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Period problems for mean curvature one surfaces in H^3 (with applications to surfaces of low total curvature)

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§1. Introduction

There is a wide body of knowledge about minimal surfaces in Euclidean 3-space $\boldsymbol{R}^3$, and there is a canonical local isometric correspondence (sometimes called the Lawson correspondence) between minimal surfaces in $\boldsymbol{R}^3$ and CMC-1 (constant mean curvature one) surfaces in hyperbolic 3-space H^3 (the complete simply-connected 3-manifold of constant sectional curvature -1). This has naturally led to the recent interest in and development of CMC-1 surfaces in H^3 in the last decade. There are now many known examples, and it is a natural next step to classify all such surfaces with low total absolute curvature.

By this canonical local isometric correspondence, minimal immersions in $\boldsymbol{R}^3$ are locally equivalent to CMC-1 immersions in H^3. But there are interesting differences between these two types of immersions on the global level. There are period problems on non-simply-connected domains of the immersions, which might be solved for one type of immersion but not the other. Solvability of the period problems is usually more likely in the H^3 case, leading to a wider variety of surfaces there. For example, a genus 1 surface with finite total curvature and two embedded ends cannot exist as a minimal surface in $\boldsymbol{R}^3$, but it does exist as a CMC-1 surface in H^3 [RS]. And a genus 0 surface with finite total curvature and two embedded ends exists as a minimal surface in $\boldsymbol{R}^3$ only if it is a surface of revolution, but it may exist as a CMC-1 surface in

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H^3 without being a surface of revolution (see Example 4.3). So there are many more possibilities for CMC-1 surfaces in H^3 than there are for minimal surfaces in $\boldsymbol{R}^3$. This means that it is more difficult to classify CMC-1 surfaces with low total curvature in H^3.

To find complete CMC-1 surfaces in H^3 with low total curvature, we must first determine the meromorphic data in the Bryant representation of the surfaces that can admit low total curvature, and then we must analyse when the parameters in the data can be adjusted to solve the period problems. Generally, finding the data is the easier step, and solving the period problems is the more difficult step. As the period problems are generally the crux of the problem, we have chosen the title of this paper to reflect this.

The total absolute curvature of a minimal surface in $\boldsymbol{R}^3$ is equal to the area of the image (counted with multiplicity) of the Gauss map of the surface, and complete minimal surfaces in $\boldsymbol{R}^3$ with total curvature at most 8π have been classified (see Lopez [Lop] and also Table 2). Furthermore, as the Gauss map of a complete conformally parametrized minimal surface is meromorphic, and has a well-defined limit at each end when the surface has finite total curvature, the area of the Gauss image must be an integer multiple of 4π.

However, unlike the case of minimal surfaces in $\boldsymbol{R}^3$, when searching for CMC-1 surfaces in H^3 with low total absolute curvature, we have a choice of two different Gauss maps: the *hyperbolic Gauss map* G and the *secondary Gauss map* g. So there are two ways to pose the question in H^3, with two very different answers. One way is to consider the true total absolute curvature, which is the area of the image of g, but since g might not be single-valued on the surface, the total curvature might not be an integer multiple of 4π, and this allows for many more possibilities. Furthermore, the Osserman inequality does not hold for the true total absolute curvature. The weaker Cohn-Vossen inequality is the best general lower bound for true absolute total curvature (with equality never holding [UY1]). So the true total absolute curvature is difficult to analyse, but it is important because of its clear geometric meaning.

The second way is to study the area of the image of G, which we call the *dual* total absolute curvature, as it is the true total curvature of the dual CMC-1 surface (which we define in Section 3) in H^3. This way has the advantage that G is single-valued on the surface, and so the dual total absolute curvature is always an integer multiple of 4π, like the case of minimal surfaces in $\boldsymbol{R}^3$. Furthermore, the dual total curvature satisfies not only the Cohn-Vossen inequality, but also the Osserman inequality [UY5, Yu2] (see also (3.13) in Section 3). So the dual total curvature shares more properties with the total curvature of minimal surfaces in $\boldsymbol{R}^3$, motivating our interest in it.

We shall refer to the true total absolute curvature of a CMC-1 immersion $f\colon M \to H^3$ of a Riemann surface as $\mathrm{TA}(f)$, and the dual total absolute curvature as $\mathrm{TA}(f^\#)$.

We review the classification results for surfaces with $\mathrm{TA}(f) \leq 4\pi$ or $\mathrm{TA}(f^\#) \leq 4\pi$ in Section 2, which are results from [RUY4] and [RUY3]. An inequality for $\mathrm{TA}(f)$ stronger than the Cohn-Vossen inequality [RUY4] (for surfaces of genus zero with odd number of ends) is also introduced. In Section 3, we review basic notions and terminology. We introduce some important examples of CMC-1 surfaces in Section 4. Section 6 is devoted to describing the results in [RUY3], a partial classification of CMC-1 surfaces with $\mathrm{TA}(f^\#) \leq 8\pi$. In the final section 7, we introduce new results on partial classification of CMC-1 surfaces with $\mathrm{TA}(f) \leq 8\pi$. Since the proofs of these results are more technical and delicate than those of the results on $\mathrm{TA}(f^\#)$, we include them in Appendix A.

§2. The cases $\mathrm{TA}(f)$ or $\mathrm{TA}(f^\#) \leq 4\pi$, and a natural extension

In [RUY4] the following theorem was proven:

Theorem 2.1. *Let $f\colon M \to H^3$ be a complete* CMC-1 *immersion of total absolute curvature* $\mathrm{TA}(f) \leq 4\pi$. *Then f is either*

- *a horosphere (Example 4.1),*
- *an Enneper cousin (Example 4.2),*
- *an embedded catenoid cousin ($0 < l < 1$, $\delta = 1$ and $b = 0$ in Example 4.3),*
- *a finite δ-fold covering of an embedded catenoid cousin ($\delta \geq 2$, $0 < l \leq 1/\delta$ and $b = 0$ in Example 4.3), or*
- *a warped catenoid cousin with injective secondary Gauss map ($l = 1$, $\delta \in \boldsymbol{Z}^+$ and $b > 0$ in Example 4.3).*

The horosphere is the only flat (and consequently totally umbilic) CMC-1 surface in H^3. The catenoid cousins are the only CMC-1 surfaces of revolution [Bry]. The Enneper cousins are isometric to minimal Enneper surfaces [Bry]. The warped catenoid cousins [UY1, RUY3] are less well known and are described more precisely in Section 4, as well as the other above three examples.

Although this theorem is simply stated, for the reasons given in the introduction the proof is more delicate than it would be if the condition $\mathrm{TA}(f) \leq 4\pi$ is replaced with $\mathrm{TA}(f^{\#}) \leq 4\pi$, or if minimal surfaces in $\boldsymbol{R}^3$ with $\mathrm{TA} \leq 4\pi$ are considered. CMC-1 surfaces f with $\mathrm{TA}(f^{\#}) \leq 4\pi$ are classified in Theorem 2.3 below. It is well-known that the only complete minimal surfaces in $\boldsymbol{R}^3$ with $\mathrm{TA} \leq 4\pi$ are the plane, the Enneper surface, and the catenoid (see Table 2).

We extend the above result in Section 7 to find an inclusive list of possibilities for CMC-1 surfaces with $\mathrm{TA}(f) \leq 8\pi$, and we consider which possibilities we can classify or find examples for, see Table 3. (Minimal surfaces in $\boldsymbol{R}^3$ with $\mathrm{TA} \leq 8\pi$ are classified by Lopez [Lop]. See Table 2.)

For a complete CMC-1 immersion f in H^3, equality in the Cohn-Vossen inequality never holds ([UY1, Theorem 4.3]). In particular, if f is of genus 0 with n ends, then

$$\mathrm{TA}(f) > 2\pi(n-2)\ . \tag{2.1}$$

When $n = 2$, the catenoid cousins show that (2.1) is sharp. However, we see from the above theorem that

$$\mathrm{TA}(f) > 4\pi \quad \text{for} \quad n = 3\ ,$$

which is stronger than the Cohn-Vossen inequality (2.1). The following theorem, which extends the above theorem and is proven in [RUY4], gives a sharper inequality than the Cohn-Vossen inequality when n is any odd integer:

Theorem 2.2. *Let* $f\colon \boldsymbol{C} \cup \{\infty\} \setminus \{p_1, \dots, p_{2m+1}\} \to H^3$ *be a complete conformal genus* 0 CMC-1 *immersion with* $2m+1$ *ends,* $m \in \boldsymbol{Z}^+$*. Then* $\mathrm{TA}(f) \geq 4\pi m$.

Remark. When $m = 1$, we know that the lower bound 4π in the theorem is sharp (see Example 4.4). However, we do not know if it is sharp for general m. For genus 0 CMC-1 surfaces with an even number $n \geq 4$ of ends, it is still an open question whether there exists any stronger lower bounds than that of the Cohn-Vossen inequality. It should be remarked that in Section 4 we have numerical examples with $n = 4$ whose total absolute curvature tends to 4π.

For the case of $\mathrm{TA}(f^{\#})$, the following theorem was proven in [RUY3]:

Theorem 2.3. *A complete* CMC-1 *immersion f with* $\mathrm{TA}(f^{\#}) \leq 4\pi$ *is congruent to one of the following*:

(1) a horosphere (Example 4.1*),*

(2) an Enneper cousin dual (Example 4.2*),*

(3) a catenoid cousin ($\delta = 1$, $l \neq 1$ and $b = 0$ in Example 4.3*), or*

(4) a warped catenoid cousin with embedded ends and injective hyperbolic Gauss map ($\delta = 1$, $l \in \boldsymbol{Z}$, $l \geq 2$ and $b > 0$ in Example 4.3*).*

§3. Preliminaries

Before we can state any results for the cases of higher $\mathrm{TA}(f)$ and higher $\mathrm{TA}(f^{\#})$, we must give some preliminaries here.

Let $f\colon M \to H^3$ be a conformal CMC-1 immersion of a Riemann surface M into H^3. Let ds^2, dA and K denote the induced metric, induced area element and Gaussian curvature, respectively. Then $K \leq 0$ and $d\sigma^2 := (-K)\, ds^2$ is a conformal pseudometric of constant curvature 1 on M. We call this pseudometric's developing map $g\colon \widetilde{M}(:=$ the universal cover of $M) \to \boldsymbol{CP}^1 = \boldsymbol{C} \cup \{\infty\}$ the *secondary Gauss map* of f. Namely, g is a conformal map so that its pull-back of the Fubini-Study metric of $\boldsymbol{CP}^1$ equals $d\sigma^2$:

$$d\sigma^2 = (-K)\, ds^2 = \frac{4\, dg\, d\bar{g}}{(1+g\bar{g})^2}\,. \tag{3.1}$$

Such a map g is determined by $d\sigma^2$ uniquely up to the change

$$g \mapsto a \star g := \frac{a_{11}g + a_{12}}{a_{21}g + a_{22}}\,, \qquad a = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in \mathrm{SU}(2)\,. \tag{3.2}$$

Since $d\sigma^2$ is invariant under the deck transformation group $\pi_1(M)$, there is a representation

(3.3)

$$\rho_g\colon \pi_1(M) \longrightarrow \mathrm{PSU}(2) \quad \text{such that} \quad g \circ \tau^{-1} = \rho_g(\tau) \star g \quad \bigl(\tau \in \pi_1(M)\bigr)\,,$$

where $\mathrm{PSU}(2) = \mathrm{SU}(2)/\{\pm \mathrm{id}\}$. The metric $d\sigma^2$ is called *reducible* if the image of ρ_g can be diagonalized simultaneously, and is called *irreducible* otherwise. In the case $d\sigma^2$ is reducible, we call it is *$\mathcal{H}^3$-reducible* if the image of ρ_g is the identity, and is called *$\mathcal{H}^1$-reducible* otherwise. We call a CMC-1 immersion $f\colon M \to H^3$ $\mathcal{H}^1$-reducible (respectively, $\mathcal{H}^3$-reducible) if the corresponding pseudometric $d\sigma^2$ is $\mathcal{H}^1$-reducible (respectively, $\mathcal{H}^3$-reducible). For details on reducibility, see Section 5.

In addition to g, two other holomorphic invariants G and Q are closely related to geometric properties of CMC-1 surfaces. The *hyperbolic Gauss map* $G\colon M \to \boldsymbol{CP}^1$ is holomorphic and is defined geometrically by identifying the ideal boundary of H^3 with $\boldsymbol{CP}^1$: $G(p)$ is the asymptotic class of the normal geodesic of $f(M)$ starting at $f(p)$ and oriented in the mean curvature vector's direction. The *Hopf differential* Q is a holomorphic symmetric 2-differential on M such that $-Q$ is the $(2,0)$-part of the complexified second fundamental form. The Gauss equation implies

$$ds^2 \cdot d\sigma^2 = 4\,Q \cdot \overline{Q}\,, \tag{3.4}$$

where $\cdot$ means the symmetric product. Moreover, these invariants are related by

$$S(g) - S(G) = 2Q\,, \tag{3.5}$$

where $S(\cdot)$ denotes the Schwarzian derivative:

$$S(h) := \left[\left(\frac{h''}{h'}\right)' - \frac{1}{2}\left(\frac{h''}{h'}\right)^2\right] dz^2 \qquad \left(' = \frac{d}{dz}\right)$$

with respect to a local complex coordinate z on M.

In terms of g and Q, the induced metric ds^2 and complexification of the second fundamental form $I\!I$ are

$$ds^2 = (1+|g|^2)^2 \left|\frac{Q}{dg}\right|^2\,, \qquad I\!I = -Q - \overline{Q} + ds^2\,.$$

Since $K \le 0$, we can define the *total absolute curvature* as

$$\mathrm{TA}(f) := \int_M (-K)\,dA \in [0, +\infty]\,.$$

Then $\mathrm{TA}(f)$ is the area of the image of M in $\boldsymbol{CP}^1$ of the secondary Gauss map g. $\mathrm{TA}(f)$ is generally not an integer multiple of 4π; for catenoid cousins [Bry, Example 2] and their δ-fold covers, $\mathrm{TA}(f)$ can be *any* positive real number.

For each conformal CMC-1 immersion $f\colon M \to H^3$, there is a holomorphic null immersion $F\colon \widetilde{M} \to \mathrm{SL}(2,\boldsymbol{C})$, the *lift* of f, satisfying the differential equation

$$dF = F\begin{pmatrix} g & -g^2 \\ 1 & -g \end{pmatrix}\omega\,, \qquad \omega = \frac{Q}{dg} \tag{3.6}$$

so that $f = FF^*$, where $F^* = {}^t\overline{F}$ [Bry, UY1]. Here we consider

$$H^3 = \mathrm{SL}(2,\boldsymbol{C})/\,\mathrm{SU}(2) = \{aa^* \mid a \in \mathrm{SL}(2,\boldsymbol{C})\}\,.$$

We call a pair (g,ω) the *Weierstrass data* of f. The lift F is said to be *null* because $\det F^{-1}dF$, the pull-back of the Killing form of $\mathrm{SL}(2,\boldsymbol{C})$ by F, vanishes identically on $\widetilde{M}$. Conversely, for a holomorphic null immersion $F\colon \widetilde{M} \to \mathrm{SL}(2,\boldsymbol{C})$, $f := FF^*$ is a conformal CMC-1 immersion of $\widetilde{M}$ into H^3. If $F = (F_{ij})$, equation (3.6) implies

$$g = -\frac{dF_{12}}{dF_{11}} = -\frac{dF_{22}}{dF_{21}}\,, \tag{3.7}$$

and it is shown in [Bry] that

$$G = \frac{dF_{11}}{dF_{21}} = \frac{dF_{12}}{dF_{22}}\,. \tag{3.8}$$

The inverse matrix F^{-1} is also a holomorphic null immersion, and produces a new CMC-1 immersion $f^\# = F^{-1}(F^{-1})^*\colon \widetilde{M} \to H^3$, called the *dual* of f [UY5]. The induced metric $ds^{2\#}$ and the Hopf differential $Q^\#$ of $f^\#$ are

$$ds^{2\#} = (1+|G|^2)^2 \left|\frac{Q}{dG}\right|^2, \qquad Q^\# = -Q\,. \tag{3.9}$$

So $ds^{2\#}$ and $Q^\#$ are well-defined on M itself, even though $f^\#$ might be defined only on $\widetilde{M}$. This duality between f and $f^\#$ interchanges the roles of the hyperbolic Gauss map G and secondary Gauss map g. In particular, one has

$$dF\,F^{-1} = -(F^{-1})^{-1}d(F^{-1}) = \begin{pmatrix} G & -G^2 \\ 1 & -G \end{pmatrix} \frac{Q}{dG}\,. \tag{3.10}$$

Hence dFF^{-1} is single-valued on M, whereas $F^{-1}dF$ generally is not.

Since $ds^{2\#}$ is single-valued on M, we can define the *dual total absolute curvature*

$$\mathrm{TA}(f^\#) := \int_M (-K^\#)\,dA^\#,$$

where $K^\#$ (≤ 0) and $dA^\#$ are the Gaussian curvature and area element of $ds^{2\#}$, respectively. As

$$d\sigma^{2\#} := (-K^\#)ds^{2\#} = \frac{4\,dG\,d\overline{G}}{(1+|G|^2)^2} \tag{3.11}$$

is a pseudo-metric of constant curvature 1 with developing map G, $\mathrm{TA}(f^\#)$ is the area of the image of G on $\boldsymbol{CP}^1 = S^2$. The following assertion is important for us:

Lemma 3.1 ([UY5, Yu2]). *The metric $ds^{2\#}$ is complete (respectively, nondegenerate) if and only if ds^2 is complete (respectively, nondegenerate).*

We now assume that the induced metric ds^2 (and consequently $ds^{2\#}$) on M is complete and that either $\mathrm{TA}(f) < \infty$ or $\mathrm{TA}(f^\#) < \infty$, hence there exists a compact Riemann surface $\overline{M}_\gamma$ of genus γ and a finite set of points $\{p_1, \dots, p_n\} \subset \overline{M}_\gamma$ $(n \geq 1)$ so that M is biholomorphic to $\overline{M}_\gamma \setminus \{p_1, \dots, p_n\}$ (see Theorem 9.1 of [Oss]). We call the points p_j the *ends* of f.

Unlike the Gauss map for minimal surface with TA $< \infty$ in $\boldsymbol{R}^3$, the hyperbolic Gauss map G of the surface might not extend to a meromorphic function on $\overline{M}_\gamma$, as the Enneper cousin (Example 4.2) shows. However, the Hopf differential Q does extend to a meromorphic differential on $\overline{M}_\gamma$ [Bry]. We say an end p_j $(j = 1, \dots, n)$ of a CMC-1 immersion is *regular* if G is meromorphic at p_j. When $\mathrm{TA}(f) < \infty$, an end p_j is regular precisely when the order of Q at p_j is at least -2, and otherwise G has an essential singularity at p_j [UY1]. Moreover, the pseudometric $d\sigma^2$ as in (3.1) has a *conical singularity* at each end p_j [Bry]. For a definition of conical singularity, see Section 5 (see also [UY3, UY7]).

Thus the orders of Q at the ends p_j are important for understanding the geometry of the surface, so we now introduce a notation that reflects this. We say a CMC-1 surface is of *type* $\mathbf{\Gamma}(d_1, \dots, d_n)$ if it is given as a conformal immersion $f\colon \overline{M}_\gamma \setminus \{p_1, \dots, p_n\} \to H^3$, where $\mathrm{ord}_{p_j} Q = d_j$ for $j = 1, \dots, n$ (for example, if $Q = z^{-2}dz^2$ at $p_1 = 0$, then $d_1 = -2$). We use $\mathbf{\Gamma}$ because it is the capitalized form of γ, the genus of $\overline{M}_\gamma$. For instance, the class $\mathbf{I}(-4)$ means the class of surfaces of genus 1 with 1 end so that Q has a pole of order 4 at the end, and the class $\mathbf{O}(-2, -3)$ is the class of surfaces of genus 0 with two ends so that Q has a pole of order 2 at one end and a pole of order 3 at the other.

Analogue of the Osserman inequality. For a CMC-1 surface of genus γ with n ends, the second and third authors showed that the equality of the Cohn-Vossen inequality for the total absolute curvature never holds [UY1]:

$$\frac{1}{2\pi}\,\mathrm{TA}(f) > -\chi(M) = 2\gamma - 2 + n\,. \tag{3.12}$$

The catenoid cousins (Example 4.3) show that this inequality is the best possible.

On the other hand, the dual total absolute curvature satisfies an Osserman-type inequality [UY5]:

$$\frac{1}{2\pi}\,\mathrm{TA}(f^{\#}) \geq -\chi(M) + n = 2(\gamma + n - 1)\,. \tag{3.13}$$

Moreover, equality holds exactly when all the ends are embedded: This follows by noting that equality is equivalent to all ends being regular and embedded ([UY5]), and that any embedded end must be regular (proved recently by Collin, Hauswirth and Rosenberg [CHR1] and Yu [Yu3]).

Effects of transforming the lift *F*. Here we consider the change $\hat{F} = aFb^{-1}$ of the lift F, where $a, b \in \mathrm{SL}(2, \boldsymbol{C})$. Then $\hat{F}$ is also a holomorphic null immersion, and the hyperbolic Gauss map $\hat{G}$, the secondary Gauss map $\hat{g}$ and the Hopf differential $\hat{Q}$ of $f = \hat{F}\hat{F}^*$ are given by (see [UY3])

$$\hat{G} = a \star G = \frac{a_{11}G + a_{12}}{a_{21}G + a_{22}}\,, \quad \hat{g} = b \star g = \frac{b_{11}g + b_{12}}{b_{21}g + b_{22}}\,, \quad \text{and} \quad \hat{Q} = Q\,, \tag{3.14}$$

where $a = (a_{ij})$ and $b = (b_{ij})$. In particular, the change $\hat{F} = aF$ moves the surface by a rigid motion of H^3, and does not change g and Q.

SU(2)-monodromy conditions. Here we recall from [RUY1] the construction of CMC-1 surfaces with given hyperbolic Gauss map G and Hopf differential Q. Let $\overline{M}_\gamma$ be a compact Riemann surface and $M := \overline{M}_\gamma \setminus \{p_1, \ldots, p_n\}$. Let G and Q be a meromorphic function and meromorphic 2-differential on $\overline{M}_\gamma$. We assume the pair (G, Q) satisfies the following two compatibility conditions:

(3.15)
For all $q \in M$, $\mathrm{ord}_q\, Q$ is equal to the branching order of G, and

(3.16)
for each end p_j, (branching order of G)$-d_j \geq 2$.

The first condition implies that the metric $ds^{2\#}$ as in (3.9) is non-degenerate at $q \in M$. The second condition implies that the metric $ds^{2\#}$ is complete at $p_j \in \overline{M}_\gamma$ $(j = 1, \ldots, n)$. Our goal is to get a CMC-1 immersion $f\colon M \to H^3$ with hyperbolic Gauss map G and Hopf differential Q. If such an immersion exists, the induced metric ds^2 of f is non-degenerate and complete, by Lemma 3.1.

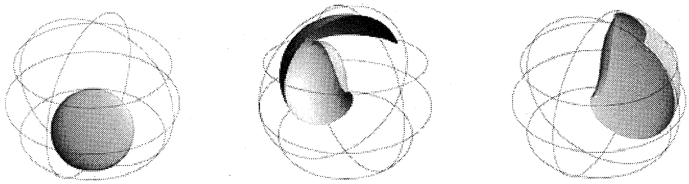


Fig. 1. A horosphere, and fundamental pieces (one-fourth of the surfaces with the ends cut away) of an Enneper cousin and the dual of an Enneper cousin. Figures are shown in the Poincaré model of H^3.

Since a pair (G, Q) satisfies (3.15) and (3.16), the differential equation (3.10) may have singularities at $\{p_1, \ldots, p_n\}$, but is regular on M. Then there exists a solution $F\colon \widetilde{M} \to \mathrm{SL}(2, \boldsymbol{C})$, where $\widetilde{M}$ is the universal cover of M. Since the solution F of (3.10) is unique up to the change $F \mapsto Fa$ $(a \in \mathrm{SL}(2, \boldsymbol{C}))$, there exists a representation $\rho_F\colon \pi_1(M) \to \mathrm{SL}(2, \boldsymbol{C})$ such that

$$F \circ \tau = F\rho_F(\tau) \qquad \left(\tau \in \pi_1(M)\right) . \tag{3.17}$$

Here we consider an element τ of the fundamental group $\pi_1(M)$ as a deck transformation on $\widetilde{M}$. Thus:

Proposition 3.2. *If there exists a solution $F\colon \widetilde{M} \to \mathrm{SL}(2, \boldsymbol{C})$ of* (3.10) *for (G, Q) satisfying* (3.15) *and* (3.16), *then $f := FF^*$ is a complete conformal* CMC-1 *immersion into H^3 which is well-defined on M if $\rho_F(\tau) \in \mathrm{SU}(2)$ for all $\tau \in \pi_1(M)$. Moreover, the hyperbolic Gauss map and the Hopf differential of f are G and Q, respectively.*

§4. Important Examples with $\mathrm{TA}(f)$ or $\mathrm{TA}(f^{\#}) \leq 8\pi$

In this section, we shall introduce several important CMC-1 surfaces with $\mathrm{TA}(f) \leq 8\pi$ or $\mathrm{TA}(f^{\#}) \leq 8\pi$.

Example 4.1 (Horosphere). A horosphere (Figure 1) is the only surface of type $\mathbf{O}(0)$, with Weierstrass data given by

$$g = 0, \qquad \omega = a\,dz \qquad (a \in \boldsymbol{C} \setminus \{0\}) .$$

The holomorphic lift $F\colon \boldsymbol{C} \to \mathrm{SL}(2, \boldsymbol{C})$ of the surface with initial condition $F(0) = \mathrm{id}$ is given by

$$F = \begin{pmatrix} 1 & 0 \\ az & 1 \end{pmatrix} .$$

In particular the hyperbolic Gauss map is a constant function, as well as the secondary Gauss map $g = 0$. This surface is flat and totally umbilic. In particular, the total curvature and the dual total curvature of the surface are both equal to zero. Any flat or totally umbilic CMC-1 surfaces are parts of this surface. Planes in $\boldsymbol{R}^3$ are the corresponding minimal surfaces with the same Weierstrass data $(g, \omega) = (0, a\,dz)$.

Example 4.2 (Enneper cousin and dual of Enneper cousin). The Enneper cousin is given in [Bry] (Figure 1), with the same Weierstrass data as the Enneper surface in $\boldsymbol{R}^3$:

$$g = z, \qquad \omega = a\,dz \qquad \left(a \in \boldsymbol{C} \setminus \{0\}\right) .$$

The holomorphic lift $F\colon \boldsymbol{C} \to \mathrm{SL}(2, \boldsymbol{C})$ of the surface with initial condition $F(0) = \mathrm{id}$ is given by

$$F = \begin{pmatrix} \cosh(az) & a^{-1}\sinh(az) - z\cosh(az) \\ a\sinh(az) & \cosh(az) - az\sinh(az) \end{pmatrix} .$$

In particular the hyperbolic Gauss map G is given by

$$G = a^{-1}\tanh(az) .$$

The Enneper cousin is in the class $\mathbf{O}(-4)$ and has a complete induced metric of total absolute curvature 4π. If one takes the inverse of F, one gets the dual of the Enneper cousin (Figure 1). Since

$$Fd(F^{-1}) = -dFF^{-1} = \begin{pmatrix} -a\cosh(az)\sinh(az) & \sinh^2(az) \\ -a^2\cosh^2(az) & a\cosh(az)\sinh(az) \end{pmatrix} ,$$

the Weierstrass data $(g^\#, \omega^\#)$ of the dual of the Enneper cousin given by

$$g^\# = a^{-1}\tanh(az), \qquad \omega^\# = a^2\cosh^2(az)\,dz .$$

This surface is also in the class $\mathbf{O}(-4)$ and has a complete induced metric of infinite total absolute curvature (see Lemma 3.1).

Example 4.3 (Catenoid cousins and warped catenoid cousins). CMC-1 surfaces of type $\mathbf{O}(-2, -2)$ are classified in Theorem 6.2 in

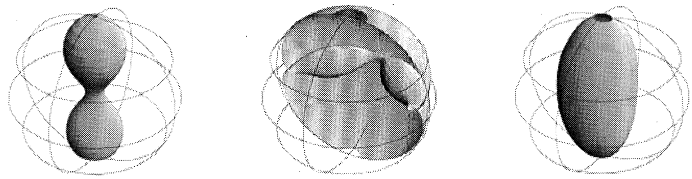


Fig. 2. A catenoid cousin with $l = 0.8$, and warped catenoid cousins with $(l, \delta, b) = (4, 1, 1/2)$ and $(1, 2, 1/2)$. The third surface has $\mathrm{TA}(f) = 4\pi$ because $l = 1$ even though its ends are not embedded.

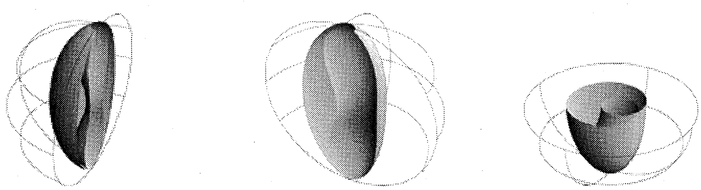


Fig. 3. Cut-away views of the third warped catenoid cousin in Figure 2.

[UY1]. Here we give a slightly refined version from [RUY4]: A complete conformal CMC-1 immersion $f\colon M = \boldsymbol{C} \setminus \{0\} \to H^3$ with regular ends have the following Weierstrass data

$$g = \frac{\delta^2 - l^2}{4l} z^l + b\,, \qquad \omega = \frac{Q}{dg} = z^{-l-1} dz\,, \tag{4.1}$$

with $l > 0$, $\delta \in \boldsymbol{Z}^+$, and $l \neq \delta$, and $b \geq 0$, where the case $b > 0$ occurs only when $l \in \boldsymbol{Z}^+$. When $b = 0$ and $\delta = 1$, the surface is called a *catenoid cousin*, which is rotationally symmetric. (The Weierstrass data of the catenoid cousin is often written as $g = z^\mu$ and $\omega = (1 - \mu^2) z^{-\mu-1}\, dz/(4\mu)$. This is equivalent to (4.1) for $b = 0$ and $\delta = 1$ and $l = \mu$ by a coordinate change $z \mapsto ((1 - \mu^2)/4\mu)^{(1/\mu)} z$.) Catenoid cousins are embedded when $0 < l < 1$ and have one curve of self-intersection when $l > 1$. When $b = 0$, f is a δ-fold cover of a catenoid cousin. When $b > 0$ (then automatically l is a positive integer), we call f a *warped catenoid cousin*, and its discrete symmetry group is the natural $\boldsymbol{Z}_2$ extension of the dihedral group D_l. Furthermore, the warped

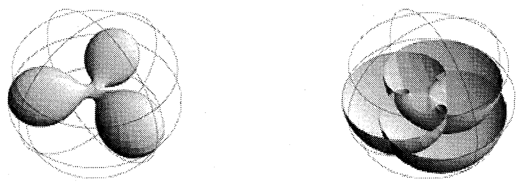


Fig. 4. Two different CMC-1 trinoids (proven to exist in [UY3]). Although these surfaces are proven to exist, and numerical experiments show that some of them are embedded (as one of the pictures here is), none have yet been proven to be embedded.

catenoid cousins can be written explicitly as $f = FF^*, F = F_0 B$, where

$$F_0 = \sqrt{\frac{\delta^2 - l^2}{\delta}} \begin{pmatrix} \frac{1}{l-\delta} z^{(\delta - l)/2} & \frac{\delta - l}{4l} z^{(l+\delta)/2} \\ \frac{1}{l+\delta} z^{-(l+\delta)/2} & \frac{-(l+\delta)}{4l} z^{(l-\delta)/2} \end{pmatrix}, \quad B = \begin{pmatrix} 1 & -b \\ 0 & 1 \end{pmatrix}.$$

In particular, the hyperbolic Gauss map and Hopf differential are given by

$$G = z^\delta, \qquad Q = \frac{\delta^2 - l^2}{4z^2}\, dz^2 ,$$

which are equal to the Gauss map and Hopf differential of the catenoids in $\boldsymbol{R}^3$. The dual total curvature of a catenoid cousin is 4π, but its total curvature is $4\pi l$ $(l > 0)$, which can take any value in $(0, 4\pi) \cup (4\pi, \infty)$. On the other hand, the total absolute curvature and the dual total absolute curvature of warped catenoid cousins are always integer multiples of 4π. (See Figures 2 and 3).

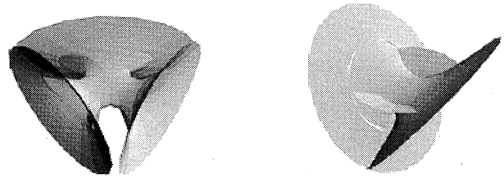
Example 4.4 (Irreducible trinoids). We take three real numbers μ_1, μ_2, $\mu_3 > -1$ such that

$$\cos^2 B_1 + \cos^2 B_2 + \cos^2 B_3 + 2\cos B_1 \cos B_2 \cos B_3 < 1, \tag{4.2}$$

where $B_j = \pi(\mu_j + 1)$ $(j = 1, 2, 3)$. We also assume

$$c_1^2 + c_2^2 + c_3^2 - 2(c_1 c_2 + c_2 c_3 + c_3 c_1) \neq 0, \tag{4.3}$$

where $c_j = -\mu_j(\mu_j + 2)/2 \in \boldsymbol{R}$ $(j = 1, 2, 3)$. Then it is shown in [UY7] that there exists a unique CMC-1 surface $f_{\mu_1, \mu_2, \mu_3} \colon \boldsymbol{C} \setminus \{0, 1\} \to H^3$ of type $\mathbf{O}(-2, -2, -2)$ such that the pseudometric $d\sigma^2 = (-K) ds^2$ defined by (3.1) is irreducible and has conical singularities of orders μ_1, μ_2, μ_3



Type P Type N

Fig. 5. Minimal trinoids of types P and N. The graphics were made by S. Tanaka of Hiroshima University.

at $z = 0, 1, \infty$, respectively. Moreover, any irreducible CMC-1 surface of type $\mathbf{O}(-2,-2,-2)$ whose ends are all embedded is congruent to some f_{μ_1,μ_2,μ_3}. All ends of these surfaces are asymptotic to catenoid cousin ends. The inequality (4.2) implies μ_1, μ_2 and μ_3 are all non-integers.

If we allow equality in (4.2), one of the μ_1, μ_2, μ_3 must be an integer. The corresponding CMC-1 surface might not exist for such μ_1, μ_2, μ_3 in general [UY7]. If it exists, its induced pseudometric $d\sigma^2$ must be reducible (see Lemmas A.1 and A.2).

The Hopf differential Q of f_{μ_1,μ_2,μ_3} is given by

$$Q = \frac{1}{2}\left(\frac{c_3 z^2 + (c_2 - c_1 - c_3)z + c_1}{z^2(z-1)^2}\right) dz^2 \,. \tag{4.4}$$

Let q_1 and q_2 be zeros of Q, that is

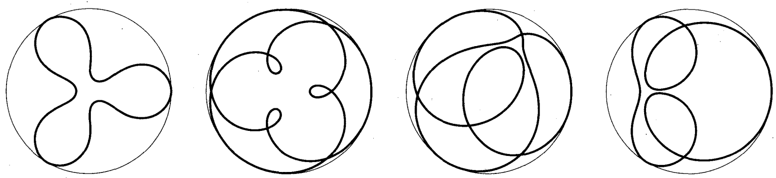
$$c_3 q_l^2 + (c_2 - c_1 - c_3)q_l + c_1 = 0 \qquad (l = 1, 2) \,. \tag{4.5}$$

By (4.3), $q_1 \neq q_2$ holds. The hyperbolic Gauss map is then given by

$$G = z + \frac{(q_1 - q_2)^2}{2\{2z - (q_1 + q_2)\}} \,. \tag{4.6}$$

In particular, all of these surfaces have dual total absolute curvature 8π. On the other hand, the total curvature is equal to $2\pi(4+\mu_1+\mu_2+\mu_3)$. If we set $\mu = \mu_1 = \mu_2 = \mu_3$, the condition (4.2) implies that $\mu > -2/3$, and then there exist $f_{\mu,\mu,\mu}$ for any μ arbitrarily close to $-2/3$, whose total curvatures tend to 4π. This implies Theorem 2.2 is sharp for $m = 1$.

It is interesting to compare these surfaces with minimal trinoids in $\boldsymbol{R}^3$. Minimal trinoids with three catenoid ends are classified in Barbanel [Bar], Lopez [Lop] and Kato [Kat]. Here, we adopt Kato's notation



Type $(+,+,+)$ Type $(-,-,-)$ Type $(+,-,-)$ Type $(+,+,-)$

Fig. 6. Profile curves of trinoids f_{μ_1,μ_2,μ_3}.

[Kat]: The Weierstrass data of these trinoids $x_0 : \boldsymbol{C} \cup \{\infty\} \setminus \{0, p_1, p_2\} \to \boldsymbol{R}^3$ are given by

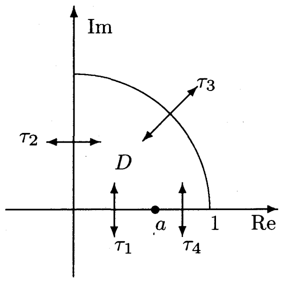
$$g = z - \frac{b(p_1^2 p_2{}^2 + p_1^2 + p_2{}^2 - p_1 p_2)}{f(z)}, \qquad \omega = -f(z)^2\, dz \qquad (b \in \boldsymbol{R})$$

where p_1 and p_2 are real numbers such that $(p_1 - p_2)(1 + p_1 p_2) \neq 0$ and

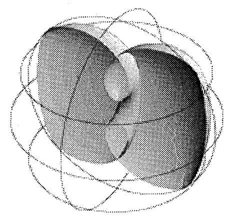
$$f(z) := b\left(\frac{p_1(p_1 - p_2)}{z - p_1} + \frac{p_2(p_2 - p_1)}{z - p_2} + \frac{p_1 p_2(p_1 p_2 + 1)}{z}\right).$$

If the coefficients of $1/z^2$, $1/(z - p_1)^2$, $1/(z - p_2)^2$ in the Laurent expansion of the Hopf differential $Q = \omega\, dg$ at $z = 0, p_1, p_2$ are all the same signature, the surface is called of *type* P and otherwise it is called of *type* N. Type P surface are all Alexandrov-embedded. On the other hand, type N surfaces are not. (For a definition of Alexandrov embedded, see Cosín and Ros [CR].) These two classes consist of the two connected components of the set of minimal trinoids (Tanaka [Tan]; see Figure 5). In the case of CMC-1 trinoids in H^3, we would like to group the surfaces according to the signatures of c_1, c_2, c_3. For example, f_{μ_1,μ_2,μ_3} is called of type $(+,+,+)$ if c_1, c_2, c_3 are all positive, and it is called of type $(-,+,+)$ if one of c_1, c_2, c_3 is negative and the other two are positive. By numerical experiment, we see that these four types $(+,+,+)$, $(-,+,+)$, $(-,-,+)$ and $(-,-,-)$ are topologically distinct (see Figure 6). Surfaces of type $(+,+,+)$ have total curvature less than 8π, and it seems that only surfaces in this class can be embedded.

Example 4.5 (4-noids with $\mathrm{TA}(f) < 8\pi$). A CMC-1 surface of genus 0 with 4 ends satisfies the Cohn-Vossen inequality $\mathrm{TA}(f) > 4\pi$ (see (3.12)). Though genus 0 surfaces with an odd number of ends satisfy a sharper inequality (Theorem 2.2), it seems that the Cohn-Vossen inequality is sharp for 4-noids, by numerical experiment: Let $a \in (0,1)$



Fundamental region of a 4-noid



A 4-noid with $\mathrm{TA}(f) = 5\pi$

Fig. 7. 4-noid

be a real number and $M = \boldsymbol{C} \cup \{\infty\} \setminus \{a, -a, a^{-1}, -a^{-1}\}$. We set

$$G := \frac{pz^3 - z}{z^2 - p}\,,$$
$$Q := -\frac{\mu(\mu+2)a^2(a^2 - a^{-2})^2}{(pa^4 - (3p^2-1)a^2 + p)}\,\frac{(pz^4 - (3p^2-1)z^2 + p)}{(z^2 - a^2)^2(z^2 - a^{-2})^2}\,dz^2\,,$$

where $\mu > -1$ and $p \in \boldsymbol{R} \setminus \{0, 1\}$ with $pa^4 - (3p^2-1)a^2 + p \neq 0$. If there exists a CMC-1 immersion $f\colon M \to H^3$ with hyperbolic Gauss map G and Hopf differential Q, then $\mathrm{TA}(f) = 4\pi(2\mu + 3)$. We shall solve the period problems using the method in [RUY1]: Let $D := \{z = re^{i\theta} \in \boldsymbol{C} \mid 0 < r < 1, 0 < \theta < \pi/2\}$. Then the Riemann surface M is obtained by reflection of D about ∂D. Let τ_1, τ_2, τ_3 and τ_4 be the reflections on the universal cover $\widetilde{M}$ of M, which are the lifts of the reflections on M about the segment $(0, a)$ on the real axis, the segment $(0, i)$ on the imaginary axis, the unit circle $|z| = 1$, and the segment $(a, 1)$ on the real axis, respectively (see Figure 7, left). Let $F\colon \widetilde{M} \to \mathrm{SL}(2, \boldsymbol{C})$ be a solution of (3.10). Since $\overline{G \circ \tau_j} = \sigma_j \star G$, $\overline{Q \circ \tau_j} = Q$, $j = 1, 2, 3, 4$, where

$$\sigma_1 = \sigma_4 = \mathrm{id}, \qquad \sigma_2 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix},$$

there exist matrices $\rho_F(\tau_j) \in \mathrm{SL}(2, \boldsymbol{C})$, $j = 1, 2, 3, 4$ such that $\overline{F \circ \tau_j} = \sigma_j F \rho_F(\tau_j)$, $j = 1, 2, 3, 4$. Moreover, by a similar argument to that in [RUY1, pp. 462–464], one can choose F such that

$$\rho_F(\tau_1) = \mathrm{id}, \quad \rho_F(\tau_2) = \sigma_2, \quad \rho_F(\tau_j) = \begin{pmatrix} q_j & i\gamma_j^1 \\ i\gamma_j^2 & \bar{q}_j \end{pmatrix} \quad (j = 3, 4)\,,$$

where $\gamma_j^k \in \boldsymbol{R}$ and $q_j\bar{q}_j + \gamma_j^1\gamma_j^2 = 1$. Assume $\gamma_3^1\gamma_3^2 > 0$. Then there exists a unique solution F of (3.10) such that

$$\rho_F(\tau_1) = \mathrm{id},\ \rho_F(\tau_2) = \sigma_2,\ \rho_F(\tau_3) = \begin{pmatrix} q & i\gamma \\ i\gamma & \bar{q} \end{pmatrix},\ \rho_F(\tau_3) = \begin{pmatrix} q_4 & i\gamma_4^1 \\ i\gamma_4^2 & \bar{q}_4 \end{pmatrix}.$$

For given μ and a, if one can choose p so that $\gamma_4^1 = \gamma_4^2$, that is $\rho_F(\tau_j) \in \mathrm{SU}(2)$, then there exists a CMC-1 immersion f of M into H^3 with hyperbolic Gauss map G and Hopf differential Q, by Proposition 4.7 in [RUY1].

By numerical calculation, for $\mu = -0.5$ and $a = 0.8$, there exists $p \simeq 1.4$ such that the period problem is solved. This surface thus has $\mathrm{TA}(f) = 8\pi$, and by continuity of the solvability of the period problems, clearly there exist surfaces with $\mathrm{TA}(f) < 8\pi$. Moreover, there exist such parameters a and p for $\mu \simeq -1$. So it seems that the Cohn-Vossen inequality for genus-zero 4-ended CMC-1 surfaces is sharp. Figure 7 shows the half cut of the surface with $\mathrm{TA}(f) = 5\pi$.

§5. Reducibility

To state the results for higher $\mathrm{TA}(f)$ or $\mathrm{TA}(f^{\#})$, we review the notion of *reducibility*. For details, see [UY3, UY7, RUY1].

Metrics with conical singularities. Let $\overline{M}$ be a compact Riemann surface. A pseudometric $d\sigma^2$ on $\overline{M}$ is said to be an element of $\mathrm{Met}_1(\overline{M})$ if there exists a finite set of points $\{p_1, \dots, p_n\} \subset \overline{M}$ such that

(1) $d\sigma^2$ is a conformal metric of constant curvature 1 on $\overline{M}\setminus\{p_1, \dots, p_n\}$, and

(2) $\{p_1, \dots, p_n\}$ is the set of *conical singularities* of $d\sigma^2$, that is, for each $j = 1, \dots, n$, there exists a real number $\beta_j > -1$ so that $d\sigma^2$ is asymptotic to $c|z - p_j|^{2\beta_j}\, dz\, d\bar{z}$, where z is a complex coordinate of $\overline{M}$ around p_j and c is a positive constant.

We call the real number β_j the *order* of the conical singularity p_j, and write $\beta_j = \mathrm{ord}_{p_j} d\sigma^2$. The formal sum

$$\beta_1 p_1 + \cdots + \beta_n p_n \tag{5.1}$$

is called the *divisor corresponding to* $d\sigma^2$.

Let $d\sigma^2 \in \mathrm{Met}_1(\overline{M})$ with divisor as in (5.1) and set $M := \overline{M} \setminus \{p_1, \dots, p_n\}$. Then there exists a holomorphic map $g\colon \widetilde{M} \longrightarrow \boldsymbol{C}\cup\{\infty\} =$

$\boldsymbol{CP}^1$ defined on the universal cover $\widetilde{M}$ of M such that

$$d\sigma^2 = \frac{4\,dg\,d\bar{g}}{(1+|g|^2)^2} = g^* ds_0^2 \,, \tag{5.2}$$

where ds_0^2 is the Fubini-Study metric of $\boldsymbol{CP}^1$. We call g the *developing map* of $d\sigma^2$. The developing map is unique up the change

$$g \longmapsto a \star g \qquad (a \in \mathrm{PSU}(2)) \,, \tag{5.3}$$

where $a \star g$ denotes the Möbius transformation of g with respect to a as in (3.2). Here we write $a \in \mathrm{PSU}(2)$ as a 2×2 matrix in $\mathrm{SU}(2)$ and identify a with $-a$.

For each deck transformation $\tau \in \pi_1(M)$ on $\widetilde{M}$, $d\sigma^2 = d\sigma^2 \circ \tau$ holds. So there exists a representation

(5.4)

$$\rho_g \colon \pi_1(M) \longrightarrow \mathrm{PSU}(2) \quad \text{such that} \quad \rho\circ\tau^{-1} = \rho_g(\tau)\star g \quad \text{for } \tau \in \pi_1(M) \,.$$

By a change of g as in (5.3), the corresponding representation changes by conjugation:

$$\rho_{a\star g} = a\rho_g a^{-1} \,. \tag{5.5}$$

Let τ_j be a deck transformation induced from a small loop on $\overline{M}$ surrounding a singularity p_j. Then by (5.5), one can choose the developing map g such that $\rho_g(\tau_j)$ is diagonal:

$$\rho_g(\tau_j) = \begin{pmatrix} e^{\pi i \nu_j} & 0 \\ 0 & e^{-\pi i \nu_j} \end{pmatrix} \qquad (\nu_j \in \boldsymbol{R}) \,,$$

namely, $g\circ\tau_j = e^{2\pi i \nu_j} g$. This implies that $(z-p_j)^{-\nu_j} g$ is single-valued on a neighborhood of p_j, where z is a complex coordinate around p_j. Then, replacing ν_j with $\nu_j + m$ ($m \in \boldsymbol{Z}$) if necessary, we can normalize

$$g = (z-p_j)^{\nu_j}\left(g_0 + g_1(z-p_j) + g_2(z-p_j)^2 + \dots\right) \qquad (g_0 \neq 0) \,. \tag{5.6}$$

By definition of the order and by equation (5.2), we have

$$\nu_j = \beta_j + 1 \qquad \text{or} \qquad -\beta_j - 1 \,.$$

Definition 5.1. A pseudometric $d\sigma^2 \in \mathrm{Met}_1(\overline{M})$ is called *reducible* if the representation ρ_g can be diagonalized simultaneously, where g is the developing map of $d\sigma^2$. More precisely, a reducible metric $d\sigma^2$ is called $\mathcal{H}^3$*-reducible* if the representation is trivial, and called $\mathcal{H}^1$*-reducible* otherwise. A pseudometric $d\sigma^2$ is called *irreducible* if it is not reducible.

By definition, a developing map g of an $\mathcal{H}^3$-reducible metric is a meromorphic function on $\overline{M}$ itself. Moreover, by (5.6), all conical singularities have integral orders, which coincide with the branching orders of the meromorphic function g. In this case, for any $a \in \mathrm{PSL}(2, \boldsymbol{C})$, $g_a := a \star g$ induces a new metric $d\sigma_a^2 := g_a^\star ds_0^2 \in \mathrm{Met}_1(\overline{M})$ with the same divisor as $d\sigma^2$. Since $d\sigma_a^2 = d\sigma^2$ if $a \in \mathrm{PSU}(2)$, we have a non-trivial deformation of $d\sigma^2$ preserving the divisor parametrized by a real 3-dimensional space $\mathcal{H}^3 = \mathrm{PSL}(2, \boldsymbol{C}) / \mathrm{PSU}(2)$, which is the hyperbolic 3-space.

On the other hand, assume $d\sigma^2 \in \mathrm{Met}_1(\overline{M})$ is $\mathcal{H}^1$-reducible. Then there exists a developing map g such that the image of ρ_g consists of diagonal matrices. Let t be a positive real number and set

$$g_t := tg = \begin{pmatrix} t^{1/2} & 0 \\ 0 & t^{-1/2} \end{pmatrix} \star g \,.$$

Then by (5.5), $\rho_{g_t} = \rho_g$ holds. Thus, g_t induces a new metric $d\sigma_t^2 \in \mathrm{Met}_1(\overline{M})$. So we have one parameter family of pseudometrics $\{d\sigma_t^2\}$ preserving the corresponding divisor. This family is considered as a deformation of pseudometric parameterized by a geodesic line in $\mathcal{H}^3$. For details, see the Appendix in [RUY1].

We introduce a criterion for reducibility:

Lemma 5.2. *A metric $d\sigma^2 \in \mathrm{Met}_1(\overline{M})$ is reducible if and only if there exists a developing map such that $d \log g$ is a meromorphic 1-form on $\overline{M}$.*

Proof. Assume $d\sigma^2$ is reducible. Then one can choose the developing map g such that ρ_g is diagonal. Then for each deck transformation $\tau \in \pi_1(M)$,

$$g \circ \tau^{-1} = \begin{pmatrix} e^{\pi i \nu_\tau} & 0 \\ 0 & e^{-\pi i \nu_\tau} \end{pmatrix} \star g = e^{2\pi i \nu_\tau} g \qquad (\nu_\tau \in \boldsymbol{R})$$

holds. Hence we have $\log g \circ \tau = g + 2\pi i \nu_\tau$. Differentiating this, $d \log g \circ \tau = d \log g$ holds. Hence $d \log g$ is single-valued on $\overline{M}$.

Conversely, we assume $d \log g$ is well-defined on $\overline{M}$ for a developing map g. Then $\log g \circ \tau - \log g$ is a constant. Hence we have $g \circ \tau = \lambda_\tau g$ for some constant λ_τ. Then ρ_g is diagonal. Q.E.D.

Relationship with CMC-1 surfaces. Let $f \colon \overline{M}_\gamma \setminus \{p_1, \dots, p_n\} \to H^3$ be a complete conformal CMC-1 immersion, where $\overline{M}_\gamma$ is a compact

Riemann surface. If $\mathrm{TA}(f) < \infty$, then the pseudometric $d\sigma^2$ as in (3.1) is considered as an element of $\mathrm{Met}_1(\overline{M}_\gamma)$ (see [Bry]), and the secondary Gauss map g is the developing map of $d\sigma^2$. Let $\{q_1, \dots, q_m\}$ be the set of umbilic points of f, that is the zeros of Q and set $\xi_k := \operatorname{ord}_{q_k} Q$ $(k = 1, \dots, m)$. Then by (3.4), $d\sigma^2$ has a conical singularity of order ξ_k for each $k = 1, \dots, m$. Hence the divisor of $d\sigma^2$ is in the form

$$(5.7) \qquad \mu_1 p_1 + \cdots + \mu_n p_n + \xi_1 q_1 + \cdots + \xi_m q_m \ ,$$

where the μ_j $(j = 1, \dots, n)$ are the branch orders of g at each p_j.

Let F be a holomorphic lift of f as in (3.6). Then there exists a representation $\rho_F \colon \pi_1(M) \to \mathrm{SU}(2)$ as in (3.17). By (3.7), the secondary Gauss map g of F changes as $g \circ \tau^{-1} = \rho_F(\tau) \star g$ for each deck transformation $\tau \in \pi_1(M)$. Hence the representation ρ_g defined in (5.4) satisfies

$$(5.8) \qquad \rho_g(\tau) = \pm \rho_F(\tau) \qquad \bigl(\tau \in \pi_1(M)\bigr) \ .$$

The immersion f is called $\mathcal{H}^3$-reducible (respectively, $\mathcal{H}^1$-reducible) if the corresponding pseudometric $d\sigma^2$ is $\mathcal{H}^3$-reducible (respectively, $\mathcal{H}^1$-reducible).

Lemma 5.3. *A* CMC-1 *immersion $f \colon M \to H^3$ is $\mathcal{H}^3$-reducible if and only if the dual immersion $f^\#$ is well-defined on M.*

Proof. Let F be a lift of f. Then $f^\# = F^{-1}(F^{-1})^*$ is well-defined on M if and only if $\rho_F = \pm\,\mathrm{id}$. This is equivalent to ρ_g being the trivial representation, by (5.8). Q.E.D.

§6. The case $\mathrm{TA}(f^\#) \le 8\pi$

We now have enough notation and facts to describe results on the case $\mathrm{TA}(f^\#) \le 8\pi$ [RUY3].

Let $f \colon \overline{M}_\gamma \setminus \{p_1, \dots, p_n\} \to H^3$ be a complete, conformal CMC-1 immersion, where $\overline{M}_\gamma$ is a Riemann surface of genus γ. Now we assume $\mathrm{TA}(f^\#) \le 8\pi$. If the hyperbolic Gauss map G has an essential singularity at any end p_j, then $\mathrm{TA}(f^\#) = +\infty$, since $\mathrm{TA}(f^\#)$ is the area of the image of G. So G is meromorphic on all of $\overline{M}_\gamma$. In particular, $\mathrm{TA}(f^\#) = 4\pi \deg G = 0$, 4π, or 8π.

Since $f^\#$ has finite total curvature, the Hopf differential $Q^\# = -Q$ can be extended to $\overline{M}_\gamma$ as a meromorphic 2-differential [Bry, Proposition

Type	TA($f^{\#}$)	Reducibility	Status	c.f.
$\mathbf{O}(0)$	0	$\mathcal{H}^3$-red.	classified^0	Horosphere
$\mathbf{O}(-4)$	4π	$\mathcal{H}^3$-red.	classified	Duals of Enneper cousins [RUY1, Example 5.4]
$\mathbf{O}(-2,-2)$	4π	reducible	classified	Catenoid cousins and warped catenoid cousins with embedded ends (i.e. $\delta = 1$) [Bry, Example 2], [UY1, RUY3, RUY4]
$\mathbf{O}(-5)$	8π	$\mathcal{H}^3$-red.	classified	[RUY3]
$\mathbf{O}(-6)$	8π	$\mathcal{H}^3$-red.	classified	[RUY3]
$\mathbf{O}(-2,-2)$	8π	red.	classified	Double covers of catenoid cousins and warped catenoid cousins with $\delta = 2$ [UY1, Theorem 6.2], [RUY3, RUY4]
$\mathbf{O}(-1,-4)$	8π	$\mathcal{H}^3$-red.	classified^0	[RUY3]
$\mathbf{O}(-2,-3)$	8π	$\mathcal{H}^1$-red.	classified	[RUY3]
$\mathbf{O}(-2,-4)$	8π	$\mathcal{H}^1$-red. $\mathcal{H}^3$-red.	classified classified	[RUY3] [RUY3]
$\mathbf{O}(-3,-3)$	8π	red.	existence	[RUY3]
$\mathbf{O}(-1,-1,-2)$	8π	$\mathcal{H}^3$-red.	classified^0	[RUY3]
$\mathbf{O}(-1,-2,-2)$	8π	$\mathcal{H}^1$-red. $\mathcal{H}^3$-red.	classified classified	[RUY3] [RUY3]
$\mathbf{O}(-2,-2,-2)$	8π	irred. $\mathcal{H}^1$-red. $\mathcal{H}^3$-red.	classified existence$^+$ existence$^+$	[UY6, Theorem 2.6] [RUY3] [RUY3]
$\mathbf{I}(-3)$	8π		unknown	
$\mathbf{I}(-4)$	8π		existence	Chen-Gackstatter cousins [RUY3]
$\mathbf{I}(-1,-1)$	8π		unknown$^+$	[RUY3]
$\mathbf{I}(-2,-2)$	8π		existence	Genus 1 catenoid cousins [RS]

Table 1. CMC-1 surfaces in H^3 with TA($f^{\#}$) $\leq 8\pi$ [RUY3].

5]. Hence $d_j = \mathrm{ord}_{p_j} Q$ is finite for each $j = 1, \ldots, n$. Our results from [RUY3] are shown in Table 1. In the table,

- *classified* means the complete list of the surfaces in such a class is known (and this means not only that we know all the possibilities for the form of the data (G, Q), but that we also know exactly for which (G, Q) the period problems of the immersions are solved).

Type	TA	The surface	c.f.
$\mathbf{O}(0)$	0	Plane	
$\mathbf{O}(-4)$	4π	Enneper's surface	
$\mathbf{O}(-5)$	8π		[Lop, Theorem 6]
$\mathbf{O}(-6)$	8π		[Lop, Theorem 6]
$\mathbf{O}(-2,-2)$	4π 8π	Catenoid Double cover of the catenoid	
$\mathbf{O}(-1,-3)$	8π		[Lop, Theorem 5]
$\mathbf{O}(-2,-3)$	8π		[Lop, Theorem 4, 5]
$\mathbf{O}(-2,-4)$	8π		[Lop, Theorem 5]
$\mathbf{O}(-3,-3)$	8π		[Lop, Theorem 4]
$\mathbf{O}(-1,-2,-2)$	8π		[Lop, Theorem 5]
$\mathbf{O}(-2,-2,-2)$	8π		[Lop, Theorem 5]
$\mathbf{I}(-4)$	8π	Chen-Gackstatter surface	[Lop, Theorem 5], [CG]

Table 2. The classification of complete minimal surfaces in $\boldsymbol{R}^3$ with TA $\leq 8\pi$ ([Lop]), for comparison with Table 1.

- *classified*^{0} means there exists a unique surface (up to isometries of H^3 and deformations that come from its reducibility).
- *existence* means that examples exist, but they are not yet classified.
- *existence*$^+$ means that all possibilities for the data (G, Q) are determined, but the period problems are solved only for special cases.
- *unknown* means that neither existence nor non-existence is known yet.
- *unknown*$^+$ means that all possibilities for the data (G, Q) are determined, but the period problems are still unsolved.

Any class and type of reducibility not listed in Table 1 cannot contain surfaces with $\mathrm{TA}(f^\#) \leq 8\pi$. For example, any irreducible or $\mathcal{H}^3$-reducible surface of type $\mathbf{O}(-2,-3)$ must have dual total absolute curvature at least 12π.

Table 2 shows the corresponding results for minimal surfaces in $\boldsymbol{R}^3$, the classification of complete minimal surfaces with TA $\leq 8\pi$ [Lop]. Comparing these two tables, one sees differences between the classes of minimal surfaces with TA $\leq 8\pi$ and the classes of CMC-1 surfaces with $\mathrm{TA}(f^\#) \leq 8\pi$. For example, there exist no mimimal surfaces of classes $\mathbf{O}(-1,-4)$ and $\mathbf{O}(-1,-1,-2)$ with TA $\leq 8\pi$, but CMC-1 surfaces of such types do exist.

§7. The case of $\mathrm{TA}(f) \leq 8\pi$

In the remainder of this paper, we shall give new results on the case of higher $\mathrm{TA}(f)$.

Preliminaries. First, we give further notation and facts that will be needed in our discussion.

Orders of the Gauss maps. Let $\overline{M}_\gamma$ be a compact Riemann surface of genus γ. For a complete conformal CMC-1 immersion $f\colon M = \overline{M}_\gamma \setminus \{p_1, \dots, p_n\} \to H^3$ with $\mathrm{TA}(f) < \infty$, we define μ_j and $\mu_j^\#$ to be the branching orders of the Gauss maps g and G, respectively, at an end p_j. Then the pseudometric $d\sigma^2$ as in (3.1) has a conical singularity of order $\mu_j > -1$ at each end p_j $(j = 1, \dots, n)$. Let $d_j = \mathrm{ord}_{p_j} Q$ $(j = 1, \dots, n)$. Then an end p_j is regular if and only if $d_j \geq -2$ (see Section 3, or [UY1]). If an end p_j is irregular, then $\mu_j^\# = \infty$. At a regular end p_j, the relation (3.5) implies that the Hopf differential Q expands as

$$Q = \left(\frac{1}{2} \frac{c_j - c_j^\#}{(z - p_j)^2} + \dots \right) dz^2 , \tag{7.1}$$

where

$$c_j = -\frac{1}{2}\mu_j(\mu_j + 2), \qquad c_j^\# = -\frac{1}{2}\mu_j^\#(\mu_j^\# + 2) \tag{7.2}$$

and z is a local complex coordinate around p_j.

Let $\{q_1, \dots, q_m\} \subset M$ be the m umbilic points of the surface, and let $\xi_k = \mathrm{ord}_{q_k} Q$. Since the total order of a holomorphic 2-differential is $-2\chi(\overline{M}_\gamma)$, we have

$$\sum_{j=1}^n d_j + \sum_{k=1}^m \xi_k = 4\gamma - 4, \qquad \text{in particular,} \quad \sum_{j=1}^n d_j \leq 4\gamma - 4 . \tag{7.3}$$

By (3.4) and (3.5), one has

(7.4)

$$\xi_k = [\text{the branching order of } G \text{ at } q_k] = [\text{the branching order of } g \text{ at } q_k] = \mathrm{ord}_{q_k} d\sigma^2 = \mathrm{ord}_{q_k} Q .$$

As in (2.4) of [RUY3], the Gauss-Bonnet theorem for $(\overline{M}_\gamma, d\sigma^2)$ implies

$$\frac{\mathrm{TA}(f)}{2\pi} = \chi(\overline{M}_\gamma) + \sum_{j=1}^n \mu_j + \sum_{k=1}^m \xi_k = (2\gamma - 2) + \sum_{j=1}^n \mu_j + \sum_{k=1}^m \xi_k \tag{7.5}$$

as well as

(7.6)
$$\frac{\mathrm{TA}(f^{\#})}{2\pi} = \chi(\overline{M}_\gamma) + \sum_{j=1}^{n} \mu_j^{\#} + \sum_{k=1}^{m} \xi_k \,,$$

which is obtained from the Gauss-Bonnet theorem for $d\sigma^{2\#} = (-K^{\#})ds^{2\#}$ [RUY3]. Combining this with (7.3), we have

(7.7)
$$\frac{\mathrm{TA}(f)}{2\pi} = 2\gamma - 2 + \sum_{j=1}^{n}(\mu_j - d_j) \,.$$

Proposition 4.1 in [UY1] implies that

(7.8) $\mu_j - d_j > 1,$ in particular, $\mu_j - d_j \geq 2$ if $\mu_j \in \boldsymbol{Z}$.

An end p_j is regular if and only if $d_j \geq -2$, and then G is meromorphic at p_j. Thus

(7.9) $\mu_j^{\#}$ is a non-negative integer if $d_j \geq -2$.

In this case, one has (Lemma 3 of [UY5])

(7.10)
$\mu_j^{\#} - d_j \geq 2$ and the equality holds if and only if p_j is embedded.

By Proposition 4 of [Bry],

(7.11)
$$\mu_j > -1 \,,$$

hence equation (7.1) implies

(7.12)
$$\mu_j = \mu_j^{\#} \qquad \text{if} \quad d_j \geq -1 \,.$$

Finally, we note that

(7.13)
any meromorphic function on a Riemann surface $\overline{M}_\gamma$ of genus $\gamma \geq 1$ has at least three distinct branch points.

To prove this, let φ be a meromorphic function on $\overline{M}_\gamma$ with branch points $\{q_1, \ldots, q_N\}$ with branching order ν_k at q_k. Then the Riemann-Hurwitz relation implies

$$2\deg\varphi = 2 - 2\gamma + \sum_{k=1}^{N} \nu_k \,.$$

On the other hand, since the multiplicity of φ at q_k is $\nu_k + 1$, $\deg\varphi \geq \nu_k + 1$ $(k = 1, \ldots, m)$. Thus $(N-2)\deg\varphi \geq 2(\gamma - 1) + N$. If $\gamma \geq 1$, then $\deg\varphi \geq 2$, and so $N \geq 3$.

Remark. Facts (7.7) and (7.8) imply that, for CMC-1 surfaces, equality never holds in the Cohn-Vossen inequality (see (3.12) and [UY1]).

Flux for CMC-1 surfaces. Let $f\colon \overline{M}_\gamma \setminus \{p_1, \dots, p_n\} \to H^3$ be a complete CMC-1 immersion. For each end p_j, the *flux* at p_j is defined as ([RUY2])

$$(7.14)\quad \mathcal{Fl}_j := \frac{1}{2\pi i}\int_{\tau_j} \begin{pmatrix} G & -G^2 \\ 1 & -G \end{pmatrix} \frac{Q}{dG} \in \mathfrak{sl}(2, \boldsymbol{C}) \qquad (j = 1, \dots, n)\,,$$

where G and Q are the hyperbolic Gauss map and the Hopf differential of f respectively, and τ_j is a loop surrounding the end p_j. Then the following balancing formula holds (Theorem 1 in [RUY2]):

$$(7.15)\qquad \sum_{j=1}^{n} \mathcal{Fl}_j = 0\,.$$

Moreover, one has (Proposition 2 and Corollary 5 in [RUY2]):

Proposition 7.1. *Let* $f\colon \overline{M}_\gamma \setminus \{p_1, \dots, p_n\} \to H^3$ *be a complete* CMC-1 *immersion.*

(1) If an end p_j *is regular and* $\operatorname{ord}_{p_j} Q = -2$, *then* $\mathcal{Fl}_j \neq 0$.

(2) If an end p_j *is regular and embedded,* $\mathcal{Fl}_j = 0$ *if and only if* $\operatorname{ord}_{p_j} Q \geq 0$.

Then by the balancing formula (7.15), we have

Corollary 7.2. *There exists no complete* CMC-1 *surface of finite total curvature with only one end* p *that is regular, such that either one of the following holds*:

(1) $\operatorname{ord}_p Q = -2$.

(2) $\operatorname{ord}_p Q < 0$ *and the end is embedded.*

Results for TA(f) $\leq 8\pi$. First, we prepare the following lemma.

Lemma 7.3. *Let* $f\colon M \to H^3$ *be a complete* CMC-1 *immersion of genus* γ *and* n *ends with* $\mathrm{TA}(f) \leq 2\pi\rho$. *If* f *is not totally umbilic (not a horosphere), then the following hold*:

(1) $2\gamma < \rho + 1$ *and* $1 \leq n < \rho - 2\gamma + 2$.

(2) If $n = 1$, *then* $2\gamma - \rho - 3 < d_1 \leq 4\gamma - 4$ *and* $d_1 \neq -2$.

(3) If $\gamma = n = 1$, *then* $-\rho - 1 < d_1 \leq -3$.

(4) If $2 \leq n = \rho + 1 - 2\gamma$*, then* $d_j = -2$ *at all ends.*

(5) If $1 = n = \rho + 1 - 2\gamma$*, then* $d_1 \geq 0$ *and* $\mu_1 = 2 + d_1$*.*

Proof. The first item of the lemma is obtained from the Cohn-Vossen inequality (3.12). In particular, if $n = 1$, (7.7), (7.11) and (7.3) imply $\rho \geq 2\gamma - 2 + \mu_1 - d_1 > 2\gamma - 3 - d_1$ and $d_1 \leq 4\gamma - 4$. In this case, by the balancing formula (7.15), the flux $\mathcal{Fl}_1$ must vanish. Hence by Corollary 7.2, $d_1 \neq -2$ holds, and the second item of the theorem follows. Even more particularly, if $\gamma = n = 1$, then $-\rho - 1 < d_1 \leq 0$ and $d_1 \neq -2$. Assume $d_1 \geq -1$. In this case, the end is regular, and then G is a meromorphic function on $\overline{M}_\gamma$. On the other hand, (7.3) implies that there is at most one umbilic point. Since a branch point of G is an umbilic point or an end, this implies that the number of branch points of G is at most 2, which contradicts (7.13). Hence the third item is proven.

Suppose $n = \rho - 2\gamma$. Then (7.3) implies

$$n + 1 \geq \sum_{j=1}^{n} (\mu_j - d_j) , \tag{7.16}$$

and we consider two cases:

Case 1 If $n \geq 2$, then (7.8) implies that $1 < \mu_j - d_j < 2$ for all j, so $\mu_j \notin \boldsymbol{Z}$ for all j, and hence (7.12) implies that $d_j \leq -2$ for all j. But by (7.16) and (7.11), we have $-2n \leq \sum_{j=1}^{n} d_j$, and so $d_j = -2$ for all j.

Case 2 If $n = 1$, then $1 < \mu_1 - d_1 \leq 2$ holds because of (7.16) and (7.8). Hence by (7.11), $d_1 \geq -2$. But Corollary 7.2 implies $d_1 \geq -1$. Then by (7.12), $\mu_1 \in \boldsymbol{Z}$ and $\mu_1 - d_1 = 2$ holds. Suppose $d_1 = -1$. Then $\mu_1^{\#} = \mu_1 = d_1 + 2 = 1$, and then by (7.10), the only end p_1 is regular and embedded. This contradicts (2) and (7.15). Hence $d_1 \geq 0$. Q.E.D.

Type	TA(f)	Reducibility	Status	cf.
$\mathbf{O}(0)$	0	$\mathcal{H}^3$-red.	classified	Horosphere
$\mathbf{O}(-4)$	4π	$\mathcal{H}^3$-red.	classified	Enneper cousins [Bry]
$\mathbf{O}(-5)$	8π	$\mathcal{H}^3$-red.	classified	Same as "dual" case
$\mathbf{O}(-6)$	8π	$\mathcal{H}^3$-red.	classified	Same as "dual" case
$\mathbf{O}(-2,-2)$	$(0,8\pi]$	$\mathcal{H}^1$-red.	classified	Catenoid cousins and their δ-fold covers [Bry, Ex. 2],[UY1]
$\mathbf{O}(-2,-2)$	4π 8π	$\mathcal{H}^3$-red.	classified	Warped cat. cous. $l=1$ Warped cat. cous. $l=2$ [UY1, Thm 6.2], Exa. 4.3
$\mathbf{O}(-1,-4)$	8π	$\mathcal{H}^3$-red.	classified	Same as "dual" case
$\mathbf{O}(-2,-4)$	8π $(4\pi,8\pi)$	$\mathcal{H}^3$-red. $\mathcal{H}^1$-red.	classified existence	Same as "dual" case Remark A.9
$\mathbf{O}(-2,-5)$	8π	$\mathcal{H}^1$-red.	existence	Remarks A.10, A.12
$\mathbf{O}(-3,-3)$		reducible	unknown	Remark A.11
$\mathbf{O}(-3,-4)$	8π	reducible	unknown	Remark A.12
$\mathbf{O}(0,-2,-2)$	$(4\pi,8\pi)$	$\mathcal{H}^1$-red.	classified	Proposition A.15
$\mathbf{O}(-1,-2,-3)$	8π	$\mathcal{H}^1$-red.	unknown	
$\mathbf{O}(-1,-1,-2)$	8π	$\mathcal{H}^3$-red.	classified	Same as "dual" case
$\mathbf{O}(-1,-2,-2)$	8π $(4\pi,8\pi)$ 8π	$\mathcal{H}^3$-red. $\mathcal{H}^1$-red. $\mathcal{H}^1$-red.	classified classified classified	Same as "dual" case Proposition A.16 Proposition A.17
$\mathbf{O}(-2,-2,-2)$	$(4\pi,8\pi]$		existence	Classified for irred. embedded end case [UY6]
$\mathbf{O}(-2,-2,-3)$		irred./$\mathcal{H}^1$-red.	unknown	
$\mathbf{O}(-2,-2,-4)$	8π	irred./$\mathcal{H}^1$-red.	unknown	
$\mathbf{O}(-2,-3,-3)$	8π	irred./$\mathcal{H}^1$-red.	unknown	
$\mathbf{O}(-2,-2,-2,-2)$			existence	Example 4.5
$\mathbf{O}(-2,-2,-2,0)$	8π		existence	Remark A.19
$\mathbf{O}(-2,-2,-2,d)$ $d=-3,-2,-1,1$	8π when $d\geq -1$		unknown	

$\mathbf{O}(-2,-2,-2,-2,-2)$	8π		unknown	Corollary 7.4
$\mathbf{I}(-3)$			unknown	
$\mathbf{I}(-4)$			unknown	
$\mathbf{I}(-1,-1)$	8π		unknown	
$\mathbf{I}(-2,-2)$			unknown	Remark A.20
$\mathbf{I}(-2,-3)$			unknown	
$\mathbf{I}(-2,-2,-2)$			unknown	Remark A.21

Table 3. Classification of CMC-1 surfaces in H^3 with $\mathrm{TA}(f) \leq 8\pi$

Lemma 7.3 gives the following corollary:

Corollary 7.4. *If $f\colon M \to H^3$ is a complete* CMC-1 *immersion with* $\mathrm{TA}(f) \leq 8\pi$*, then it is either*

(1) a surface of genus 0 *with at most* 5 *ends. (if it has* 5 *ends, then all* 5 *ends are regular with $d_1 = d_2 = d_3 = d_4 = d_5 = -2$), or*

(2) a surface of genus 1 *with at most* 3 *ends (if it has* 3 *ends, then all* 3 *ends are regular with $d_1 = d_2 = d_3 = -2$; if it has* 1 *end, then the end is irregular with $d_1 = -3$ or $d_1 = -4$).*

Proof. We only have to show that a CMC-1 surface with $\mathrm{TA}(f) \leq 8\pi$ of genus 2 and with 1 regular end satisfying $0 \leq d_1 \leq 4$ cannot exist. By (7.7), (7.12) and (7.6), such a surface would satisfy $\mathrm{TA}(f^\#) = \mathrm{TA}(f) \leq 8\pi$ and hence the hyperbolic Gauss map G is a meromorphic function on a compact Riemann surface $\overline{M}_2$ of genus 2 with $\deg G \leq 2$. Therefore $\mu_1^\#$ can be only 0 or 1, and so $d_1 \leq \mu_1^\# - 2 < 0$, a contradiction. Q.E.D.

Now we compile an *unfinished* classification of CMC-1 surfaces with $\mathrm{TA}(f) \leq 8\pi$ (see Table 3). In the "status" column of the table, *classified* means that the surfaces of such a class are completely classified (i.e. not only is the holomorphic data known, but the period problems are also completely solved), *existence* means that there exists such a surface, and *unknown* means that it is unknown if such a surface exists. Surfaces of any type not appearing in the table cannot exist with $\mathrm{TA}(f) \leq 8\pi$. The proofs of the existence and non-existence results are given in Appendix A.

§Appendix A. Detailed discussion of the case $\mathrm{TA}(f) \le 8\pi$

In this appendix, we give a precise discussion of complete CMC-1 surfaces with $\mathrm{TA}(f) \le 8\pi$.

Detailed Preliminaries. Here, we review facts which will be used to prove existence and non-existence for special cases.

Metrics in $\mathrm{Met}_1(\boldsymbol{C} \cup \{\infty\})$. In this subsection, we introduce special properties of pseudometrics in $\mathrm{Met}_1(\boldsymbol{C} \cup \{\infty\})$ (see Section 5).

Lemma A.1. *A pseudometric $d\sigma^2 \in \mathrm{Met}_1(\boldsymbol{C} \cup \{\infty\})$ with divisor as in* (5.1) *is $\mathcal{H}^3$-reducible if and only if all orders of conical singularities are integers.*

Proof. If $d\sigma^2$ is $\mathcal{H}^3$-reducible, then the developing map g is a meromorphic function on $\boldsymbol{C} \cup \{\infty\}$. So the branch orders must all be integers. Conversely, assume all conical singularities have integral orders. Then by (5.6), $\rho_g(\tau_j) = \pm\,\mathrm{id}$ for each j, where τ_j is the deck transformation on $M := \boldsymbol{C} \cup \{\infty\} \setminus \{p_1, \dots, p_n\}$ corresponding to the loop surrounding p_j. Since $\pi_1(M)$ is generated by $\tau_1, \dots, \tau_n$, ρ_g is the trivial representation. Q.E.D.

Lemma A.2. *Let $d\sigma^2 \in \mathrm{Met}_1(\boldsymbol{C} \cup \{\infty\})$ with divisor as in* (5.1). *Assume the orders β_1 and β_2 are not integers, and β_j $(j \ge 3)$ are integers. Then $d\sigma^2$ is $\mathcal{H}^1$-reducible.*

Proof. Let g be a developing map such that $\rho_g(\tau_1)$ is diagonal. Here, as in the proof of the previous lemma, we have $\rho_g(\tau_j) = \pm\,\mathrm{id}$ $(j \ge 3)$. Then we have $\rho_g(\tau_1)\rho_g(\tau_2) = \pm\,\mathrm{id}$ because $\tau_1 \circ \cdots \circ \tau_n = \mathrm{id}$. Hence $\rho_g(\tau_2)$ is also a diagonal matrix. Q.E.D.

Lemma A.3 ([RUY4, Proposition A.1]). *There exists no metric $d\sigma^2 \in \mathrm{Met}_1(\boldsymbol{C} \cup \{\infty\})$ with divisor as in* (5.1) *such that only one β_j is a non-integer and all others are integers.*

A developing map of a reducible metric in $\mathrm{Met}_1(\boldsymbol{C} \cup \{\infty\})$ can be written explicitly as follows:

Lemma A.4 ([RUY4, Proposition B.1]). *Let $d\sigma^2 \in \mathrm{Met}_1(\boldsymbol{C} \cup \{\infty\})$ be reducible with divisor as in* (5.1). *Assume*

$$p_n = \infty\,, \qquad \beta_1, \dots, \beta_m \notin \boldsymbol{Z}\,, \qquad \beta_{m+1}, \dots, \beta_{n-1} \in \boldsymbol{Z}\,.$$

Then there exists a developing map g of $d\sigma^2$ such that

$$g = (z-p_1)^{\nu_1} \dots (z-p_m)^{\nu_m}\, r(z) \qquad (\nu_1, \dots, \nu_m \in \boldsymbol{R} \setminus \boldsymbol{Z}) ,$$

where $r(z)$ is a rational function on $\boldsymbol{C} \cup \{\infty\}$.

Corollary A.5. *Let $d\sigma^2 \in \mathrm{Met}_1(\boldsymbol{C} \cup \{\infty\})$ be reducible with divisor as in* (5.1) *and $p_n = \infty$. Then there exists a developing map g such that*

(A.1)
$$dg = t\,\frac{(z-p_1)^{\alpha_1} \dots (z-p_{n-1})^{\alpha_{n-1}}}{\prod_{k=1}^{N}(z-a_k)^2}\,dz \qquad (\alpha_j = \beta_j \quad \textit{or} \quad -\beta_j - 2) ,$$

where $a_1, \dots, a_N \in \boldsymbol{C} \setminus \{p_1, \dots, p_{n-1}\}$ are mutually distinct, t is a positive real number. Moreover, one has

$$\text{(A.2)} \qquad -(\alpha_1 + \dots + \alpha_{n-1}) + 2N - 2 = \beta_n \quad \textit{or} \quad -\beta_n - 2 .$$

Proof. If $d\sigma^2$ is $\mathcal{H}^3$-reducible, g is a meromorphic function on $\boldsymbol{C} \cup \{\infty\}$ which branches at $p_1, \dots, p_n$ with branch orders $\beta_j \in \boldsymbol{Z}^+$. Hence p_j is a zero of order β_j or a pole of order $\beta_j + 2$ of dg for each $j = 1, \dots, n$. Let $\{a_1, \dots, a_N\}$ be the simple poles of g on $\boldsymbol{C} \setminus \{p_1, \dots, p_{n-1}\}$, then each a_k is a pole of order 2 of dg. (The a_j are not branch points of g.) The zeros and poles of dg are the branch points and the simple poles of g. Hence we have (A.1) for $t \in \boldsymbol{C} \setminus \{0\}$. By a suitable change $g \mapsto e^{i\theta} g$ (which is a special form of the change (5.3)), we can choose g such that $t \in \boldsymbol{R}^+$. Since $\infty = p_n$ is a zero of order β_n or a pole of order $\beta_n + 2$ of dg, we have (A.2). Next we assume $d\sigma^2$ is $\mathcal{H}^1$-reducible. Without loss of generality, we may assume $\beta_1, \dots, \beta_m \notin \boldsymbol{Z}$ and $\beta_{m+1}, \dots, \beta_{n-1} \in \boldsymbol{Z}$. Then by Lemma A.4, we can choose the developing map g as $g = (z-p_1)^{\nu_1} \dots (z-p_m)^{\nu_m}\, r(z)$, where $r(z)$ is a rational function. By (5.6), we have $\nu_j = \beta_j + 1$ or $\nu_j = -\beta_j - 1$ ($j = 1, \dots, m$). Differentiating this, we have

$$dg = (z-p_1)^{\alpha_1} \dots (z-p_m)^{\alpha_m} r_1(z)\, dz \qquad (\alpha_j = \beta_j \text{ or } \alpha_j = -\beta_j - 2)$$

where $r_1(z)$ is a rational function. Since each p_j ($j = m+1, \dots, n-1$) is a branch point of g of order $\beta_j \in \boldsymbol{Z}$, we have (A.1) by an argument similar to the $\mathcal{H}^3$-reducible case. Moreover, since $\mathrm{ord}_\infty\, d\sigma^2 = \beta_n$, we have (A.2). Q.E.D.

Remark A.6. Let $M = \boldsymbol{C} \cup \{\infty\} \setminus \{p_1, \dots, p_n\}$, and $\widetilde{M}$ the universal cover. Then there exists a meromorphic function $g \colon \widetilde{M} \to \boldsymbol{C} \cup \{\infty\}$ satisfying (A.1) if and only if all of the residues of dg at the following points vanish:

(1) a_k ($k = 1, \ldots, n$), and

(2) p_j such that α_j is a negative integer.

Construction of CMC-1 *surface from two Gauss maps.* In addition to the SU(2)-conditions for the period problem (Proposition 3.2), we introduce another method to construct CMC-1 surfaces [UY3]: Let M be a Riemann surface and $\widetilde{M}$ the universal cover of M.

Proposition A.7. *Let G and g be meromorphic functions defined on M and $\widetilde{M}$, respectively. Assume*

(1) $d\sigma^2 := 4\,dg\,d\bar{g}/(1+g\bar{g})^2$ is a pseudometric with conical singularities which is single-valued on M.

(2) The meromorphic differential $Q := (S(g) - S(G))/2$ is holomorphic on M.

(3) The metric $ds^2 := (1+|g|^2)^2|Q/dg|^2$ is a non-degenerate complete metric on M.

Then there exists a complete CMC-1 *immersion $f\colon M \to H^3$ with hyperbolic Gauss map G and secondary Gauss map g.*

Proof. By the second assumption , $\operatorname{ord}_p Q = \operatorname{ord}_p d\sigma^2$ for any point $p \in M$. Then, by Theorem 2.2 and Remark 2.3 in [UY3], there exists a CMC-1 immersion $f\colon M \to H^3$ whose hyperbolic Gauss map, secondary Gauss map and Hopf differential are G, g and Q, respectively. Moreover, by the third assumption, the induced metric is complete. Q.E.D.

Partial classification for TA(f) $\leq 8\pi$. By Corollary 7.4, a complete CMC-1 surface with TA(f) $\leq 8\pi$ is either a surface of genus 0 with at most 5 ends or a surface of genus 1 with at most 3 ends. We denote by γ and n the genus and the number of the ends, respectively.

The case $(\gamma, n) = (0,1)$. In this case, we may assume $M = \boldsymbol{C}$ and the only end is $p_1 = \infty$. Since M is simply-connected, the representation ρ_g as in (3.3) is trivial, that is, such a surface is $\mathcal{H}^3$-reducible. Then by Lemma 5.3, the dual immersion $f^\#$ is also well-defined on M. And since the dual surface of $f^\#$ is f itself, the classification reduces to that for CMC-1 surfaces with dual absolute total curvature at most 8π, which is done in [RUY3].

The case $(\gamma, n) = (0,2)$. In this case, the pseudometric $d\sigma^2$ as in (3.1) has the divisor $\mu_1 p_1 + \mu_2 p_2 + \xi_1 q_1 + \cdots + \xi_m q_m$ (see (5.7)), where p_1 and p_2 are the ends and $q_1, \ldots, q_m$ are umbilic points. Since ξ_k ($k = 1, \ldots, m$)

are integers, Lemma A.3 implies that a surface in this class satisfies either

(1) Both μ_1 and μ_2 are integers (the case of $\mathcal{H}^3$-reducible), or

(2) Both μ_1 and μ_2 are non-integral real numbers (the case of $\mathcal{H}^1$-reducible).

If both ends are regular, such surfaces are completely classified (see Example 4.3 or [UY1]), and the only possible case is $\mathbf{O}(-2,-2)$. So we may assume at least one end is irregular: $d_2 \leq -3$. If $d_1 \geq -1$, then $\mu_1 \in \boldsymbol{Z}$ by (7.12), and hence we have the case (1). Hence g is a meromorphic function on the genus 0 Riemann surface $\boldsymbol{C} \cup \{\infty\}$, and $\mathrm{TA}(f) = 4\pi \deg g$. Thus $\deg g \leq 2$, and hence μ_1 and μ_2 are 0 or 1. Then by (7.8), we have $d_1 \leq -1$ and $\mu_1 = 1$. Moreover, by (7.7), we have $d_2 \geq -4$. On the other hand, if $d_1 \leq -2$, by (7.3), (7.7) and (7.11), we have $-7 \leq d_1 + d_2 \leq -4$. Hence the possible cases are $(d_1, d_2) = (-1,-3), (-1,-4), (-2,-3), (-2,-4), (-2,-5), (-3,-3)$ and $(-3,-4)$. Throughout this subsection, we set $M = \boldsymbol{C} \setminus \{0\}$.

Proposition A.8. *There exists no complete* CMC-1 *immersion* $f\colon \boldsymbol{C}\setminus\{0\} \to H^3$ *with* $\mathrm{TA}(f) \leq 8\pi$ *and of class* $\mathbf{O}(-1,-3)$ *or* $\mathbf{O}(-2,-3)$.

Proof. Assume f is of class $\mathbf{O}(-1,-3)$. In this case, $\mu_1 \in \boldsymbol{Z}$ by (7.12), and then f is $\mathcal{H}^3$-reducible (the case (1) above). Then the dual immersion $f^{\#}$ is also well-defined on M whose dual absolute total curvature is not greater than 8π. Such a surface cannot exist because of the results in [RUY3] (see Table 1).

Now suppose f is of class $\mathbf{O}(-2,-3)$. If $\mu_1 \in \boldsymbol{Z}$, then for the same reason as in the $\mathbf{O}(-1,-3)$ case, such a surface does not exist. Now assume $\mu_1 \notin \boldsymbol{Z}$. Then the surface is of type (2): $\mu_2 \notin \boldsymbol{Z}$. By the same argument as in the case $d_1 + d_2 = -5$ of $(\gamma, n) = (0,2)$ in the proof in [RUY4] of Theorem 2.1 (in this paper), such a surface cannot exist. Q.E.D.

By a similar argument to that in the proof of Proposition A.8, if $\mu_1 \in \boldsymbol{Z}$, the classification is the same as the dual case in [RUY3]. Hence the case $\mathbf{O}(-1,-4)$, and also the case $\mathbf{O}(-2,-4)$ with $\mu_1 \in \boldsymbol{Z}$ ($\mathcal{H}^3$-reducible), are classified. Furthermore, for the same reason, the $\mathbf{O}(-2,-5)$ case with $\mu_1 \in \boldsymbol{Z}$ and $\mathrm{TA}(f) \leq 8\pi$ does not exist.

Remark A.9. In the case $\mathbf{O}(-2,-4)$ with (2) holding, we have the following examples: Let

$$dg = t\, z^{\mu} \frac{z^2 - a^2}{(z^2-1)^2}\, dz\,, \qquad Q = \theta \frac{z^2 - a^2}{z^2} dz^2\,,$$

where

$$a^2 = \frac{\mu+1}{\mu-1}\,, \qquad \theta = \frac{\mu(\mu+2)(\mu-1)}{4(\mu+1)}\,, \qquad -1 < \mu < 0\,.$$

Here t is a positive real number corresponding to the one parameter deformation coming from reducibility (see Section 5). Then the residues of dg at -1 and 1 vanish, and there exists the meromorphic function g defined on the universal cover of $\boldsymbol{C} \setminus \{0\}$ (see Remark A.6). We set $\omega = Q/dg$. Then by Theorem 2.4 in [UY1], one can check that there exists an immersion $f\colon \boldsymbol{C}\setminus\{0\} \to H^3$ with data (g, Q). For this example, $\mu_1 = \mu_2 = |\mu+1| - 1 = \mu$ because $-1 < \mu < 0$ (see Corollary A.5). Then, $\mathrm{TA}(f)/2\pi = 2(\mu+2) \in (4\pi, 8\pi)$.

Remark A.10. For the $\mathbf{O}(-2,-5)$ case, the following data gives examples: We set

$$dg = t\, z^{\mu} \frac{z^3 - a^3}{(z^3-1)^2}\, dz,\ Q = \theta \frac{z^3 - a^3}{z^2}\, dz^2,\ a^3 = \frac{\mu+1}{\mu-2},\ \theta = \frac{\mu(\mu^2-4)}{4(\mu+1)},$$

where $\mu \in \boldsymbol{R}\setminus\{0,-1,\pm 2\}$, $t \in \boldsymbol{R}^+$. Here t is a parameter corresponding to a deformation which comes from reducibility (see Section 5). The ends are $0, \infty$ and the umbilic points are $a, ae^{2\pi/3i}, ae^{4\pi/3i}$.

In this case, we have $\mu_1 = |\mu+1| - 1$ and $\mu_2 = |\mu| - 1$. Hence $\mu_1 + \mu_2 = |\mu+1| + |\mu| - 2 \geq -1$, where equality holds if and only if $-1 \leq \mu \leq 0$. Thus the total absolute curvature is $\mathrm{TA}(f) = 2\pi(-2 + \mu_1 + \mu_2 - d_1 - d_2) \geq 8\pi$ and equality holds if and only if $-1 < \mu < 0$.

Remark A.11. For the cases $\mathbf{O}(-3,-3)$ and $\mathbf{O}(-3,-4)$, all ends are irregular, and then one cannot solve the period problem immediately. In the dual total curvature case, a deformation procedure as in [RUY1] can be used to construct examples of type $\mathbf{O}(-3,-3)$ [RUY3]. Unfortunately, this procedure cannot be used here, because the hyperbolic Gauss map is not a rational function.

Remark A.12. In the cases of $\mathbf{O}(-3,-4)$ and $\mathbf{O}(-2,-5)$, it can be shown that $\mathrm{TA}(f) \geq 8\pi$. In fact, in these cases, the divisor corresponding the pseudometric $d\sigma^2$ is $\mu_1 p_1 + \mu_2 p_2 + \xi_1 q_1 + \cdots + \xi_m q_m$, and by (7.3), we have $\xi_1 + \cdots + \xi_m = 3$ is an odd integer. Then by Corollary 4.7 of [RUY4], we have $\mu_1 + \mu_2 \geq -1$. This shows that $\mathrm{TA}(f) \geq 8\pi$.

The case $(\gamma,n)=(0,3)$. If μ_1, μ_2 and μ_3 are integers, then by Lemma A.1 and Lemma 5.3, the surface is $\mathcal{H}^3$-reducible and its dual is also well-defined on M with dual total absolute curvature at most 8π. By [RUY3], these must be of type $\mathbf{O}(-1,-1,-2)$, $\mathbf{O}(-1,-2,-2)$, or $\mathbf{O}(-2,-2,-2)$, and the first two cases are classified. Also, examples exist in the third case as well [RUY3, Example 4.4]. Moreover, for any surface of type $\mathbf{O}(-1,-1,-2)$, μ_1 and μ_2 are integers, by (7.12). Then, by Lemma A.3, μ_3 is also an integer. Thus, surfaces of type $\mathbf{O}(-1,-1,-2)$ must be $\mathcal{H}^3$-reducible and are completely classified.

Next, we assume all $\mu_j \notin \boldsymbol{Z}$. Then (7.3), (7.7) and (7.11) imply that $-8 \le d_1+d_2+d_3 \le -4$, and (7.12) implies that $d_j \le -2$ $(j=1,2,3)$. Hence the possible cases are $\mathbf{O}(-2,-2,-2)$, $\mathbf{O}(-2,-2,-3)$, $\mathbf{O}(-2,-2,-4)$ and $\mathbf{O}(-2,-3,-3)$. For the case $\mathbf{O}(-2,-2,-2)$, that is, for surfaces with three regular ends, the second and third authors classified the irreducible ones with embedded ends ([UY7], see Example 4.4). For the cases $\mathbf{O}(-2,-3,-3)$ and $\mathbf{O}(-2,-2,-4)$, the sum of the orders of the umbilic points are an even integer, by (7.3). Then by Corollary 4.7 in [RUY4] and (7.7), we have $\mathrm{TA}(f) \ge 8\pi$, hence $\mathrm{TA}(f)=8\pi$. By Lemma A.3, there exists no surface with only one non-integer μ_j. Then the remaining case is to assume that one μ_j, say μ_1, is an integer and $\mu_2,\mu_3 \notin \boldsymbol{Z}$. Then by (7.12), d_2, $d_3 \le -2$. Also, by (7.7), (7.8) and (7.11), we have $-5 \le d_2+d_3$. Hence we have two possibilities: $(d_2,d_3)=(-2,-2)$ or $(d_2,d_3)=(-2,-3)$. When $(d_2,d_3)=(-2,-2)$, by (7.7) and (7.8), we have $\mu_1-d_1=2$ or 3. And by (7.3), $d_1\le 0$. Hence we have the possibilities $\mathbf{O}(-3,-2,-2)$, $\mathbf{O}(-2,-2,-2)$, $\mathbf{O}(-1,-2,-2)$ and $\mathbf{O}(0,-2,-2)$. Similarly, when $(d_2,d_3)=(-2,-3)$, we have $\mu_1-d_1=2$ and $d_1\le 1$. Hence the possibilities are $\mathbf{O}(-2,-2,-3)$, $\mathbf{O}(-1,-2,-3)$, $\mathbf{O}(0,-2,-3)$, $\mathbf{O}(1,-2,-3)$. In this case, the corresponding divisor of the pseudometric $d\sigma^2$ is

$$\mu_1p_1+\mu_2p_2+\mu_3p_3+\sum_{k=1}^{m}\xi_kq_k=\mu_2p_2+\mu_3p_3+(2+d_1)p_1+\sum_{k=1}^{m}\xi_kq_k\ ,$$

where the q_k $(k=1,\dots,m)$ are the umbilic points and ξ_k is the order of Q at q_k (see (5.7)). Here, by (7.3), $\xi_1+\cdots+\xi_m=1-d_1$, so $\mu_1+\sum_{k=1}^m\xi_k=d_1+2+\sum_{k=1}^m\xi_k=3$. Hence if $d_1\ge -1$ (and so $\mu_1\in\boldsymbol{Z}^+$), Corollary 4.7 of [RUY4] implies $\mu_2+\mu_3\ge -1$. This implies that $\mathrm{TA}(f)\ge 8\pi$, and so

(A.3) For a surface of type $\mathbf{O}(d,-2,-3)$ $(d\ge -1)$, $\qquad \mathrm{TA}(f)\ge 8\pi$.

Proposition A.13. *There exists no complete* CMC-1 *surface* f *with* $\mathrm{TA}(f)\le 8\pi$ *and of type* $\mathbf{O}(0,-2,-3)$.

Proof. Assume such an immersion $f\colon \boldsymbol{C}\cup\{\infty\}\setminus\{p_1,p_2,p_3\}\to H^3$ exists. By (7.3), there is the only umbilic point q_1. We set the ends $(p_1,p_2,p_3)=(0,1,\infty)$ and the umbilic point $q_1=q\in\boldsymbol{C}\setminus\{0,1\}$. Then the Hopf differential Q has a zero only at q with order 1, and two poles at 1 and ∞ with orders 2 and 3, respectively. Thus, Q can be written as

$$Q:=\theta\frac{z-q}{(z-1)^2}\,dz^2 \qquad (\theta\in\boldsymbol{C}\setminus\{0\})\,.$$

On the other hand, by (7.7), (7.11), (7.8) and (7.12), we have $\mu_1=2$, and

$$\text{(A.4)}\qquad -2<\mu_2+\mu_3\le -1 \qquad\text{and}\qquad -1<\mu_j<0 \qquad (j=2,3)\,.$$

The secondary Gauss map branches at (p_1,p_2,p_3) and q with branch orders 2, μ_2, μ_3 and 1, respectively. Then by Corollary A.5, we can take the secondary Gauss map g such that

$$dg=t\,\frac{z^{\alpha}(z-1)^{\nu}(z-q)^{\beta}}{\prod_{j=1}^{N}(z-a_j)^2}\,dz \qquad (t\in\boldsymbol{R}\setminus\{0\})\,,$$

where

$$\nu=\mu_2 \quad\text{or}\quad -\mu_2-2\,, \qquad \alpha=2 \quad\text{or}\quad -4\,, \qquad \beta=1 \quad\text{or}\quad -3\,,$$

and $a_j\in\boldsymbol{C}\setminus\{0,1,q\}$ $(j=1,\dots,N)$ are mutually distinct points.

Without loss of generality, we may assume $\nu=\mu_2$ (if not, we can take $1/g$ instead of g). Then by (A.2) in Corollary A.5, we have

$$-(\alpha+\beta)-\mu_2+2N-2=\mu_3 \quad\text{or}\quad -\mu_3-2\,,$$

so $\mu_2+\mu_3$ or $\mu_2-\mu_3$ is an odd integer. Then by (A.4), we have $\mu_2+\mu_3=-1$ and $(\alpha,\beta,N)=(2,1,2)$ or $(2,-3,0)$.

First, we assume $(\alpha,\beta,N)=(2,1,2)$, and we set $\mu_2=\mu$. Then

$$\text{(A.5)}\qquad dg=t\,\frac{z^2(z-1)^{\mu}(z-q)}{(z-a)^2(z-b)^2}\,dz \qquad (a,b\in\boldsymbol{C}\setminus\{0,1,q\},a\ne b)\,.$$

Such a g exists if and only if the residues of the right-hand side of (A.5) vanish:

$$\text{(A.6)}\qquad \frac{2}{a}+\frac{\mu}{a-1}+\frac{1}{a-q}-\frac{2}{a-b}=0\,,$$

$$\text{(A.7)}\qquad \frac{2}{b}+\frac{\mu}{b-1}+\frac{1}{b-q}-\frac{2}{b-a}=0\,.$$

Since $a \neq b$, these equations are equivalent to (A.6) − (A.7) and $b \times$ (A.6) − $a \times$ (A.7):

$$\text{(A.8)}\qquad \begin{aligned} &(\mu+1)(a^2+b^2)-(\mu q+1)(a+b)-2ab+2q=0\,,\\ &(\mu+1)(a+b)ab-2(\mu q+q+2)ab+2q(a+b)=0\,. \end{aligned}$$

On the other hand, let $\omega = Q/dg$, and consider the equation

$$\text{(A.9)}\qquad X''-(\log\hat{\omega})'X'-\hat{Q}X=0 \qquad \left(\omega=\hat{\omega}\,dz,\quad Q=\hat{Q}\,dz^2\right)\,,$$

which is named (E.1) in [UY1]. The roots of the indicial equation of (A.9) at $z=0$ are 0 and −1. By Theorem 2.2 of [UY1], the log-term coefficient of (A.9) at $z=0$ must vanish if the surface exists:

$$\text{(A.10)}\qquad \mu+2-\frac{2}{a}-\frac{2}{b}=0\,.$$

(See Appendix A of [RUY3] or Appendix A of [UY1].) Here, the solution of equations (A.8) and (A.10) is $a=b=q=4/(\mu+2)$, a contradiction. Hence the case $(\alpha,\beta,N)=(2,1,2)$ is impossible.

Next, we consider the case $(\alpha,\beta,N)=(2,-3,0)$. Then one has

$$dg=t\,\frac{z^2(z-1)^{\mu}}{(z-q)^3}\,dz \qquad (t\in \boldsymbol{C}\setminus\{0\})\,.$$

The residue at $z=q$ vanishes if and only if

$$\text{(A.11)}\qquad (\mu+2)(\mu+1)q^2-4(\mu+1)q+2=0\,.$$

On the other hand, in the same way as the first case, the log-term coefficient of (A.9) at $z=0$ vanishes if and only if

$$\text{(A.12)}\qquad \mu+2=\frac{4}{q}\,.$$

However, there is no pair (μ,q) satisfying (A.11) and (A.12) simultaneously. Hence this case is also impossible. Q.E.D.

Proposition A.14. *There exists no* CMC-1 *immersion of type* $\mathbf{O}(1,-2,-3)$ *with* $\mathrm{TA}(f)\leq 8\pi$.

Proof. Assume such an immersion $f\colon \boldsymbol{C}\cup\{\infty\}\setminus\{p_1,p_2,p_3\}\to H^3$ exists. Then we have $\mathrm{TA}(f)=8\pi$ because of (A.3), and by (7.7), (7.8) and (7.11), one has

$$\text{(A.13)}\qquad \mu_1=3 \qquad\text{and}\qquad \mu_2+\mu_3=-1, \qquad -1<\mu_j<0 \quad (j=2,3)\,.$$

We set $(p_1, p_2, p_3) = (1, 0, \infty)$. By (7.3), there are no umbilic points. Then the Hopf differential Q can be written as

$$Q = \theta \, \frac{z-1}{z^2} \, dz^2 \qquad (\theta \in \boldsymbol{C} \setminus \{0\}) \, .$$

The secondary Gauss map g branches at 0, ∞ and 1 with orders μ_2, μ_3 and 3, respectively. Then by Corollary A.5, dg can be put in the following form:

$$dg = t \, z^{\mu_2} \frac{(z-1)^\alpha}{\prod_{j=1}^N (z - a_j)^2} \, dz \, ,$$

where $a_j \in \boldsymbol{C} \setminus \{0, 1\}$ $(j = 1, \dots, N)$ are mutually distinct numbers, t is a positive real number, and $\alpha = 3$ or -5, and $-\mu_2 - \alpha + 2N - 2 = \mu_3$ or $\mu_3 - 2$. The second case is impossible because of (A.13). Hence $2N = \alpha + 1$, and then $\alpha = 3$ and $N = 2$. Thus we have the form

$$dg = t \, z^{\mu} \frac{(z-1)^3}{(z-a)^2 (z-b)^2} \, dz \qquad (\mu = \mu_2) \, .$$

Such a g exists if the residues at $z = a$ and $z = b$ vanish:

$$\frac{\mu}{a} + \frac{3}{a-1} - \frac{2}{a-b} = 0 \quad \text{and} \quad \frac{\mu}{b} + \frac{3}{b-1} - \frac{2}{b-a} = 0 \, .$$

By direct calculation, we have

$$a = \frac{-2 + \mu + \mu^2 + \sqrt{2}\sqrt{2 - \mu - \mu^2}}{(\mu+1)(\mu+2)} \, ,$$
$$b = \frac{-2 + \mu + \mu^2 - \sqrt{2}\sqrt{2 - \mu - \mu^2}}{(\mu+1)(\mu+2)} \, .$$

Consider the equation (E.1) in [UY1] with $\omega = Q/dg$. Then, the indicial equation at $z = 1$ has the two roots 0 and -1 with difference 1. By direct calculation again, the log-term at $z = 1$ vanishes if and only if

$$\mu + 2 - \frac{2}{1-a} - \frac{2}{1-b} = -\frac{1}{3}(\mu + 2) = 0 \, ,$$

which is impossible because $\mu = \mu_2 > -1$. Q.E.D.

Proposition A.15. *For a non-zero real number μ $(-1 < \mu < 0)$ and positive integer m, set*

$$\text{(A.14)} \quad G = z^{m+1} \frac{mz - (m+2)}{(m+2)z - m} \qquad \textit{and} \qquad g = z^{\mu+1} \frac{\mu z - (\mu+2)}{(\mu+2)z - \mu} \, .$$

Then there exists a one parameter family of conformal CMC-1 *immersions* $f\colon \boldsymbol{C}\setminus\{0,1\}\to H^3$ *of type* $\mathbf{O}(0,-2,-2)$ *with* $\mathrm{TA}(f)=4\pi(\mu+2)$, *whose hyperbolic Gauss map and secondary Gauss map are* G *and* tg $(t\in\boldsymbol{R}^+)$, *respectively.*

Conversely, any CMC-1 *surface of type* $\mathbf{O}(0,-2,-2)$ *with* $\mathrm{TA}(f)\le 8\pi$ *is obtained in such a manner. In particular,* $\mathrm{TA}(f)<8\pi$.

Proof. For g and G as in (A.14), set

$$Q:=\frac{1}{2}\left(S(g)-S(G)\right)=\frac{m(m+2)-\mu(\mu+2)}{4}\frac{dz^2}{z^2}\,.$$

Since $\mu\notin\boldsymbol{Z}$, the right-hand side is not identically zero. Moreover, one can easily check that the assumptions of Proposition A.7 hold. Hence there exists a complete CMC-1 immersion $f\colon \boldsymbol{C}\setminus\{0,1\}\to H^3$ with hyperbolic Gauss map G, secondary Gauss map g and Hopf differential Q. Conversely, suppose such a surface exists. Then without loss of generality, we set $M=\boldsymbol{C}\setminus\{0,1\}$ and $(p_1,p_2,p_3)=(1,0,\infty)$. By (7.3), there are no umbilic points. Then the Hopf differential Q has poles of order 2 at 0 and ∞, and has no zeros. Hence we have $Q=\theta\, z^{-2}\,dz^2$ $(\theta\in\boldsymbol{C}\setminus\{0\})$. This implies that the secondary and hyperbolic Gauss maps branch only at the ends. Hence both $S(g)$ and $S(G)$ have poles of order 2 at $z=0,1,\infty$ and holomorphic on M. More precisely, we have

$$\text{(A.15)}\qquad \begin{aligned} S(g)&=\frac{c_3z^2+(c_1-c_2-c_3)z+c_2}{z^2(z-1)^2}\,dz^2,\\ S(G)&=\frac{c_3^\# z^2+(c_1^\#-c_2^\#-c_3^\#)z+c_2^\#}{z^2(z-1)^2}\,dz^2\,,\end{aligned}$$

where c_j and $c_j^\#$ are as in (7.2). Here, $\mu_1=\mu_1^\#$ because of (7.12). Hence we have

$$\begin{aligned}2Q&=S(g)-S(G)\\ &=\left(\frac{c_2-c_2^\#}{z^2}+\frac{(c_2-c_3)-(c_2^\#-c_3^\#)}{z}-\frac{(c_2-c_3)-(c_2^\#-c_3^\#)}{z-1}\right)dz^2\,.\end{aligned}$$

Thus we have

$$\text{(A.16)}\qquad (\mu_2-\mu_3)(\mu_2+\mu_3+2)=(\mu_2^\#-\mu_3^\#)(\mu_2^\#+\mu_3^\#+2)\,.$$

On the other hand, (7.7), (7.8) and (7.11) imply that $\mu_1=\mu_1^\#=2$ or 3. If $\mu_1=3$, (7.7) implies that $\mu_2+\mu_3\le -1$. Then by (7.11),

$-1 < \mu_j < 0$ for $j = 2, 3$. Hence using (A.16), we have

$$1 > |\mu_2 - \mu_3| > |\mu_2^\# - \mu_3^\#| .$$

Here $\mu_2^\#$ and $\mu_3^\#$ are integers, hence $\mu_2^\# = \mu_3^\#$. The hyperbolic Gauss map G is a meromorphic function on $\boldsymbol{C} \cup \{\infty\}$. Then the Riemann-Hurwitz relation implies that $\boldsymbol{Z} \ni \deg G = \frac{1}{2}(2 + \mu_1^\# + \mu_2^\# + \mu_3^\#) = \mu_2^\# + 2 + \frac{1}{2}$. This is impossible.

When $\mu_1 = \mu_1^\# = 2$, by similar arguments, we have $-1 < \mu_j < 1$ ($j = 2, 3$) and $\mu_2 + \mu_3 \le 0$. This implies that $|\mu_2 - \mu_3| < 2$. Thus by (A.16), we have $|\mu_2^\# - \mu_3^\#| = 0$ or 1. We may assume that $\mu_3^\# \ge \mu_2^\#$ (if not, exchange the ends 0 and ∞). Assume $\mu_3^\# - \mu_2^\# = 1$. In this case, $\boldsymbol{Z} \ni \deg G = \frac{1}{2}(2 + \mu_1^\# + \mu_2^\# + \mu_3^\#) = \mu_2^\# + 2 + \frac{1}{2}$, which is impossible. Hence, using also (A.16), we have $\mu_3^\# - \mu_2^\# = \mu_3 - \mu_2 = 0$. Moreover, $\mu_2 + \mu_3 = 2\mu_2 \le 0$, so $\mu_2 \le 0$. Putting all this together, we have

$$\mu_1 = \mu_1^\# = 2 \, , \; -1 < \mu_2 = \mu_3 \le 0 \, , \; \mu_2^\# = \mu_3^\# \, , \text{ and } Q = \frac{c_2 - c_2^\#}{2z^2} \, dz^2 \, .$$

If $\mu_2 = 0$, the secondary Gauss map g is a meromorphic function on $\boldsymbol{C} \cup \{\infty\}$ with only one branch point, which is impossible. Hence $\mu_2 < 0$. In this case, the pseudometric $d\sigma^2$ branches on the divisor $\mu_1 p_1 + \mu_2 p_2 + \mu_3 p_3$ because there are no umbilic points. Thus the secondary Gauss map g satisfies (A.15). One possibility of such a g is in the form (A.14) with $\mu = \mu_2$. On the other hand, since the surface is $\mathcal{H}^1$-reducible, g can be normalized as in (A.14) because of Corollary A.5. Since $S(G) = S(g) - 2Q$, the Schwarzian derivative of the hyperbolic Gauss map G is uniquely determined, and G is determined up to Möbius transformations. Then such a surface is unique, with given g and G. Q.E.D.

Proposition A.16. *Let $\mu \in (-1, 0)$, $m \ge 2$ an integer,*

$$a := -\frac{m + \mu + 2}{m - \mu - 2}, \qquad p := \frac{a\mu + a - a^2}{a\mu + a - 1} \, ,$$

and $M = \boldsymbol{C} \cup \{\infty\} \setminus \{0, 1, p\}$. Then there exist a meromorphic function G on $\boldsymbol{C} \cup \{\infty\}$ and a meromorphic function g on the universal cover $\widetilde{M}$ of M such that

(A.17)

$$dG = z \frac{(z - p)^{m-2}}{(z - 1)^{m+2}} \, dz \qquad \text{and} \qquad dg = t \, z \frac{(z - 1)^{\mu}(z - p)^{-\mu - 2}}{(z - a)^2} \, dz$$

respectively, where $t \in \boldsymbol{R}^+$, and there exists a complete CMC-1 *immersion $f \colon M \to H^3$ whose hyperbolic Gauss map and secondary Gauss map* *are G and g, respectively. Moreover* $\mathrm{TA}(f) = 4\pi(\mu+2) \in (4\pi, 8\pi)$. *Conversely, an $\mathcal{H}^1$-reducible complete* CMC-1 *surface of class* $\mathbf{O}(-1,-2,-2)$ *with* $\mathrm{TA}(f) < 8\pi$ *is obtained in such a way.*

Proof. The residue of dG in (A.17) at $z=1$ and the residue of dg in (A.17) at $z=a$ vanish. Thus there exist G and g such that (A.17) hold. Moreover, by direct calculation, we have

$$Q := \frac{1}{2}\bigl(S(g) - S(G)\bigr) = \frac{4m^2\bigl(m(m+2) - \mu(\mu+2)\bigr)}{(m+\mu)^2(2-m+\mu)^2} \frac{dz^2}{z(z-1)^2(z-p)^2} .$$

Then by Proposition A.7, there exists a CMC-1 immersion $f\colon M \to H^3$. One can easily check that f is complete and $\mathrm{TA}(f) = 4\pi(\mu+2)$. Conversely, assume such a surface exists. Then by (7.3), there is only one umbilic point q_1 of order one. We set $(p_1, p_2, p_3) = (0, 1, p)$ and $q_1 = \infty$, where $p \in \boldsymbol{C} \setminus \{0,1\}$. By (7.7), (7.8), (7.12), (7.9) and (7.11), we have $\mu_1 = 1$ or 2. When $\mu_1 = 2$, by Corollary 4.7 of [RUY4], $\mu_2 + \mu_3 \geq -1$ holds, and then $\mathrm{TA}(f) \geq 8\pi$. Assume $\mu_1 = 1$ and $\mathrm{TA}(f) \leq 8\pi$. If one of μ_j $(j = 2,3)$ is an integer, by Lemma A.3, the other is also an integer, and hence the surface is $\mathcal{H}^3$-reducible. Thus both μ_2 and μ_3 are non-integers. By (7.7) and (7.11), we have

$$\text{(A.18)} \qquad -2 < \mu_2 + \mu_3 \leq 0 , \qquad -1 < \mu_j < 1 \quad (j = 2,3) .$$

Then the secondary Gauss map g branches at 0, 1, p and ∞ with orders μ_1, μ_2, μ_3 and 1, respectively. Then by Corollary A.5, g can be chosen in the form

$$dg = t\, \frac{(z-1)^{\nu_2}(z-p)^{\nu_3} z^{\alpha}}{\prod_{k=1}^{N}(z-a_k)^2}\, dz \qquad (t \in \boldsymbol{R}^+) ,$$

where $\nu_j = \mu_j$ or $-\mu_j - 2$, and $\alpha = 1$ or -3, and $\{a_1, \dots, a_N\} \subset \boldsymbol{C} \setminus \{0,1,p\}$ are mutually distinct points. We may assume $\nu_2 = \mu_2$ (if not, take $1/g$ instead of g). Then by (A.2),

$$-\mu_2 - \nu_3 - \alpha + 2N - 2 = 1 \quad \text{or} \quad -3$$

holds. This implies that $\mu_2 + \mu_3$ or $\mu_2 - \mu_3$ is an even integer. Then by (A.18), we have

$$\mu := \mu_2 = \mu_3 \in (-1, 0), \qquad \nu_3 = -\mu - 2, \qquad \alpha = 1, \quad \text{and} \quad N = 1 .$$

Hence we have

$$\text{(A.19)} \qquad dg = t\, \frac{z(z-1)^{\mu}(z-p)^{-\mu-2}}{(z-a)^2}\, dz \qquad (t \in \boldsymbol{R}^+, \quad a \in \boldsymbol{C} \setminus \{0,1,p\}) .$$

Such a map g exists on the universal cover of $\boldsymbol{C}\setminus\{0,1,p\}$ if and only if the residue at $z=a$ of the right-hand side of (A.19) vanishes, that is, if $p=(a\mu+a-a^2)/(a\mu+a-1)$. The Hopf differential of such a surface can be written in the form

$$\text{(A.20)}\qquad Q=\frac{\theta\,dz^2}{z(z-1)^2(z-p)^2}\qquad(\theta\in\boldsymbol{C}\setminus\{0\})$$

because it has poles of order 2 at $z=1$ and p, a pole of order 1 at $z=0$ and a zero of order 1 at $z=\infty$. Let $\mu_1^\#$, $\mu_2^\#$ and $\mu_3^\#$ be the branch orders of the hyperbolic Gauss map at p_1, p_2 and p_3, respectively. Then by (7.1), we have

$$\text{(A.21)}\qquad c_2-c_2^\#=\frac{2\theta}{(1-p)^2},\qquad c_3-c_3^\#=\frac{2\theta}{(1-p)^2p}\,,$$

where c_j and $c_j^\#$ are as in (7.2). Then p and θ are positive real numbers. Without loss of generality, we may assume $\mu_2^\#\geq\mu_3^\#$. Then we have

$$\text{(A.22)}\qquad 0\geq c_2^\#-c_3^\#=\frac{2\theta}{p(1-p)}\,.$$

Hence $\mu_2^\#\neq\mu_3^\#$, that is, $\mu_2^\#>\mu_3^\#$. Since $\mu_1^\#=1$ by (7.12), the hyperbolic Gauss map branches at 0, 1, p and ∞ with branching order 1, $\mu_2^\#$, $\mu_3^\#$ and 1, respectively. Then the Riemann-Hurwitz relation implies that $\deg G=2+(\mu_2^\#+\mu_3^\#)/2<\mu_2^\#+2$. On the other hand, we have $\deg G\geq\mu_2^\#+1$. Hence we have $\deg G=\mu_2^\#+1$ and $\mu_3^\#=\mu_2^\#-2$. We set $m:=\mu_2^\#$. By a suitable Möbius transformation, we may set $G(p_2)=G(1)=\infty$. Since $z=1$ is a point of multiplicity $m+1$, G has no pole except $z=1$. Then dG can be written in the form

$$dG=c\,z\frac{(z-p)^{m-2}}{(z-1)^{m+2}}\,dz\qquad(c\in\boldsymbol{C}\setminus\{0\})\,,$$

and we can choose $c=1$ by a Möbius transformation again. Moreover, the Hopf differential $Q=(S(g)-S(G))/2$ is as in (A.20) if and only if $a=-(m+\mu+2)/(m-\mu-2)$. Q.E.D.

Proposition A.17. *Let m be a positive integer and $\mu\in(-1,0)$ a real number.*

(1) If $m\geq 3$, $p:=\frac{m(m+2)-\mu(\mu+2)}{(m-2)^2-\mu^2}$, $\theta:=\frac{(\mu-3m+2)^2(m(m+2)-\mu(\mu+2))}{((m-2)^2-\mu^2)^2}$, then there exists a complete CMC-1 *immersion $f\colon M:=\boldsymbol{C}\cup\{\infty\}\setminus\{0,1,p\}\to H^3$ with hyperbolic Gauss map G and Hopf differential Q so that $dG=z^2\frac{(z-p)^{m-3}}{(z-1)^{m+2}}\,dz$, $Q=\frac{\theta}{z(z-1)^2(z-p)^2}\,dz^2$.*

(2) If $m \geq 1$, $p := \frac{\mu+m+2}{\mu+m}$, $\theta := \frac{(m-\mu)(\mu+m+2)}{(m+\mu)^2}$, *then there exists a complete* CMC-1 *immersion* $f\colon M := \boldsymbol{C} \cup \{\infty\} \setminus \{0,1,p\} \to H^3$ *with hyperbolic Gauss map* G *and Hopf differential* Q *so that*

$$dG = z^2 \frac{(z-p)^{m-1}}{(z-1)^{m+2}(z-a)^2}\, dz, \; Q = \frac{\theta}{z(z-1)^2(z-p)^2}\, dz^2,$$

where $a := \frac{m-\mu\pm\sqrt{9(m-\mu)^2+16m(\mu+1)+16\mu(m+1)}}{2(\mu+m)}$.

In each case, the immersion f *is complete, of type* $\mathbf{O}(-1,-2,-2)$, $\mathcal{H}^1$*-reducible and satisfies* $\mathrm{TA}(f) = 8\pi$. *Conversely, any* $\mathcal{H}^1$*-reducible* CMC-1 *immersion of class* $\mathbf{O}(-1,-2,-2)$ *with* $\mathrm{TA}(f) = 8\pi$ *is obtained in this way.*

Proof. Since the residue of dG as in (1) at $z=1$ vanishes, there exists a meromorphic function G. Since the metric $ds^{2\#}$ as in (3.9) is non-degenerate and complete on $M := \boldsymbol{C} \cup \{\infty\} \setminus \{0,1,p\}$, there exists a CMC-1 immersion $f\colon \widetilde{M} \to H^3$ with hyperbolic Gauss map G and Hopf differential Q as in (1), where $\widetilde{M}$ is the universal cover of M. (In fact, there exists a CMC-1 immersion $f^{\#}\colon \widetilde{M} \to H^3$ with Weierstrass data $(G, -Q/dG)$. Then taking the dual yields the desired immersion.) Let F be the lift of f. Then F is a solution of (3.10), and there exists a representation ρ_F as in (3.17).

The components F_{21} and F_{22} of F satisfy the equation $(\mathrm{E}.1)^{\#}$ in [RUY3]:
(A.23)
$$X'' - \bigl(\log(\hat{\omega}^{\#})'\bigr)X' + \hat{Q}X = 0, \; \Bigl(\omega^{\#} := \hat{\omega}^{\#}\, dz = \frac{Q}{dG},\; Q = \hat{Q}\, dz^2\Bigr)\,.$$

By a direct calculation, the roots of the indicial equation of (A.23) at $z=0$ are 0 and -2, and the log-term coefficient at $z=0$ vanishes (see Appendix A of [RUY3]). Hence F_{21} and F_{22} are meromorphic on a neighborhood of $z=0$, and then, the secondary Gauss map $g = -dF_{22}/dF_{21}$ is meromorphic at $z=0$. Hence, by (5.8), $\rho_F(\tau_1) = \pm\rho_g(\tau_1) = \pm\,\mathrm{id}$, where ρ_F is a representation corresponding to the secondary Gauss map g, and τ_1 is a deck transformation corresponding to a loop surrounding $z=0$. Moreover, the difference of the roots of the indicial equation at $z=1$ is $\mu+1 \notin \boldsymbol{Z}$. This implies that one can choose the secondary Gauss map g such that $g \circ \tau_2 = e^{2\pi i\mu} g$, where τ_2 is a deck transformation of $\widetilde{M}$ corresponding to a loop surrounding $z=1$. Then $\rho_F(\tau_2) = \pm\rho_g(\tau_2) = \mathrm{diag}\{e^{\pi i\mu}, e^{-\pi i\mu}\} \in \mathrm{SU}(2)$. Hence the representation ρ_F lies in $\mathrm{SU}(2)$, since τ_1 and τ_2 generate the fundamental group of M. Then by Proposition 3.2, the immersion f is well-defined on M, and by Lemma 3.1, f is a complete immersion. Using (7.1), we have $\mu_1 = 2, \quad \mu_2 = \mu, \quad \mu_3 = -\mu - 1$. Then by (7.7), we have $\mathrm{TA}(f) = 8\pi$.

In the case (2), we can prove the existence of f in a similar way.

Conversely, we assume a complete $\mathcal{H}^1$-reducible immersion $f\colon M \to H^3$ of type $\mathbf{O}(-1,-2,-2)$ with $\mathrm{TA}(f) = 8\pi$ exists. Without loss of generality, we may set $(p_1, p_2, p_3) = (0, 1, p)$ and the only umbilic point $q = \infty$. As shown in the proof of Proposition A.16, we have $\mu_1 = \mu_1^\# = 2$. Thus, by (7.7) and the assumption $\mathrm{TA}(f) = 8\pi$, we have $\mu_2 + \mu_3 = -1$. Hence by (7.11), we can set $\mu_2 = \mu$, $\mu_3 = -1 - \mu$, $-1 < \mu < 0$. Without loss of generality, we may assume $\mu_2^\# \ge \mu_3^\#$. Then by the Riemann-Hurwitz relation, we have

$$\text{(A.24)} \quad \deg G = \frac{1}{2}\left(2 + \mu_1^\# + \mu_2^\# + \mu_3^\# + 1\right) = \frac{5}{2} + \frac{\mu_2^\# + \mu_3^\#}{2} \le \frac{5}{2} + \mu_2^\# \,.$$

On the other hand, $\deg G \ge \mu_2^\# + 1$. Thus we have $\deg G = \mu_2^\# + 1$ or $\mu_2^\# + 2$. We set $m := \mu_2^\#$. Assume $\deg G = \mu_2^\# + 1 = m + 1$. Then by (A.24), $\mu_3^\# = m - 3$. Hence, the hyperbolic Gauss map G branches at 0, 1, p and ∞ with branch orders 2, m, $m-3$ and 1, respectively. By a suitable Möbius transformation, we assume $G(1) = \infty$. The multiplicity of G at $z = 1$ is $m + 1 = \deg G$. Then G has no other poles on $\boldsymbol{C} \cup \{\infty\}$. Thus, dG can be written in the form

$$dG = c\, z^2 \frac{(z-p)^{m-3}}{(z-1)^{m+2}}\, dz \,,$$

where $c \in \boldsymbol{C} \setminus \{0\}$. By a suitable Möbius transformation, we may set $c = 1$. On the other hand, the Hopf differential Q can be written in the form

$$Q = \frac{\theta}{z(z-1)^2(z-p)^2}\, dz^2$$

because f is type $\mathbf{O}(-1,-2,-2)$ and ∞ is the umbilic point of order 1. Thus, by (7.1), we have

$$\text{(A.25)} \qquad c_2 - c_2^\# = \frac{2\theta}{(1-p)^2}\,, \qquad c_3 - c_3^\# = \frac{2\theta}{(1-p)^2 p}\,,$$

where c_j and $c_j^\#$ are as in (7.2). Thus we have the case (1).

Next, we assume $\deg G = m + 2$. Then by (A.24), we have $\mu_3^\# = m - 1$. If we set $G(1) = \infty$, then G has only one simple pole other than the pole $z = 1$, since the multiplicity of G at $z = 1$ is $m + 1$. So dG can be written in the form

$$dG = c\,z^2 \frac{(z-p)^{m-1}}{(z-1)^{m+2}(z-a)^2}\,dz \qquad (a \in \boldsymbol{C} \setminus \{0,1,p\}) ,$$

where $c \in \boldsymbol{C} \setminus \{0\}$, which can be set to $c = 1$ by a suitable Möbius transformation. The residue of dG at $z = a$ vanishes if and only if $p = (a(m+1) + a^2)/(ma+2)$. On the other hand, the relation (A.25) also holds in this case. Thus we have the case (2). Q.E.D.

The case $(\gamma, n) = (0,4)$. If two of the μ_j are integers, (7.7) and (7.8) imply that $\mathrm{TA}(f) > 8\pi$. So at most one μ_j is an integer. By (7.7), (7.3) and (7.11), we have $-9 \leq d_1 + d_2 + d_3 + d_4 \leq -4$. When all $\mu_j \notin \boldsymbol{Z}$, all $d_j \leq -2$. Hence the possible cases are $\mathbf{O}(-2,-2,-2,-3)$ and $\mathbf{O}(-2,-2,-2,-2)$ (see Example 4.5).

Assume $\mu_1 \geq 0$ is an integer. Then $\mu_2, \mu_3, \mu_4 \notin \boldsymbol{Z}$ and $d_2, d_3, d_4 \leq -2$. In this case, by (7.7) and (7.8), we have $-6 \leq d_2 + d_3 + d_4$. Hence $d_2 = d_3 = d_4 = -2$ and $\mu_1 - d_1 = 2$. This implies that $d_1 \geq -2$. Moreover, by (7.3), we have $d_1 \leq 2$. Hence the possible cases are $\mathbf{O}(d,-2,-2,-2)$ with $-2 \leq d \leq 2$. Moreover, $\mu_1 = 2 + d$ holds and there are $2 - d$ umbilic points. Then when $d \geq -1$, we have $\mu_1 \in \boldsymbol{Z}^+$, and so by Corollary 4.7 in [RUY4], we have $\mathrm{TA}(f) \geq 8\pi$. So $\mathrm{TA}(f) = 8\pi$.

Proposition A.18. *There exist no* CMC-1 *surfaces in* H^3 *with* $\mathrm{TA}(f) \leq 8\pi$ *of class* $\mathbf{O}(2,-2,-2,-2)$.

Proof. Assume such an immersion $f\colon \boldsymbol{C} \cup \{\infty\} \setminus \{p_1, \dots, p_4\} \to H^3$ exists. Then there are no umbilic points, and by (7.7), (7.8) and (7.11), $\mu_1 = \mu_1^\# = 4$ holds. Let G be the hyperbolic Gauss map. Then $\deg G \geq 5$ because $\mu_1^\# = 4$. Hence by the Riemann-Hurwitz relation, $10 \leq 2\deg G = \sum_{j=2}^4 \mu_j^\# + \mu_1^\# + 2 = \sum_{j=2}^4 \mu_j^\# + 6$ holds. This implies that $\mu_2^\# + \mu_3^\# + \mu_4^\#$ is an even number not less than 4:

$$\text{(A.26)} \qquad \mu_2^\# + \mu_3^\# + \mu_4^\# = 2l, \qquad (l \in \boldsymbol{Z}, l \geq 2) .$$

Since $\mathrm{TA}(f) = 8\pi$, we have

$$\text{(A.27)} \qquad \mu_2 + \mu_3 + \mu_4 = -2 .$$

Hence by (7.11),

$$\text{(A.28)} \qquad -1 < \mu_j < 0 \qquad (j = 2,3,4) .$$

We set $(p_1, p_2, p_3, p_4) = (p, 0, 1, -1)$, where $p \in \boldsymbol{C} \cup \{\infty\}$. We may assume $p_1 \in \boldsymbol{C}$. In fact, if $p_1 = \infty$, the Möbius transformation

$$z \longmapsto \frac{z-1}{3z+1}\,, \tag{A.29}$$

maps the ends $(p_1, p_2, p_3, p_4) = (\infty, 0, 1, -1)$ to $(1/3, -1, 0, 1)$. The Hopf differential can be written as

$$Q = 2\theta^2 \frac{(z-p)^2}{z^2(z^2-1)^2}\, dz^2 \qquad (\theta \in \boldsymbol{C} \setminus \{0\})\,.$$

By the relation (7.1), we have

$$c_2 - c_2^\# = 4\theta^2 p^2, \quad c_3 - c_3^\# = \theta^2(1-p)^2, \quad c_4 - c_4^\# = \theta^2(1+p)^2, \tag{A.30}$$

where c_j and $c_j^\#$ are as in (7.2). Since $-1 < \mu_j < 0$ and $\mu_j^\#$ is a non-negative integer, we have

$$0 < c_j < \frac{1}{2}\,, \qquad c_j^\# \le 0\,, \tag{A.31}$$

and consequently, $c_j - c_j^\# > 0$ $(j = 2, 3, 4)$. Let

$$\alpha_2 = \theta p\,, \qquad \alpha_3 = \frac{1}{2}\theta(1-p)\,, \qquad \alpha_4 = \frac{1}{2}\theta(1+p)\,. \tag{A.32}$$

Then we have $4\alpha_j^2 = c_j - c_j^\#$, which implies that the α_j $(j = 2, 3, 4)$ are real numbers. And then $p = \alpha_2/(\alpha_3 + \alpha_4)$ and $\theta = \alpha_3 + \alpha_4$ are real numbers. Here, without loss of generality, we may set $0 < p < 1$. (In fact, if $p < 0$, applying the coordinate change $z \mapsto -z$, we have $p > 0$. Moreover, if $p > 1$, by the transformation (A.29), we have $0 < p < 1$.)

We choose the sign of θ as $\theta > 0$. Then, we have α_2, α_3 and α_4 are positive numbers. Moreover, by (A.32), we have

$$\alpha_2 + \alpha_3 = \alpha_4\,. \tag{A.33}$$

Using this, we have

$$\mu_2^\#\,, \quad \mu_3^\# \;\le\; \mu_4^\#\,. \tag{A.34}$$

In fact, by (A.33) we have $\alpha_j < \alpha_4$ for $j = 2, 3$. Then $c_j - c_j^\# < c_4 - c_4^\#$. Hence by (A.31), $-c_j^\# < \frac{1}{2} - c_4^\#$. By definition, this implies that $\mu_j^\#(\mu_j^\# + 2) < 1 + \mu_4^\#(\mu_4^\# + 2)$. Thus we have $(\mu_j^\# - \mu_4^\#)(\mu_j^\# + \mu_4^\# + 2) < 1$

for $j=2,3$. As the $\mu_j^\#$ are non-negative integers, $\mu_j^\# - \mu_4^\# \leq 0$, which implies (A.34). By (A.30) and the definition of α_j, we have

$$\mu_j(\mu_j+2) = \mu_j^\#(\mu_j^\#+2) - 8\alpha_j^2 \qquad (j=2,3,4)\,.$$

Since $\mu_j \in (-1,0)$, this implies that

(A.35)
$$\mu_j+1 = \sqrt{1+\mu_j^\#(\mu_j^\#+2)-8\alpha_j^2} = \sqrt{(\mu_j^\#+1)^2-8\alpha_j^2} \qquad (j=2,3,4)\,.$$

Now, defining $m_j := \mu_j^\#+1 \geq 1$ $(j=2,3,4)$, (A.34) and (A.33) imply

$$\text{(A.36)} \qquad m_2\,,\ m_3\ \leq\ m_4 \qquad \text{and} \qquad \alpha_2+\alpha_3=\alpha_4\,.$$

Moreover,

$$\text{(A.37)} \qquad m_2+m_3 \neq m_4$$

holds. To prove this, if $m_2+m_3=m_4$, then $\mu_2^\#+\mu_3^\#+2=\mu_4^\#+1$. This implies that $\mu_2^\#+\mu_3^\#+\mu_4^\#$ is an odd number, contradicting (A.26).

Using (A.35) and (A.36), the equality (A.27) can be written as

$$\text{(A.38)} \qquad \sqrt{m_2^2-8\alpha_2^2}+\sqrt{m_3^2-8\alpha_2^2}+\sqrt{m_4^2-8(\alpha_2+\alpha_3)^2}=1\,.$$

We shall prove that (A.38) cannot hold, making a contradiction. Let m_2, m_3, m_4 be positive integers which satisfy (A.36) and (A.37). Define

$$\varphi(\alpha_2,\alpha_3) := \sqrt{m_2^2-8\alpha_2^2}+\sqrt{m_3^2-8\alpha_3^2}+\sqrt{m_4^2-8(\alpha_2+\alpha_3)^2}\,.$$

on the closure $\overline{D}$ of the open domain

$$D := \left\{(\alpha_2,\alpha_3)\ :\ 0<\alpha_2<\frac{m_2}{\sqrt{8}}, 0<\alpha_3<\frac{m_3}{\sqrt{8}}, 0<\alpha_2+\alpha_3<\frac{m_4}{\sqrt{8}}\right\}$$

in the $\alpha_2\alpha_3$-plane. Then one has $\varphi(\alpha_2,\alpha_3)>1$ if $(\alpha_2,\alpha_3)\in D$. To prove this, note that since φ is a continuous function on a compact set $\overline{D}$, it takes a minimum on $\overline{D}$. By a direct calculation, we have

$$\frac{\partial\varphi}{\partial\alpha_3} = \frac{-8\alpha_3}{\sqrt{m_3^2-8\alpha_3^2}}+\frac{-8\alpha_2-8\alpha_3}{\sqrt{m_4^2-8(\alpha_2+\alpha_3)^2}}<0 \qquad \text{on} \quad D\,.$$

So φ does not take its minimum in the interior D of $\overline{D}$, but rather on ∂D. Similarly, $\partial\varphi/\partial\alpha_2<0$ on D, so the minimum occurs at $(\alpha_2,\alpha_3)=(m_2/\sqrt{8},m_3/\sqrt{8})$, where $\varphi(m_2/\sqrt{8},m_3/\sqrt{8}) = \sqrt{m_4^2-(m_2+m_3)^2} \geq$

1, if the line $\alpha_2 + \alpha_3 = m_4/\sqrt{8}$ does not intersect ∂D. If the line $\alpha_2 + \alpha_3 = m_4/\sqrt{8}$ does intersect ∂D, then $m_2 + m_3 > m_4$ holds, and the minimum occurs somewhere on this line with α_2 in the interval $[(m_4 - m_3)/\sqrt{8}, m_2/\sqrt{8}]$. We have

$$\varphi(\alpha_2, m_4/\sqrt{8} - \alpha_2) = \sqrt{m_2^2 - 8\alpha_2^2} + \sqrt{m_3^2 - 8(m_4/\sqrt{8} - \alpha_2)^2}\,,$$

which takes its minimum values $\sqrt{(m_3 + m_2 - m_4)(m_3 + m_4 - m_2)}$ and $\sqrt{(m_3 + m_2 - m_4)(m_2 + m_4 - m_3)} \geq 1$ at the endpoints of the interval $[(m_4 - m_3)/\sqrt{8}, m_2/\sqrt{8}]$. Hence $\varphi > 1$ on D, contradicting (A.27) and proving the theorem. Q.E.D.

Remark A.19. There exist CMC-1 surfaces of class $\mathbf{O}(-2,-2,-2,0)$ with $\mathrm{TA}(f) = 8\pi$: We set $(p_1, p_2, p_3, p_4) = (1, -1, \infty, 0)$ and $M := \boldsymbol{C} \cup \{\infty\} \setminus \{p_1, p_2, p_3, p_4\}$. Set

(A.39)
$$dg := \frac{(z^2-1)^\mu (z^2-q^2) z^2}{(z^2-a^2)^2}\, dz \qquad (a, q \in \boldsymbol{C} \setminus \{0,1\},\ a \neq \pm q,\ \mu \in \boldsymbol{R})$$

and

$$Q := \frac{-\mu(\mu+2)}{q^2-1} \frac{z^2-q^2}{(z^2-1)^2}\, dz^2\,, \tag{A.40}$$

where $\frac{2\mu a}{a^2-1} + \frac{2a}{a^2-q^2} + \frac{1}{a} = 0$. Then the residues of dg at $z = \pm a$ vanish, and thus, there exists the secondary Gauss map g as in (A.39). We assume $-1 < \mu < -\frac{1}{2}$ and $a^2 = -\frac{1-\mu-q^2}{3+\mu-3q^2}$. Then by Theorem 2.4 of [UY1], there exists a CMC-1 immersion $f\colon M \to H^3$ with given g and Q. One can check that such an immersion is complete and has $\mathrm{TA}(f) = 8\pi$.

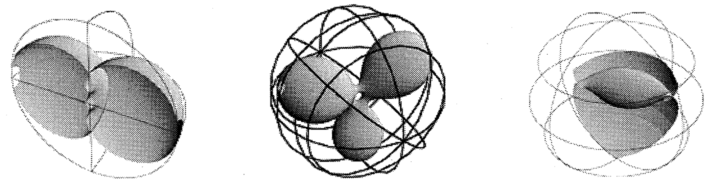
The case $(\gamma, n) = (0, 5)$. In this case, by Corollary 7.4, the only possible case is $\mathbf{O}(-2,-2,-2,-2,-2)$.

The case of $\gamma = 1$. By Corollary 7.4, a surface of this type has at most three ends, and if the surface has 3 ends, the only possible case is $\mathbf{I}(-2,-2,-2)$. If a surface has only one end, part (3) of Lemma 7.3 implies that it must be of type $\mathbf{I}(-3)$ or $\mathbf{I}(-4)$.

Now suppose there are two ends. By (7.3), (7.7) and (7.11), we have

$$-5 \leq d_1 + d_2 \leq 0\,. \tag{A.41}$$

Also, by (7.7) and (7.8), $\mathrm{TA}(f) = 8\pi$ if $d_1, d_2 \geq -1$. Suppose that both ends are regular (i.e. $d_1, d_2 \geq -2$). Then Theorem 7 of [RUY2] implies



(a) (b) (c)

Fig. 8. Examples B.1, B.2 and B.3. The first two graphics were made by Katsunori Sato of Tokyo Institute of Technology.

that if $d_1 = -2$, then also $d_2 = -2$. Furthermore, by Lemma 3 of [UY5] combined with (7.7), (7.8) and (7.12), if $d_j \geq -1$, then the end at p_j is embedded. Therefore, when $d_j \geq -1$, Proposition 7.1 implies that the flux at the end p_j is zero if and only if $d_j \geq 0$. By the balancing formula (7.15) and Proposition 7.1, we conclude that the only possibilities are $\mathbf{I}(-2,-2)$, $\mathbf{I}(-1,-1)$, and $\mathbf{I}(0,0)$. But in fact the case $\mathbf{I}(0,0)$ cannot occur, because then (7.3) and (7.4) imply that the hyperbolic Gauss map G has at most two branch points, contradicting (7.13). If the end p_1 is irregular, $d_1 \leq -3$. Then by (A.41), we have $d_2 \geq -2$. In particular, the other end p_2 is regular. When $d_2 \geq -1$, then $\mu_1, \mu_2 \in \boldsymbol{Z}$, and (7.7) and (7.8) imply $\mu_1 - d_1 = \mu_2 - d_2 = 2$. In particular $d_1 = \mu_1 - 2 > -3$, a contradiction. Hence the only possible case is $\mathbf{I}(-2,-3)$.

Remark A.20. The genus one catenoid cousin in [RS] is of type $\mathbf{I}(-2,-2)$ (Figure 8 in Appendix B). However, the total absolute curvature seems to be strictly greater than 8π.

Remark A.21. There exists an example of CMC-1 surface of type $\mathbf{I}(-2,-2,-2)$, which is so-called the genus one trinoid ([RUY1], see Figure 8 in Appendix B), which is obtained by deforming minimal surface in $\boldsymbol{R}^3$. However, the absolute total curvature of the original minimal surface is 12π, so the obtained CMC-1 surface has $\mathrm{TA}(f)$ close to 12π. Thus, surfaces obtained by deformation are far from satisfying $\mathrm{TA}(f) \leq 8\pi$.

§Appendix B. Further examples

In this appendix, we introduce examples of interesting CMC-1 surfaces which do not appear in the classification table (Table 3).

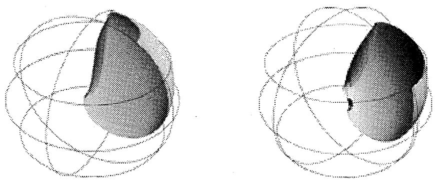


Fig. 9. Example B.4

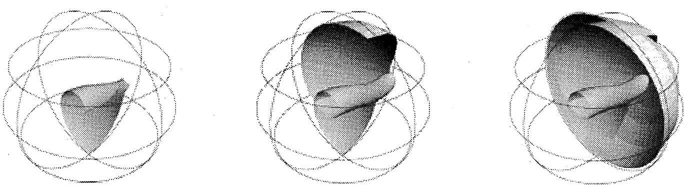


Fig. 10. Example B.5

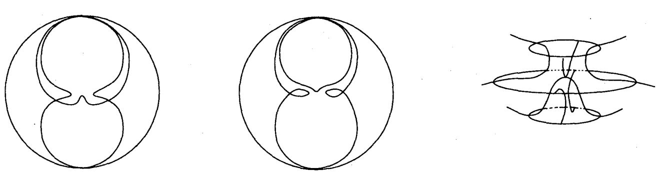


Fig. 11. Example B.6

Example B.1. There exists a genus 1 catenoid cousin in H^3 [RS] Figure 8 (a)). No corresponding minimal surface can exist, by Schoen's result [Sch]. Levitt and Rosenberg [LR] have proved that any complete properly embedded CMC-1 surface in H^3 with asymptotic boundary consisting of at most two points is a surface of revolution, which implies that this example and the last two examples in Figure 2 cannot be embedded, and we see that they are not. A CMC-1 genus 1 catenoid cousin in H^3 was proven to exist in [RS]. See Remark A.20 in Appendix A.

Example B.2. Figure 8 shows a genus 1 trinoid in H^3 proven to exist in [RUY1]. See Remark A.21 in Appendix A.

Example B.3. Figure 8 (c) shows 5 ended CMC-1 surface in H^3 found in [UY1]. Here we show only one of six congruent disks that form the surface. The full surface is constructed by reflections across planes containing boundary curves of the disk shown here.

Example B.4. Figure 9 shows genus 0 and genus 1 Enneper cousin duals. Each surface has a single end that triply wraps around its limiting point at the south pole of the sphere at infinity. These surfaces are of type $\mathbf{O}(-4)$ and $\mathbf{I}(-4)$, and have $\mathrm{TA}(f^{\#}) = 4\pi$ and $\mathrm{TA}(f^{\#}) = 8\pi$. In both cases only one of four congruent pieces (bounded by planar geodesics) of the surface is shown.

Example B.5. Figure 10 shows a CMC-1 surface in H^3, proven to exist in [UY1]. This example is interesting because the hyperbolic Gauss map has an essential singularity at one of its two ends, like the end of the Enneper cousin. And the geometric behavior of the end here is strikingly similar to that of the Enneper cousin's end (see the middle figure of Figure 1). Here we show three pictures consecutively including more of this end.

Example B.6. A CMC-1 "Costa cousin" in H^3 was proven to exist by Costa and Sousa Neto [CN]. In Figure 11, rather than showing graphics of this surface, we show two vertical cross sections by which the surface is reflectionally symmetric (including the "circles" at infinity), and a schematic of the central portion of the surface.

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