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Harmonic tori and their spectral data

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One of the earliest applications of modern integrable systems theory (or "soliton theory") to differential geometry was the solution of the problem of finding all constant mean curvature (CMC) tori in $\mathbb{R}^3$ (and therefore, by taking the Gauss map, finding all non-conformal harmonic maps from a torus to S^2). At its simplest level this proceeds from the recognition that the Gauss-Codazzi equations of a CMC torus are the elliptic sinh-Gordon equations

$$u_{z\bar{z}} + \sinh(4u) = 0, \quad z = x + iy. \tag{1}$$

It was shown in the late 1980's ([24, 1]) that each doubly periodic solution of this equation can be written down in terms of the Riemann θ-function for a compact Riemann surface X, called the spectral curve (this also follows from Hitchin's work [10] on harmonic tori in S^3, which used a distinctly different approach). That this is true relies on two observations. First, (1) has a zero-curvature (or Lax pair) representation: it is the condition that

$$[\frac{\partial}{\partial z} - U_\zeta, \frac{\partial}{\partial \bar{z}} + U^\dagger_{\bar{\zeta}^{-1}}] = 0, \; U_\zeta = \begin{pmatrix} u_z & e^{-2u}\zeta^{-1} \\ e^{2u}\zeta^{-1} & -u_z \end{pmatrix}, \; \forall \zeta \in \mathbb{C}^*,$$

where '†' denotes the Hermitian transpose. As a result this equation belongs to a hierarchy of infinitely many commuting equations, so that solutions to (1) may belong to an infinite dimensional family of deformations through solutions. These deformations are called the "higher flows" of the sinh-Gordon hierarchy. Secondly, each independent higher flow contributes to the number of independent Jacobi fields which the CMC surface admits: these belong to the kernel of the elliptic operator

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$\triangle + 4\cosh(4u)$. Thus for a torus there can only be finitely many independent higher flows. It follows that there must be a higher flow with respect to which the solution $u(z,\bar{z})$ is stationary. In this context this means there is a solution to

$$d\xi_\zeta = [\xi_\zeta, \alpha_\zeta],\ \alpha_\zeta = U_\zeta dz - U^\dagger_{\bar{\zeta}^{-1}} d\bar{z}, \tag{2}$$

in which the matrix $\xi_\zeta(z,\bar{z})$ is a Laurent polynomial in ζ: it is called a polynomial Killing field. The spectral data of the CMC torus consists of the eigenvalues and eigenlines of ξ_ζ. In particular, equation (2) means ξ_ζ is isospectral i.e. its characteristic polynomial is independent of z. This provides us a with an invariant planar algebraic curve which is essentially the Riemann surface X. Altogether the spectral data consists of the Riemann surface X, which always possesses a real involution, a rational function λ on X of degree 2, and a line bundle $\mathcal{L}$ over X satisfying a certain reality condition. The CMC surface is determined, up to Euclidean motions, by its spectral data. However, the existence of a polynomial Killing field is only a necessary condition for a CMC plane to be doubly periodic. If we call CMC planes "of finite type" when they possess a polynomial Killing field then one must still work at distinguishing the tori amongst the planes of finite type: this is a problem of closing periods on the surface (see e.g. [1, 8, 11]). This is also true for the Gauss map: the space of non-conformal harmonic maps $\varphi : \mathbb{R}^2 \to S^2$ of finite type is substantially larger than the set of non-conformal harmonic tori.

Essentially the same line of argument shows that all non-isotropic harmonic tori in $\mathbb{CP}^n, S^n$ [4, 9] and all non-conformal harmonic tori in rank 1 compact symmetric spaces [5] are of (semisimple) finite type. Although the construction of the spectral data is more complicated the principle is the same [17, 18]. However, these complications have the effect of obscuring the geometry of the original map. In [20] I proposed a more direct geometric construction of the map from the spectral data, and showed how this produces pluri-harmonic maps $\mathbb{R}^{2k} \to Gr_k(\mathbb{C}^{n+1})$ as well.

My aim here is to use the example of non-conformal harmonic maps $\varphi : \mathbb{R}^2 \to S^2$ as a way of motivating the geometric construction of [20]. To this end sections 1.1-1.5 describe the construction and properties of the spectral data for a map of semisimple finite type into S^2. The approach is more concrete than that of [17] and owes much to [9, 22, 25]. Having obtained the spectral data we examine it closely, in sections 1.6 and 1.7, to see exactly what is needed to reproduce the map. In particular, we obtain a clear understanding of the periodicity conditions by introducing a "singularisation" X' of X. Section 1.8 ties the previous discussion in with two other methods of reconstruction: the Symes' formula of [6] and the dressing orbit of the vacuum solution [7]. I give explicit formulae for computing φ from its (hyperelliptic) spectral curve. This is illustrated with the example of the bubbletons: these are CMC surfaces in $\mathbb{R}^3$ whose Gauss maps have rational nodal spectral curve. They are the solitons of CMC theory, some of which were known to geometers of the 19th century (see [21]). The calculations in section 1.8 are particularly satisfying because they allow us to compute (using Nick Schmitt's CMCLab) explicit pictures of some CMC surfaces (see figures 1 and 2).

Section 2 describes the generalization presented in [20], which constructs pluri-harmonic maps of $\mathbb{R}^{2k}$ into $Gr_k(\mathbb{C}^{n+1})$. The key point is that a pluri-harmonic map $\varphi : \mathbb{R}^{2k} \to Gr_k(\mathbb{C}^{n+1})$ of semisimple finite type arises as a composition: $\varphi = \psi \circ \gamma$ where

$$\mathbb{R}^{2k} \xrightarrow{\gamma} J(X') \xrightarrow{\psi} Gr_k(\mathbb{C}^{n+1}).$$

The middle factor is the generalized Jacobian of a singularisation X' of the spectral curve X. The map γ is a homomorphism and the map ψ is algebraic, derived from a section of a trivial $Gr_k(\mathbb{C}^{n+1})$-bundle over $J(X')$. No proofs are given here, they can be found in [20]. Nevertheless, I give the details for the construction of totally equivariant maps, which are characterized as being those whose spectral curve is the Riemann sphere.

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Notation. If V is a vector space then V^t will denote its dual, while V^* will denote $V - \{0\}$.

§1. Maps into S^2.

1.1. Maps of semisimple finite type.

Let us start with a harmonic map $\varphi : \mathbb{R}^2 \to S^2$ of semisimple finite type. To recall what this means we fix a framing $F : \mathbb{R}^2 \to SU_2$ with

$F(0) = I$ i.e. $\varphi = F \cdot T$ if we view $S^2 \simeq SU_2/T$ where T is the maximal torus of diagonal matrices. The Lie algebra $\mathfrak{su}_2$ splits into the vector space sum $\mathfrak{t} + \mathfrak{m}$ where $\mathfrak{t}$ contains all diagonal matrices and $\mathfrak{m}$ contains all off-diagonal matrices. Now define the $\mathfrak{su}_2$-valued 1-form $\alpha = F^{-1}dF$: with respect to the splitting of $\mathfrak{su}_2$ this decomposes into $\alpha_{\mathfrak{t}} + \alpha_{\mathfrak{m}}$. From these components we construct a $\mathbb{C}^*$-family of $\mathfrak{gl}_2$- valued 1-forms

$$\alpha_\zeta = \zeta^{-1}\alpha_{\mathfrak{m}}^{(1,0)} + \alpha_{\mathfrak{t}} + \zeta\alpha_{\mathfrak{m}}^{(0,1)}$$

where $\zeta \in \mathbb{C}^*$. The condition that φ is harmonic is precisely the condition that α_ζ satisfies the Maurer-Cartan equations for all ζ. In addition, it has two symmetries:

$$\alpha_{\bar{\zeta}^{-1}} = -\alpha_\zeta^\dagger, \qquad \alpha_{-\zeta} = \nu(\alpha_\zeta), \tag{3}$$

where '†' denotes the Hermitian transpose and for $A \in \mathfrak{gl}_2$, $\nu(A) = \mathrm{Ad}\tau \cdot A$ where

$$\tau = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

We say that φ is of semisimple finite type when:

(1a) there exists a smooth function $a : \mathbb{R}^2 \to \mathbb{C}^*$ and a complex coordinate z on $\mathbb{R}^2$ such that

$$\alpha_{\mathfrak{m}}(\partial/\partial z) = \mathrm{Ad}\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix};$$

(1b) there exists a smooth map $\xi_\zeta : \mathbb{R}^2 \to C^\omega(\mathbb{C}^*, \mathfrak{gl}_2)$ satisfying

(i) $d\xi_\zeta + [\alpha_\zeta, \xi_\zeta] = 0$,

(ii) ξ_ζ also possesses the two symmetries in (3),

(iii) for all $z \in \mathbb{R}^2$ there is a positive integer p such that ξ_ζ is a Laurent polynomial in ζ of order $2p + 1$.

These properties together imply

$$\xi_\zeta = \zeta^{-2p-1}\alpha_{\mathfrak{m}}(\frac{\partial}{\partial z}) + \ldots + \zeta^{2p+1}\alpha_{\mathfrak{m}}(\frac{\partial}{\partial \bar{z}}). \tag{4}$$

1.2. The symmetric spectral curve Σ.

Define, for each $z \in \mathbb{R}^2$,

$$\Sigma_A(z) = \{(\zeta, [v]) \in \mathbb{C}^* \times \mathbb{P}^1 : \exists \mu \in \mathbb{C} \text{ such that } \xi_\zeta(z)v = \mu v \}.$$

To maximise the domain of definition here, whenever ξ_ζ is either singular or zero at ζ_0 we replace it by $(\zeta - \zeta_0)^m \xi_\zeta$ where m is chosen so that this is regular and non-zero at ζ_0. It is clear that this describes an algebraic curve birationally equivalent to the planar curve with equation $\mu^2 + \det(\xi_\zeta) = 0$. Moreover $\Sigma_A(z)$ will be smooth (and unramified over the ζ-plane) at all points for which ξ_ζ (or its renormalisation) is not nilpotent. In particular this is true over the unit ζ-circle (for the symmetry conditions imply ξ is skew-Hermitian there). Further, from (4) and 1a we see that $\Sigma_A(z)$ completes to a curve $\Sigma(z)$ in $\mathbb{P}^1 \times \mathbb{P}^1$ by adding two smooth points over each of $\zeta = 0$ and $\zeta = \infty$.

This curve admits a fixed point free involution arising from one of the symmetries of ξ_ζ. Define

$$\begin{array}{rccc} \tilde{\nu}: & \mathbb{P}^1 \times \mathbb{P}^1 & \to & \mathbb{P}^1 \times \mathbb{P}^1 \\ & (\zeta, [v]) & \mapsto & (-\zeta, [\tau v]) \end{array}$$

Then $\tilde{\nu}$ induces a fixed point free involution on $\Sigma(z)$ and the quotient curve $\Sigma(z)/\tilde{\nu}$ is smooth wherever $\Sigma(z)$ is.

1.3. The quotient spectral curve X.

Here we construct a model of the quotient curve $\Sigma(z)/\tilde{\nu}$. First, for any $\eta_\zeta \in C^\omega(\mathbb{C}^*, \mathfrak{gl}_2)$ satisfying $\nu(\eta_\zeta) = \eta_{-\zeta}$ define

$$\hat{\eta} = \mathrm{Ad}\kappa \cdot \eta_\zeta, \qquad \kappa = \begin{pmatrix} 1 & 0 \\ 0 & \zeta \end{pmatrix}.$$

It is easy to check that $\hat{\eta}(-\zeta) = \hat{\eta}(\zeta)$ so that it is a function of $\lambda = \zeta^2$. Therefore, with an abuse of notation, let us use the notation

$$\eta_\lambda = \mathrm{Ad}\kappa \cdot \eta_\zeta, \quad \lambda = \zeta^2. \tag{5}$$

Now define

$$X_A(z) = \{(\lambda, [w]) \in \mathbb{C}^* \times \mathbb{P}^1 : \xi_\lambda w = \mu w\}$$

with the same convention at singular points or zeroes of ξ_λ as earlier. An easy computation shows that

$$\xi_\lambda = \lambda^{-p-1} \begin{pmatrix} 0 & a^2 \\ 0 & 0 \end{pmatrix} + \ldots + \lambda^{p+1} \begin{pmatrix} 0 & 0 \\ -\bar{a}^2 & 0 \end{pmatrix}.$$

Therefore $X_A(z)$ is completed in $\mathbb{P}^1 \times \mathbb{P}^1$ by adding the points $P_0 = (0, [1,0])$ and $P_\infty = (\infty, [0,1])$. We will call this complete curve $X(z)$.

Lemma 1. *$X(z)$ is isomorphic to the quotient curve $\Sigma(z)/\tilde{\nu}$.*

Proof. Let $f : \mathbb{C}^* \times \mathbb{P}^1 \to \mathbb{C}^* \times \mathbb{P}^1$ be given by $f(\zeta, [v]) = (\zeta^2, [\kappa v])$. Since $\xi_\lambda = \mathrm{Ad}\kappa \cdot \xi_\zeta$ this maps $\Sigma_A(z)$ onto $X_A(z)$ and exhibits it as an unramified double cover. Further, it is easy to check that $f \circ \tilde{\nu} = f$ so that $\Sigma_A/\tilde{\nu} \simeq X_A$. Finally, one readily checks that the restriction of f to Σ_A extends to Σ with image X. Q.E.D.

We deduce from this that $X(z)$ is smooth at both P_0 and P_∞.

Lemma 2. *$X(0) \simeq X(z)$ for all $z \in \mathbb{R}^2$.*

Proof. By 1b we have $d(\mathrm{Ad}F_\lambda \cdot \xi_\lambda(z)) = 0$, where F_λ is given by $F_\lambda^{-1} dF_\lambda = \alpha_\lambda$ and $F_\lambda(0) = I$. Hence

$$\mathrm{Ad}F_\lambda \cdot \xi_\lambda(z) = \xi_\lambda(0). \tag{6}$$

It follows that the map

$$X_A(0) \to X_A(z); \quad (\lambda, [v]) \mapsto (\lambda, [F_\lambda^{-1} v]) \tag{7}$$

is an isomorphism. To see that this extends to the complete curves we follow [9].

Define

$$H_+ = \exp(-z\lambda^p \xi_\lambda(0)) F_\lambda; \qquad H_- = \exp(-\bar{z}\lambda^{-p}\xi_\lambda(0)) F_\lambda.$$

Then

$$\begin{aligned} H_+^{-1} dH_+ &= -\mathrm{Ad}F_\lambda^{-1} \cdot \lambda^p \xi_\lambda(0) dz + \alpha_\lambda \\ &= -\lambda^p \xi_\lambda(z) + \alpha_\lambda \end{aligned}$$

which is polynomial in λ. Therefore H_+ is holomorphic in λ. A similar computation shows that H_- is holomorphic in λ^{-1}. Whenever $\xi_\lambda(0)v = \mu v$ we see that

$$F_\lambda^{-1} v = H_+^{-1} \exp(z\lambda^p \xi_\lambda(0)) v = e^{z\lambda^p \mu} H_+^{-1} v$$

so that the line $[F_\lambda^{-1} v]$ equals $[H_+^{-1} v]$. Similarly we can show that $[F_\lambda^{-1} v] = [H_-^{-1} v]$. Now, we also have

$$\xi_\lambda(z) = \mathrm{Ad}F_\lambda^{-1} \cdot \xi_\lambda(0) = \mathrm{Ad}H_\pm^{-1} \cdot \xi_\lambda(0)$$

and it follows that the isomorphism (7) extends to give $X(0) \simeq X(z)$. Q.E.D.

Remark. Notice that this proof shows that $H_+|_{\lambda=0}$ is upper triangular since the isomorphism fixes the point $(0, [1, 0])$. Likewise, $H_-|_{\lambda=0}$ must be lower triangular.

1.4. The eigenline bundle $\mathcal{E}$ and its dual $\mathcal{L}$.

Let $\mathcal{E}_z$ denote the eigenline bundle of $\xi_\lambda(z)$: it is the pullback to $X(z)$ of the tautological bundle over $\mathbb{P}^1$, using the projection $(\lambda, [v]) \mapsto [v]$. We will denote its restriction to X_A by $\mathcal{E}_{z,A}$ and, to avoid too many superscripts, we will denote the dual bundle by $\mathcal{L}_z$. The inclusion $\mathcal{E}_z \hookrightarrow \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{C}^2$ pulls back the canonical coordinates e_1, e_2 on $\mathbb{C}^2$ to give two independent globally holomorphic sections of $\mathcal{L}_z$, which we also denote e_1, e_2 (or e_1^z, e_2^z when necessary). Notice that $e_1((0,1)^t) = 0$ and $e_2((1,0)^t) = 0$ from which it follows that $e_1 \in \Gamma(\mathcal{L}(-P_\infty))$ and $e_2 \in \Gamma(\mathcal{L}(-P_0))$.

Our next aim is to show that these sections span the space of global holomorphic sections of $\mathcal{L}$ and this characterizes $\mathcal{L}$. Indeed, one reason for working over X is that on Σ we do not have such a straightforward relationship between points and the sections arising from the coordinates e_1, e_2. First let $A = \mathbb{C}[X_A]$, the coordinate ring of X_A, let $B = \mathbb{C}[\lambda, \lambda^{-1}] \subset A$ and for any ring R use $R\langle\cdot\rangle$ to denote an R-module presented by its generators.

Theorem 1. *(i)* $\Gamma(\mathcal{L}_A) = B\langle e_1, e_2\rangle$, *(ii)* $\Gamma(\mathcal{L}) = \mathbb{C}\langle e_1, e_2\rangle$, *(iii)* $\deg \mathcal{L} = g + 1$ *(where g is the genus of X).*

Proof. Let Y be the completion of the planar curve with equation $\mu^2 + \det(\xi_\lambda) = 0$, with Y_A that part of the curve lying over $\lambda \neq 0, \infty$. Then Y_A has coordinate ring $A_Y = \mathbb{C}[\lambda, \lambda^{-1}, \mu] \subset A$ and there is a degree 1 morphism $\alpha : X \to Y$. Set $M = \Gamma(X_A, \mathcal{L}) = \Gamma(Y_A, \alpha_*\mathcal{L})$. First we will show that $B\langle e_1, e_2\rangle \subset M$ is an A_Y-submodule. For any $v \in \Gamma(Y_A, \alpha_*\mathcal{E}) = \Gamma(X_A, \mathcal{E})$ we have $v = (e_1(v), e_2(v))^t$ and $\xi_\lambda v = \mu v$ implies $\mu e_i(v) = a_i e_1(v) + b_i e_2(v)$ for some $a_i, b_i \in B$. It follows that $\mu e_i \in B\langle e_1, e_2\rangle$. So $B\langle e_1, e_2\rangle$ is an A_Y-module.

Now it can only be a proper A_Y-module if its localisation at every maximal ideal $\mathfrak{p}$ is also proper in the corresponding localisation $M_\mathfrak{p}$. But at any smooth point $P \in Y_A$, with maximal ideal $\mathfrak{p}$, $M_\mathfrak{p}$ is the stalk $\mathcal{L}_P$ of $\mathcal{L}$ at P, and all its proper submodules are contained in $\mathcal{L}_P(-P)$ (the local sections which vanish at P). But in that case every section in M must vanish at P. This means that for every $v \in \mathcal{E}_P$ both $e_1(v)$ and $e_2(v)$ vanish at P, which is absurd (there is always a non-zero eigenvector). Since Y_A must have at least one smooth point we deduce that $M = B\langle e_1, e_1\rangle$.

(ii) Given (i) it suffices to show that if $\lambda^n e_i \in \Gamma(\mathcal{L})$ for $n \in \mathbb{Z}$ then $n = 0$. Since e_1 does not vanish at P_0 and e_2 does not vanish at P_∞ it suffices to show that neither λe_1 nor $\lambda^{-1}e_2$ are globally holomorphic. Consider first λe_1: it is globally holomorphic if and only if $\lambda e_1(v)$ is holomorphic at P_∞ for all locally holomorphic sections v of $\mathcal{E}$ about P_∞. By definition,

$$[v] = \left[\begin{pmatrix} 1 & 0 \\ 0 & \pm\zeta \end{pmatrix} w\right]$$

where $\zeta^{-2p-1}\xi_\zeta w = \mu\zeta^{-2p-1}w$. We may assume, without loss of generality, that

$$w = \begin{pmatrix} \bar{a}^{-1} + O(\zeta^{-1}) \\ \pm\bar{a} + O(\zeta^{-1}) \end{pmatrix}.$$

Therefore

$$v = \begin{pmatrix} \zeta^{-1}\bar{a}^{-1} + O(\zeta^{-2}) \\ \bar{a} + O(\zeta^{-1}) \end{pmatrix}.$$

Hence $\lambda e_1(v)$ has a first order pole at P_∞. A similar calculation shows for v a locally holomorphic section about P_0 we can take

$$v = \begin{pmatrix} a + O(\zeta) \\ \zeta a^{-1} + O(\zeta^2) \end{pmatrix}$$

and therefore $\lambda^{-1}e_2(v)$ has a first order pole at P_0.

(iii) Since λ has divisor $2P_0 - 2P_\infty$, (i) and (ii) imply $\Gamma(\mathcal{L}(-2P_0)) = 0$. So applying the Riemann-Roch formula gives $\deg \mathcal{L} \leq g+1$. Now we show $\deg \mathcal{L} \geq g+1$. For n any suitably large positive integer $\mathcal{L}(2nP_\infty)$ must be non-special so that $\dim \Gamma(\mathcal{L}(2nP_\infty)) = \deg \mathcal{L} + 2n + 1 - g$. But

$$\mathbb{C}\langle e_1, e_2, \lambda e_1, \lambda e_2, \ldots, \lambda^n e_1, \lambda^n e_2\rangle \subset \Gamma(\mathcal{L}(2nP_\infty))$$

so $\deg \mathcal{L} + 2n + 1 - g \geq 2n + 2$. Q.E.D.

1.5. The real structure of $\Gamma(\mathcal{L})$.

An important property of $\Gamma(\mathcal{L})$ is that it possesses a Hermitian inner product: this comes from a reality condition on $\mathcal{L}$ and is essential since we intend to identify $\mathbb{P}\Gamma(\mathcal{L})$ with $S^2 \cong \mathbb{CP}^1$ as a Hermitian symmetric space. This reality condition arises as follows.

The real symmetry $\xi_{\bar{\lambda}^{-1}} = -\xi_\lambda^\dagger$ induces a real involution ρ on X for which $\overline{\rho_*\lambda} = \lambda^{-1}$ and $\overline{\rho_*\mu} = -\mu$. Notice that, since ξ_λ is skew-Hermitian over $|\lambda| = 1$, μ is pure imaginary there so ρ fixes all points over $|\lambda| = 1$; this will prove to be important later. Consequently the eigenline bundle $\mathcal{E}$ also satisfies a reality condition.

Proposition 1. *$\overline{\rho_*\mathcal{E}} \simeq \mathcal{L}(-R)$ where R is the ramification divisor of $\lambda : X \to \mathbb{P}^1$.*

Proof. Since $\overline{\rho_*\xi_\lambda} = -\xi_\lambda^t$, sections of $\overline{\rho_*\mathcal{E}}$ correspond to solutions of $\xi_\lambda^t w = \mu w$. Take any proper open subset $U \subset X$ for which $U = \lambda^{-1} \circ \lambda(U)$, and let $v : U \to \mathcal{E}$ be a trivialising section. If σ denotes the hyperelliptic involution on X then $\sigma_*\mu = -\mu$ and clearly $v, \sigma_* v$ are linearly independent at $P \in X$ if and only if P is not a ramification point. Take V to be the matrix with columns $v, \sigma_* v$, then we have $\det(V)$ vanishing only at ramification points. Define $W = \det(V).V^{-1t}$, then W is holomorphic in U and

$$\xi_\lambda^t W = W \begin{pmatrix} \mu & 0 \\ 0 & -\mu \end{pmatrix}.$$

It is easy to check that the columns of W are given by $w, -\sigma_* w$ where $e_1(w) = e_2(\sigma_* v)$ and $e_2(w) = -e_1(\sigma_* v)$. Therefore w corresponds to a trivialising section for $\overline{\rho_*\mathcal{E}}$ over U. Now consider the injective homomorphism of $\mathcal{O}_U$-modules

$$\begin{array}{rcl} \mathcal{O}_U\langle w\rangle & \to & \mathrm{Hom}(\mathcal{O}_U\langle v\rangle, \mathcal{O}_U) \\ f.w & \mapsto & (h.v \mapsto fh.w^t v) \end{array}$$

for $f, h \in \mathcal{O}_U$. Since $w^t v = \det(V)$ we see that the induced sequence of sheaves is

$$0 \to \overline{\rho_*\mathcal{E}} \to \mathcal{L} \to \mathcal{O}_R \to 0$$

where $\mathcal{O}_R$ is the skyscraper sheaf for the divisor R. Therefore $\overline{\rho_*\mathcal{E}} \simeq \mathcal{L}(-R)$. Q.E.D.

Consequently we have $\mathcal{L} \otimes \overline{\rho_*\mathcal{L}} \simeq \mathcal{O}_X(R)$. The inner product on $\mathbb{C}^2$ corresponds (at least over the unit circle) to the section

$$e_1 \otimes \overline{\rho_* e_1} + e_2 \otimes \overline{\rho_* e_2} \tag{8}$$

(which maps (v, w) to $\overline{\rho_* v}^t w$). Up to scaling this corresponds to an inner product on $\Gamma(\mathcal{L})$ determined in the following manner.

We first take any $s \in \Gamma(\mathcal{L})$ to identify $\mathcal{L}$ with the divisor line bundle $\mathcal{O}_X(D)$ where D is the divisor of zeroes of s. Second we fix a rational function f with divisor $D + \rho_* D - R$ for which $\overline{\rho_* f} = f$ and f is positive over $|\lambda| = 1$ (this is always possible since ρ fixes all points over the unit circle). Now we define

$$\begin{aligned} &h : \Gamma(\mathcal{L}) \times \Gamma(\mathcal{L}) \to \mathbb{C}; \\ &h(s_1, s_2) = \tfrac{1}{2}\textstyle\sum_{j=1}^{2} f(O_j)(s_1/s)(O_j)\overline{\rho_*(s_2/s)}(O_j) \end{aligned} \tag{9}$$

where O_1, O_2 are the two points over $\lambda = 1$. The proper interpretation of the right hand side is in terms of the trace map $\mathrm{Tr} : \mathcal{O}_X(R) \to \mathbb{C}$ which I will not explain in detail here (see, for example, [26]). But it is worth noting for future reference that this inner product clearly makes the subspaces $\Gamma(\mathcal{L}(-O_1)), \Gamma(\mathcal{L}(-O_2)) \subset \Gamma(\mathcal{L})$ orthogonal.

1.6. What the frame does.

Let us introduce $\hat{\Gamma}(\cdot)$ for spaces of analytic sections and let $\hat{A}$ denote the ring of analytic functions on X_A while $\hat{B}$ denotes the analytic functions on the punctured λ-plane $\mathbb{P}^1_\lambda \setminus \{0, \infty\}$. The map

$$\hat{\Gamma}(\mathcal{E}_{z,A}) \to \hat{\Gamma}(\mathcal{E}_{0,A}); \quad v \mapsto F_\lambda v$$

is clearly an isomorphism of $\hat{A}$-modules. Therefore it corresponds to a family of trivialising sections

$$\theta_z \in \hat{\Gamma}(\mathcal{E}_{0,A} \otimes \mathcal{L}_{z,A}).$$

Let $J(X)$ denote the Jacobian of X — the abelian variety of isomorphism classes of line bundles of degree zero. If $J_R(X)$ denotes the real subgroup of degree zero line bundles L for which $\overline{\rho_* L} \simeq L^{-1}$ then we deduce from the previous section that $\mathcal{E}_0 \otimes \mathcal{L}_z$ belongs to $J_R(X)$ for all z.

Proposition 2. *(i) Define $L : \mathbb{R}^2 \to J_R(X)$ by $L_z = \mathcal{E}_0 \otimes \mathcal{L}_z$. Then L is $\mathbb{R}$-linear (i.e. a homomorphism of real abelian groups). (ii) The section $\theta_z \exp(z\lambda^p \mu)$ is holomorphic and non-vanishing over P_0, while $\theta_z \exp(\bar{z}\lambda^{-p}\mu)$ is holomorphic and non-vanishing over P_∞.*

Proof. Observe that (ii) implies (i) since we deduce from it that L_z corresponds to the transition functions $\exp(z\lambda^p\mu)$ and $\exp(\bar{z}\lambda^{-p}\mu)$ patching from X_A to U_0 and U_∞ respectively, where the latter are open neighbourhoods of P_0 and P_∞ respectively.

To prove (ii) we recall from the proof of lemma 2 that if v_0 is a holomorphic section of $\mathcal{E}_0$ about P_0 then $F_\lambda^{-1} v_0 = H_+^{-1} \exp(z\lambda^p\mu) v_0$ so that $\exp(-z\lambda^p\mu) F_\lambda^{-1} v_0$ is a holomorphic section of $\mathcal{E}_z$ about P_0. But $F_\lambda^{-1} v_0$ corresponds to $v_0 \otimes \theta_z^{-1}$ so tensoring with $\exp(-z\lambda^p\mu)\theta_z^{-1}$ preserves holomorphicity about P_0. A similar argument using H_- about P_∞ proves the second part of (ii). Q.E.D.

Let $L_{z,A}$ denote the restriction to X_A of L_z. We want to make explicit the representation

$$\mathcal{F} : \hat{\Gamma}(L_A) \to \hat{B} \otimes \mathfrak{gl}_2$$

which gives us $\mathcal{F}(\theta) = F_\lambda$. It arises from the composite isomorphism

$$\epsilon_z : \hat{\Gamma}(\mathcal{E}_{z,A}(R)) \to \hat{B} \otimes \Gamma(\mathcal{L}_z)^t \to \hat{B} \otimes \mathbb{C}^2. \tag{10}$$

The second arrow is just the identification $\Gamma(\mathcal{L}_z)^t \to \mathbb{C}^2$ determined by e_1^z, e_2^z. The first arrow is the $\hat{B}$-module isomorphism dual to $\hat{B} \otimes \Gamma(\mathcal{L}) \simeq \hat{\Gamma}(\mathcal{L}_A)$, from theorem 1. This uses the fact, implicit in the proof of proposition 1, that $\lambda_*\mathcal{L}$ is dual to $\lambda_*\mathcal{E}(R)$. It follows that to any $\phi \in \hat{\Gamma}(L_A)$ there is some $\mathcal{F}(\phi) \in \hat{B} \otimes \mathfrak{gl}_2$ so that the following diagram commutes:

$$\begin{array}{ccc} \hat{\Gamma}(\mathcal{E}_{z,A}(R)) & \xrightarrow{\phi} & \hat{\Gamma}(\mathcal{E}_{0,A}(R)) \\ \epsilon_z \downarrow & & \downarrow \epsilon_0 \\ \hat{B} \otimes \mathbb{C}^2 & \xrightarrow{\mathcal{F}(\phi)} & \hat{B} \otimes \mathbb{C}^2. \end{array} \tag{11}$$

Next we will show that θ is almost completely determined by its behaviour at the points P_0, P_∞. First observe that from (4) we have

$$\mu^2 = -\det \xi_\zeta = \zeta^{-4p-2} + \ldots + \zeta^{4p+2} = \lambda^{-2p-1} + \ldots + \lambda^{2p+1}.$$

Since $\overline{\rho_*\mu} = -\mu$ whereas $\overline{\rho_*\zeta} = \zeta^{-1}$ we find, with the right sign choice for ζ, $\mu = \zeta^{-2p-1} + \ldots - \zeta^{2p+1}$. Therefore $\lambda^p\mu - \zeta^{-1}$ is holomorphic about P_0 while $\lambda^{-p}\mu + \zeta$ is holomorphic about P_∞. Consequently, as a corollary of proposition 2 we have:

Corollary 1. *θ_z is determined up to sign, amongst trivialising sections of L_A, by the properties that: (a) $\theta_z \exp(z\zeta^{-1})$ is holomorphic and non-vanishing over P_0 while $\theta_z \exp(-\bar{z}\zeta)$ is holomorphic and non-vanishing over P_∞ and, (b)* $\det(\mathcal{F}(\theta_z)) = 1$.

Proof. If ϕ is any other trivialising section with these properties then $\phi\theta_z^{-1}$ is a globally holomorphic function and therefore a constant, k say. But clearly $\det(\mathcal{F}(k\theta_z)) = k^2$ so that $k = \pm 1$. Q.E.D.

Remark. The unitary nature of $\mathcal{F}(\theta_z)$ on the unit circle is a reflection of the fact that $\overline{\rho_*\theta} = \theta^{-1}$.

Finally, let us use this corollary to display a simple characterisation for the map L. Since it is linear it is completely determined by $dL_0(\partial/\partial z)$ which lies in $T_1^{1,0}J(X)$ (here 1 denotes the identity in $J(X)$). By the corollary above $L(z)$ corresponds to the cohomology class $[c(z)]$ in $H^1(X, \mathcal{O}^*)$ for the 1-cocycle

$$c(z) = \{(e^{z\zeta^{-1}}, X_A, U_0), (e^{-\bar{z}\zeta}, X_A, U_\infty)\} \tag{12}$$

for the open cover X_A, U_0, U_∞, where now U_0, U_∞ are (disjoint) parameter discs (i.e. domains for ζ, ζ^{-1}). Therefore

$$\frac{\partial[c]}{\partial z}|_{z=0} = [(\zeta^{-1}, X_A, U_0), (1, X_A, U_\infty)] \in H^1(X, \mathcal{O}).$$

Now recall the isomorphism $H^1(X,\mathcal{O}) \simeq \Gamma(\Omega_X)^t$: it identifies $\partial[c]/\partial z$ with the map $f : \omega \mapsto res_{P_0}\zeta^{-1}\omega$ for $\omega \in \Gamma(\Omega_X)$. But now observe that

$$res_{P_0}\zeta^{-1}\omega = (\omega/d\zeta)(P_0) = \frac{\partial}{\partial\zeta}\int_{P_0}^{\zeta}\omega.$$

Hence $f = d\mathcal{A}_{P_0}(\partial/\partial\zeta)$ where $\mathcal{A}_{P_0} : X \to J(X)$ is the Abel map with base point P_0. Thus we learn:

Lemma 3. *The linear map $L : \mathbb{R}^2 \to J_R(X)$ is uniquely determined by the property that $dL_0(\partial/\partial z) = d\mathcal{A}_{P_0}(\partial/\partial\zeta)$.*

1.7. Periodicity conditions.

We have seen that the non-conformal doubly periodic harmonic map $\varphi : \mathbb{R}^2 \to S^2$ yields us spectral data $(X, \lambda, \mathcal{L})$ and it is easy to see how to reverse this procedure to reconstruct the map from this data. We first construct the linear map $L : \mathbb{R}^2 \to J_R(X)$ given by lemma 3 and define $\mathcal{L}_z = \mathcal{L} \otimes L_z$. By theorem 1 $\Gamma(\mathcal{L}_z)$ comes equipped with a frame e_1^z, e_2^z determined by the points P_∞, P_0: this frame is chosen to be unitary according to the trace inner product described above. With the frame we recover the map $\mathcal{F}$ in (11). Now we equip L_z with the unique (up to sign) trivialising section θ_z over X_A given by corollary 1. Thus we obtain the extended frame $F_\lambda = \mathcal{F}(\theta_z)$ and the map φ is recovered as $F_1 \circ [1,0]$ where $[1,0] \in \mathbb{CP}^1$.

However, we do not need the frame itself to obtain φ: it is clear that each line $\varphi(z) \in \mathbb{CP}^1$ corresponds to the line $\Gamma(\mathcal{L}_z(-P_\infty)) \in \mathbb{P}\Gamma(\mathcal{L}_z)$ where $\mathbb{P}\Gamma(\mathcal{L}_z)$ is identified with $\mathbb{CP}^1$ using θ_z. There is an invariant way of describing this identification which avoids explicit reference to θ_z and this helps us understand the periodicity conditions. To obtain this let us first consider the expression for φ in homogeneous coordinates: it can be written as

$$\varphi = [(e_1^0 f_1^z\theta_z)|_{\lambda=1}, (e_2^0 f_1^z\theta_z)|_{\lambda=1}],$$

where $f_j^z \in \Gamma(\mathcal{E}_{z,A}(R))$ is the B-module generator dual to to e_j^z. Now it is clear that if we choose some other (unitary) basis v_1, v_2 of $\Gamma(\mathcal{L})$ we obtain, up to isometry of S^2, the same map. In particular, following the remarks made earlier, we could choose this new basis such that v_1 vanishes at O_2 and v_2 vanishes at O_1. In that case

$$\varphi = [(f_1^z \theta_z)|_{O_1}, \alpha(f_1^z \theta_z)|_{O_2}],$$

where $\alpha : \mathcal{L}|O_2 \to \mathcal{L}|O_1$ is the fibre identification induced by the choice of v_1, v_2 (i.e. $\alpha(v_2|_{O_2}) = v_1|_{O_1}$).

Recall that up to scaling f_1 is determined purely by the vanishing of e_1 at P_∞. It follows that φ has periodicity $\varphi(z+\tau) = \varphi(z)$ precisely when both equations $L(z+\tau) = L(z)$ and

$$(\theta_{z+\tau}\theta_z^{-1})|_{O_2} = (\theta_{z+\tau}\theta_z^{-1})|_{O_1}$$

are satisfied. The latter condition is more simply interpreted as saying that the fibre identification $L_z|O_2 \to L_z|O_1$ given by $\theta_z|_{O_2} \mapsto \theta_z|_{O_1}$ is τ-periodic. This identication determines at each z a line bundle L'_z over X', the singular curve obtained from X by identifying O_1 with O_2 to obtain a node. Thus we have a τ-periodic map L' from $\mathbb{R}^2$ to $J(X')$, the (generalized) Jacobi variety for X'. Recall that the pullback of line bundles along $X \to X'$ induces a surjective homomorphism $\pi : J(X') \to J(X)$ whose fibre at L is $L|O_2 \otimes L^{-1}|O_1 \cong \mathbb{C}^*$. In fact $\overline{\rho_* L'} \cong L'^{-1}$ so L' takes values in a real subgroup $J_R(X')$ of $J(X')$. It can be shown that, when X is smooth of genus g, this group is a real compact torus of dimension $g+1$.

Lemma 4. *The map $L' : \mathbb{R}^2 \to J_R(X')$ defined above is linear and is uniquely determined by the property that $dL'_0(\partial/\partial z) = d\mathcal{A}'_{P_0}(\partial/\partial\zeta)$, where $\mathcal{A}'_{P_0} : X' - \{O\} \to J(X')$ is the Abel map for X' based at P_0.*

Proof. Since θ_z arises from the 1-cocycle $c(z)$ in (12) L'_z has 1-cocycle

$$c'(z) = \{(e^{z\zeta^{-1}}, X'_A, U_0), (e^{-\bar{z}\zeta}, X'_A, U_\infty)\}.$$

Now recall (from e.g. [26]) that

$$J(X') \simeq \Gamma(\Omega'_X)^t / H_1(X - \{O_1, O_2\}, \mathbb{Z}),$$

where Ω'_X is the sheaf of regular differentials on X': each such differential can be identified with a meromorphic differential on X whose only poles are simple ones at O_1 and O_2 i.e. $\Omega'_X \cong \Omega_X(O_1 + O_2)$. The Abel map for X' is defined by

$$\mathcal{A}'_{P_0} : X' - \{O\} \to J(X'); \quad P \mapsto \int_{P_0}^{P},$$

where O is the nodal point lying under O_1, O_2. To compute dL_0' we simply repeat the computation prior to lemma 3 using $c'(z)$. Q.E.D.

Corollary 2. *The harmonic map $\varphi : \mathbb{R}^2 \to S^2$ with spectral data $X, \lambda, \mathcal{L}$ has period τ if and only if the related map $L' : \mathbb{R}^2 \to J_R(X')$ has period τ. This depends only on the data X, λ.*

In particular, if X has genus $g \leq 1$ the harmonic map is necessarily doubly periodic since $J_R(X')$ is topologically S^1 or $S^1 \times S^1$. These examples yield the Gauss maps of all Delaunay surfaces in $\mathbb{R}^3$ (i.e. the constant mean curvature surfaces of revolution) with the case $g = 0$ corresponding to the Gauss map of the cylinder. A more interesting class of singly periodic examples are the Gauss maps of the "bubbletons". The bubbletons are periodic CMC surfaces whose ends are asymptotic to the standard cylinder (see figure 2). They get their name because they correspond to soliton solutions of the sinh-Gordon equation, which governs the behaviour of the metric. As with KdV solitons, these solutions have rational nodal spectral curves. Using the theory above we can characterize these spectral curves as follows.

Proposition 3. *Let X be the rational nodal curve of arithmetic genus $g = 2r$ with equation*

$$\mu^2 = \lambda \prod_{j=1}^{r} (\lambda - a_j)^2 (1 - a_j \lambda)^2, \; a_j \in \mathbb{R}, \; 0 < a_j < 1. \tag{13}$$

Then X, λ is the spectral data for a singly periodic non-conformal harmonic map $\varphi : \mathbb{R}^2 \to S^2$ if and only if there exist positive integers $p_0, p_1, \ldots, p_r$ for which

$$a_j = \left(\frac{p_j}{p_0} \pm \sqrt{\frac{p_j^2}{p_0^2} - 1} \right)^2, \; j = 1, \ldots, r. \tag{14}$$

Proof. Let us set $\zeta = \sqrt{\lambda}$: this is a rational coordinate on X. Thus we identify X with the singularization of the Riemann sphere $\mathbb{P}_\zeta$ with the points $\pm\zeta_j$ identified, where $\zeta_j^2 = a_j$ and $\zeta_{r+j}^2 = a_j^{-1}$ for $j = 1, \ldots, r$. Notice that $\zeta_j \in \mathbb{R}$ since $a_j > 0$. X' is the further singularization obtained by additionally identifying $\pm\zeta_0$, where $\zeta_0 = 1$. We may assume the real involution is $\rho_* \zeta = \bar{\zeta}^{-1}$. A basis for $\Gamma(\Omega'_X)$ is given by

$$\omega_j = \frac{1}{2\pi i} \left(\frac{1}{\zeta - \zeta_j} - \frac{1}{\zeta + \zeta_j} \right) d\zeta, \; j = 0, \ldots, 2r,$$

Now X' is obtained from its normalisation (a smooth curve of genus $2r+1$) by shrinking half the homology generators to zero, hence $H_1(X',\mathbb{Z})$ is generated by γ_j, $j=0,\ldots,2r$ where each of these is the boundary of a small positively oriented disc containing ζ_j. It follows that $\oint_{\gamma_j}\omega_k=\delta_{jk}$. The real group $J_R(X')$ is isomorphic to

$$\{\omega\in\Gamma(\Omega'_X):\rho_*\omega=-\bar{\omega}\}^t/\{\gamma\in H_1(X',\mathbb{Z}):\rho_*\gamma\sim-\gamma\}$$

which we will write more simply as V^t/Γ. It is not hard to check that a basis for V is given by

$$v_0=\omega_0, v_j=\frac{1}{2}(\omega_j+\omega_{r+j}), v_{r+j}=\frac{i}{2}(\omega_j-\omega_{r+j}),\ j=1,\ldots,r$$

and generators for $\Gamma\subset V^t$ can be given by

$$\oint_{\gamma_0},\oint_{\gamma_j+\gamma_{r+j}},\ j=1,\ldots,r.$$

With respect to this basis for V the dual isomorphism $V^t\cong\mathbb{R}^{2r+1}$ identifies the generators for Γ with the first $r+1$ standard basis vectors for $\mathbb{R}^{2r+1}$.

The map $L':\mathbb{R}^2\to J_R(X')$ described above is covered by

$$\ell:\mathbb{R}^2\to V^t;\ \ell(z,\bar{z})=z\,res_0\zeta^{-1}-\bar{z}\,res_\infty\zeta$$

where e.g. $res_0\zeta^{-1}:V\to\mathbb{C}$ takes the residue of $\zeta^{-1}\omega$ at $\zeta=0$. In terms of the dual basis for V^t this has coordinates

$$\ell:\mathbb{R}^2\to\mathbb{R}^{2r+1};\ \ell(z,\bar{z})=zU+\bar{z}\bar{U}$$

where $U\in\mathbb{C}^{2r+1}$ has coordinates

$$U=\frac{-1}{2\pi i}(2,\ldots,\alpha_j^{-1}+\alpha_j,\ldots,i(\alpha_j^{-1}-\alpha_j),\ldots)$$

where $\alpha_j^2=a_j$. The map is periodic precisely when there exists $z\in\mathbb{C}$ for which

$$zU+\bar{z}\bar{U}=(p_0,p_1,\ldots,p_r,0\ldots,0),\ p_j\in\mathbb{Z}.$$

If we write $z=x+iy$ then these $2r+1$ equations become

$$2y=-\pi p_0,\ y(\alpha_j+\alpha_j^{-1})=-\pi p_j,\ x(\alpha_j-\alpha_j^{-1})=0,\ j=1,\ldots,r.$$

These equations have a solution for $a_j<1$ if and only if $x=0$ and

$$\alpha_j^2-\frac{2p_j}{p_0}\alpha_j+1=0.$$

Q.E.D.

Remark. The reader may wonder why we only consider $a_j \in \mathbb{R}$. The more general case of complex nodes also leads to periodic maps $\varphi : \mathbb{R}^2 \to S^2$. The conditions are that, writing $a_j = r_j^2 e^{2\theta_j}$, there must be positive integers $p_0, \ldots, p_r$ for which

$$r_j^2 - \frac{2p_j}{p_0}\cos(\theta_j)r_j + 1 = 0.$$

However, these are not the Gauss maps of periodic CMC surfaces unless $\theta_j = 0$. Indeed, it is not obvious even then that we obtain periodic CMC surfaces since none of the discussion above accounts for the extra condition that the CMC surface must also have a period when its Gauss map does. That this happens when (14) is satisfied follows from an argument I learned from Martin Kilian and Nick Schmitt, which exploits the dressing construction. Unfortunately to describe this closing argument would take us too far afield, although I will say something about the dressing construction in the next section.

1.8. Two reconstructions of the harmonic map: Symes' method and dressing the vacuum.

I know of three approaches to reconstructing the harmonic map from its spectral data. The first of these, which I will describe in a more general context later, boils down to writing the map down in terms of the θ-functions for X' (cf. [1]). The other two methods use a loop group and require one to be able to perform a certain loop group factorization. Until recently this had only theoretical interest, but with the advent of Nick Schmitt's CMCLab software it is now possible to perform explicit calculations involving the (approximate) factorization, so I want to take this opportunity to explain how to reproduce the map $\varphi : \mathbb{R}^2 \to S^2$ (and hence its associated family of CMC surfaces) from its spectral data. Before I begin we must recall some fundamentals about the application of loop groups to the construction of harmonic maps.

First, set $G^{\mathbb{C}} = SL_2(\mathbb{C})$ and let G denote its compact real form SU_2. For $\epsilon \in \mathbb{R}^+$ with $0 < \epsilon < 1$ we let C be the union of circles $\{\zeta : |\zeta| = \epsilon \text{ or } |\zeta| = \epsilon^{-1}\}$ on the Riemann sphere $\mathbb{C} \cup \{\infty\}$ and consider it as the common boundary of the two open sets

$$E = \{\zeta : \epsilon < |\zeta| < \epsilon^{-1}\},\ I = \{\zeta : |\zeta| < \epsilon \text{ or } |\zeta| > \epsilon^{-1}\}.$$

We will work with the loop group (of "twisted loops")

$$\Lambda_C G = \{C^\omega \text{ maps } g : C \to G^C | g_{\bar{\zeta}^{-1}} = g_\zeta^\dagger,\ \nu(g_\zeta) = g_{-\zeta}\}.$$

This loop group contains in particular $\Lambda_E G$, the subgroup of those $g \in \Lambda_C G$ which extend holomorphically into E, and $\Lambda_I G$, the subgroup of those $g \in \Lambda_C G$ which extend holomorphically into I such that g_0 is upper triangular with positive real diagonal entries. It is well known (see [16]) that every $g \in \Lambda_C G$ factorizes uniquely into $g_E g_I$ where $g_E \in \Lambda_E G$ and $g_I \in \Lambda_I G$: this is sometimes called the Iwasawa decomposition for $\Lambda_C G$.

The relevance of these groups to our harmonic maps can be encapsulated in the following theorem. First, notice that the simplest non-conformal map $\varphi : \mathbb{R}^2 \to S^2$, which has been dubbed the "vacuum solution", maps onto a great circle and is framed by the homomorphism

$$F^{(0)} : \mathbb{R}^2 \to SU_2;\ F^{(0)} = \exp(zA - \bar{z}A);\ A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

This has extended frame

$$F_\zeta^{(0)} : \mathbb{R}^2 \to \Lambda_E G;\ F_\zeta^{(0)} = \exp(\zeta^{-1} zA - \zeta\bar{z}A).$$

Theorem 2. [6, 7] *Let $\varphi : \mathbb{R}^2 \to S^2$ be a non-conformal harmonic map of finite type with polynomial Killing field $\xi_\zeta(z)$, in the form (4).*

(1) *φ has an extended frame given by*

$$F_\zeta = \exp(z\zeta^{2p}\xi_\zeta(0))_E.$$

This is "Symes' formula" [6].

(2) *For some $0 < \epsilon < 1$ there exists $g_\zeta \in \Lambda_I G$ so that φ has an extended frame given by*

$$F_\zeta = (g_\zeta F_\zeta^{(0)})_E.$$

This is "dressing the vacuum solution" [7].

Since in both formulae the frame satisfies the same Maurer-Cartan equations with $F_\zeta(0) = I$, each method gives the same extended frame. Now I will describe how to compute the polynomial Killing field ξ_ζ and the dressing matrix g_ζ corresponding to the spectral data $X, \lambda, \mathcal{L}$ for a particularly amenable choice of $\mathcal{L}$.

Proposition 4. *Let X, λ correspond to the curve with affine equation*

$$y^2 = \lambda \prod_{j=1}^{g} (\lambda - a_j)(1 - \bar{a}_j\lambda);\ 0 < |a_j| < 1.$$

and let $\mathcal{L} = \mathcal{O}_X(R_+)$ *where* R_+ *is the divisor* $P_0 + \sum_{j=1}^g R_j$ *for* $\lambda(R_j) = a_j$. *Then the non-conformal map* $\varphi : \mathbb{R}^2 \to S^2$ *with spectral data* $X, \lambda, \mathcal{L}$ *arises from:*

(1) *Symes' formula using* $\xi_\zeta(0) = \eta_\zeta - \eta^\dagger_{\bar\zeta^{-1}}$ *where*

$$\eta_\zeta = \begin{pmatrix} 0 & \zeta\prod_{j=1}^g(1-\bar a_j\zeta^2) \\ \zeta\prod_{j=1}^g(\zeta^2 - a_j) & 0 \end{pmatrix}; \tag{15}$$

(2) *dressing the vacuum solution by*

$$g_\zeta = \begin{pmatrix} h^{-1/4} & 0 \\ 0 & h^{1/4} \end{pmatrix}; \quad h = \prod_{j=1}^g \left(\frac{\zeta^2 - a_j}{1 - \bar a_j \zeta^2}\right). \tag{16}$$

Proof. 1. Given an orthonormal basis e_1, e_2 for $\Gamma(\mathcal{L})$ we obtain a B-module morphism

$$K : \{f \in \mathbb{C}[X_A] : \overline{\rho^* f} = -f\} \to \{\xi_\zeta(z) : d\xi = [\xi, \alpha],\ \xi_{\bar\zeta^{-1}} = -\xi_\zeta^\dagger\}$$

in which each ξ is algebraic (indeed, a Laurent polynomial) in λ. In fact this map is an isomorphism for real algebraic ξ [19]. It arises from the commutative diagram

$$\begin{array}{ccc} \hat\Gamma(\mathcal{E}_{z,A}(R)) & \xrightarrow{\times f} & \hat\Gamma(\mathcal{E}_{z,A}(R)) \\ \epsilon_z \downarrow & & \downarrow \epsilon_z \\ \hat B \otimes \mathbb{C}^2 & \xrightarrow{\xi(f)} & \hat B \otimes \mathbb{C}^2 \end{array}$$

This gives $\xi_\lambda(z) = K(f)$ for each z, where we recall from (5) that $\xi_\lambda = \mathrm{Ad}\kappa \cdot \xi_\zeta$. Since $\theta f \theta^{-1} = f$ it follows, by combining this diagram and the diagram (11), that $\xi(0) = F\xi(z)F^{-1}$ whence $d\xi = [\xi, \alpha]$. For the purposes of Symes' formula we want to compute $K(f)$ at $z = 0$ for $f = y - \overline{\rho_* y}$. Since

$$K(\overline{\rho_* y}) = \overline{\rho_* K(y)}^t$$

it suffices to compute $K(y)$ at $z = 0$. A simple computation shows that with respect to the trace inner product (9) $\Gamma(\mathcal{O}_X(R_+))$ has an orthonormal basis given by

$$e_1 = \frac{y}{\lambda\prod_{j=1}^g(\lambda - a_j)}, \quad e_2 = 1. \tag{17}$$

Here we are identifying holomorphic sections of $\mathcal{L}$ with rational functions on X whose divisor of poles is no worse than R_+. Notice that e_1

generates $\Gamma(\mathcal{L}(-P_\infty))$ while e_2 generates $\Gamma(\mathcal{L}(-P_0))$. Now $K(y)$ is the matrix

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

where

$$ye_1 = \alpha e_1 + \beta e_2, \quad ye_2 = \gamma e_1 + \delta e_2$$

so that at $z = 0$

$$K(y) = \begin{pmatrix} 0 & \prod_{j=1}^g (1 - \bar{a}_j \lambda) \\ \lambda \prod_{j=1}^g (\lambda - a_j) & 0 \end{pmatrix}.$$

Finally, let η_ζ be the twisted loop $\mathrm{Ad}\kappa^{-1} \cdot K(y)$ to obtain the formula (15).

2. Let us consider the geometric meaning of the equation (16). If we write these loops in their untwisted form, then $g_\lambda F_\lambda^{(0)} = F_\lambda b_\lambda$ where F_λ extends holomorphically to an annulus on the λ-sphere (which we will call E despite the abuse of notation) and b_λ extends holomorphically to a pair of discs about $\lambda = 0, \infty$ (which we will call I) and is upper triangular at $\lambda = 0$. A little thought shows that the columns of F^{-1t} represent $e_1^z \theta^{-1}, e_2^z \theta^{-1}$, thought of as sections of the rank two vector bundle $\lambda_* \mathcal{L}$ over E, with respect to the global frame e_1^0, e_2^0. Let ϕ_E denote the trivialisation of $\lambda_* \mathcal{L}$ determined by this global frame, then the equation $g_\lambda^{-1t} b_{\lambda, z=0}^{-1t} = I$ expresses the fact that there is some local trivialisation ϕ_I for $\lambda_* \mathcal{L}$ over I for which the transition relation on $E \cap I$ is

$$g^{-1t} \phi_I = \phi_E.$$

Therefore g_λ^t is the matrix whose columns are $\phi_I(e_1^0), \phi_I(e_2^0)$. Now we recall from [15] that ϕ_I is obtained by direct image from a trivialisation of $\mathcal{L}$ over $\lambda^{-1}(I)$ in the following way. Let s_I be a non-vanishing holomorphic section of $\mathcal{L}$ over $\lambda^{-1}(I)$. By definition $\Gamma(I, \lambda_* \mathcal{L}) = \Gamma(\lambda^{-1}, \mathcal{L})$ and s_I induces the trivialisation

$$\phi_I : \Gamma(I, \lambda_* \mathcal{L}) \to \mathrm{Hol}(I, \mathbb{C}^2); \quad s \mapsto (s_1, s_2)$$

where $s/s_I = s_1(\zeta^2) + \zeta s_2(\zeta^2)$. Any trivialisation ϕ_I obtained this way and which gives $\det(g_\lambda) = 1$ will provide a suitable matrix g_λ (the freedom here is right multiplication of g_λ by any element of $\Lambda_I G$ which commutes with $F_\lambda^{(0)}$ for all z). To calculate g_λ we let e_1, e_2 be the basis (17) and initially take $s_I = e_1$: this is appropriate since as a function it has a simple pole at P_0 and does not vanish at P_∞, therefore it represents a non-vanishing section of $\mathcal{L}$ over I provided I is small enough.

Now we write

$$e_1/e_1 = 1 + \zeta.0, \quad e_2/e_1 = 0 + \zeta. \prod_{j=1}^{g} (\frac{\zeta^2 - a_j}{1 - \bar{a}_j \zeta^2})^{1/2}.$$

However, this choice of s_I does not give $\det(g_\lambda) = 1$, so it remains to rescale s_I by the appropriate non-vanishing function to obtain (16). Q.E.D.

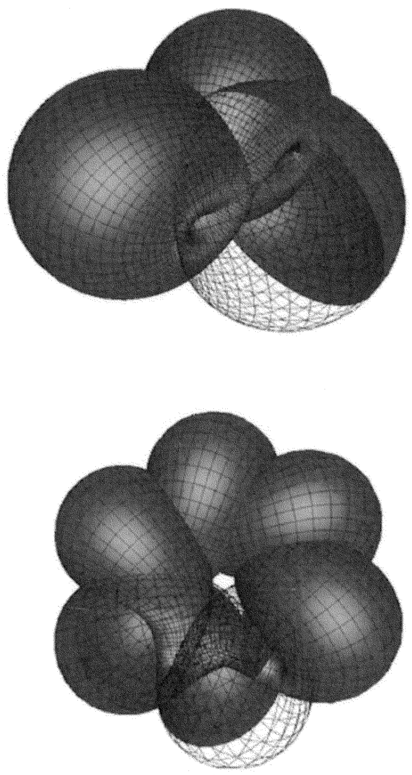

Fig. 1. Wente torus (top), twisty torus (bottom).

Remark 1. For simplicity define $\bar{\eta}_\zeta = -\eta^\dagger_{\bar{\zeta}^{-1}}$. It suffices to use $\bar{\eta}_\zeta$ instead of $\xi_\zeta(0)$ in Symes' formula, since $[\eta, \bar{\eta}] = 0$ and $\exp(z\zeta^{2g}\eta_\zeta)_E = I$ (since η_ζ is polynomial in ζ), therefore $\exp(z\zeta^{2g}\xi_\zeta(0))_E = \exp(z\zeta^{2g}\bar{\eta}_\zeta)_E$.

Moreover, by combining the extended frame with the Sym-Bobenko formula [2, 14] we can produce CMC tori once we know a choice of branch points for the spectral curve which satisfies the double periodicity condition (not just the periodicity condition above, which only makes the Gauss map periodic, but the full CMC periodicity condition described in [1]). The following examples for a genus two curve are due to Matthias Heil (private communication):

$$a_1 = 0.1413 + 0.1018i,\ a_2 = 0.1413 - 0.1018i,\ \text{(Wente torus)};$$
$$a_1 = 0.124 + 0.1485i,\ a_2 = 0.4387 - 0.071i\ \text{(twisty torus)}.$$

The corresponding CMC tori are drawn in figure 1.

Remark 2. In fact we can use the dressing construction to produce all harmonic maps with spectral data X, λ. For even though the dressing matrix (16) corresponds to the line bundle $\mathcal{O}_X(R_+)$ every other line bundle satisfying the reality condition is of the form $\mathcal{O}_X(R_+) \otimes L$ where $L \in J_R(X)$. It was shown in [17] that the full family of these is swept out by the "higher flows" described in [7]. That means an extended frame for the harmonic map with data $X, \lambda, \mathcal{O}_X(R_+) \otimes L$ is given by dressing the vacuum by

$$g_\zeta \exp\Big(\sum_{j=1}^{\infty}(t_j\zeta^j A^j - \bar{t}_j\zeta^{-j}A^{-j})\Big)$$

for some sequence $t_j \in \mathbb{C}$. Moreover, for a map of finite type only finitely many of the higher flows are independent, so there is no need for an infinite sum here. It can be shown that it suffices to have only $t_1, t_3, \ldots, t_{2g-1}$ taking any values and all other parameters zero: the first flow t_1 is just a z-translation of the surface domain.

Remark 3. By combining proposition 3 with proposition 4 we can compute the one and two bubbletons in figure 2. These have respectively $r = 1, g = 2$ and $r = 2, g = 4$. Using the previous remark we obtain a real g-parameter family of deformations of these surfaces. Each bubble can be moved relative to any other (or the cylinder) by a translation along the cylinder or a rotation about its circumference. Thus each bubble contributes two real parameters: there are r bubbles altogether. This demonstrates that $J_R(X') \cong (\mathbb{R} \times S^1)^r$. It is interesting to note that we can also think of the bubbletons as being constructed by dressing the vacuum by a rational loop on the ζ-sphere. We can re-scale the matrix g_ζ in (16) so that for a nodal curve (13) it becomes

$$\prod_{j=1}^{r}\begin{pmatrix} \bar{a}_j\zeta^2 - 1 & 0 \\ 0 & \zeta^2 - a_j \end{pmatrix},\quad r = g/2. \tag{18}$$

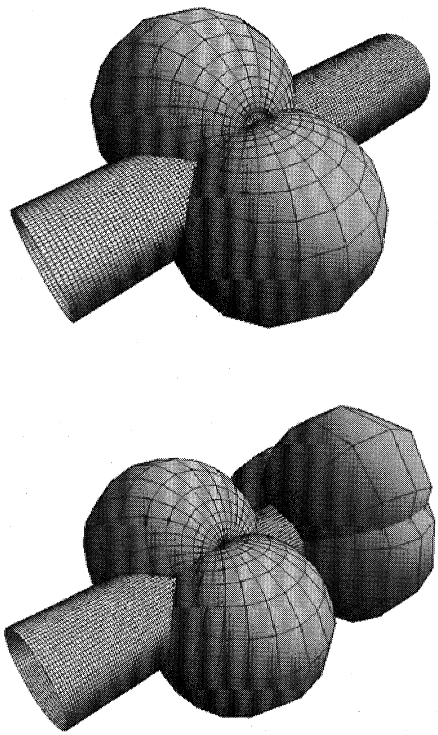

Fig. 2. One bubbleton and two bubbleton.

This dressing matrix produces the same surface and resembles a product of Bäcklund transforms in the sense of [28]. Although proposition 3 only proves that the Gauss map of the CMC surface is periodic (given the conditions (14) on each a_j) it turns out that the CMC surface itself is periodic. This can be shown by examining the effect on the monodromy matrix of $F_\zeta^{(0)}$ of dressing by any factor in the product (18). This approach was explained to me by Martin Kilian and Nick Schmitt. Their approach also explains the geometric significance of the positive integers $p_0, \ldots, p_n$ appearing in (14). The integer p_0 determines the number of times the cylindrical end of the bubbleton wraps around itself, while p_j is the number of "lobes" the j-th bubble possesses.

§2. Harmonic and pluri-harmonic maps into $Gr_k(\mathbb{C}^{n+1})$.

Let $k \leq (n+1)/2$. Here I will briefly recount the theory given in [20] for constructing pluri-harmonic maps $\varphi : \mathbb{R}^{2k} \to Gr_k(\mathbb{C}^{n+1})$ which generalizes the construction given above (recall that a map is pluri-harmonic if it is harmonic on any holomorphic curve: here $\mathbb{R}^{2k}$ is given the usual complex structure). At the end I will illustrate this with the example $X \cong \mathbb{P}^1$.

We assume that the spectral data here consists of a smooth compact Riemann surface X (of genus g) with real involution ρ together with a degree $n+1$ function λ on X and a line bundle $\mathcal{L}$ over X. We require: $\overline{\rho_*\lambda} = \lambda^{-1}$; the ramification divisor R of λ has no support over $|\lambda| = 1$; and ρ fixes every point over $|\lambda| = 1$. In that case $R = R_+ + \rho_* R_+$, where R_+ is the divisor of ramification over $|\lambda| > 1$. We can choose $\mathcal{L}$ to satisfy the reality condition $\overline{\rho_*\mathcal{L}} \cong \mathcal{L}^t(R)$ by taking any element of the compact real connected g-dimensional torus

$$\mathcal{N} = \{\mathcal{O}_X(R_+) \otimes L : \ L \in J_R\}$$

where J_R is the identity component of $\{L \in J(X) : \ L \cong \overline{\rho_* L}^{-1}\}$. It can be shown that for any such bundle $\lambda_*\mathcal{L}$ is a trivial rank $n+1$ bundle so $\dim(\Gamma(\mathcal{L})) = n+1$. Further, the trace pairing equips $\Gamma(\mathcal{L})$ with a Hermitian inner product.

As before, the geometry of the construction is best understood by working with the singularisation X' of X obtained by identifying the $n+1$ points $O_1, \ldots, O_{n+1}$ lying over $\lambda = 1$ together to obtain a nodal singularity O on X'. A line bundle $\mathcal{L}'$ over X' is best thought of as a line bundle $\mathcal{L}$ over X equipped with a linear identification of the fibres over $O_1, \ldots, O_{n+1}$: we can think of this as assigning a non-zero element to each stalk $\mathcal{L}|O_j$. In $Pic(X')$ (the algebraic group of all holomorphic line bundles over X') we distinguish the real variety

$$\mathcal{N}' = \{\mathcal{O}_{X'}(R_+) \otimes L' : L' \in J'_R\},$$

where $J'_R = \{L' \in J(X') : L' \cong \overline{\rho_* L'}^{-1}\}$. Let $\pi : Pic(X') \to Pic(X)$ be the natural epimorphism for which $\pi(\mathcal{L}') = \mathcal{L}$. An element of $\Gamma(\mathcal{L}')$ is a global section of $\mathcal{L}$ which "takes the same value" at each O_j using the fibre identification with which $\mathcal{L}'$ is equipped. For $\mathcal{L}' \in \mathcal{N}'$, since $\lambda_*\mathcal{L}$ is trivial, there is no non-zero global section of $\mathcal{L}$ which vanishes at every O_j, therefore $\dim(\Gamma(\mathcal{L}')) = 1$. Thus any non-zero global section of $\mathcal{L}'$ gives us a convenient representation for the fibre identification carried by $\mathcal{L}'$.

Over $\mathcal{N}'$ there exists a natural rank $n+1$ bundle E' whose fibre at $\mathcal{L}'$ is $\Gamma(\mathcal{L})$. For any k the Grassmann bundle $Gr_k(E')$ possesses a canonical trivialisation given pointwise as follows. Let

$$[e_1 \wedge \ldots \wedge e_k] \in Gr_k(E').$$

By taking any non-zero $s_{\mathcal{L}} \in \Gamma(\mathcal{L}')$ we can identify

$$e_j \mapsto v_j = (\frac{e_j|O_1}{s_{\mathcal{L}}|O_1}, \ldots, \frac{e_j|O_{n+1}}{s_{\mathcal{L}}|O_{n+1}}) \in \mathbb{C}^{n+1}$$

and this is projectively dependent only on $\mathcal{L}'$. Thus we have a natural map

$$[e_1 \wedge \ldots \wedge e_k] \mapsto [v_1 \wedge \ldots \wedge v_k] \in Gr_k(\mathbb{C}^{n+1}).$$

The relevance of this is that by taking a suitable section of $Gr_k(E')$ and applying this trivialisation we obtain a map $J'_R \cong \mathcal{N}' \rightarrow Gr_k(\mathbb{C}^{n+1})$ whose restriction to suitable subgroups of J'_R is (pluri)-harmonic. This result is true for any choice of isomorphism $J'_R \cong \mathcal{N}'$, so in fact we obtain not just one map but a family of them — these correspond to the deformations made available by the higher flows discussed earlier.

Now I must explain which section of $Gr_k(E')$ yields (pluri)-harmonic maps. Although we could discuss the construction of maps of any isotropy order we will stick with the simplest case of lowest isotropy order i.e. non-conformal maps. For this we take λ to have (at least) k double zeroes $P_1, \ldots, P_k$. Consequently the divisor of λ has the form

$$(\lambda) = 2P_1 + \ldots + 2P_k + E_0 - 2Q_1 - \ldots 2Q_k - E_\infty$$

where E_0, E_∞ are positive divisors of degree $n+1-2k$. Let D_∞ denote the positive divisor $Q_1 + \ldots Q_k + E_\infty$ of degree $n+1-k$, then D_∞ gives us a section of $Gr_k(E')$ by assigning to each $\mathcal{L}'$ the k-plane $\Gamma(\mathcal{L}(-D_\infty))$. Thus by our canonical trivialisation we have map

$$\psi : J'_R \cong \mathcal{N}' \rightarrow Gr_k(\mathbb{C}^{n+1}).$$

Now let $\gamma : \mathbb{R}^{2k} \rightarrow J'_R$ be the real homomorphism uniquely determined up to scalings by:

$$\partial\gamma/\partial z_j = \partial\mathcal{A}'_{P_j}/\partial\zeta_{P_j}$$

where $z_1, \ldots, z_k$ denote complex coordinates on $\mathbb{R}^{2k}$, ζ_{P_j} is a local coordinate about P_j and $\mathcal{A}'_{P_j}$ denotes the Abel map for X' with base point P_j.

Theorem 3. [20] *The map* $\varphi = \psi \circ \gamma : \mathbb{R}^{2k} \to Gr_k(\mathbb{C}^{n+1})$ *given above is pluri-harmonic. Indeed, the harmonic map obtained by restriction of* φ *to the complex line with tangent* $\sum a_j \partial/\partial z_j$ *is harmonic: it is also nowhere conformal iff* $\sum a_j^2 \neq 0$.

Remark. According to [17, 18] this theorem accounts for all non-conformal harmonic maps $\varphi : \mathbb{R}^2 \to \mathbb{CP}^n$ of semisimple finite type (and therefore all non-conformal tori). Indeed I believe it will account for all maps of semisimple finite type into $Gr_k(\mathbb{C}^{n+1})$ using a similar argument. The main unanswered question is to what extent the non-conformal (or more generally, non-isotropic) harmonic tori in $Gr_k(\mathbb{C}^{n+1})$ are accounted for by the tori of semisimple finite type. Some progress has been made in this direction (see [29]) but the problem is not yet settled.

2.1. Explicit formulae in terms of Riemann θ-functions.

In the construction above there is, up to scalings, a natural basis $e_1, \ldots, e_k$ for the k-plane $\Gamma(\mathcal{L}(-D_\infty)$. For each $j = 1, \ldots, k$ let D_j be the positive divisor $D_\infty + \sum_{k \neq j} Q_j$, which has degree n, and notice that for any j the divisor of poles of λ is $D_j + Q_j$. Since $\lambda_* \mathcal{L}$ is trivial the subspace $\Gamma(\mathcal{L}(-D_j) \subset \Gamma(\mathcal{L}(-D_\infty))$ is one dimensional and $\mathcal{L}(-D_j)$ is non-speciali of degree g. This means we can obtain a non-zero section of it using Riemann's θ-function. To obtain a formula for ψ we then have to understand the behaviour of the fibre identifications. It turns out that these can be incorporated by pulling back the θ-line bundle over $J(X)$ to $J(X')$ using π. An explicit formula for ψ is then obtained as follows. Throughout this discussion we take $\mathcal{L} = \mathcal{O}_X(R_+)$: any other choice of $\mathcal{L}$ simply amounts to a translation in the argument of the θ-function with no loss of generality.

We know that we can make identifications

$$J(X') \cong H^0(\Omega_X(\mathfrak{o}))^t / H_1(X \setminus \mathfrak{o}, \mathbb{Z}) \simeq \mathbb{C}^{g+n}/\Lambda', \tag{19}$$

where $\Omega_X(\mathfrak{o})$ is the sheaf of mermorphic differentials on X with divisor of poles no worse than $\mathfrak{o} = O_1 + \ldots + O_{n+1}$ and Λ' is a lattice on $2g+n$ generators. We choose coordinates so that $\pi : J(X') \to J(X)$ is covered by the map

$$\pi : \mathbb{C}^{g+n} \to \mathbb{C}^g; \quad \tilde{W} = (w_1, \ldots, w_{g+n}) \mapsto W = (w_1, \ldots, w_g).$$

Now let us define $\theta_0(\tilde{W}) = \theta(W)$ and for $j = 1, \ldots, n$ define

$$\theta_j(\tilde{W}) = \exp(2\pi i w_{g+j})\theta(W + \mathcal{A}(O_{j+1} - O_1)),$$

where θ is the classical Riemann θ-function on $\mathbb{C}^g$ corresponding to the induced isomorphism $J(X) \simeq \mathbb{C}^g/\pi(\Lambda')$. Each of $\theta_0, \dots, \theta_n$ represents a global holomorphic section of the pullback by π of the θ-line bundle over $J(X)$ [20].

For $l = 1, \dots, k$ let $\tilde{D}_l$ be the unique positive divisor (of degree g) in the linear system of $\mathcal{L}(-D_l)$ and let $\kappa_l \in \mathbb{C}^g$ be the appropriate translation for which $\theta(\mathcal{A}(P) + \kappa_l)$ has divisor of zeroes $\tilde{D}_l$. Finally, let f_l be a rational function on X with divisor $(f_l) = R_+ - D_l - \tilde{D}_l$ so that $f_l(P)\theta(\mathcal{A}(P) + \kappa_l)$ has divisor $R_+ - D_l$.

Proposition 5. [20] *Let $v_l : \mathbb{C}^{g+n} \to \mathbb{C}^{n+1}$ be defined by*

$$v_l(\tilde{W}) = (f_l(O_1)\theta_0(\tilde{W} + \kappa_l), \dots, f_l(O_{n+1})\theta_n(\tilde{W} + \kappa_l))$$

Then, taking the base point $\mathcal{O}_{X'}(R_+)$ on $\mathcal{N}'$ for the identification $J'_R \cong \mathcal{N}'$, the map $\psi : J'_R \to Gr_k(\mathbb{C}^{n+1})$ above is explicitly given by the Λ'-periodic map

$$\psi(\tilde{W}) = [v_1(\tilde{W}) \wedge \dots \wedge v_k(\tilde{W})].$$

An explicit formula for the function f can be obtained using Fay's prime form (see e.g. [23]). It remains to combine this with the real homomorphism $\gamma : \mathbb{R}^{2k} \to \mathbb{C}^{g+n}/\Lambda'$ which we have essentially computed earlier (cf. [20]). For illustration I will do these calculations explicitly for $X \cong \mathbb{P}^1$ in the next section.

2.2. Example: X is the Riemann sphere.

Let ζ be a rational parameter on $X \cong \mathbb{P}^1$ and define the real involution to be $\rho_*\zeta = \bar{\zeta}^{-1}$, then to satisfy all our conditions λ must be of the form

$$\lambda = \alpha \prod_{j=1}^{k} \frac{(\zeta - P_j)^2}{(\zeta - \bar{P}_j^{-1})^2} \prod_{i=1}^{n+1-2k} \frac{(\zeta - E_j)}{(\zeta - \bar{E}_j^{-1})}, \tag{20}$$

where the points $P_1, \dots, P_k, E_1, \dots, E_{n+1-2k}$ all lie inside $|\zeta| < 1$ (cf. [27]). The constant α is chosen so that $|\lambda| = 1$ over $|\zeta| = 1$.

First we construct the homomorphism $\gamma : \mathbb{R}^{2k} \to J'_R$. To fix the isomorphism (19) we choose the basis $\omega_1, \dots, \omega_n$ of $H^0(\Omega(\mathfrak{o}))$ given by

$$\omega_m = \frac{1}{2\pi i}\left(\frac{1}{\zeta - O_{m+1}} - \frac{1}{\zeta - O_1}\right)d\zeta, \quad m = 1, \dots, n.$$

Let $a_m \in H_1(X \setminus \mathfrak{o}, \mathbb{Z})$ be the class of a positively oriented cycle about O_{m+1} only, so that $\oint_{a_l} \omega_m = \delta_{lm}$. With these bases we have

$$J(X') \cong \mathbb{C}^n/\mathbb{Z}^n \stackrel{\exp(2\pi i \cdot)}{\rightarrow} (\mathbb{C}^*)^n.$$

Take $\zeta_{P_j} = \zeta - P_j$ for the local parameter about P_j and recall from earlier that as an element of $H^0(\Omega(\mathfrak{o}))^t \cong T_0 J(X')$

$$\frac{\partial \mathcal{A}'_{P_j}}{\partial \zeta_{P_j}} : \omega_m \mapsto res_{P_j} \zeta_{P_j}^{-1} \omega_m.$$

In our coordinates this is the vector $\frac{1}{2\pi i} U_j$ where $U_j \in \mathbb{C}^n$ has m-th coordinate

$$U_{jm} = \frac{1}{P_j - O_m} - \frac{1}{P_j - O_1}.$$

The map $\gamma : \mathbb{R}^{2k} \rightarrow (\mathbb{C}^*)^n$ is given by

$$\gamma(z_1, \ldots, z_k) = \exp(\sum_{j=1}^{k} (U_j z_j - \bar{U}_j \bar{z}_j)).$$

Now to apply proposition 5 we notice that since $J(X)$ is the trivial group we can take $\theta \equiv 1$. So for $\tilde{W} = (w_1, \ldots, w_n)$ we have simply

$$\theta_0(\tilde{W}) = 1, \ \theta_1(\tilde{W}) = \exp(2\pi i w_1), \ldots, \ \theta_n(\tilde{W}) = \exp(2\pi i w_n).$$

Finally, we need the divisors

$$D_l = 2Q_1 + \ldots + Q_l + \ldots 2Q_k + E_\infty, \quad l = 1, \ldots, k,$$

where $Q_j = \bar{P}_j^{-1}$ and $E_\infty = \bar{E}_1^{-1} + \ldots + \bar{E}_{n+1-2k}^{-1}$. Let f_l be any rational function with divisor $R_+ - D_l$ and define $v_l : \mathbb{C}^k \rightarrow \mathbb{C}^{n+1}$ by

(21) $$v_l(z_1, \ldots, z_k) = (f_l(O_1), f_l(O_2)\gamma_1, \ldots, f_l(O_{n+1})\gamma_n)$$

where $\gamma_m = \exp(\sum_{j=1}^k (z_j U_{jm} - \bar{z}_j \bar{U}_{jm})$.

Proposition 6. [20] *The pluri-harmonic map $\varphi : \mathbb{R}^{2k} \rightarrow Gr_k(\mathbb{C}^{n+1})$ with spectral data $X \cong \mathbb{P}^1$ and λ given by (20) is given by*

$$\varphi(z_1, \ldots, z_k) = [v_1 \wedge \ldots \wedge v_k].$$

This map is totally equivariant i.e. it can be framed by a homomorphism $\mathbb{R}^{2k} \rightarrow U_{n+1}$.

By a result of Kenmotsu [13] (see also [3]) the minimal (i.e. conformal harmonic) totally equivariant maps $\mathbb{R}^2 \to \mathbb{CP}^n$ include those minimal totally real maps which are isometric for the flat metric on $\mathbb{R}^2$. A study of their periodicity can be found in [12]. To pass from non-conformal to conformal maps in our construction (in the case $k = 1$ i.e. $\mathbb{CP}^n$) one insists that λ has a zero of degree 3 at P_1. In particular, this requires $n \geq 2$.

Remark. There is a geometric interpretation behind the form of v_l. Suppose $\varphi : \mathbb{R}^{2k} \to Gr_k(\mathbb{C}^{n+1})$ is totally equivariant with frame

$$F = \exp(z \cdot A - \bar{z} \cdot A^\dagger), \quad z \cdot A = \sum_{j=1}^{k} z_j A_j,$$

where $A_1, \ldots, A_k \in \mathfrak{gl}_{n+1}(\mathbb{C})$ are mutually commuting normal matrices. We will assume φ is based so that $\varphi(0) = [e_1 \wedge \ldots \wedge e_k]$ where the e_j are the standard basis vectors for $\mathbb{C}^{n+1}$. The matrices A_j and their Hermitian transposes may be simultaneously diagonalized by a unitary matrix: $MA_jM^{-1} = D_j$ where M is unitary and each D_j is diagonal. Therefore

$$\begin{aligned} M \circ \varphi &= MFM^{-1}M \circ [e_1 \wedge \ldots \wedge e_k] \\ &= \exp(z \cdot D - \bar{z} \cdot \bar{D}) \circ [u_1 \wedge \ldots \wedge u_k] \end{aligned}$$

where $u_1, \ldots, u_k$ are the first k columns of M. Thus $M \circ \varphi = [v_1 \wedge \ldots \wedge v_k]$ where $v_l = \exp(z \cdot D - \bar{z} \cdot \bar{D}) \circ u_l$. Notice that this is essentially the form of the map we derived above, using

$$D_j = \mathrm{diag}(1, U_{j1}, \ldots, U_{jn}), \; j = 1, \ldots, k.$$

The v_l appearing in (21) span the same k-plane as these but are not necessarily orthonormal.

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