo-Anosov flow (with singularities!). Later we describe other classes of examples transverse to certain classes of foliations.

What good is the flow ideal boundary? Let us review what we have done so far: At this point one has an ideal compactification $\widetilde{M} \cup \mathcal{R}$ of $\widetilde{M}$ which is constructed using the dynamics of the pseudo-Anosov flow. Keeping in mind 3-dimensional topology and the Gromov ideal boundary if $\pi_1(M)$ is Gromov hyperbolic, one would certainly like to relate these two approaches to an ideal boundary of $\widetilde{M}$!

In order to do this, we want to analyse the properties of $\mathcal{R}$ and $\widetilde{M} \cup \mathcal{R}$. This is because, as we will see below, certain topological properties completely characterize the Gromov ideal boundary of a group. The fundamental group of M acts on $\mathcal{R}$ and $\widetilde{M} \cup \mathcal{R}$ and the action captures the essence of these objects, as M is compact and is $\widetilde{M}/\pi_1(M)$. One important question to ask is: which properties do the actions have? We have actions of $\pi_1(M)$ on a circle $\partial\mathcal{O}$ and a sphere $\mathcal{R}$ (besides the actions on $\widetilde{M}$ and $\mathcal{O}$ and the singular foliations $\widetilde{\Lambda}^s, \widetilde{\Lambda}^u$). Motivated by a lot of of previous work on 2 and 3-dimensional topology, one asks whether these actions are convergence group actions. For example a group that acts as a uniform convergence group in the circle is topologically conjugate to a Möbius group [Ga4, Ca-Ju]. with fundamental consequences for 3-manifold theory. In particular this is related to the Seifert fibered conjecture for 3-manifolds [Ga4, Ca-Ju]. Also a fundamental question of Cannon asks whether a uniform convergence group acting on a sphere is conjugate to a cocompact Kleinian group. This is an extremely important question which is related to the geometrization of 3-manifolds.

A compactum is a compact Hausdorff space. A group γ acts as a *convergence group* [Bo2] on a metrisable compactum Z if for any sequence $(\gamma_n), n \in \mathbf{N}$ of distinct elements in Γ, there is a subsequence $(\gamma_{n_i}), i \in \mathbf{N}$ and a source/sink pair y, x so that $(\gamma_{n_i}), i \in \mathbf{N}$ converges uniformly to x in compact sets of $Z - \{y\}$. This is equivalent to Γ acts properly discontinuously on the set of distinct triples $\Theta_3(Z)$ of Z. In addition the action is *uniform* if the quotient of $\Theta_3(Z)$ by the action of Γ is compact. If Z is perfect (no isolated points) then the additional condition is equivalent to every point of Z being a conical limit point for the action. A point x in Z is a *conical limit point* if there is a sequence $(\gamma_n), n \in \mathbf{N}$ in Γ and b, c distinct in Z, with $\gamma_n(x)$ converging to c, but for every other point y in Z then $\gamma_n(y)$ converges to b.

It is easy to see that the action of $\pi_1(M)$ on $\partial\mathcal{O}$ is not a convergence group (expected because $\pi_1(M)$ is not a surface group). But the action on $\mathcal{R}$ has excellent properties. This turns out to be a key result:

Theorem 4. [Fe8] *Let Φ be a pseudo-Anosov flow without perfect fits and not topologically conjugate to a suspension Anosov flow. Let $\mathcal{R}$ be the associated ideal boundary sphere with corresponding compactification $\widetilde{M} \cup \mathcal{R}$ of the universal cover. Then the action of $\pi_1(M)$ on $\mathcal{R}$ is a uniform convergence group action. The action of $\pi_1(M)$ on $\widetilde{M} \cup \mathcal{R}$ is also a convergence group action (not uniform).*

The main part of the proof is to prove uniform convergence action on $\mathcal{R}$. Here there is a strong interplay between 2-dimensional dynamics of the action of $\pi_1(M)$ on $\mathcal{R}$ and the action of $\pi_1(M)$ on $\partial\mathcal{O}$, which describes the other action. In other words 1-dimensional dynamics (action on a circle) completely encodes 2-dimensional dynamics.

We mention that when there are perfect fits it is not clear what will be the resulting structure of the quotient space $\mathcal{R}$. For example consider an **R**-covered Anosov flow. This means that the stable foliation of the flow is **R**-covered (and hence so is the unstable foliation [Ba, Fe2]). There are infinitely many examples where the manifold is hyperbolic [Fe2]. Then the quotient $\mathcal{R}$ (of the circle $\mathcal{O}$) is the union of a circle and two special points: each special point is non separated from every point in the circle. Hence $\mathcal{R}$ is not even Hausdorff and clearly in this case the quotient $\mathcal{R}$ does not provide the expected ideal boundary of $\widetilde{M}$ which should be a sphere!

In the situation of theorem 4, one can then use the excellent dynamical properties of the action on $\mathcal{R}$ and $\widetilde{M} \cup \mathcal{R}$ to relate them to the large scale geometry of the manifold. This will have geometric consequences for flows and foliations.

The key tool here is the following result: Bowditch [Bo1], following ideas of Gromov, proved the very interesting theorem that if Γ acts as a uniform convergence group on a perfect, metrisable compactum Z, then Γ is Gromov hyperbolic, Z is homeomorphic to the Gromov ideal boundary $\partial\Gamma$ of Γ and the action of Γ on Z is equivariantly topologically conjugate to the action of Γ on its Gromov ideal boundary $\partial\Gamma$. Theorem 4 then immediately implies the following:

Theorem 5. [Fe8] *Let Φ be a pseudo-Anosov flow without perfect fits and not topologically conjugate to a suspension Anosov flow. Let $\mathcal{R}$ be the associated flow ideal boundary of $\widetilde{M}$ and let $\widetilde{M} \cup \mathcal{R}$ be the flow ideal compactification. Then $\pi_1(M)$ acts as a uniform convergence group on $\mathcal{R}$ and Bowditch's theorem implies that $\pi_1(M)$ is Gromov hyperbolic and the action of $\pi_1(M)$ on $\mathcal{R}$ is topologically conjugate to the action on the Gromov ideal boundary S^2_∞. In addition the actions on $\widetilde{M} \cup \mathcal{R}$ and $\widetilde{M} \cup S^2_\infty$ are also topologically conjugate – by a homeomorphism which is the identity in $\widetilde{M}$.*

The first conclusion of theorem 5 is not new. The fact that $\pi_1(M)$ is Gromov hyperbolic can also be obtained by previous results of Gabai and Kazez [Ga-Ka] and the author [Fe6]. The reason is the following: if M is a toroidal manifold with a pseudo-Anosov flow, then either there is a free homotopy between closed orbits of the flow or the flow is topologically conjugate to a suspension Anosov flow [Fe6]. The second option is disallowed by hypothesis. If there is a free homotopy between closed orbits of the flow then there are perfect fits [Fe3, Fe4]. This is also disallowed by hypothesis, hence it follows that M is atoroidal. Start with the stable foliation of Φ and blow up each singular leaf to produce solid tori complementary components. This produces an essential lamination which is genuine [Ga-Ka]. Since M is atoroidal, the main result of [Ga-Ka] implies then that $\pi_1(M)$ is Gromov hyperbolic. The interest in theorem 5 is that the construction here gives a very explicit description of the Gromov ideal boundary of $\widetilde{M}$ – first as a purely dynamical systems object and a posteriori implying that $\pi_1(M)$ is Gromov hyperbolic and then totally relating the two boundaries. The result of [Ga-Ka] provides no direct relationship with the Gromov ideal boundary. Gabai and Kazez show that least area disks satisfy a linear isoperimetric inequality.

We stress that the flow ideal boundary constructed in theorem 3 and analysed in theorem 4 is obtained using only dynamical systems properties and has no geometric hypothesis. The geometric properties will come as a consequence of the flow behavior.

When Φ is a suspension (singular) pseudo-Anosov flow (fiber is hyperbolic), theorem 5 provides a new proof that if M fibers over the circle with pseudo-Anosov monodromy then $\pi_1(M)$ is Gromov hyperbolic.

These results have some important geometric consequences for flows and foliations. Flow objects (flowlines, stable/unstable leaves, foliations transverse to the flow) behave very well in the flow ideal compactification $\widetilde{M} \cup \mathcal{R}$. This is due to the explicit structure of the flow compactification $\widetilde{M} \cup \mathcal{R}$. We can show for example that flow lines of $\widetilde{\Phi}$ converge to unique points in $\mathcal{R}$ in both forward and backwards direction [Fe8]. One then uses the important fact that since the flow ideal compactification is equivalent to the Gromov compactification, then flow lines also have good asymptotic behavior in the Gromov compactification. Using results in [Fe-Mo] this implies the following:

Theorem 6. [Fe8] *Let Φ be a pseudo-Anosov flow not topologically conjugate to a suspension flow and which does not have perfect fits. By theorem* 5*, $\pi_1(M)$ is Gromov hyperbolic. Then Φ is a quasigeodesic flow in M. In addition Λ^s, Λ^u are quasi-isometric singular foliations in M.*

Finally we come back to full circle and apply these results to foliations and their asymptotic properties. This also will show that pseudo-Anosov flow without perfect fits are very common. We use the geometric tools developed above and prove:

Theorem 7. [Fe8] *Let $\mathcal{F}$ be an $\mathbf{R}$-covered foliation in M^3 acylindrical and atoroidal. Up to a double cover $\mathcal{F}$ is transverse to a pseudo-Anosov flow Φ without perfect fits and not conjugate to a suspension Anosov flow. Then Φ is quasigeodesic and consequently $\mathcal{F}$ satisfies the continuous extension property. In addition the stable/unstable foliations of Φ are quasi-isometric.*

In order to prove theorem 7, start with a pseudo-Anosov flow transverse to $\mathcal{F}$ as constructed previously by the author [Fe5] and independently by Calegari [Cal1]. We show that Φ is not conjugate to a suspension Anosov flow and has no perfect fits. By theorem 6, the flow Φ is quasigeodesic and its stable/unstable foliations are quasi-isometric. Theorem 1 then implies that $\mathcal{F}$ has the continuous extension property. This result includes the cases of fibrations and uniform foliations/slitherings.

Another class of foliations that can be analysed is the following. Given a Reebless foliation $\mathcal{F}$ in M^3 closed then the leaf space $\mathcal{H}$ of $\widetilde{\mathcal{F}}$ is a simply connected 1-manifold, which has a countable basis and no boundary. If $\mathcal{H}$ is Hausdorff then it is homeomorphic to the real numbers and $\mathcal{F}$ is $\mathbf{R}$-covered. On the other hand if $\mathcal{H}$ is not Hausdorff, then it is still orientable, as it is simply connected. Hence there is a notion of positive and negative sides of a leaf of $\widetilde{\mathcal{F}}$. The non separated points in $\mathcal{H}$ correspond to branching of $\widetilde{\mathcal{F}}$ in the negative direction if they are non separated from each other on their positive sides. Similarly for branching in the positive direction. A foliation $\mathcal{F}$ has *one sided branching* if the branching in $\widetilde{\mathcal{F}}$ is only in one direction (positive or negative). We prove the following:

Theorem 8. [Fe8] *Let $\mathcal{F}$ be a foliation with one sided branching in M^3 atoroidal, acylindrical. Then $\mathcal{F}$ is transverse to a pseudo-Anosov flow without perfect fits and not conjugate to a suspension Anosov flow. Then Φ is quasigeodesic, its stable/unstable foliations are quasi-isometric and consequently $\mathcal{F}$ has the continuous extension property. If F is a leaf of $\widetilde{\mathcal{F}}$ then the limit set of F is not the whole sphere.*

Under the conditions of this theorem, Calegari [Cal2] proved that $\mathcal{F}$ is transverse to a pseudo-Anosov flow Φ. We show that such Φ does not have perfect fits nor is conjugate to a suspension Anosov flow. By theorem 6, the flow Φ is quasigeodesic. As seen in theorem 7 this implies that $\mathcal{F}$ has the continuous extension property. The second statement follows from the metric properties of leaves of Λ^s, Λ^u.

In this way we one obtains the continuous extension property for these two classes of foliations: **R**-covered and with one sided branching assuming only topological conditions. These are conditions on the leaf spaces of $\widetilde{\mathcal{F}}$. Using the tools of flow ideal boundaries and uniform convergence groups we are able to analyse the continuous extension property for these foliations.

The open case for the continuous extension property now is foliations with two sided branching and not finite depth. We should remark that this is the generic case for foliations. It may be that the techniques mentioned here are useful in this situation.

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Florida State University
Tallahassee, FL 32306-4510, USA
and
Princeton University
Princeton, NJ 08540, USA
E-mail address: fenley@mail.math.fsu.edu