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Quantum product, topological recursion relations, and the Virasoro conjecture

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The Virasoro conjecture predicts that the generating function of the Gromov-Witten invariants is annihilated by infinitely many differential operators which form a half branch of the Virasoro algebra. This conjecture was proposed by Eguchi, Hori and Xiong [EHX] and S. Katz (cf. [CK] [EJX]). It is a natural generalization of a conjecture of Witten (cf. [W2] [Ko] [W2]) and provides a powerful tool in the computation of Gromov-Witten invariants. The genus-0 Virasoro conjecture was proved in [LT] (cf. [DZ] and [G3] for alternative proofs). The genus-1 Virasoro conjecture for manifolds with semisimple quantum cohomology was proved in [DZ]. Without assuming semisimplicity, the genus-1 Virasoro conjecture was reduced to the genus-1 L_1-constraint on the small phase space in [L1]. It was also proved in [L1] that the genus-1 Virasoro conjecture holds if the quantum cohomology is not too degenerate (a condition weaker than semisimplicity). The essential part of the results in [L1] was extended to the genus-2 Virasoro conjecture in [L2]. The study of the genus-2 Virasoro conjecture is important because this is the first case where we do not have a formula to reduce the problem to the small phase space. The behavior of the Virasoro conjecture in this case will provide much needed insight in what we should expect in the higher genera cases. The techniques developed in [L2] could be easily adapted to the study of the higher genera Virasoro conjecture.

In this expository article, we will explain how to apply the main ideas in [L2] to the study of the Virasoro conjecture in all genera. In particular, we will explain how to use the quantum product on the big

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phase space to interpret topological recursion relations and the Virasoro conjecture. The quantum product on the big phase space does not have an identity element. The string vector field is the closest vector field to an identity. In our view point, various topological recursion relations are just different ways to express how close the string vector field is to an identity for the quantum product on the big phase space. Using such topological recursion relations, we can rephrase the Virasoro conjecture in terms of quantum powers of the Euler vector fields. More precisely, for $g \geq 1$, the genus-g Virasoro constraints compute the derivatives of the genus-g generating function along a sequence of vector fields constructed from (twisted) quantum powers of the Euler vector field. When the quantum cohomology is semisimple, quantum powers of the Euler vector field span the space of the primary vector fields. Therefore in this case, the Virasoro conjecture is strong enough to determine the genus-g generating function in terms of data of genus less than g. Such an interpretation of the Virasoro conjecture is very useful both in resolving the conjecture and in applying it to solve other problems.

To keep the article brief, we will emphasize the main ideas and omit most proofs. Interested readers are referred to [L2] for more details. The major exception to this rule is a complete proof of the genus-0 Virasoro conjecture, which simplifies the arguments in [LT]. The argument presented here is new and has not appeared elsewhere.

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§1. Quantum product on the big phase space and topological recursion relations

1.1. Gromov-Witten invariants

For simplicity, we assume that V is a smooth projective variety with $H^{\text{odd}}(V;\mathbb{C}) = 0$. All results in this paper should also be true for compact symplectic manifolds except those concerning Virasoro constraints of genus bigger than 0. Gromov-Witten invariants are defined via the intersection theory of moduli spaces of stable maps from Riemann surfaces to V. For any element $A \in H_2(V,\mathbb{Z})$ and non-negative integers g and k, the moduli space $\overline{\mathcal{M}}_{g,k}(V,A)$ is defined to be the collection of all data $(C; x_1, \ldots, x_k; f)$ where C is a genus-g projective connected curve over $\mathbb{C}$ whose only possible singularities are simple double points, where $x_1, \ldots, x_k$ are smooth points on C (called marked points), and when f is an algebraic map from C to V which is stable with respect to $(C; x_1, \ldots, x_k)$, (i.e. there is no infinitesimal deformation for this data). Each marked point x_i defines a map, called the i-th evaluation map,

$$\begin{array}{cccc} ev_i : & \overline{\mathcal{M}}_{g,k}(V,A) & \longrightarrow & V \\ & (C; x_1, \ldots, x_k; f) & \longmapsto & f(x_i). \end{array}$$

It also defines a line bundle over $\overline{\mathcal{M}}_{g,k}(V,A)$, denoted by E_i, whose fiber over $(C; x_1, \ldots, x_k; f)$ is $T^*_{x_i}C$. For any cohomology classes $\gamma_1, \ldots, \gamma_k \in H^*(V, \mathbb{C})$ and non-negative integers $n_1, \ldots, n_k$, the corresponding descendant Gromov-Witten invariants are defined by

$$\langle \tau_{n_1}(\gamma_1) \cdots \tau_{n_k}(\gamma_k) \rangle_g := \\ \sum_A q^A \int_{[\overline{\mathcal{M}}_{g,k}(V,A)]^{\mathrm{virt}}} c_1(E_1)^{n_1} \cup \mathrm{ev}_1^*(\gamma_1) \cup \cdots \cup c_1(E_k)^{n_k} \cup \mathrm{ev}_k^*(\gamma_k),$$

where $[\overline{\mathcal{M}}_{g,k}(V,A)]^{\mathrm{virt}}$ is the virtual fundamental class of $\overline{\mathcal{M}}_{g,k}(V,A)$ (cf. [LiT]) and q^A belongs to the Novikov ring (i.e. the multiplicative ring spanned by monomials $q^A = q_1^{a_1} \cdots q_r^{a_r}$ over the ring of rational numbers, where $\{q_1, \cdots, q_r\}$ is a fixed basis of $H_2(V, \mathbb{Z})$ and $A = \sum_{i=1}^r a_i q_i$). When all the n_i are zero, the corresponding invariants are called primary Gromov-Witten invariants.

1.2. Notational conventions

We will use d to denote the complex dimension of V and N the dimension of the space of cohomology classes $H^*(V, \mathbb{C})$. To define the generating functions, we need to fix a basis $\{\gamma_1, \ldots, \gamma_N\}$ of $H^*(V, \mathbb{C})$ with γ_1 equal to the identity for the cohomology ring of V and $\gamma_\alpha \in H^{p_\alpha, q_\alpha}(V, \mathbb{C})$ for every α. We also arrange the basis in such a way that the dimension of γ_α is non-decreasing with respect to α, and, if two cohomology classes have the same dimension, we also require that the holomorphic dimension p_α is non-decreasing.

Lower case Greek characters will be used to index the cohomology classes. The range of these indices is from 1 to N. Lower case Roman characters will be used to index the level of descendants. Their range is the set of all non-negative integers, $\mathbb{Z}_+$. All summations are over the entire ranges of the indices unless otherwise indicated. Let

$$\eta_{\alpha\beta} = \int_V \gamma_\alpha \cup \gamma_\beta$$

be the intersection form on $H^*(V,\mathbb{C})$. We will use $\eta = (\eta_{\alpha\beta})$ and $\eta^{-1} = (\eta^{\alpha\beta})$ to lower and raise indices. For example, $\gamma^\alpha := \eta^{\alpha\beta}\gamma_\beta$. Here we are using the summation convention that repeated indices (in this formula, β) are summed over their entire ranges. Let $\mathcal{C} = (\mathcal{C}_\alpha^\beta)$ be the matrix of multiplication by the first Chern class $c_1(V)$ in the ordinary cohomology ring, i.e. $c_1(V) \cup \gamma_\alpha = \mathcal{C}_\alpha^\beta \gamma_\beta$. Since we are dealing only with even dimensional cohomology classes, the $\mathcal{C}^k\eta$ are symmetric matrices for all $k \geq 0$, where the entries of $\mathcal{C}^k\eta$ are given by

$$(\mathcal{C}^k)_{\alpha\beta} = \int_V c_1(V)^k \cup \gamma_\alpha \cup \gamma_\beta.$$

Let $b_\alpha = p_\alpha - \frac{1}{2}(d-1)$. The following simple observations will be used throughout the calculations without mention: If $\eta^{\alpha\beta} \neq 0$ or $\eta_{\alpha\beta} \neq 0$, then $b_\alpha = 1 - b_\beta$. $\mathcal{C}_\alpha^\beta \neq 0$ implies $b_\beta = 1 + b_\alpha$, and $\mathcal{C}_{\alpha\beta} \neq 0$ implies $b_\beta = -b_\alpha$.

1.3. Generating functions and correlation functions

The genus-g generating function is defined to be

$$F_g = \sum_{k\geq 0} \frac{1}{k!} \sum_{\substack{\alpha_1,\ldots,\alpha_k \\ n_1,\ldots,n_k}} t_{n_1}^{\alpha_1} \cdots t_{n_k}^{\alpha_k} \langle \tau_{n_1}(\gamma_{\alpha_1}) \tau_{n_2}(\gamma_{\alpha_2}) \ldots \tau_{n_k}(\gamma_{\alpha_k}) \rangle_g,$$

where $\{t_n^\alpha \mid n \in \mathbb{Z}_+, \alpha = 1, \cdots, N\}$ is an infinite set of parameters. We can think of these parameters as coordinates on an infinite dimensional vector space, called the *big phase space.* The finite dimensional subspace defined by $\{t_n^\alpha = 0 \text{ if } n > 0\}$ is called the *small phase space.* The function F_g is to be understood as a formal power series of t_n^α. For convenience, we will always identify the symbol $\tau_n(\gamma_\alpha)$ with the tangent vector field $\frac{\partial}{\partial t_n^\alpha}$ on the big phase space. We also consider $\tau_n(\gamma_\alpha)$ with $n < 0$ as the zero operator. For each α, we will abbreviate $\tau_0(\gamma_\alpha)$ as γ_α. We call a (formal) vector field $\mathcal{W} = \sum_{m,\alpha} f_{m,\alpha}\tau_m(\gamma_\alpha)$ a *primary vector field* if $f_{m,\alpha} = 0$ whenever $m > 0$, a *descendant vector field* if $f_{m,\alpha} = 0$ whenever $m = 0$.

Instead of coordinates $\{t_m^\alpha \mid m \in \mathbb{Z}_+, \alpha = 1, \ldots, N\}$, it is very convenient to use the following shifted coordinates on the big phase space

$$\tilde{t}_m^\alpha = t_m^\alpha - \delta_{m,1}\delta_{\alpha,1} = \begin{cases} t_m^\alpha - 1, & \text{if } m = \alpha = 1, \\ t_m^\alpha, & \text{otherwise.} \end{cases}$$

As in [LT], it is convenient to introduce a k-tensor $\langle\langle \underbrace{\cdots\cdots}_{k} \rangle\rangle$ defined by $\langle\langle \mathcal{W}_1\mathcal{W}_2\cdots\mathcal{W}_k \rangle\rangle_g :=$

$$\sum_{m_1,\alpha_1,\ldots,m_k,\alpha_k} f^1_{m_1,\alpha_1}\cdots f^k_{m_k,\alpha_k} \frac{\partial^k}{\partial t^{\alpha_1}_{m_1}\partial t^{\alpha_2}_{m_k}\cdots\partial t^{\alpha_k}_{m_k}} F_g,$$

for (formal) vector fields $\mathcal{W}_i = \sum_{m,\alpha} f^i_{m,\alpha}\frac{\partial}{\partial t^{\alpha}_m}$ where the $f^i_{m,\alpha}$ are (formal) functions on the big phase space. We can also view this tensor as the k-th covariant derivative of F_g. This tensor is called the *k-point (correlation) function.*

1.4. Quantum product

For any vector fields $\mathcal{U}$ and $\mathcal{W}$ on the big phase, define the **quantum product** of $\mathcal{U}$ and $\mathcal{W}$ by

$$\mathcal{U}\bullet\mathcal{W} := \langle\langle \mathcal{U}\mathcal{W}\gamma^{\alpha}\rangle\rangle_0\,\gamma_{\alpha}.$$

By definition, the quantum product of two vector fields is always a primary vector field. This product is evidently commutative. It is also associative due to the *generalized WDVV equation*

$$\langle\langle \mathcal{W}_1\,\mathcal{W}_2\,\gamma^{\alpha}\rangle\rangle_0\,\langle\langle \gamma_{\alpha}\,\mathcal{W}_3\,\mathcal{W}_4\rangle\rangle_0 = \langle\langle \mathcal{W}_1\,\mathcal{W}_3\,\gamma^{\alpha}\rangle\rangle_0\,\langle\langle \gamma_{\alpha}\,\mathcal{W}_2\,\mathcal{W}_4\rangle\rangle_0\,,$$

which follows in turn from the genus-0 topological recursion relation

$$\langle\langle \tau_m(\gamma_\alpha)\;\tau_n(\gamma_\beta)\;\tau_k(\gamma_\mu)\rangle\rangle_0 = \langle\langle \tau_{m-1}(\gamma_\alpha)\;\gamma_\sigma\rangle\rangle_0\,\langle\langle \gamma^{\sigma}\;\tau_n(\gamma_\beta)\;\tau_k(\gamma_\mu)\rangle\rangle_0$$

for $m > 0$ (cf. [W1]). When restricted to tangent vector fields on the small phase, this is precisely the product in the quantum cohomology of V (called the big quantum cohomology by some authors). For any vector field $\mathcal{W}$ on the big phase space, we define $\mathcal{W}^k$ to be the k-th quantum power of $\mathcal{W}$. i.e., $\mathcal{W}^k = \underbrace{\mathcal{W}\bullet\mathcal{W}\bullet\cdots\mathcal{W}}_{k}$, for $k > 0$.

For the quantum product on the small phase space, the constant vector field γ_1, which was chosen to be the identity for the ordinary cohomology ring, is also the identity for the quantum cohomology. However on the big phase space, there is no identity vector field for the quantum product. Instead, the *string vector field*

$$\mathcal{S} = -\sum_{m,\alpha}\tilde{t}^{\alpha}_m\tau_{m-1}(\gamma_\alpha)$$

can be considered as a sort of identity in the following sense: First, the *string equation* can be written as

$$\langle\!\langle \mathcal{S} \rangle\!\rangle_g = \frac{1}{2}\delta_{g,0}\eta_{\alpha\beta}t_0^\alpha t_0^\beta.$$

Taking second derivatives of the genus-0 string equation, we obtain

$$\langle\!\langle \gamma_\alpha \mathcal{S} \gamma^\beta \rangle\!\rangle_0 = \delta_\alpha^\beta$$

for any α and β. This equation can be interpreted as $\mathcal{S} \bullet \mathcal{W} = \mathcal{W}$ for any primary vector field $\mathcal{W}$. Second, since $\mathcal{U} \bullet \mathcal{W}$ is always a primary field, the associativity of the quantum product implies

$$\mathcal{S} \bullet \mathcal{U} \bullet \mathcal{W} = \mathcal{U} \bullet \mathcal{W} \tag{1}$$

for all vector fields $\mathcal{U}$ and $\mathcal{W}$. Define $\overline{\mathcal{W}} := \mathcal{S} \bullet \mathcal{W}$ for any vector field $\mathcal{W}$. Then

$$\mathcal{W} \bullet \mathcal{V} = \overline{\mathcal{W}} \bullet \mathcal{V} \tag{2}$$

for any vector field $\mathcal{V}$. If we restrict the quantum product to the space of primary vector fields, then it has an identity $\overline{\mathcal{S}}$.

1.5. Topological recursion relations

In some sense, the topological recursion relations tells us how far $\mathcal{S}$ is from being an identity for the quantum product on the big phase space. To describe this interpretation, we introduce the following linear transformations (that is, linear with respect to multiplication by functions) on the space of vector fields on the big phase space:

Definition 1.1. For any vector field $\mathcal{W} = \sum_{n,\alpha} f_{n,\alpha}\tau_n(\gamma_\alpha)$, define

$$\tau_+(\mathcal{W}) := \sum_{n,\alpha} f_{n,\alpha}\tau_{n+1}(\gamma_\alpha), \qquad \tau_-(\mathcal{W}) := \sum_{n,\alpha} f_{n,\alpha}\tau_{n-1}(\gamma_\alpha),$$

$$T(\mathcal{W}) := \tau_+(\mathcal{W}) - \langle\!\langle \mathcal{W}\gamma^\alpha \rangle\!\rangle_0 \gamma_\alpha, \qquad \pi(\mathcal{W}) := \sum_\alpha f_{0,\alpha}\gamma_\alpha.$$

Then $\tau_+\tau_-(\mathcal{W}) = \mathcal{W} - \pi(\mathcal{W})$, $\tau_-\tau_+(\mathcal{W}) = \tau_- T(\mathcal{W}) = \mathcal{W}$. Moreover the second derivatives of the genus-0 string equation have the following form (cf. [LT, Lemma 1.1 (3)]):

$$\langle\!\langle \mathcal{W}\,\mathcal{S}\,\mathcal{V} \rangle\!\rangle_0 = \langle\!\langle \tau_-(\mathcal{W})\,\mathcal{V} \rangle\!\rangle_0 + \langle\!\langle \mathcal{W}\,\tau_-(\mathcal{V}) \rangle\!\rangle_0 + \frac{1}{2}\nabla^2_{\mathcal{W},\mathcal{V}}\eta_{\alpha\beta}t_0^\alpha t_0^\beta \tag{3}$$

where $\mathcal{W}$ and $\mathcal{V}$ are arbitrary vector fields and $\nabla^2_{\mathcal{W},\mathcal{V}} = \nabla_{\mathcal{W}}\nabla_{\mathcal{V}} - \nabla_{\nabla_{\mathcal{W}}\mathcal{V}}$ is the second covariant derivative. In particular, this implies that

$$(4) \qquad \overline{\mathcal{W}} = \langle\!\langle \tau_-(\mathcal{W})\gamma^\alpha \rangle\!\rangle_0 \gamma_\alpha + \pi(\mathcal{W}).$$

An immediate consequence of this formula is the following:

$$(5) \qquad T(\mathcal{W}) = \tau_+(\mathcal{W}) - \mathcal{S} \bullet \tau_+(\mathcal{W}).$$

The genus-0 topological recursion relation implies

$$(6) \qquad T(\mathcal{W}) \bullet \mathcal{V} = 0$$

for any vector fields $\mathcal{W}$ and $\mathcal{V}$. The converse of this equation is also true (cf. [L2]), i.e.

$$\mathcal{W} \bullet \mathcal{V} = 0 \text{ for all } \mathcal{V} \iff \mathcal{W} = T(\mathcal{U}) \text{ for some } \mathcal{U}.$$

It follows from the first derivatives of the genus-0 string equation that $T(\mathcal{S}) = \mathcal{D}$ where $\mathcal{D}$ is the *dilaton vector field* $\mathcal{D} = -\sum_{m,\alpha} \tilde{t}^\alpha_m \, \tau_m(\gamma_\alpha)$. The genus-0 *dilaton equation* and its first two derivatives have the following form (cf. [LT, Lemma 1.2])

$$(7) \quad \langle\!\langle \mathcal{D} \rangle\!\rangle_0 = -2F_0, \qquad \langle\!\langle \mathcal{D}\mathcal{W} \rangle\!\rangle_0 = -\langle\!\langle \mathcal{W} \rangle\!\rangle_0, \qquad \langle\!\langle \mathcal{D}\mathcal{W}\mathcal{V} \rangle\!\rangle_0 = 0$$

for all vector fields $\mathcal{W}$ and $\mathcal{V}$. Equation (6) in particular implies that $\mathcal{D}\bullet\mathcal{W} = 0$ for all $\mathcal{W}$, which also follows from (7). Moreover the equation $T(\mathcal{D})\bullet\mathcal{W} = 0$ for all $\mathcal{W}$ is equivalent to [LT, Lemma 5.2 (2)], which is also equivalent to the genus-0 $\widetilde{\mathcal{L}}_1$ constraint. Similar reasoning also applies to the genus-0 $\widetilde{\mathcal{L}}_2$ constraint by considering the vector field $T(R(\mathcal{D}))$ where R is defined in Definition 2.2.

Equation (6) has the following generalization, which is obtained by taking derivatives of the equation $T(\mathcal{W}) \bullet \mathcal{V} = 0$,

$$(8) \qquad \left\langle\!\!\left\langle T^k(\mathcal{W})\, \mathcal{V}_1 \cdots \mathcal{V}_{k+1} \right\rangle\!\!\right\rangle_0 = 0$$

for any vector fields $\mathcal{W}$ and $\mathcal{V}_1, \ldots, \mathcal{V}_{k+1}$ with $k \geq 1$. Note that the power of T in this formula is sharp in the sense that if we replace T^k by T^i for $i < k$, the formula is no longer correct. The same is also true for (10). For example,

$$\langle\!\langle T(\mathcal{V})\, \mathcal{W}_1\, \mathcal{W}_2\, \gamma^\alpha \rangle\!\rangle_0 \gamma_\alpha = \mathcal{W}_1 \bullet \mathcal{W}_2 \bullet \mathcal{V}$$

for any vector fields $\mathcal{W}_1$, $\mathcal{W}_2$, and $\mathcal{V}$ (cf. [L2, Corollary 1.6]).

If $\mathcal{S}$ were an identity for the quantum product on the big phase space, $T(\mathcal{W})$ would always be a zero vector field because of (5). However, from the definition of T, we know $T(\mathcal{W}) \neq 0$ unless $\mathcal{W} = 0$. On the other hand, equation (6) tells us that at the genus-0 level, the operator T always annihilates vector fields from the viewpoint of the quantum product. This equation can be thought of as another way to interpret the genus-0 topological recursion relation. Similar interpretations can also be given for the higher genus topological recursion relations.

The genus-1 topological recursion relation is the following (cf. [DW]):

$$\langle\langle \tau_{n+1}(\gamma_\alpha) \rangle\rangle_1 = \langle\langle \tau_n(\gamma_\alpha)\, \gamma^\mu \rangle\rangle_0 \, \langle\langle \gamma_\mu \rangle\rangle_1 + \frac{1}{24} \langle\langle \tau_n(\gamma_\alpha)\, \gamma^\mu\, \gamma_\mu \rangle\rangle_0 .$$

This formula is equivalent to

$$\langle\langle T(\mathcal{W}) \rangle\rangle_1 = \frac{1}{24} \langle\langle \mathcal{W}\, \gamma^\mu\, \gamma_\mu \rangle\rangle_0 \tag{9}$$

for any vector field $\mathcal{W}$. For $g > 0$, we call a vector field $\mathcal{W}$ *trivial at the genus-g level* if $\langle\langle \mathcal{W} \rangle\rangle_g$ can be represented by data of genera less than g. Then the genus-1 topological recursion relation just means that $T(\mathcal{W})$ is trivial at the genus-1 level for all $\mathcal{W}$.

Equation (9) and its derivatives imply the following

$$\langle\langle T^{k+2}(\mathcal{W})\, \mathcal{V}_1 \cdots \mathcal{V}_k \rangle\rangle_1 = 0 \tag{10}$$

for $k \geq 0$ and any vector fields $\mathcal{W}$, $\mathcal{V}_1, \ldots, \mathcal{V}_k$.

The genus-2 topological recursion relations are much more complicated than genus-0 and genus-1 topological recursion relations. Two genus-2 topological recursion relations were given in [G2]. The first one was derived by using a formula due to Mumford:

$$\begin{aligned}
\langle\langle \tau_{i+2}(x) \rangle\rangle_2 \;=\; & \langle\langle \tau_{i+1}(x)\, \gamma^\alpha \rangle\rangle_0 \, \langle\langle \gamma_\alpha \rangle\rangle_2 + \langle\langle \tau_i(x)\, \gamma^\alpha \rangle\rangle_0 \, \langle\langle \tau_1(\gamma_\alpha) \rangle\rangle_2 \\
& - \langle\langle \tau_i(x)\, \gamma^\alpha \rangle\rangle_0 \, \langle\langle \gamma_\alpha\, \gamma^\beta \rangle\rangle_0 \, \langle\langle \gamma_\beta \rangle\rangle_2 + \frac{7}{10} \langle\langle \tau_i(x)\, \gamma^\alpha\, \gamma^\beta \rangle\rangle_0 \, \langle\langle \gamma_\alpha \rangle\rangle_1 \, \langle\langle \gamma_\beta \rangle\rangle_1 \\
& + \frac{1}{10} \langle\langle \tau_i(x)\, \gamma^\alpha\, \gamma^\beta \rangle\rangle_0 \, \langle\langle \gamma_\alpha\, \gamma_\beta \rangle\rangle_1 - \frac{1}{240} \langle\langle \tau_i(x)\, \gamma_\alpha \rangle\rangle_1 \, \langle\langle \gamma^\alpha\, \gamma_\beta\, \gamma^\beta \rangle\rangle_0 \\
& + \frac{13}{240} \langle\langle \tau_i(x)\, \gamma_\alpha\, \gamma^\alpha\, \gamma^\beta \rangle\rangle_0 \, \langle\langle \gamma_\beta \rangle\rangle_1 + \frac{1}{960} \langle\langle \tau_i(x)\, \gamma^\alpha\, \gamma_\alpha\, \gamma^\beta\, \gamma_\beta \rangle\rangle_0 .
\end{aligned}$$

This formula can be written in the following form: For any vector field $\mathcal{W}$ on the big phase space,

$$\begin{aligned} \langle\!\langle T^2(\mathcal{W}) \rangle\!\rangle_2 = & \frac{7}{10} \langle\!\langle \gamma_\alpha \rangle\!\rangle_1 \langle\!\langle \{\gamma^\alpha \bullet \mathcal{W}\} \rangle\!\rangle_1 + \frac{1}{10} \langle\!\langle \gamma_\alpha \{\gamma^\alpha \bullet \mathcal{W}\} \rangle\!\rangle_1 \\ & - \frac{1}{240} \langle\!\langle \mathcal{W} \{\gamma_\alpha \bullet \gamma^\alpha\} \rangle\!\rangle_1 + \frac{13}{240} \langle\!\langle \mathcal{W} \gamma_\alpha \gamma^\alpha \gamma^\beta \rangle\!\rangle_0 \langle\!\langle \gamma_\beta \rangle\!\rangle_1 \\ & + \frac{1}{960} \langle\!\langle \mathcal{W} \gamma^\alpha \gamma_\alpha \gamma^\beta \gamma_\beta \rangle\!\rangle_0, \end{aligned} \tag{11}$$

where $T^2(\mathcal{W}) := T(T(\mathcal{W}))$. Another topological recursion relation in [G2] is a formal consequence of this formula and a formula in [BP] (cf. [L2]).

Unlike in the genus-1 case, $T(\mathcal{W})$ is no longer trivial at the genus-2 level in general. In fact, the genus-2 dilaton equation implies $\langle\!\langle T(\mathcal{S}) \rangle\!\rangle_2 = \langle\!\langle \mathcal{D} \rangle\!\rangle_2 = 2F_2$. Unless F_2 can be expressed as a function of F_0 and F_1, $T(\mathcal{S})$ is not trivial at the genus-2 level. However, the topological recursion relation (11) tells us that $T^2(\mathcal{W})$ is trivial at the genus-2 level for all vector fields $\mathcal{W}$. The topological recursion relations for genus $g = 1, 2$ are derived by using a formula for expressing the tautological class ψ_1^g (i.e. $c_1(E_1)^g$ according to the notation in Section 1.1) on the moduli space of stable curves $\overline{\mathcal{M}}_{g,1}$ in terms of boundary classes. It was conjectured in [G2] that polynomials of degree g in the tautological classes ψ_i are boundary classes on $\overline{\mathcal{M}}_{g,n}$. This conjecture was proved in [Io]. A somewhat stronger version of a special case of this conjecture would be that ψ_1^g is equal to a boundary class in $\overline{\mathcal{M}}_{g,1}$ without genus-g components. This would imply the following:

(12) $T^g(\mathcal{W})$ is trivial at the genus$-g$ level for $g \geq 1$.

The following topological recursion relation for all $g \geq 1$ was derived in [EX] under the assumption that the genus-g generating function is a function of the derivatives of genus-0 generating function:

$$\langle\!\langle \tau_{n+3g-1}(\gamma_\alpha) \rangle\!\rangle_g = \sum_{j=0}^{3g-2} \langle\!\langle \tau_{n+3g-2-j}(\gamma_\alpha) \; \gamma_\beta \rangle\!\rangle_0 \, A_j^\beta$$

where $A_0^\beta = \langle\!\langle \gamma^\beta \rangle\!\rangle_g$ and $A_j^\beta = \langle\!\langle \tau_j(\gamma^\beta) \rangle\!\rangle_g - \sum_{k=0}^{j-1} \langle\!\langle \tau_k(\gamma^\beta) \; \gamma_\mu \rangle\!\rangle_0 \, A_{j-1-k}^\mu$. An easy induction on j shows that $\langle\!\langle T^j(\gamma^\beta) \rangle\!\rangle_g = A_j^\beta$ and a similar induction argument also shows that this topological recursion relation is precisely

$$\langle\!\langle T^{3g-1}(\mathcal{W}) \rangle\!\rangle_g = 0 \tag{13}$$

for all vector fields $\mathcal{W}$. For $g = 1$ and 2, this equation follows from equations (8), (10), and (11). For general g, it follows from $\psi_1^{3g-1} = 0$ on $\overline{\mathcal{M}}_{g,1}$ since the complex dimension of $\overline{\mathcal{M}}_{g,1}$ is $3g - 2$. This was first observed by Getzler.

To apply these topological recursion relations, we observe that

$$\mathcal{W} = \overline{\mathcal{W}} + \sum_{i=1}^{k-1} T^i(\overline{\tau_-^i(\mathcal{W})}) + T^k(\tau_-^k(\mathcal{W})) \tag{14}$$

for any vector field $\mathcal{W}$ and $k \geq 1$ (cf. [L2]). Since $\overline{\tau_-^i(\mathcal{W})}$ is a primary vector field, the descendant level of $T^i(\overline{\tau_-^i(\mathcal{W})})$ is at most i. In applications, we can choose k large enough to apply suitable topological recursion relations to get rid of the last term on the right hand side of (14). For example, to apply (12), we would choose $k = g$. To apply (13), we would choose $k = 3g - 1$.

§2. Virasoro conjecture

Let

$$Z := \exp \sum_{g \geq 0} \lambda^{2g-2} F_g.$$

This is the partition function in the topological sigma model coupled to gravity. In the case that the underlying manifold is a point, Witten conjectured that Z is a τ-function of the KdV hierarchy (cf. [W1]). Witten's conjecture was proved in [Ko] and [W2]. In [DVV] [FKN] [KS], it was proved that a function is a τ function of the KdV hierarchy and satisfies the string equation if and only if it is annihilated by a specific sequence of differential operators satisfying the Virasoro bracket relation. Eguchi, Hori, and Xiong [EHX] defined a sequence of differential operators L_n, $n \geq -1$, which satisfy the Virasoro type relation

$$[L_m, L_n] = (m - n)L_{m+n}$$

for a class of compact symplectic manifolds whose Chern numbers satisfy a certain non-trivial equation. For general projective varieties V, these operators were modified by S. Katz so that the Virasoro relation is always satisfied. The *Virasoro conjecture* predicts that

$$L_n Z = 0$$

for $n \geq -1$. The genus-g Virasoro conjecture predicts that the coefficient of λ^{2g-2} in the Laurent polynomial $Z^{-1} L_n Z$ in terms of λ vanishes.

The Virasoro conjecture is one natural generalization of Witten's conjecture. Another way to generalize Witten's conjecture is to construct an integrable system to govern the Gromov-Witten theory for each projective variety. In [EY], it was conjectured that the integrable system corresponding to $\mathbb{P}^1$ is the Toda hierarchy. For general projective varieties, the corresponding integrable systems are not clear at the time of writing. The study of the Virasoro conjecture should shed more light in this direction. In fact, it was conjectured in [DZ] that the Virasoro constraints can be used to construct KdV type integrable systems for semisimple Frobenius manifolds. If this program is successful, we can then obtain an integrable system for each projective variety with semisimple quantum cohomology provided that the Virasoro conjecture can be proved.

The Virasoro operators L_n defined in [EHX] are very complicated. A basic idea in [LT] is that we can study the Virasoro conjecture through the study of certain globally defined vector fields on the big phase space. For this reason, we will not give the explicit form of these operators here. Instead, we will give a recursive description of the relevant vector fields following [L2].

2.1. Virasoro vector fields

We first define a new product on the space of vector fields on the big phase space.

Definition 2.1. *For* $\mathcal{W} = \sum_{m,\alpha} f_{m,\alpha}\tau_m(\gamma_\alpha)$, $\mathcal{V} = \sum_{m,\alpha} g_{m,\alpha}\tau_m(\gamma_\alpha)$, *we define* $\mathcal{W} * \mathcal{V} = \sum_{m,\alpha} f_{m,\alpha}g_{m,\alpha}\tau_m(\gamma_\alpha)$.

We can think of "$*$" as a commutative associative product on the space of vector fields with the identity

$$\mathcal{Z} := \sum_{m,\alpha} \tau_m(\gamma_\alpha).$$

Another important vector field for this product is $\mathcal{G} := \sum_{m,\alpha}(m + b_\alpha)\tau_m(\gamma_\alpha)$. We also need the following linear (over the ring of functions) transformations on the space of vector fields:

Definition 2.2. *For any vector field* $\mathcal{W} = \sum_{m,\alpha} f_{m,\alpha}\tau_m(\gamma_\alpha)$, *define* $C(\mathcal{W}) = \sum_{m,\alpha,\beta} f_{m,\alpha}\mathcal{C}_\alpha^\beta\tau_m(\gamma_\beta)$ *and* $R(\mathcal{W}) = \mathcal{G} * T(\mathcal{W}) + C(\mathcal{W})$.

As pointed out in [LT], the most important vector field in studying the Virasoro conjecture is the *Euler vector field*, which is defined by

$$\mathcal{X} := -\sum_{m,\alpha} (m + b_\alpha - b_1 - 1)\, \tilde{t}^\alpha_m\, \tau_m(\gamma_\alpha) - \sum_{m,\alpha,\beta} \mathcal{C}^\beta_\alpha \tilde{t}^\alpha_m\, \tau_{m-1}(\gamma_\beta).$$

This vector field satisfies the following *quasi-homogeneity equation*

$$\langle\langle \mathcal{X} \rangle\rangle_g = (3-d)(1-g)F_g + \frac{1}{2}\delta_{g,0} \sum_{\alpha,\beta} \mathcal{C}_{\alpha\beta} t^\alpha_0 t^\beta_0 - \frac{1}{24}\delta_{g,1} \int_V c_1(V) \cup c_{d-1}(V).$$

The second derivatives of the genus-0 quasi-homogeneity equation have the following form (cf. [LT, lemma 1.4 (3)]):

$$\langle\langle \mathcal{W}\, \mathcal{X}\, \mathcal{V} \rangle\rangle_0 = \langle\langle Q(\mathcal{W})\, \mathcal{V} \rangle\rangle_0 + \langle\langle \mathcal{W}\, Q(\mathcal{V}) \rangle\rangle_0 + \frac{1}{2}\nabla^2_{\mathcal{W},\mathcal{V}} \mathcal{C}_{\alpha\beta} t^\alpha_0 t^\beta_0 \tag{15}$$

where

$$Q(\mathcal{W}) := \mathcal{G} * \mathcal{W} + C\mathcal{T}_-(\mathcal{W}) \tag{16}$$

for any vector field $\mathcal{W}$. This equation implies the following (see [L2])

Theorem 2.3. *For any vector field* $\mathcal{W}$, $\mathcal{W} \bullet \mathcal{X} = \mathcal{S} \bullet R(\mathcal{W}) = \overline{R(\mathcal{W})}$.

Proof: Since both sides of this equation are tensors with respect to $\mathcal{W}$, we only need to prove the theorem for $\mathcal{W} = \tau_m(\gamma_\alpha)$. By definition, $R(\tau_m(\gamma_\alpha)) = (m + 1 + b_\alpha)\tau_{m+1}(\gamma_\alpha) - b_\beta \langle\langle \tau_m(\gamma_\alpha)\, \gamma^\beta \rangle\rangle_0 \gamma_\beta + \mathcal{C}^\beta_\alpha \tau_m(\gamma_\beta)$. Equation (3) implies that $\langle\langle \mathcal{S}\, R(\tau_m(\gamma_\alpha))\, \gamma^\mu \rangle\rangle_0 = (m + 1 + b_\alpha - b_\mu) \langle\langle \tau_m(\gamma_\alpha)\, \gamma^\mu \rangle\rangle_0 + \mathcal{C}^\beta_\alpha \langle\langle \tau_{m-1}(\gamma_\beta)\, \gamma^\mu \rangle\rangle_0 + \delta^0_m \mathcal{C}^\mu_\alpha$. The right hand side of this equation is precisely $\langle\langle \mathcal{X}\, \tau_m(\gamma_\alpha)\, \gamma^\mu \rangle\rangle_0$ by (15). The theorem is thus proved. $\Box$

Theorem 2.3 implies that $\overline{R(\mathcal{W})} = \overline{R(\mathcal{V})}$ if $\overline{\mathcal{W}} = \overline{\mathcal{V}}$. Another consequence of the quasi-homogeneity equation is that

$$[\overline{\mathcal{X}}^m, \overline{\mathcal{X}}^k] = (k - m)\, \overline{\mathcal{X}}^{m+k-1} \tag{17}$$

for all $m, k \geq 0$. This can be proved by using an argument similar to the one on the small phase space given in [L1].

Define $\mathcal{L}_k := -R^{k+1}(\mathcal{S})$ for $k \geq -1$. We call $\mathcal{L}_k$ the k-th *Virasoro vector field* due to its intimate connection with the Virasoro conjecture. Since $\mathcal{D} = T(\mathcal{S})$, $\mathcal{L}_0 = -\mathcal{X} - (b_1 + 1)\mathcal{D}$. Therefore $\overline{\mathcal{L}}_0 = -\overline{\mathcal{X}}$ because $\overline{\mathcal{D}} = 0$. So Theorem 2.3 implies

$$\overline{\mathcal{L}}_k = -\overline{\mathcal{X}}^{k+1} \tag{18}$$

for $k \geq -1$. Here and thereafter, we will understand $\mathcal{X}^0$ as the string vector field $\mathcal{S}$ and $\overline{\mathcal{X}}^0$ as $\overline{\mathcal{S}}$.

Let ∇ be the covariant derivative defined by

$$\nabla_{\mathcal{V}} \mathcal{W} = \sum_{m,\alpha} (\mathcal{V} f_{m,\alpha})\, \tau_m(\gamma_\alpha)$$

for any vector fields $\mathcal{V}$ and $\mathcal{W} = \sum_{m,\alpha} f_{m,\alpha}\, \tau_m(\gamma_\alpha)$. It is straightforward to check

$$R\tau_-(\mathcal{W}) = \tau_- R(\mathcal{W}) - \mathcal{G} * \overline{\mathcal{W}} - \mathcal{W} \tag{19}$$

and

$$\nabla_{\mathcal{V}}(R(\mathcal{W})) = R(\nabla_{\mathcal{V}} \mathcal{W}) - \mathcal{G} * (\mathcal{V} \bullet \mathcal{W}) \tag{20}$$

for any vector fields $\mathcal{V}$ and $\mathcal{W}$ (cf. [L2]). By induction on k and using Theorem 2.3, we obtain

$$R^k \tau_-(\mathcal{W}) = \tau_- R^k(\mathcal{W}) - k\, R^{k-1}(\mathcal{W}) - \sum_{i=0}^{k-1} R^i \left(\mathcal{G} * (\mathcal{X}^{k-1-i} \bullet \mathcal{W})\right) \tag{21}$$

and

$$\nabla_{\mathcal{V}}(R^k(\mathcal{W})) = R^k(\nabla_{\mathcal{V}} \mathcal{W}) - \sum_{i=0}^{k-1} R^i \left(\mathcal{G} * (\mathcal{X}^{k-1-i} \bullet \mathcal{V} \bullet \mathcal{W})\right) \tag{22}$$

for $k \geq 1$. We can also use (21) to simplify (22) and obtain

$$\nabla_{\mathcal{V}}(R^k(\mathcal{W})) = R^k(\nabla_{\mathcal{V}} \mathcal{W}) - \tau_- R^k(\mathcal{V} \bullet \mathcal{W}) + k\, R^{k-1}(\mathcal{V} \bullet \mathcal{W}) \tag{23}$$

since $\tau_-(\mathcal{V} \bullet \mathcal{W}) = 0$. Using these equations, we can prove the following

Lemma 2.4. $[\mathcal{L}_m, \mathcal{L}_k] = (m-k)\mathcal{L}_{m+k}$.

Proof: It is straightforward to check that $\nabla_{\mathcal{W}} \mathcal{S} = -\tau_-(\mathcal{W})$ for any vector field $\mathcal{W}$. Therefore, applying (22) to $\mathcal{L}_k = -R^{k+1}(\mathcal{S})$ first, then using (1) and (21), we have

$$\nabla_{\mathcal{W}}\, \mathcal{L}_k = \tau_- R^{k+1}(\mathcal{W}) - (k+1) R^k(\mathcal{W}) \tag{24}$$

for any vector field $\mathcal{W}$. In particular, if $\mathcal{W} = \mathcal{L}_m$, we have $\nabla_{\mathcal{L}_m} \mathcal{L}_k = \tau_-(\mathcal{L}_{m+k+1}) - (k+1)\mathcal{L}_{m+k}$. Therefore $[\mathcal{L}_m, \mathcal{L}_k] = \nabla_{\mathcal{L}_m} \mathcal{L}_k - \nabla_{\mathcal{L}_k} \mathcal{L}_m = (m-k)\mathcal{L}_{m+k}$. $\square$

2.2. Quantum powers of the Euler vector field

From (18), we can see that the quantum powers of the Euler vector field are very important in the study of the Virasoro vector fields. In this section, we compute $\langle\langle \mathcal{W}\,\mathcal{X}^k\,\mathcal{V} \rangle\rangle_0$ for $k \geq 1$ and arbitrary vector fields $\mathcal{W}$ and $\mathcal{V}$.

For any $k \geq 2$, define

$$Q_k := Q(Q-1)\cdots(Q-k+1)\tau_+^{k-1}$$

where Q is defined by (16). For convenience, we also define

$$Q_0 := \tau_- \qquad \text{and} \qquad Q_1 := Q.$$

These operators have the following properties

Lemma 2.5.

$$\begin{aligned}
&(i)\ \tau_+ Q_k = (Q-1)\cdots(Q-k)\,\tau_+^k - C^k\pi \quad \text{for } k \geq 1,\\
&(ii)\ \tau_+^k\, Q\,\tau_+ = (Q-k)\,\tau_+^{k+1},\\
&(iii) Q\,\tau_+\, Q_k = Q_{k+1} - QC^k\pi,\\
&(iv) Q_k\,(Q-1)\,\tau_+ = Q_{k+1}.
\end{aligned}$$

The first two formulae follow from a straightforward induction on k. The last two formulae are direct consequences of the first two.

The aim of this section is to prove the following

Proposition 2.6. *For any vector fields $\mathcal{W}$ and $\mathcal{V}$ and $k \geq 1$,*

$$\begin{aligned}
\langle\langle \mathcal{W}\,\mathcal{X}^k\,\mathcal{V} \rangle\rangle_0 &= \langle\langle Q_k(\mathcal{W})\,\mathcal{V} \rangle\rangle_0 + \langle\langle \mathcal{W}\,Q_k(\mathcal{V}) \rangle\rangle_0 + \frac{1}{2}\nabla^2_{\mathcal{W},\mathcal{V}}(\mathcal{C}^k)_{\alpha\beta}\,t_0^\alpha t_0^\beta \\
&\quad + \sum_{i=1}^{k-1} \langle\langle \mathcal{W}\,Q_i(\gamma^\alpha) \rangle\rangle_0 \langle\langle \{(Q_{k-i} - Q_{k-1-i}C)(\gamma_\alpha)\}\,\mathcal{V} \rangle\rangle_0 .
\end{aligned}$$

Note that this proposition also makes sense for $k = 0$ and it is precisely (3) if one takes $\mathcal{X}^0 = \mathcal{S}$ and $(\mathcal{C}^0)_{\alpha\beta} = \eta_{\alpha\beta}$. By the genus-0 topological recursion relation, $\langle\langle \mathcal{W}\,\gamma^\alpha \rangle\rangle_0 \langle\langle \gamma_\alpha\,\mathcal{S}\,\tau_+(\mathcal{V}) \rangle\rangle_0 = \langle\langle \tau_+(\mathcal{W})\,\mathcal{S}\,\tau_+(\mathcal{V}) \rangle\rangle_0$. Applying (3) to both sides of this equation and observing that we have $\nabla^2_{\tau_+(\mathcal{W}),\mathcal{V}}\, t_0^\alpha\, t_0^\beta = 0$, we obtain

$$\langle\langle \mathcal{W}\,\gamma^\alpha \rangle\rangle_0 \langle\langle \gamma_\alpha\,\mathcal{V} \rangle\rangle_0 = \langle\langle \tau_+(\mathcal{W})\ \mathcal{V} \rangle\rangle_0 + \langle\langle \mathcal{W}\ \tau_+(\mathcal{V}) \rangle\rangle_0 . \tag{25}$$

This is precisely [LT, equation (10)].

For $k=1$, Proposition 2.6 is precisely (15). By the genus-0 topological recursion relation, $\langle\!\langle \mathcal{W}\,\mathcal{X}\,\gamma^\alpha \rangle\!\rangle_0 \langle\!\langle \gamma_\alpha\,\mathcal{V} \rangle\!\rangle_0 = \langle\!\langle \mathcal{W}\,\mathcal{X}\,\tau_+(\mathcal{V}) \rangle\!\rangle_0$. Applying (15) to both sides of this equation, then using (25) and Lemma 2.5 (i) and observing that $\frac{1}{2}\left(\nabla^2_{\mathcal{W},\gamma^\mu}(\mathcal{C}^k)_{\alpha\beta}\, t_0^\alpha\, t_0^\beta\right)\gamma_\mu = C^k\pi(\mathcal{W})$, we obtain
(26)
$$\langle\!\langle \mathcal{W}\,Q(\gamma^\alpha) \rangle\!\rangle_0 \langle\!\langle \gamma_\alpha\,\mathcal{V} \rangle\!\rangle_0 = -\langle\!\langle \{(Q-1)\tau_+(\mathcal{W})\}\;\mathcal{V} \rangle\!\rangle_0 + \langle\!\langle \mathcal{W}\;\{Q\tau_+(\mathcal{V})\} \rangle\!\rangle_0 .$$

We remark here that [LT, Lemma 3.2 amd 3.3] are simple consequences of this formula and (25). Note that by using Proposition 2.6 for $k=2$ in a similar way, we also obtain [LT, Lemma 4.4], which was needed in the proof of the genus-0 Virasoro conjecture in [LT]. But it will not be needed in this paper.

Proof of Proposition 2.6: We prove this proposition by induction on k. When $k=1$, it is just (15). For $k \geq 2$, we have for any vector fields $\mathcal{W}$, $\mathcal{V}$,

$$\langle\!\langle \mathcal{W}\,\mathcal{X}^k\,\mathcal{V} \rangle\!\rangle_0 = \langle\!\langle \mathcal{W}\,\mathcal{X}\,\gamma^\alpha \rangle\!\rangle_0 \langle\!\langle \gamma_\alpha\,\mathcal{X}^{k-1}\,\mathcal{V} \rangle\!\rangle_0 . \tag{27}$$

If $\mathcal{V} = \gamma^\beta$ for some β, this formula is just the associativity of the quantum product. The general case follows from this special case and the genus-0 topological recursion relation.

Applying (15) to the first term on the right hand side of (27) and then using the genus-0 topological recursion relation, $\langle\!\langle \mathcal{W}\,\mathcal{X}^k\,\mathcal{V} \rangle\!\rangle_0 =$

$$\langle\!\langle \{(\tau_+ Q + C\pi)(\mathcal{W})\}\;\mathcal{X}^{k-1}\;\mathcal{V} \rangle\!\rangle_0 + \langle\!\langle \mathcal{W}\;Q(\gamma^\alpha) \rangle\!\rangle_0 \langle\!\langle \gamma_\alpha\,\mathcal{X}^{k-1}\,\mathcal{V} \rangle\!\rangle_0 .$$

The proposition then follows from applying the induction hypothesis to the right hand side of this equation and simplifying the resulting expression using (26), Lemma 2.5 and the following simple observation:

$$\langle\!\langle \cdots \{P_1 C(\gamma^\alpha)\} \rangle\!\rangle_0 P_2(\gamma_\alpha) = \langle\!\langle \cdots \{P_1(\gamma^\alpha)\} \rangle\!\rangle_0 P_2 C(\gamma_\alpha) \tag{28}$$

for any linear transformations P_1, P_2 on the space of vector fields. $\square$

If we set $\mathcal{W} = \mathcal{V} = \mathcal{D}$ in Proposition 2.6 and use the genus-0 dilaton equation (7), we obtain

Corollary 2.7. $\langle\!\langle Q_k(\mathcal{D}) \rangle\!\rangle_0 =$

$$\frac{1}{2}\sum_{i=1}^{k-1} \langle\!\langle Q_i(\gamma^\alpha) \rangle\!\rangle_0 \langle\!\langle \{(Q_{k-i} - Q_{k-1-i}C)(\gamma_\alpha)\} \rangle\!\rangle_0 + \frac{1}{2}(\mathcal{C}^k)_{\alpha\beta}\, t_0^\alpha t_0^\beta .$$

2.3. Genus-0 Virasoro conjecture

In this section we compute $\langle\langle \mathcal{L}_k \rangle\rangle_0$ using Corollary 2.7 and then relate it to the genus-0 Virasoro conjecture. For this purpose, we need to study relations between the operators R, T, and Q.

First, replacing $\mathcal{W}$ by $RT(\mathcal{W})$ in (14) with $k=1$, then using Theorem 2.3 and (19), we obtain $RT = T(R+T)$ (cf. [L2, Lemma 4.6]). By induction on k, we have

$$T^k R = (R - kT)T^k \tag{29}$$

for $k \geq 0$. By the definitions of R and Q, $R = QT$. Writing $R^k = R^{k-1}R$, then by induction on k and using (29), we obtain

$$R^k = Q(Q-1)\cdots(Q-k+1)T^k \tag{30}$$

for $k \geq 1$. Therefore

$$\begin{aligned} \mathcal{L}_k &= -R^{k+1}(\mathcal{S}) = -Q(Q-1)\cdots(Q-k)T^k(\mathcal{D}) \\ &= -Q_{k+1}(\mathcal{D}) - Q(Q-1)\cdots(Q-k)(T^k - \tau_+^k)(\mathcal{D}). \end{aligned} \tag{31}$$

On the other hand, writing

$$T^k(\mathcal{W}) = T^{k-1}(T(\mathcal{W})) = T^{k-1}\left(\tau_+(\mathcal{W}) - \langle\langle \mathcal{W}\gamma^\alpha \rangle\rangle_0 \gamma_\alpha\right),$$

then by induction on k and using (25), we obtain

$$T^k(\mathcal{W}) = \tau_+^k(\mathcal{W}) + \sum_{i=0}^{k-1}(-1)^{i+1}\langle\langle \mathcal{W}\tau_+^i(\gamma^\alpha) \rangle\rangle_0 \tau_+^{k-1-i}(\gamma_\alpha) \tag{32}$$

for any vector field $\mathcal{W}$. In particular, replacing $\mathcal{W}$ by $\mathcal{D}$ and using (7), we have

$$(T^k - \tau_+^k)(\mathcal{D}) = \sum_{i=0}^{k-1}(-1)^i \langle\langle \tau_+^i(\gamma^\alpha) \rangle\rangle_0 \tau_+^{k-1-i}(\gamma_\alpha). \tag{33}$$

Together with (31) and Corollary 2.7, we can use this equation to compute $\langle\langle \mathcal{L}_k \rangle\rangle_0$. To simplify the result, we also need the following

Lemma 2.8. *For any linear (over the ring of functions) transformation P on the space of vector fields and $k \geq 1$,*

$$\begin{aligned} &Q(Q-1)\cdots(Q-k)\left(\sum_{i=0}^{k-1}(-1)^i \langle\langle \{P\tau_+^i(\gamma^\alpha)\} \rangle\rangle_0 \tau_+^{k-1-i}(\gamma_\alpha)\right) \\ &= -\sum_{i=1}^{k}\langle\langle \{PQ_i(\gamma^\alpha)\} \rangle\rangle_0 (Q_{k+1-i} - Q_{k-i}C)(\gamma_\alpha). \end{aligned}$$

Proof: Let $\mathcal{W}_{P,k} := \sum_{i=0}^{k-1}(-1)^i \langle\!\langle \{P\tau_+^i(\gamma^\alpha)\} \rangle\!\rangle_0 \tau_+^{k-1-i}(\gamma_\alpha)$. We prove the lemma by induction on k. The lemma holds for $k=1$ because $Q(Q-1)(\mathcal{W}_{P,1}) = \langle\!\langle P(\gamma^\alpha) \rangle\!\rangle_0 (b_\alpha - 1)Q(\gamma_\alpha) = -\langle\!\langle \{PQ(\gamma^\alpha)\} \rangle\!\rangle_0 Q(\gamma_\alpha)$. By Lemma 2.5 (i) and (ii), $Q\,\tau_+^k = \tau_+^k\,(Q+k) + \tau_+^{k-1} C\pi$ for $k \geq 1$. Using this equation and (28), it is straightforward to check that

$$
\begin{aligned}
&(Q-k)\left(\langle\!\langle \{P\tau_+^i(\gamma^\alpha)\} \rangle\!\rangle_0 \tau_+^{k-1-i}(\gamma_\alpha)\right) = \\
&-\langle\!\langle \{P(Q\tau_+^i - \tau_+^{i-1}C)(\gamma^\alpha)\} \rangle\!\rangle_0 \tau_+^{k-1-i}(\gamma_\alpha) + \langle\!\langle \{P\tau_+^i C(\gamma^\alpha)\} \rangle\!\rangle_0 \tau_+^{k-2-i}(\gamma_\alpha).
\end{aligned}
$$

Consequently, we have

$$(Q-k)(\mathcal{W}_{P,k}) = \sum_{i=0}^{k-1}(-1)^{i+1} \langle\!\langle \{PQ\tau_+^i(\gamma^\alpha)\} \rangle\!\rangle_0 \tau_+^{k-1-i}(\gamma_\alpha).$$

Therefore, for $k \geq 2$, $Q(Q-1)\cdots(Q-k)(\mathcal{W}_{P,k}) =$

$$Q(Q-1)\cdots(Q-k+1)\left(\mathcal{W}_{(PQ\tau_+),\,(k-1)}\right) - \langle\!\langle \{PQ(\gamma^\alpha)\} \rangle\!\rangle_0 Q_k(\gamma_\alpha).$$

The lemma then follows from induction on k and by using Lemma 2.5 (iii). $\square$

Combining Corollary 2.7, equations (31) and (33), and lemma 2.8 for P equal to the identity transformation, we obtain

Theorem 2.9. $\langle\!\langle \mathcal{L}_k \rangle\!\rangle_0 =$

$$\frac{1}{2}\sum_{i=1}^{k} \langle\!\langle Q_i(\gamma^\alpha) \rangle\!\rangle_0 \langle\!\langle \{(Q_{k+1-i} - Q_{k-i}C)(\gamma_\alpha)\} \rangle\!\rangle_0 - \frac{1}{2}(\mathcal{C}^{k+1})_{\alpha\beta}\, t_0^\alpha t_0^\beta.$$

Remark: The right hand side of this equation is equal to $\rho_{0,k}$ as defined in [L2, Section 5.1]. One can show this by using the following properties of the operator R_+ defined in [L2]: $R_+^k = Q_k\,\tau_+$ for $k \geq 1$.

Now we relate Theorem 2.9 to the genus-0 Virasoro conjecture. Since

$$\langle\!\langle \mathcal{L}_k\,\gamma^\alpha \rangle\!\rangle_0 = \gamma^\alpha \langle\!\langle \mathcal{L}_k \rangle\!\rangle_0 - \langle\!\langle \{\nabla_{\gamma^\alpha}\mathcal{L}_k\} \rangle\!\rangle_0,$$

we can use Theorem 2.9 and (24) to compute $\langle\!\langle \mathcal{L}_k\,\gamma^\alpha \rangle\!\rangle_0$. We can then compute $\mathcal{L}_{k+1} = R(\mathcal{L}_k)$ from $\mathcal{L}_k$ by using these formulae. We already know that the vector fields $\mathcal{L}_{-1}$ and $\mathcal{L}_0$ are precisely the first derivative parts of the Virasoro operators L_{-1} and L_0 defined in [EHX]. Therefore using the above method, we obtain $\mathcal{L}_1 =$

$$\sum_{m,\alpha}(m+b_\alpha)(m+b_\alpha+1)\tilde{t}_m^\alpha \tau_{m+1}(\gamma_\alpha) + \sum_{m,\alpha,\beta}(2m+2b_\alpha+1)\mathcal{C}_\alpha^\beta \tilde{t}_m^\alpha \tau_m(\gamma_\beta)$$
$$+\sum_{m,\alpha,\beta}(\mathcal{C}^2)_\alpha^\beta \tilde{t}_m^\alpha \tau_{m-1}(\gamma_\beta) - \sum_\alpha b_\alpha(b_\alpha - 1)\langle\langle\, \gamma^\alpha \,\rangle\rangle_0\, \gamma_\alpha$$

and

$$\begin{aligned}
\mathcal{L}_2 = &\sum_{m,\alpha}(m+b_\alpha)(m+b_\alpha+1)(m+b_\alpha+2)\tilde{t}_m^\alpha \tau_{m+2}(\gamma_\alpha) \\
&+\sum_{m,\alpha,\beta}\left\{3(m+b_\alpha)^2+6(m+b_\alpha)+2\right\}\mathcal{C}_\alpha^\beta \tilde{t}_m^\alpha \tau_{m+1}(\gamma_\beta) \\
&+\sum_{m,\alpha,\beta}3(m+b_\alpha+1)(\mathcal{C}^2)_\alpha^\beta \tilde{t}_m^\alpha \tau_m(\gamma_\beta) + \sum_{m,\alpha,\beta}(\mathcal{C}^3)_\alpha^\beta \tilde{t}_m^\alpha \tau_{m-1}(\gamma_\beta) \\
&-\sum_\alpha b_\alpha(b_\alpha^2-1)\{\langle\langle\, \tau_1(\gamma_\alpha) \,\rangle\rangle_0\, \gamma^\alpha + \langle\langle\, \gamma^\alpha \,\rangle\rangle_0\, \tau_1(\gamma_\alpha)\} \\
&-\sum_{\alpha,\,\beta}(3b_\beta^2-1)\mathcal{C}_\beta^\alpha \,\langle\langle\, \gamma_\alpha \,\rangle\rangle_0\, \gamma^\beta.
\end{aligned}$$

These vector fields are slightly different from the vector field $\mathcal{L}_1$ and $\mathcal{L}_2$ in [LT]. The linear parts of these vector fields are the corresponding vector fields in [LT], but $\mathcal{L}_1$ and $\mathcal{L}_2$ here have extra non-linear terms. These vector fields do agree with the corresponding vector field in [DZ] and [G3]. Therefore Theorem 2.9 is precisely the genus-0 Virasoro conjecture as proved in [LT]. So the arguments presented here give a new proof of the genus-0 Virasoro conjecture. We also notice that equation (18) for $k = 1, 2$ is equivalent to Lemma 3.1 and Lemma 4.2 of [LT] because of (3). We can interpret them as the second derivatives of the genus-0 L_1 and L_2 constraints. As explained in [LT], they are eventually also equivalent to the genus-0 L_1 and L_2 constraints because of the dilaton equation.

Moreover, as a consequence of Lemma 2.4, we obtain

$$\mathcal{L}_k = \lim_{\lambda\to 0}\; Z^{-1}\circ L_k \circ Z,$$

where L_k is the k-th Virasoro operator defined in [EHX], since the sequence of vector fields on the right hand side of this equality also satisfy the same Virasoro type bracket relation as in Lemma 2.4 (cf. [G3]).

2.4. Higher genus Virasoro conjecture

For convenience, we introduce the following notation: For two functions f_1 and f_2 on the big phase space, we say $f_1 \overset{g}{\approx} f_2$ if $f_1 - f_2$ can be expressed as a function of $F_0, \ldots, F_{g-1}$ and their derivatives. For $g \geq 1$, the genus-g Virasoro conjecture gives an explicit formula for computing $\langle\!\langle \mathcal{L}_n \rangle\!\rangle_g$ in terms of data with genus less than g for $n \geq -1$. The precise way to express $\langle\!\langle \mathcal{L}_n \rangle\!\rangle_g$ in terms of data of genus less than g can be explicitly written out using the Virasoro operators L_n defined in [EHX] (see, for example, [L2, Section 5.1] for a description using recursive operators). A weaker form of this conjecture

$$\langle\!\langle \mathcal{L}_n \rangle\!\rangle_g \overset{g}{\approx} 0. \tag{34}$$

In other words, this conjecture predicts that the sequence of vector fields $\mathcal{L}_k$ are trivial at the genus-g level for all $g \geq 1$. Since the genus-g Virasoro conjecture only differs from its weak version (34) at the lower genus level, (34) captures the major difficulty as well as the computational power of the original Virasoro conjecture. Moreover, conjecture (34) makes sense for all compact symplectic manifolds while the original Virasoro conjecture requires a non-trivial topological condition for the underlying manifolds.

To compute $\langle\!\langle \mathcal{L}_n \rangle\!\rangle_g$, we can use either (12) or (13) together with formula (14). Because of these formulae, the computation of $\langle\!\langle \mathcal{L}_n \rangle\!\rangle_g$ is equivalent to computing the derivative of F_g along the vector field $\sum_{m=0}^{k-1} T^m(\overline{\tau_-^m(\mathcal{L}_n)})$ where $k = g$ or $3g-1$ depending on whether we are using (12) or (13). For this purpose, we first need to compute $\overline{\tau_-^m(\mathcal{L}_k)}$. For $m = 0$, $\overline{\tau_-^m(\mathcal{L}_n)} = \overline{\mathcal{L}}_n = -\overline{\mathcal{X}}^{n+1}$ by (18). Using (19), we can prove inductively that $\tau_-^m(\mathcal{L}_{n+1}) = R(\tau_-^m(\mathcal{L}_n)) + m\,\tau_-^{m-1}(\mathcal{L}_n) + \mathcal{G} * \overline{\tau_-^{m-1}(\mathcal{L}_n)}$ for all $m \geq 1$ and $n \geq -1$. Therefore, by Theorem 2.3, we have the following (cf. [L2])

Theorem 2.10.

$$\overline{\tau_-^m(\mathcal{L}_{n+1})} = \mathcal{X} \bullet \overline{\tau_-^m(\mathcal{L}_n)} + m\,\overline{\tau_-^{m-1}(\mathcal{L}_n)} + \mathcal{G} * \overline{\tau_-^{m-1}(\mathcal{L}_n)}$$

for all $m \geq 1$ *and* $n \geq -1$.

Using this recursion formula, we can express $\overline{\tau_-^m(\mathcal{L}_n)}$ in terms of twisted quantum powers of the Euler vector field (here the twisting is given by the operation $\mathcal{G}*$) and vector fields $\overline{\tau_-^i(\mathcal{S})}$ where $i = 1, \ldots, m$. For example, when $m = 1$, $\overline{\tau_-(\mathcal{L}_n)} = -\overline{\mathcal{X}}^{n+1} \bullet \overline{\tau_-(\mathcal{S})} - (n+1)\overline{\mathcal{X}}^n -$

$\sum_{j=0}^{n} \overline{\mathcal{X}}^{j} \bullet (\mathcal{G} * \overline{\mathcal{X}}^{n-j})$ for all $n \geq 0$. By the genus-1 topological recursion relation (9) and (14),

$$\langle\langle \mathcal{L}_n \rangle\rangle_1 = \langle\langle \overline{\mathcal{L}}_n \rangle\rangle_1 + \langle\langle T(\tau_-(\mathcal{L}_n)) \rangle\rangle_1$$
$$= - \langle\langle \overline{\mathcal{X}}_{n+1} \rangle\rangle_1 + \frac{1}{24} \left\langle\left\langle \overline{\tau_-(\mathcal{L}_n)} \, \gamma_\alpha \gamma^\alpha \right\rangle\right\rangle_0 .$$

This immediately explains the rather mysterious formulae for the genus-1 Virasoro conjecture in [L1]. We would also like to point out that for $i \geq 1$, $\tau_-^i(\mathcal{S})$ is zero when restricted to the small phase space. In this way, we can get rid of such vector fields in applications.

From the point of view of applications, the computational power of the Virasoro conjecture depends on how large the space spanned by the vector fields $\sum_{m=0}^{k-1} T^m(\overline{\tau_-^m(\mathcal{L}_n)})$ (with $n \geq -1$) is. In general, this space could be very small. For example, for all algebraic curves, the space spanned by the quantum powers of the Euler vector fields always has dimension 2, while the dimension of the small phase space can be arbitrarily large if the genus of the curve is large enough. If the space spanned by the quantum powers of the Euler vector field is small compared to the small phase space, we do not expect that the Virasoro conjecture will determine the generating functions. In the other extreme case, when the quantum cohomology is semisimple, the quantum powers of the Euler vector fields span the entire tangent space of the small phase space. Therefore in this case, the generating functions can be computed by using the Virasoro conjecture.

For genus $g \geq 1$, the topological recursion relation is not sufficient to prove the Virasoro conjecture. For $g = 1$ and 2, we need at least the equations derived from the algebraic relations in $H^4(\overline{\mathcal{M}}_{1,4})$ and $H^4(\overline{\mathcal{M}}_{2,3})$ which were proved in [G1] and [BP] respectively. In [L1] and [L2], we wrote these equations as equations for certain global tensors on the big phase space. Studying these tensors for quantum powers of the Euler vector field, we proved that for all $n \geq 3$,

$$\left\langle\left\langle \overline{\mathcal{X}}^{n} \right\rangle\right\rangle_1 \overset{1}{\approx} \frac{n}{2} \overline{\mathcal{X}}^{n-1} \left\langle\left\langle \overline{\mathcal{X}}^{2} \right\rangle\right\rangle_1 \tag{35}$$

and

$$\psi_n \overset{2}{\approx} \frac{n}{2(n-2)} T(\overline{\mathcal{X}}) \psi_{n-1} \tag{36}$$

where $\psi_n := \left\langle\left\langle \overline{\mathcal{X}}^{n} \right\rangle\right\rangle_2 - \left\langle\left\langle T(\overline{\tau_-(\mathcal{L}_{n-1})}) \right\rangle\right\rangle_2$. The difference of the two sides of each equation can be written down explicitly in terms of the lower genus data. The topological recursion relations (9) and (11) imply that $\langle\!\langle \mathcal{L}_n \rangle\!\rangle_1 \stackrel{1}{\approx} -\left\langle\!\!\left\langle \overline{\mathcal{X}}^{n+1} \right\rangle\!\!\right\rangle_1$, $\langle\!\langle \mathcal{L}_n \rangle\!\rangle_2 \stackrel{2}{\approx} -\psi_{n+1}$ for all n. Therefore equations (35) and (36) imply that the conjecture (34) of genus 1 and 2 can be reduced to the corresponding L_1-constraint. The same is also true for the original Virasoro conjecture (cf. [L1] and [L2]). Moreover, since the space of primary vector fields is finite dimensional, the infinite sequence of vector fields $\{\overline{\mathcal{X}}^k \mid k \geq 0\}$ must be linearly dependent at each point. This adds extra information to equations (35) and (36). In the case that the quantum cohomology is not too degenerate (which is a condition weaker than semisimplicity), this enables us to solve for $\left\langle\!\!\left\langle \overline{\mathcal{X}}^n \right\rangle\!\!\right\rangle_1$ and ψ_n from these equations and thus obtain the corresponding Virasoro conjecture (the weak version only for the genus-2 case). The reader is referred to [L1] and [L2] for details. In [L2] we also explicitly found the genus-2 generating function F_2 in terms of F_1 and F_0 when the quantum cohomology is not too degenerate. For genus bigger than 2, relations analogous to the ones in [G1] and [BP] are missing. Provided with such equations, the structures described here combined with the techniques developed in [LT], [L1] and [L2] could lead to the final resolution of the Virasoro conjecture for all genera.

Update added, August 2007: The first version of this paper was submitted in January 2001. Much progress on the Virasoro conjecture has been made since then. In particular, the genus-2 Virasoro conjecture for manifolds with semisimple quantum cohomology was proved in [L3]. When the underlying manifold has a torus action with isolated fixed points and also has semisimple quantum cohomology, Givental gave a scheme to reduce the Virasoro conjecture to the so-called R-conjecture, which only depends on genus-0 equivariant Gromov-Witten invariants. An outline for the proof of the R-conjecture for projective spaces was given in [Gi]. The R-conjecture has been verified for flag manifolds in [JK] and for Grassmannians in [BCK]. Moreover, the Virasoro conjecture for algebraic curves was proved in [OP].

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