Advanced Studies in Pure Mathematics 51, 2008
Surveys on Geometry and Integrable Systems
pp. 113–141

Timelike surfaces with harmonic inverse mean curvature

Atsushi Fujioka and Jun-ichi Inoguchi

Introduction

In this paper we introduce the notion of *timelike surfaces with harmonic inverse mean curvature* in 3-dimensional Lorentzian space forms, and study their fundamental properties.

In classical differential geometry, surfaces of constant mean curvature (CMC surfaces) have been studied extensively [1]. As a generalization of CMC surfaces, Bobenko [2] introduced the notion of *surface with harmonic inverse mean curvature* (HIMC surface). He showed that HIMC surfaces admit a Lax representation with *variable* spectral parameter. In [5], Bobenko, Eitner and Kitaev showed that the Gauss equations of θ-isothermic HIMC surfaces reduce to the ordinary differential equation:

$$(*) \qquad \left(\frac{q''(t)}{q'(t)}\right)' - q'(t) = \mathcal{S}(t)\left(2 - \frac{q^2(t)+c}{q'(t)}\right), \quad q'(t) < 0,$$

with $c = \theta^2 > 0$. Here the coefficient function $\mathcal{S}(t)$ is $1/\sin^2(2t)$, $1/\sinh^2(2t)$ or $1/t^2$. This ordinary differential equation is called the *generalized Hazzidakis equation.* Bobenko, Eitner and Kitaev [5] solved $(*)$ in terms of Painlevé transcendents P_{V} and P_{VI}.

For $c < 0$, solutions to $(*)$ do not describe surfaces in Euclidean 3-space. It seems to be interesting to find "corresponding surfaces" to such solutions.

2000 *Mathematics Subject Classification.* Primary 37K25; Secondary 53C42, 53C50.

Received January 12, 2001.

Revised May 22, 2001.

Fujioka: Partially supported by Grant-in-Aid for Encouragement of Young Scientists No. 12740037, JSPS; Inoguchi: Partially supported by Grant-in-Aid for Encouragement of Young Scientists No. 12740051, JSPS.

The first author extended the notion of HIMC surface in Euclidean 3-space to that of Riemannian 3-space forms [7]. Moreover he generalized a theorem due to Lawson (the *Lawson correspondence*) to HIMC surfaces. By using the Lawson correspondence for HIMC surfaces, we have classified Bonnet surfaces with constant curvature in Riemannian 3-space forms [8]. Corresponding results for spacelike surfaces in Lorentzian 3-space forms are obtained in [10].

On the other hand, very little is known about (integrable) timelike surfaces of nonconstant mean curvature in Lorentzian 3-space forms. Timelike Bonnet surfaces were investigated by the present authors very recently [11].

In this paper we introduce the notion of timelike surfaces with harmonic inverse mean curvature (THIMC surface) in Lorentzian 3-space forms. We shall show that every solution to the generalized Hazzidakis equation with $c < 0$ describes a THIMC surface in Minkowski 3-space. This is one of the motivations to study THIMC surfaces.

Because of the indefiniteness of the metric, timelike surface geometry has many aspects different from Euclidean surface geometry. For instance, there exist timelike (HIMC) surfaces with imaginary principal curvatures. Moreover there exist non totally umbilical timelike surfaces with real repeated principal curvatures. Such surfaces have no counterparts in Euclidean surface geometry and spacelike surface geometry. Thus the geometry of THIMC surfaces has its own interest.

The second motivation of the present study is to give new examples of Lax equations with variable spectral parameter, namely, Lax equations whose spectral parameters depend on the variables. Burtsev, Zakharov and Mikhailov [6] exhibited some examples of Lax equations with variable spectral parameter which appeared in theoretical physics. In differential geometry, HIMC surfaces and Bianchi surfaces are known examples. (See [2], [15] and [16].)

We shall show that THIMC surfaces Lorentzian 3-space forms admit a Lax representation with variable spectral parameter. Moreover we shall show that in de Sitter 3-space or anti-de Sitter 3-space, THIMC surfaces admit a Lax representation with *two* independent variable spectral parameters.

This paper is organized as follows. After recalling fundamental facts on Lorentzian geometry, we introduce the notion of THIMC surface in Minkowski 3-space in Section 3. We give a Lax representation and an immersion formula (Sym formula) for THIMC surfaces. Some elementary examples will be given in Section 3. In Section 4, we introduce the notion of $\pm$ isothermic timelike surface. We shall give a duality between timelike Bonnet surfaces and $\pm$ isothermic THIMC surfaces.

In Section 5, we shall investigate the normal forms of the Gauss equations of THIMC surfaces. More precisely we show that (θ-isothermic or anti-θ-isothermic) THIMC surfaces in Minkowski 3-space are derived from solutions to the generalized Hazzidakis equation with $c = -\theta^2 < 0$.

In Section 6, we shall generalize the notion of THIMC surface to Lorentzian 3-space forms and establish a Lawson-type correspondence for THIMC surfaces.

1 Lorentzian space forms

1.1 First of all, we shall describe *Lorentzian* 3*-space forms*, i.e., complete and connected Lorentzian 3-manifolds $\mathfrak{M}^3_1(c)$ of constant curvature c explicitly. Without loss of generality, we may assume that $c = 0$ or ± 1.

We equip the Cartesian 4-space $\mathbf{R}^4$, with the following scalar product $\langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle_c$:

$$\langle a, b\rangle_c = -a_0 b_0 + a_1 b_1 + a_2 b_2 + a_3 b_3, \quad c = 1,$$
$$\langle a, b\rangle_c = -a_0 b_0 - a_1 b_1 + a_2 b_2 + a_3 b_3, \quad c = 0,\ -1.$$

The resulting semi-Euclidean 4-space $(\mathbf{R}^4, \langle \cdot, \cdot \rangle)$ is of index 1 for $c = 1$ and of index 2 for $c = 0$ or -1 respectively. The Lorentzian 3-space forms $\mathfrak{M}^3_1(c)$ are embedded in the semi-Euclidean space $(\mathbf{R}^4, \langle \cdot, \cdot \rangle_c)$ as

$$\mathfrak{M}^3_1(0) = \{p \in (\mathbf{R}^4, \langle \cdot, \cdot \rangle_0) \mid p_0 = 0\} = \mathbf{E}^3_1, \text{Minkowski 3-space},$$
$$\mathfrak{M}^3_1(1) = \{p \in (\mathbf{R}^4, \langle \cdot, \cdot \rangle_1) \mid \langle p, p\rangle_1 = 1\ \} = S^3_1, \text{de Sitter 3-space},$$
$$\mathfrak{M}^3_1(-1) = \{p \in (\mathbf{R}^4, \langle \cdot, \cdot \rangle_{-1}) \mid \langle p, p\rangle_1 = -1\ \} = H^3_1, \text{anti-de Sitter 3-space}.$$

For more details on semi-Riemannian geometry, we refer to O'Neill [18].

Next we recall 2 by 2 matrix models of $\mathfrak{M}^3_1(c)$ for later use.

First the semi-Euclidean 4-space $\mathbf{E}^4_2 = (\mathbf{R}^4, \langle \cdot, \cdot \rangle_{-1})$ is identified with the linear space $\mathrm{M}_2\mathbf{R}$ of all 2 by 2 real matrices via the isomorphism:

(1.1)
$$p = (p_0, p_1, p_2, p_3) \longleftrightarrow p_0\mathbf{1} + p_1\mathbf{i} + p_2\mathbf{j}' + p_3\mathbf{k}' = \begin{pmatrix} p_0 - p_3 & -p_1 + p_2 \\ p_1 + p_2 & p_0 + p_3 \end{pmatrix}.$$

The semi-Euclidean metric of $\mathbf{E}^4_2$ corresponds to the following scalar product on $\mathrm{M}_2\mathbf{R}$.

$$(1.2)\qquad \langle X,Y\rangle = \frac{1}{2}\{\mathrm{tr}(XY) - \mathrm{tr}(X)\mathrm{tr}(Y)\},\quad X,Y \in \mathrm{M}_2\mathbf{R}.$$

Under the identification (1.1), the Minkowski 3-space $\mathbf{E}^3_1(p_1,p_2,p_3)$ is identified with the Lie algebra $\mathfrak{g} = \mathfrak{sl}_2\mathbf{R} = \{X \in \mathrm{M}_2\mathbf{R} \mid \mathrm{tr}\, X = 0\ \}$ with metric $\langle X,Y\rangle = \frac{1}{2}\mathrm{tr}(XY),\ X,Y \in \mathfrak{g}$.

Next, since $\langle X,X\rangle = -\det X$ for all $X \in \mathrm{M}_2\mathbf{R}$, the anti-de Sitter 3-space $H^3_1 \subset \mathbf{E}^4_2$ corresponds to the real special linear group:

$$G = \mathrm{SL}_2\mathbf{R} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{M}_2\mathbf{R} \mid ad - bc = 1 \right\}.$$

Since the Lorentzian metric of G is bi-invariant, the product group $G \times G$ acts transitively and isometrically on H^3_1 as follows:

$$\mu_H : (G \times G) \times H^3_1 \longrightarrow H^3_1,\quad \mu_H(g_1,g_2)X = g_1\ X\ g_2^{-1}$$

for $(g_1,g_2) \in G\times G,\ X \in H^3_1$. The isotropy subgroup Δ of $G\times G$ at $\mathbf{1}$ is the diagonal subgroup of $G \times G$, that is, $\Delta = \{(g_1,g_1)\ |g_1 \in G\}$. Hence the anti-de Sitter 3-space H^3_1 is represented by $H^3_1 = (G \times G)/\Delta$ as a Lorentzian symmetric space. The natural projection $p_H : G \times G \to H^3_1$ is given explicitly by $p_H(g_1,g_2) = g_1\ g_2^{-1},\ (g_1,g_2) \in G \times G$.

Moreover G acts isometrically on $\mathbf{E}^3_1$ via the Ad-action:

$$\mathrm{Ad} : G \times \mathbf{E}^3_1 \to \mathbf{E}^3_1;\ \mathrm{Ad}(a)X = aXa^{-1},\ a \in G,\ X \in \mathbf{E}^3_1.$$

Finally we recall a 2 by 2 matrix model of S^3_1. The Minkowski 4-space $\mathbf{E}^4_1 = (\mathbf{R}^4, \langle\cdot,\cdot\rangle_1)$ is identified with the space $\mathbb{H}$ of all Hermitian 2-matrices via the following isomorphism:

$$(1.4)\qquad p = (p_0,p_1,p_2,p_3) \longleftrightarrow \begin{pmatrix} p_0 + p_1 & p_3 - \sqrt{-1}p_2 \\ p_3 + \sqrt{-1}p_2 & p_0 - p_1 \end{pmatrix} \in \mathbb{H}.$$

Under the identification (1.4), the scalar product $\langle\cdot,\cdot\rangle_1$ of $\mathbf{E}^4_1$ corresponds to the following scalar product on $\mathbb{H}$:

$$(1.5)\quad \langle X,Y\rangle = -\frac{1}{2}\mathrm{tr}(\mathbf{i}'X\ \mathbf{i}'Y^t),\ X,Y \in \mathbb{H},\ \ \mathbf{i}' = \begin{pmatrix} 0 & -\sqrt{-1} \\ \sqrt{-1} & 0 \end{pmatrix}.$$

In particular $\det X = -\langle X, X\rangle_1$ under (1.4). Thus the de Sitter 3-space S^3_1 is represented by $S^3_1 = \{X \in \mathbb{H} \mid \det X = -1\}$. The complex special linear group $\mathrm{SL}_2\mathbf{C}$ acts transitively and isometrically on S^3_1 by $\mu_S : \mathrm{SL}_2\mathbf{C} \times S^3_1 \to S^3_1$, $\mu_S(g) = gXg^*$. Here g^* denotes the transposed complex conjugate of g. The isotropy subgroup of $\mathrm{SL}_2\mathbf{C}$ at $\mathbf{i}'$ is $\mathrm{SL}_2\mathbf{R}$. Hence the de Sitter 3-space S^3_1 is represented by $S^3_1 = G^{\mathbf{C}}/G$ as a Lorentzian symmetric space. The natural projection $p_S : G^{\mathbf{C}} \to G^{\mathbf{C}}/G$ is given explicitly by $p_S(g) = \mu(g)\ \mathbf{i}' = g\ \mathbf{i}'\ g^*,\ g \in G^{\mathbf{C}}$.

2 Timelike surfaces in Lorentzian space forms

We start with some preliminaries on the geometry of timelike surfaces in Lorentzian space forms $\mathfrak{M}^3_1(c)$.

Let M be a connected 2-manifold and $F : M \to \mathfrak{M}^3_1(c)$ an immersion. The immersion F is said to be *timelike* if the induced metric I of M is Lorentzian. Hereafter we assume that M is an orientable timelike surface in $\mathfrak{M}^3_1(c)$ immersed by F. The induced Lorentzian metric I of M determines a Lorentzian conformal structure on M. We treat M as a Lorentz surface with respect to this conformal structure and F as a conformal immersion. Our general reference on Lorentz surfaces is Weinstein [21].

On a timelike surface M, there exists a local coordinate system (x, y) such that

$$I = e^{\omega}(-dx^2 + dy^2). \tag{2.1}$$

Such a local coordinate system (x, y) is called a *Lorentz isothermal coordinate system.*

Let (u, v) be the local *null coordinate system* of M derived from (x, y). Namely (u, v) is defined by $u = x + y,\ v = -x + y$. Then the induced metric I can be written as

$$I = e^{\omega} du dv. \tag{2.2}$$

Now, let N be a unit normal vector field to M. The *second fundamental form* $I\!I$ of (M, F) derived from N is defined by $I\!I = -\langle dF, dN\rangle$. The *shape operator* S of (M, F) relative to N is defined by $S = -dN$. The (complex) eigenvalues of S are called *principal curvatures* of (M, F). The *mean curvature* H of (M, F) is defined by $H = \mathrm{tr}\ S/2$. The Gaussian curvature K of (M, I) is given by the formula $K = \det S$.

The Gauss-Codazzi equations of (M, F) have the following form:

$$(\mathrm{G}_c) \qquad \omega_{uv} + \frac{1}{2}(H^2 + c)e^{\omega} - 2QRe^{-\omega} = 0,$$

$$(\mathrm{C}_c) \qquad H_u = 2e^{-\omega}Q_v, \quad H_v = 2e^{-\omega}R_u.$$

Here the functions $Q = \langle F_{uu}, N\rangle$ and $R = \langle F_{vv}, N\rangle$ define global null 2-differentials $Q^{\#} = Qdu^2$ and $R^{\#} = Rdv^2$ on M. These two differentials are called the *Hopf differentials* of M. The Gauss equation implies

$$(2.3) \qquad K = -2\,\omega_{uv}\,e^{-\omega}.$$

Let us denote by $\mathcal{D}$ the discriminant of the characteristic equation $\det(t\mathrm{I} - S) = 0$ for the shape operator S. Here I is the identity transformation of the tangent bundle TM of M. Then by the Gauss equation, we have

$$(2.4) \qquad \mathcal{D} = H^2 - K + c = 4e^{-2\omega}\,QR.$$

The first and second fundamental forms are related by the formula

$$II - HI = Q^{\#} + R^{\#}.$$

This formula implies that the common zero of Q and R coincides with the umbilic point of (M, F). Even if S has real and equal eigenvalues, (M, F) is not necessarily totally umbilic. In fact, there exist timelike surfaces with $QR = 0$ but $II - HI \neq 0$. See Example 3.3.

In the study of timelike surfaces, we also use the following local coordinate system:

Lemma 2.1. *Let $F : M \to \mathfrak{M}^3_1(c)$ be a timelike surface. Then there exists a local coordinate system $(\check{u}, \check{v})$ such that*

$$(2.5) \qquad I = -e^{\check{\omega}}d\check{u}d\check{v}.$$

With respect to this coordinate system, the Gauss-Codazzi equations are

$$(\mathrm{G}_c^-) \qquad \check{\omega}_{\check{u}\check{v}} - \frac{1}{2}(H^2 + c)e^{\check{\omega}} + 2\check{Q}\check{R}e^{-\check{\omega}} = 0,$$

$$(\mathrm{C}_c^-) \qquad H_{\check{u}} = -2e^{-\check{\omega}}\check{Q}_{\check{v}}, \quad H_{\check{v}} = -2e^{-\check{\omega}}\check{R}_{\check{u}}$$

for $\check{Q} = \langle F_{\check{u}\check{u}}, N\rangle, \check{R} = \langle F_{\check{v}\check{v}}, N\rangle$.

We call the local coordinate system $(\check{u}, \check{v})$ an *anti-isothermal coordinate system.* Anti-isothermal coordinate systems will be used for introducing the notion of the Christoffel transformation of an anti-isothermic surface. See Proposition 4.14.

3 Timelike HIMC surfaces in Minkowski 3-space

In this section we shall consider a generalization of timelike CMC surfaces in Minkowski 3-space in terms of integrability theory.

We start by recalling the Lax representation for timelike surfaces in $\mathbf{E}^3_1$. Hereater we assume $H \neq 0$.

Let $F : M \to \mathbf{E}^3_1$ be a timelike surface. Let us take an $\mathrm{SL}_2\mathbf{R}$-valued framing Φ defined by $\mathrm{Ad}(\Phi)(\mathbf{i},\ \mathbf{j}',\ \mathbf{k}') = (e^{-\frac{\omega}{2}}F_x,\ e^{-\frac{\omega}{2}}F_y,\ N)$. Thus we get the following Lax representation of the Gauss-Codazzi equations:

$$\frac{\partial}{\partial u}\Phi = \Phi U,\ \frac{\partial}{\partial v}\Phi = \Phi V, \tag{3.1}$$

$$U = \begin{pmatrix} -\frac{1}{4}\omega_u & -Qe^{-\frac{\omega}{2}} \\ \frac{H}{2}e^{\frac{\omega}{2}} & \frac{1}{4}\omega_u \end{pmatrix},\quad V = \begin{pmatrix} \frac{1}{4}\omega_v & -\frac{H}{2}e^{\frac{\omega}{2}} \\ Re^{-\frac{\omega}{2}} & -\frac{1}{4}\omega_v \end{pmatrix}. \tag{3.2}$$

Now we shall insert a *variable spectral parameter* λ , *i.e.*, an additional *real* parameter λ which depends on the coordinates (u, v) into the Lax pair (3.2) in the following way:

$$U_\lambda = \begin{pmatrix} -\frac{1}{4}\omega_u & -Qe^{-\frac{\omega}{2}} \\ \frac{H}{2}\lambda e^{\frac{\omega}{2}} & \frac{1}{4}\omega_u \end{pmatrix},\quad V_\lambda = \begin{pmatrix} \frac{1}{4}\omega_v & -\frac{H}{2}\lambda^{-1}e^{\frac{\omega}{2}} \\ Re^{-\frac{\omega}{2}} & -\frac{1}{4}\omega_v \end{pmatrix}. \tag{3.3}$$

Then the compatibility condition

$$\frac{\partial}{\partial u}V_\lambda - \frac{\partial}{\partial v}U_\lambda + [\ U_\lambda, V_\lambda\ ] = 0 \tag{3.4}$$

for the deformed Lax pair $\{U_\lambda, V_\lambda\}$ yields

$$\omega_{uv} + \frac{1}{2}H^2e^\omega - 2QRe^{-\omega} = 0, \tag{G$_0$}$$

$$Q_v = \frac{e^\omega}{2}(H\lambda^{-1})_u, \quad R_u = \frac{e^\omega}{2}(H\lambda)_v. \tag{3.5}$$

The Lax pair $\{U_\lambda, V_\lambda\}$ describes a timelike surface in $\mathbf{E}^3_1$ if and only if the equations (3.5) are consistent with the Codazzi equations (C_0). The equations (3.5) are consistent with (C_0) if and only if

$$\frac{\partial}{\partial v}\{H(1-\lambda)\} = 0, \quad \frac{\partial}{\partial u}\{H(1-\lambda^{-1})\} = 0. \tag{3.6}$$

These equations (3.6) can be solved easily as follows:

$$H = \frac{1}{f(u)+g(v)}, \qquad \lambda = -\frac{g(v)}{f(u)}, \tag{3.7}$$

where $f(u)$ and $g(v)$ are smooth functions. It is easy to see that the mean curvature H is invariant under the one parametric deformation

$$f \longmapsto f + \frac{1}{2\tau}, \quad g \longmapsto g - \frac{1}{2\tau}, \quad \tau \in \mathbf{R}^*.$$

Under this deformation, the spectral parameter λ is transformed as

$$\lambda = -\frac{g}{f} \longmapsto \lambda(u,v;\tau) = \frac{1-2\tau g}{1+2\tau f}, \quad \tau \in \mathbf{R}.$$

Note that $\lambda(u,v;0) \equiv 1$. The form (3.7) of H is equivalent to the Lorentz-harmonicity of $1/H$, *i.e.*, $(1/H)_{uv} = 0$. As in Euclidean surface geometry [2] and spacelike surface geometry [10], we shall call a timelike surface M in $\mathbf{E}^3_1$, a *timelike surface with harmonic inverse mean curvature* (THIMC surface) if $1/H$ is a Lorentz-harmonic function.

Here we would like to exhibit three elementary examples of THIMC surfaces.

Example 3.1. (THIMC cylinders.) Let $a(y) = (a_2(y), a_3(y))$ be a curve in Euclidean plane $\mathbf{E}^2(\xi_2, \xi_3)$ parametrized by the arclength parameter $y \in \mathcal{I}$. Here $\mathcal{I}$ is an interval. *A timelike cylinder over the curve* a is a flat timelike surface in $\mathbf{E}^3_1$ defined by the immersion $F : \mathcal{I} \times \mathbf{R} \longrightarrow \mathbf{E}^3_1$; $F(x,y) = (x, a_2(y), a_3(y))$. It is straightforward to see that the mean curvature of the cylinder is $H = \kappa(y)/2$. Here κ is the curvature of a. Thus the cylinder F is a THIMC surface if and only if the base curve has the curvature $\frac{1}{C_1 y + C_2}$, $C_1, C_2 \in \mathbf{R}$. It is well known that curves with curvature $\frac{1}{C_1 y + C_2}$ are logarithmic spirals or circles. Hence all the THIMC cylinders over a Euclidean curve are cylinders over a logarithmic spiral or a circular cylinder.

Example 3.2. (THIMC cylinders over timelike curves.) Let $a(x) = (a_1(x), a_2(x))$ be a timelike curve in Minkowski plane $\mathbf{E}^2_1(\xi_1, \xi_2)$ parametrized by the proper time parameter x defined on an interval $\mathcal{I}$. *A timelike cylinder over the timelike curve a* is a flat timelike surface in $\mathbf{E}^3_1$ defined by the immersion $F : \mathcal{I} \times \mathbf{R} \longrightarrow \mathbf{E}^3_1;\ \ F(x, y) = (a_1(x), a_2(x), y)$. The mean curvature of F is $H = \kappa(x)/2$. Here κ is the curvature of a. The cylinder F is THIMC if and only if $1/\kappa = C_1 x + C_2$, $C_1, C_2 \in \mathbf{R}$.

We can see that timelike curves with curvature $\frac{1}{C_1 x + C_2}$ are logarithmic pseudo-spirals or timelike hyperbolas (*cf.* the Appendix of [10]). Hence all the THIMC cylinders are cylinders over a logarithmic pseudo-spiral or a timelike hyperbola.

Example 3.3. (B-scrolls.) A curve $\gamma(s)$ in $\mathbf{E}^3_1$ is said to be a *null Frenet curve* if it admits a frame field $\mathcal{L} = (A, B, C)$ along γ (called a *null frame field*) such that $A = \gamma'$, $\langle A, A\rangle = \langle B, B\rangle = 0$, $\langle A, B\rangle = 1$, $\langle C, C\rangle = 1$, $\langle A, C\rangle = \langle B, C\rangle = 0$ and

$$\frac{d}{ds}\mathcal{L} = \mathcal{L}\begin{pmatrix} 0 & 0 & -\tau \\ 0 & 0 & -\kappa \\ \kappa & \tau & 0 \end{pmatrix}.$$

The functions κ and τ are called the *curvature* and *torsion* of γ respectively. The ruled surface $F(s, t) = \gamma(s) + tB(s)$ is called the *B-scroll* of γ. (See Graves [12] and McNertney [17]). The mean curvature of F is the torsion $\tau(s)$. It is straightforward to check that for any null Frenet curve with $\tau \neq 0$, its B-scroll is a THIMC surface.

Remark. The Gaussian curvature of the B-scroll is τ^2. Thus every B-scroll satisfies $QR = 0$ but is not totally umbilical ($II - HI \neq 0$). The property $QR = 0$ implies that every B-scroll is a timelike Bonnet surface. Here *timelike Bonnet surfaces* are timelike surfaces which admit nontrivial isometric deformation preserving mean curvature [11]. Conversely we have proved that every timelike Bonnet surface with $QR = 0$ is a B-scroll [11].

In [14], we have obtained a one-parameter "isometric" deformation of timelike surfaces with constant mean curvature (TCMC surfaces). For THIMC surfaces in $\mathbf{E}^3_1$, we get the following one-parameter family of "conformal" deformations.

Proposition 3.4. *Let $F : M \to \mathbf{E}^3_1$ be a timelike surface with harmonic inverse mean curvature. Let us express the mean curvature H as*

$$H = \frac{1}{f(u) + g(v)}$$

in terms of null coordinate system (u, v). Here $f(u)$ and $g(v)$ are smooth functions. Then F admits the following Lax representation with variable spectral parameter $\lambda(u, v; \tau) = (1 - 2\tau g(v))/(1 + 2\tau f(u))$, $\tau \in \mathbf{R}$:

$$\frac{\partial}{\partial u}\Phi_\lambda = \Phi_\lambda U_\lambda, \ \frac{\partial}{\partial v}\Phi_\lambda = \Phi_\lambda V_\lambda, \tag{3.8}$$

$$U_\lambda = \begin{pmatrix} -\frac{1}{4}\omega_u & -Qe^{-\frac{\omega}{2}} \\ \frac{H}{2}\lambda e^{\frac{\omega}{2}} & \frac{1}{4}\omega_u \end{pmatrix}, \quad V_\lambda = \begin{pmatrix} \frac{1}{4}\omega_v & -\frac{H}{2}\lambda^{-1}e^{\frac{\omega}{2}} \\ Re^{-\frac{\omega}{2}} & -\frac{1}{4}\omega_v \end{pmatrix}.$$

Let $\Phi_\lambda(u, v)$ be a solution of (3.8). Then

$$F_\lambda = -\frac{\partial}{\partial \tau}\Phi_\lambda \cdot \Phi_\lambda^{-1} \tag{3.9}$$

describes a family of THIMC surfaces through $F = F_\lambda|_{\tau=0}$ with Gauss map $N_\lambda = \mathrm{Ad}(\Phi_\lambda)\ \mathbf{k}'$. The fundamental associated quantities of F_λ are given as follows:

$$I_\lambda = \frac{e^\omega du dv}{(1 + 2\tau f)^2(1 - 2\tau g)^2}, \tag{3.10}$$

$$\frac{1}{H_\lambda} = f_\lambda + g_\lambda, \ f_\lambda = \frac{f}{(1 + 2\tau f)}, \ g_\lambda = \frac{g}{(1 - 2\tau g)}, \tag{3.11}$$

$$Q_\lambda = \frac{Q}{(1 + 2\tau f)^2}, \quad R_\lambda = \frac{R}{(1 - 2\tau g)^2}, \tag{3.12}$$

$$K_\lambda = (1 + 2\tau f)(1 - 2\tau g)K, \tag{3.13}$$

$$H_\lambda^2/K_\lambda \equiv H^2/K. \tag{3.14}$$

The formula (3.14) implies that the members of the one parameter family F_λ have the same ratio of principal curvatures.

4 Timelike HIMC surfaces in Minkowski 3-space

In the study of HIMC surfaces in Riemannian space forms, *isothermic surfaces* play a fundamental role. In this section we shall consider such surfaces in timelike surface geometry.

Definition 4.1. Let $F : M \to \mathfrak{M}^3_1(c)$ be a timelike surface. Then (M, F) is said to be *isothermic* if there exists a local isothermal–curvature line coordinate system around any point of M.

An isothermal–curvature line coordinate system is a local Lorentz–isothermal coordinate system such that both parameter curves are curvature lines. It should be remarked that the isothermic property implies the positivity of the discriminant $\mathcal{D}$ for the characteristic equation for the shape operator S. Equivalently, every isothermic timelike surface has real distinct principal curvatures.

The isothermic property for timelike surfaces in $\mathfrak{M}^3_1(c)$ can be reformulated in terms of *associated null coordinate system* as follows.

Proposition 4.2. *A timelike surface (M, F) is isothermic if and only if there exists a local null coordinate system (u, v) around any point of M such that the Hopf differentials take the following form:*

$$(4.1)\quad Q(u,v) = \frac{1}{2}\mathfrak{q}(u,v)\varrho(u), \quad R(u,v) = \frac{1}{2}\mathfrak{q}(u,v)\sigma(v), \quad \varrho > 0, \ \sigma > 0.$$

Here $\mathfrak{q}$ is a real smooth function and ϱ and σ are positive Lorentz holomorphic and anti-holomorphic functions respectively.

Remark. On a Lorentz surface M with null coordinate system (u, v), a smooth function f on M depending only on u [resp. v] is called a *Lorentz holomorphic function* [resp. *Lorentz anti-holomorphic function*].

Hereafter we shall call a null coordinate system derived from an isothermic coordinate system simply an *isothermic coordinate system.*

Remark. Isothermic timelike surfaces in $\mathbf{E}^3_1$ correspond to solutions of the Zoomeron equation studied in soliton theory. Note that the Zoomeron equation is related to the Davey-Stewartson III-equation. See Schief [19, p. 97].

Typical examples of isothermic timelike surfaces are timelike surfaces of revolution in $\mathbf{E}^3_1$. Let us recall the notion of timelike surfaces of revolution in $\mathbf{E}^3_1$. A *revolution of* $\mathbf{E}^3_1$ is a linear isometry which lies in the identity component $\mathrm{O}_1^{++}(3)$ of the Lorentz group $\mathrm{O}_1(3)$. Every revolution fixes a line pointwise. Such a fixed line of a revolution is called the *axis of* revolution. Hence revolutions of $\mathbf{E}^3_1$ can be characterised by the causal character of the axis.

By a *timelike surface of revolution* in $\mathbf{E}^3_1$ we mean a timelike surface obtained by revolving about an axis a regular curve lying in some plane containing the axis [17].

Example 4.3. (Spacelike axis and Euclidean profile curve.) Let $F : M \longrightarrow \mathbf{E}^3_1$ be a timelike surface of revolution with spacelike axis and Euclidean profile curve. Then there exists an isothermic parametrization

$$F(x,y) = \frac{1}{a}\left(e^{\frac{\omega(y)}{2}}\sinh(ax), e^{\frac{\omega(y)}{2}}\cosh(ax), c(y)\right), \quad a \in \mathbf{R}^*$$

so that $c'(y)^2 e^{-\omega(x)} + \left(\frac{\omega'(y)}{2}\right)^2 = a^2$. With respect to this isothermic coordinate system, the mean curvature is given by

$$H(y) = \frac{1}{8c'(y)}\{4a^2 - \omega'(y)^2 - 2\omega''(y)\}, \ c''(y) = e^{\omega(y)}\omega'(y)H(y).$$

Example 4.4. (Spacelike axis and timelike profile curve.) Let $F : M \longrightarrow \mathbf{E}^3_1$ be a timelike surface of revolution with spacelike axis and timelike profile curve. Then there exists an isothermic parametrization

$$F(x,y) = \frac{1}{a}\left(e^{\frac{\omega(x)}{2}}\cosh(ay), e^{\frac{\omega(x)}{2}}\sinh(ay), c(x)\right), \quad a \in \mathbf{R}^*$$

so that $-c'(x)^2 e^{-\omega(x)} + \left(\frac{\omega'(x)}{2}\right)^2 = a^2$. With respect to this isothermic coordinate system, the mean curvature is given by

$$H(x) = \frac{1}{8c'(x)}\{4a^2 - \omega'(x)^2 - 2\omega''(x)\}, \ c''(x) = -e^{\omega(x)}\omega'(x)H(x).$$

Example 4.5. (Timelike axis.) Let $F : M \longrightarrow \mathbf{E}^3_1$ be a timelike surface of revolution with timelike axis. Then there exists an isothermic parametrization

$$F(x,y) = \frac{1}{a}\left(c(x), e^{\frac{\omega(x)}{2}}\cos(ay), e^{\frac{\omega(x)}{2}}\sin(ay),\right), \quad a \in \mathbf{R}^*$$

so that $c'(x)^2 e^{-\omega(x)} - \left(\frac{\omega'(x)}{2}\right)^2 = a^2$. With respect to this isothermic coordinate system, the mean curvature is given by

$$H(x) = -\frac{1}{8c'(x)}\{2\omega''(x) + \omega'(x)^2 + 4a^2\}, \quad c''(x) = -e^{\omega(x)}\omega'(x)H(x).$$

Example 4.6. (Null axis.) Let $F : M \longrightarrow \mathbf{E}^3_1$ be a timelike surface of revolution with null axis. Then there exists a null basis $\{L_1, L_2, L_3\}$ of $\mathbf{E}^3_1$ and an isothermic parametrization

$$F(x,y) = \left(a(x), b(x) - \frac{y^2}{2}a(x), ya(x)\right), \quad a \in \mathbf{R}^*$$

relative to the null basis $\{L_1, L_2, L_3\}$ so that $2a'(x)b'(x) = -a(x)^2$. Here a linear null frame means a basis of $\mathbf{E}^3_1$ such that $\langle L_1, L_1\rangle = \langle L_2, L_2\rangle = 0$, $\langle L_1, L_2\rangle = 1$, $\langle L_3, L_3\rangle = 1$, $\langle L_1, L_3\rangle = \langle L_2, L_3\rangle = 0$. With respect to this isothermic parametrization, the mean curvature of F is given by

$$H = \frac{a''(x)a(x) + a'(x)^2}{4a(x)^2 a'(x)}.$$

Proposition 4.7. *For any THIMC surface of revolution with non-constant mean curvature, there exists an isothermic coordinate system* (x, y) *such that* $H(x) = 1/x$ *or* $H(y) = 1/y$.

Proposition 4.8. *Let* $F : M \longrightarrow \mathbf{E}^3_1$ *be a timelike surface of revolution with spacelike axis and Euclidean profile curve parametrized as in Example* 4.3 *with harmonic inverse mean curvature* $1/H = y$ *and* $a = 2$. *Then there exists a real valued function* ϕ *such that*

$$e^{\omega(y)} = \frac{y^2}{4}\left\{\phi'(y) + 2\sin\phi(y)\right\}^2, \quad c(y) = -\frac{y^2}{4}\{\phi'(y)^2 - 4\sin^2\phi(y)\}.$$

Furthermore ϕ *is a solution to the third Painlevé equation of trigonometric form:*

$$y\{\phi''(y) - 2\sin(2\phi(y))\} + \phi'(y) + 2\sin\phi(y) = 0. \tag{4.2}$$

Proposition 4.9. *Let $F : M \longrightarrow \mathbf{E}^3_1$ be a timelike surface of revolution with spacelike axis and timelike profile curve parametrized as in Example* 4.4 *with harmonic inverse mean curvature* $1/H = x$ *and* $a = 2$*. Then there exists a real valued function* ϕ *such that*

$$e^{\omega(x)} = \frac{x^2}{4}\left\{\phi'(x) - 2\sinh\phi(x)\right\}^2, \quad c(x) = \frac{x^2}{4}\{\phi'(x)^2 - 4\sinh^2\phi(x)\}.$$

Furthermore ϕ *is a solution to the third Painlevé equation of hyperbolic form:*

$$x\left\{\phi''(x) - 2\sinh(2\phi(x))\right\} + \phi'(x) \mp 2\sinh\phi(x) = 0. \tag{4.3}$$

Proposition 4.10. *Let $F : M \longrightarrow \mathbf{E}^3_1$ be a timelike surface of revolution with timelike axis parametrized as in Example* 4.5 *with harmonic inverse mean curvature* $1/H = x$ *and* $a = 2$*. Then there exists a real valued function* ϕ *such that*

$$e^{\omega(x)} = \frac{x^2}{4}\left\{\phi'(x) + 2\cosh\phi(x)\right\}^2, \quad c(x) = \frac{x^2}{4}\{\phi'(x)^2 - 4\cosh^2\phi(x)\}.$$

Furthermore ϕ *is a solution to the ordinary differential equation*

$$x\left\{\phi''(x) - 2\sinh(2\phi(x))\right\} - \phi'(x) - 2\cosh\phi(x) = 0. \tag{4.4}$$

Remark 4.11. The ordinary differential equations (4.2) and (4.3) are related to the third Painlevé equation. More precisely let $w = w(x)$ be a solution to the third Painlevé equation:

$$w'' - \frac{1}{w}(w')^2 + \frac{w'}{x} - \frac{\alpha w^2 - \alpha}{x} - \frac{\gamma}{w^3} - \frac{\gamma}{w} = 0 \tag{P_{III}}$$

with unit modulus, *i.e.*, $w(x) = e^{\sqrt{-1}\psi(x)}$ for some real valued function $\psi(x)$. Then (P_{III}) is equivalent to the following ordinary differential equation:

$$x\left\{\psi''(x) + 2\gamma\sin(2\psi(x))\right\} + \psi'(x) + 2\alpha\sin\psi(x) = 0.$$

If we choose $\alpha = \gamma = 1$ then we get (4.2). In addition, if we complexify the above third Painlevé equation in trigonometric form and put $\psi = \sqrt{-1}\phi$ then ϕ satisfies

$$x\left\{\phi''(x) + 2\gamma\sinh(2\phi(x))\right\} + \phi'(x) + 2\alpha\sinh\phi(x) = 0.$$

If we choose $\alpha = \mp 1$ and $\gamma = -1$ then we get (4.3).

Timelike HIMC surfaces of revolution with null axis can be classified as follows:

Proposition 4.12. *Let $F : M \longrightarrow \mathbf{E}^3_1$ be a timelike surface of revolution with null axis parametrized as in Example* 4.6 *with harmonic inverse mean curvature $1/H = 4x$. Then the function $a(x)$ is a solution to the following ordinary differential equation:*

$$x\left\{a''(x)a(x) + a'(x)^2\right\} = a^2(x)a'(x). \tag{4.5}$$

This ordinary differential equation can be explicitly solved by quadratures. In fact the solution $a(x)$ is given as follows.

$$12\int \frac{a}{2a^3 + 3a^2 + c_1}da = 2\log|x| + c_2, \ \ c_1, c_2 \in \mathbf{R}. \tag{4.6}$$

Next, to study timelike surfaces with imaginary principal curvatures we shall introduce the notion of *anti-isothermic surface.*

Definition 4.13. Let $F : M \rightarrow \mathfrak{M}^3_\nu(c)$ be a timelike surface. A null coordinate system (u, v) is said to be *anti-isothermic* if its Hopf differentials take the following form:

$$Q(u,v) = \frac{1}{2}\mathfrak{q}(u,v)\varrho(u), \quad R(u,v) = -\frac{1}{2}\mathfrak{q}(u,v)\sigma(v), \quad \varrho > 0, \sigma > 0. \tag{4.7}$$

In addition (M, F) is said to be *anti-isothermic* if there exists an anti-isothermic coordinate system around any point of M.

Note that anti-isothermic property implies that M has imaginary principal curvatures. In $\mathfrak{M}^3_1(c)$, $c \geq 0$, anti-isothermic surfaces have non negative Gaussian curvature. (See (2.4).)

The following result plays a fundamental role in the study of isothermic timelike surfaces and anti-isothermic timelike surfaces in $\mathbf{E}^3_1$. We write these alternatives together as $\pm$ isothermic.

Proposition 4.14. *Let (M, F) be a $\pm$ isothermic timelike surface in $\mathbf{E}^3_1$ and $(\mathfrak{D}; u, v)$ a simply connected $\pm$ isothermic coordinate region so that*

$$I = e^\omega du dv, \quad Q = \frac{1}{2}\mathfrak{q}(u,v)\varrho(u), \quad R = \pm\frac{1}{2}\mathfrak{q}(u,v)\sigma(v), \quad \varrho > 0, \sigma > 0.$$

Then the formulas

$$F^*_u = e^{-\omega}\varrho F_v,\ F^*_v = \pm e^{-\omega}\sigma F_u,\ N^* = N \tag{4.8}$$

define a $\pm$ *isothermic timelike immersion* $F^* : \mathfrak{D} \to \mathbf{E}^3_1$. *The conformal structure of* $\mathfrak{D}$ *induced by* F^* *is anti-conformal to the original conformal structure determined by* I. *The fundamental quantities of* F^* *are given as follows*:

(4.9)

$$I^* = \pm e^{\omega^*} dudv = \pm e^{-\omega}\varrho\sigma dudv, \quad H^* = \mathfrak{q}, \quad Q^* = \varrho H/2, \quad R^* = \pm\sigma H/2.$$

The new immersion F^* *is called the Christoffel transform of* F *or dual of* F.

In particular for $\pm$ isothermic THIMC surfaces, we have the following.

Corollary 4.15. *Every* $\pm$ *isothermic THIMC surface in* $\mathbf{E}^3_1$ *is dual to a* $\pm$ *timelike Bonnet surface in* $\mathbf{E}^3_1$ *and vice versa.*

5 The Hazzidakis equation

In this section we shall investigate normal forms of the Gauss equation for THIMC surfaces.

5.1 Timelike surfaces with $\pm$ holomorphic inverse mean curvature

Let $F : M \to \mathbf{E}^3_1$ be a $\pm$ isothermic timelike surface with $\pm$ holomorphic inverse mean curvature.

Without loss of generality we may assume that $1/H = g(v)$. Take a $\pm$ isothermic coordinate system (u, v) such that $Q = \varepsilon R = \mathfrak{q}(u, v)/2$. Here ε denotes the signature $+$ or $-$. The Codazzi equations (C_0) become

$$\mathfrak{q} = \mathfrak{q}(u),\ e^{\omega} = -\frac{\varepsilon g^2 \mathfrak{q}_u}{g_v}. \tag{5.1}$$

Hence we get $\omega_{uv} = 0$ and hence M is flat by (2.3). On the other hand the Gauss equation (G_0) implies

$$e^{2\omega} = \frac{4\varepsilon Q^2}{H^2} = \frac{\varepsilon \mathfrak{q}^2}{H^2}. \tag{5.2}$$

Hence M is isothermic. Moreover (5.1) and (5.2) imply that $g^2\mathfrak{q}_u^2 = \mathfrak{q}^2 g_v^2$. Hence $g\mathfrak{q}_u = \pm\mathfrak{q}g_v$. Thus we have $g(v) = C_1 e^{\alpha v}$, $\mathfrak{q}(u) = C_2 e^{\mu\alpha u}$, $\mu = \pm 1$, $\alpha \in \mathbf{R}$, $C_1, C_2 \in \mathbf{R}^*$, $C_1C_2/\mu < 0$. These formulas show that timelike surfaces with $\pm$ holomorphic inverse mean curvature are the flat Bonnet surfaces with $\pm$holomorphic mean curvature described in [11, Theorem 3.1]. In particular the case $\alpha = 0$ corresponds to timelike CMC cylinders.

Proposition 5.1. *Let M be a timelike surface in $\mathbf{E}_1^3$ with $\pm$ holomorphic inverse mean curvature. If M is $\pm$ isothermic then M is a flat isothermic timelike Bonnet surface.*

The notion of $\pm$ isothermic surface can be generalized to the notion of "(ε, ϑ)-isothermic surface" in the following way:

Definition 5.2. A timelike surface (M, F) is *(ε, ϑ)-isothermic* if there exists a local null coordinate system (u, v) around any point of M such that the Hopf differentials Q and R have the following form:
(5.3)

$$Q(u,v) = \frac{1}{2}(\mathfrak{q}(u,v)+\vartheta)\varrho(u), \quad R(u,v) = \frac{\varepsilon}{2}(\mathfrak{q}(u,v)-\vartheta)\sigma(v), \quad \varrho > 0, \ \sigma > 0.$$

Here $\mathfrak{q}$ is a real smooth function, ϱ and σ are $\pm$ Lorentz-holomorphic functions and ϑ is a real constant. If $\varepsilon = +$ [resp. $\varepsilon = -$], then we call M a *ϑ–isothermic* surface [resp. an anti-*ϑ–isothermic* surface].

Note that the constant ϑ has no global meaning, in fact, ϑ depends on the choice of (u, v).

Proposition 5.3. *Let M be an (ε, ϑ)–isothermic timelike surface with $\vartheta \neq 0$. Then M is $\pm$ isothermic if and only if M is a timelike Bonnet surface.*

Proposition 5.1 can be generalized as follows:

Theorem 5.4. *Let M be an (ε, ϑ)–isothermic timelike surface with Lorentz anti-holomorphic inverse mean curvature $1/H = g(v)$, $\vartheta \neq 0$. Then M is flat and has real distinct principal curvatures.*

(1) *If M is ϑ–isothermic then $g(v) = Ce^{\alpha v}$, $\mathfrak{q}(u) = \vartheta\cosh(\alpha u + \beta)$,*

(2) *If M is anti-ϑ-isothermic then $g(v) = Ce^{\alpha v}$, $\mathfrak{q}(u) = \vartheta\sin(\alpha u + \beta)$, $C \in \mathbf{R}^*$, $\alpha, \beta \in \mathbf{R}$.*

For any (ε, ϑ)–isothermic THIMC surface in $\mathbf{E}^3_1$, we can consider the *dual* Bonnet surface in H^3_1 or S^3_1.

Proposition 5.5. *Let (M, F) be an (ε, ϑ)-isothermic timelike surface in $\mathbf{E}^3_1$ and $(\mathfrak{D}; u, v)$ a simply connected (ε, ϑ)-isothermic coordinate region such that the Hopf differentials take the following forms:*

$$Q = \frac{1}{2}(\mathfrak{q}(u,v) + \vartheta), \;\; R = \frac{\varepsilon}{2}(\mathfrak{q}(u,v) - \vartheta).$$

Then

(1) *if $\varepsilon = +$, there exists a timelike immersion*

$$F^* : \mathfrak{D} \longrightarrow \begin{cases} H^3_1(\frac{1}{|\vartheta|}), & \vartheta \neq 0, \\ \mathbf{E}^3_1, & \vartheta = 0. \end{cases}$$

(2) *if $\varepsilon = -$, there exists a timelike immersion*

$$F^* : \mathfrak{D} \longrightarrow \begin{cases} S^3_1(\frac{1}{|\vartheta|}), & \vartheta \neq 0, \\ \mathbf{E}^3_1, & \vartheta = 0. \end{cases}$$

The timelike immersion F^* is called a dual surface of F. In particular if F is a THIMC surface then F^* is a timelike Bonnet surface and vice versa.

Remark. In Section 6, we shall prove a Lawson correspondence between THIMC surfaces in Lorentzian space forms. Combining the duality in the preceding proposition and Lawson correspondence, we get a duality between THIMC surfaces and timelike Bonnet surfaces in H^3_1.

5.2 Timelike surfaces with non ± holomorphic inverse mean curvature

Let $F : M \to \mathbf{E}^3_1$ be a THIMC surface parametrized by a null coordinate system $(\bar{u}, \bar{v})$. Since the reciprocal of the mean curvature of (M, F) is harmonic, the mean curvature H can be written as

$$\frac{1}{H} = f(\bar{u}) + g(\bar{v}). \tag{5.4}$$

Inserting (5.4) in the Codazzi equation (C_0) we get

$$f_{\bar{u}} R_{\bar{u}} = g_{\bar{v}} Q_{\bar{v}}. \tag{5.5}$$

Inserting this formula into the Gauss equation (G_0) we get

$$f_{\bar{u}}\left(\frac{Q_{\bar{u}\bar{v}}}{Q_{\bar{v}}}\right)_{\bar{v}} - Q_{\bar{v}} = \frac{f_{\bar{u}}g_{\bar{v}}}{(f+g)^2}\left(2f_{\bar{u}} - \frac{QR}{R_{\bar{u}}}\right). \tag{5.6}$$

Thanks to (5.5), the equation (5.6) is equivalent to

$$g_{\bar{v}}\left(\frac{R_{\bar{u}\bar{v}}}{R_{\bar{u}}}\right)_{\bar{u}} - R_{\bar{u}} = \frac{f_{\bar{u}}g_{\bar{v}}}{(f+g)^2}\left(2g_{\bar{v}} - \frac{QR}{Q_{\bar{v}}}\right). \tag{5.7}$$

As long as $f_{\bar{u}} \neq 0$, $g_{\bar{v}} \neq 0$, we may assume $\xi := f(\bar{u})$, $\eta := g(\bar{v})$ is a local null coordinate system. With respect to (ξ, η), the Gauss-Codazzi equations (G_0) and (C_0) become:

$$\left(\frac{Q_{\xi\eta}}{Q_\eta}\right)_\eta - Q_\eta = \frac{1}{(\xi+\eta)^2}\left(2 - \frac{QR}{R_\xi}\right), \quad Q_\eta = R_\xi. \tag{5.8}$$

We should remark that every solution $\{Q, R\}$ to

$$2 - \frac{QR}{R_\xi} = 0 \tag{5.9}$$

solves (5.8). Let $\{Q, R\}$ be a solution to (5.9). Then by the Codazzi equations (C_0) and the formula $1/H = \xi + \eta$, we get

$$e^{\omega(\xi,\eta)} = -2(\xi+\eta)^2 R_\xi = -(\xi+\eta)^2 Q(\xi,\eta)R(\xi,\eta).$$

Hence the solution $\{Q, R\}$ to (5.9) defines a THIMC surface if and only if $QR < 0$. Such THIMC surfaces have no Euclidean counterparts. (Compare with the Euclidean case [5, p. 203].)

Hereafter we restrict our attention to (ε, ϑ)–isothermic THIMC surfaces. Namely we assume

$$Q(\xi,\eta) = \frac{1}{2}(\mathfrak{q}(\xi,\eta)+\vartheta)\varrho(\xi), \quad R(\xi,\eta) = \frac{\varepsilon}{2}(\mathfrak{q}(\xi,\eta)-\vartheta)\sigma(\eta), \quad \varrho > 0, \sigma > 0. \tag{5.10}$$

To adapt our computations to [5] and [10], and avoid unnecessary 1/2's, we shall use the following convention:

$$q(u,v) := \frac{\varepsilon}{2}\mathfrak{q}(u,v), \quad \theta := \frac{1}{2}\vartheta.$$

We call (ξ, η) an *(ε, θ)–isothermic coordinate system.*

Inserting (5.10) into (5.5), we get

$$\varepsilon\,\sigma(\eta)q_\xi(\xi,\eta)=\varrho(\xi)q_\eta(\xi,\eta). \tag{5.11}$$

Now we introduce a new null coordinate system (u,v) by

$$u=\int\varrho(\xi)d\xi,\quad v=\int\sigma(\eta)d\eta.$$

Then the formula (5.11) implies that q depends only on $t:=\varepsilon u+v$.

We consider two cases: (1) $2-QR/R_\xi=0$, (2) $2-QR/R_\xi\neq 0$.

Case 1: $2-QR/R_\xi=0$. In this case, the Hopf differentials are given by

$$Q(\xi,\eta)=\varrho(\xi)(\varepsilon q(t)+\theta),\quad R(\xi,\eta)=\sigma(\eta)(q(t)-\varepsilon\theta).$$

$$q(t)=\begin{cases}-\theta\tanh\left(\theta t/2\right), & \theta\neq 0\\ -2/t, & \theta=0.\end{cases} \tag{5.12}$$

Inserting (5.12) into (C_0), we have

$$e^{\omega(u,v)}=\begin{cases}\varepsilon\theta^2\left(\xi(u)+\eta(v)\right)^2/\cosh^2(\theta t/2), & \theta\neq 0,\\ -4\varepsilon\left(\xi(u)+\eta(v)\right)^2/t^2, & \theta=0.\end{cases}$$

These formulas imply that $\varepsilon=+$ for $\theta\neq 0$ and $\varepsilon=-$ for $\theta=0$.

Proposition 5.7. *Let (M,F) be an (ε,θ)–isothermic THIMC surface in $\mathbf{E}^3_1$ with (ε,θ)–isothermic coordinate (ξ,η) of the form* (5.10) *and* (5.11). *If $2R_\xi-QR=0$, then $\varepsilon=+$ for $\theta\neq 0$ and $\varepsilon=-$ for $\theta=0$. The fundamental quantities of (M,F) are given by*

$$Q(u,v)=\frac{\varepsilon q(t)+\theta}{\varrho(\xi(u))},\ R(u,v)=\frac{q(t)-\varepsilon\theta}{\sigma(\eta(v))},\ q(t)=\begin{cases}-\theta\tanh(\frac{\theta t}{2}), & \theta\neq 0\\ -2/t, & \theta=0,\end{cases}$$

$$H(u,v)=\frac{1}{\xi(u)+\eta(v)},\quad I=\begin{cases}\theta^2(\xi(u)+\eta(v))^2dudv/\cosh^2(\frac{\theta t}{2}), & \theta\neq 0,\\ 4(\xi(u)+\eta(v))^2\ dudv/t^2, & \theta=0.\end{cases}$$

The dual surface of (M,F) is given by the following formulas :

(1) If $\theta\neq 0$ then the dual surface F^* in $H^3_1(1/(2|\theta|))$ is defined by the data

$$e^{\omega^*(u,v)}=\frac{\cosh^2(\frac{\theta t}{2})}{\theta^2(\xi(u)+\eta(v))^2},$$

$$Q^*(u,v) = R^*(u,v) = \frac{1}{2(\xi(u)+\eta(v))}, \quad H^*(u,v) = -2\theta \tanh(\frac{\theta t}{2}).$$

The dual surface F^* is an isothermic timelike Bonnet surface in $H_1^3(1/(2|\theta|))$.

(2) If $\theta = 0$ then the dual surface F^* in $\mathbf{E}_1^3$ is defined by the data:

$$e^{\omega^*(u,v)} = \frac{t^2}{4(\xi(u)+\eta(v))^2},$$

$$Q^*(u,v) = -R^*(u,v) = \frac{1}{2(\xi(u)+\eta(v))}, \quad H^*(u,v) = \frac{-4}{t}.$$

The dual surface F^* is an anti-isothermic timelike Bonnet surface in $\mathbf{E}_1^3$.

We call a THIMC surface (M,F) *generic* if (M,F) does not correspond to a solution of $2R_\xi - QR = 0$.

Case 2 : $2R_\xi - QR \neq 0$. In this case, inserting $Q(\xi,\eta) = \varrho(\xi)(\varepsilon q(t) + \theta)$, $R(\xi,\eta) = \sigma(\eta)(q(t) - \varepsilon\theta)$, into (5.8) and by the assumption $2R_\xi - QR \neq 0$, we can define the following function

$$\mathcal{S}(t) = \frac{1}{\varrho(\xi(u))\sigma(\eta(v))(\xi(u)+\eta(v))^2}. \tag{5.13}$$

The following theorem is proved as in [5] and [10].

Theorem 5.8. *There exist three classes– A, B and C– of associated families of generic (ε,θ)–isothermic THIMC surfaces in $\mathbf{E}_1^3$. The immersion function of each family is given by the Sym formula* (3.8) *and* (3.9) *in Proposition* 3.4, *where the data (ω, Q, R, H) in* (3.8) *are determined by*

$$e^{\omega(u,v)} = -2\varepsilon q'(t)(\xi(u)+\eta(v))^2,$$

$$Q(u,v) = \frac{\varepsilon q(t)+\theta}{\varrho(\xi(u))}, \quad R(u,v) = \frac{q(t)-\varepsilon\theta}{\sigma(\eta(v))}, \quad H(u,v) = \frac{1}{\xi(u)+\eta(v)}.$$

Here $q(t)$ is a solution to the generalized Hazzidakis equation:

$$(\bigstar^{-\varepsilon}_{-\theta^2}) \qquad \left(\frac{q''(t)}{q'(t)}\right)' - q'(t) = \mathcal{S}(t)\left(2 - \frac{q^2(t)-\theta^2}{q'(t)}\right), \quad -\varepsilon q'(t) > 0.$$

Here the coefficient function $\mathcal{S}(t)$ in the generalized Hazzidakis equation is given by

Family	*Coefficient*
A-family	$\mathcal{S}(t) = 1/\sin^2(2t)$
B-family	$\mathcal{S}(t) = 1/\sinh^2(2t)$
C-family	$\mathcal{S}(t) = 1/t^2$

Any generic (ε, θ)–isothermic THIMC surface belongs to one of these families A, B or C.

Via the duality between ± 1–isothermic THIMC surfaces in $\mathbf{E}^3_1$ and isothermic timelike Bonnet surfaces in H^3_1, the generalized Hazzidakis equation $(\bigstar^-_{-1})$ coincides with that for isothermic timelike Bonnet surfaces in H^3_1 obtained in [11, Theorem 6.1].

Moreover the generalized Hazzidakis equation $(\bigstar^-_{-1})$ coincides with that for Bonnet surfaces in hyperbolic 3-space H^3. (See [4, Theorem 3.3.1] and [20].) Thus $(\bigstar^-_{-1})$ for A or B-family [respectively, C-family] is solved by the Painlevé transcendents P_{VI} [respectively, P_{V}]. See [4, Theorem 3.5.1, 3.5.2]. Hence Bonnet surfaces in H^3 of non-Willmore type, $(\varepsilon, \pm 1)$–isothermic THIMC surfaces in $\mathbf{E}^3_1$ and (generic) timelike Bonnet surfaces in H^3_1 are derived from P_{V} and P_{VI}. Note that $(\bigstar^+_{\theta^2})$ coincides with generalized Hazzidakis equation for θ–isothermic spacelike HIMC surfaces in $\mathbf{E}^3_1$ (and hence spacelike Bonnet surfaces in H^3_1) [10].

6 Timelike HIMC surfaces in $\mathfrak{M}^3_1(c)$

In this section we shall generalize the notion of THIMC surface in Minkowski 3-space to that of $\mathfrak{M}^3_1(c)$.

Proposition 6.1. *Let $I[c]$ be a 1-dimensional Riemannian manifold defined by*

$$I[c] = \begin{cases} (\mathbf{R}, g[c]) & c = 0,\ 1, \\ (\mathbf{R} \setminus \{\pm 1\}, g[c]) & c = -1, \end{cases} \quad g[c] = \frac{dt^2}{(1+ct^2)^2}.$$

Let $\varphi : M \to I[c]$ be a smooth map from a Lorentz surface M. Then φ is a (Lorentzian) harmonic map if and only if

$$\frac{\partial^2 \varphi}{\partial u \partial v} - \frac{2c\varphi}{1+c\varphi^2} \frac{\partial \varphi}{\partial u} \frac{\partial \varphi}{\partial v} = 0 \tag{6.1}$$

with respect to any (and hence all) null coordinate systems (u, v).

The harmonic map equation (6.1) may be considered as a nonlinear generalization of the classical linear wave equation $\varphi_{uv} = 0$. As is well known, the classical linear wave equation can be solved by the *d'Alembert formula*. The following may be regarded as a nonlinear d'Alembert formula for (6.1).

Proposition 6.2. *The harmonic map equation* (6.1) *can be solved as follows*:

$$\varphi(u,v) = \begin{cases} f(u) + g(v), & c = 0, \\ \frac{f(u)+g(v)}{1-cf(u)g(v)} \text{ or } \frac{1-cf(u)g(v)}{f(u)+g(v)}, & c = \pm 1. \end{cases}$$

The following definition is a generalization of that in Section 3.

Definition 6.3. Let $F : M \to \mathfrak{M}^3_1(c)$ be a timelike surface. Then M is said to be a *timelike surface with harmonic inverse mean curvature* (THIMC surface) if $1/H$ is a harmonic map into $I[c]$.

Hereafter we assume that M is *simply connected.* We denote by $\mathcal{C}_H$ the *moduli space* of conformal immersions of M into $\mathfrak{M}^3_1(c)$ with prescribed mean curvature H:

$\mathcal{C}_H = \{F : M \to \mathfrak{M}^3_1(c) \mid$ a conformal timelike immersion with mean curvature $H\ \}/\mathfrak{I}_0(c)$.

Here $\mathfrak{I}_0(c)$ is the identity component of the full isometry group of $\mathfrak{M}^3_1(c)$. Then we can deduce (by the fundamental theorem of surface theory) that

$$\mathcal{C}_H \cong \{(\omega, Q, R) \mid \text{a solution to } (\mathrm{G}_c) \text{ and } (\mathrm{C}_c) \text{ with mean curvature } H\ \}.$$

Theorem 6.4. (Generalized Lawson Correspondences) *Let M be a simply connected Lorentz surface, f a holomorphic function and g an anti-holomorphic function on M. We define a function H_c by $H_c := (1 - cfg)/(f + g)$. Then the three moduli spaces $\mathcal{C}_{H_0}$, $\mathcal{C}_{H_1}$, $\mathcal{C}_{H_{-1}}$ are mutually isomorphic.*

Proof. Let (ω, Q, R, H_c) be a solution of (G_c) and (C_c) for $c = \pm 1$.

Then $(\tilde{\omega}, \tilde{Q}, \tilde{R}, H_0)$ defined by

$$e^{\tilde{\omega}} := (1+cf^2)(1+cg^2)\, e^{\omega}, \quad \tilde{Q} := (1+cf^2)Q, \quad \tilde{R} := (1+cg^2)R$$

is a solution to (G_c) and (C_c). Note that in the case $c=-1$, the function $(1+cf^2)(1+cg^2)$ is positive if and only if $H_{-1}^2 > 1$. ■

Theorem 6.4 may be considered as a generalization of the so-called *Lawson correspondences* for timelike CMC surfaces.

Remark. In the Riemannian case, (non CMC) HIMC surfaces in H^3 have Lawson correspondents if and only if $H^2 > 1$. On the other hand, in the spacelike case, (non CMC) spacelike HIMC surfaces in S_1^3 have Lawson correspondents if and only if $H^2 > 1$. See [7], [10].

Using the Lawson correspondences described above, we can give an immersion formulas for THIMC surfaces in $\mathfrak{M}_1^3(c)$, $c = \pm 1$. Before doing so, we point out the following invariance of (6.1). Let φ be a solution to (6.1) of the form:

$$\varphi(u,v) = \frac{f(u)+g(v)}{1-cf(u)g(v)}.$$

Then the replacements $f \longmapsto 2\tau f$, $g \longmapsto 2\tau g$, $\tau \in \mathbf{R}^*$ produce a new solution to (6.1). More precisely, the function $\varphi[\tau]$ defined by

$$\varphi[\tau](u,v) = \frac{2\tau(f(u)+g(v))}{1-4c\tau^2 f(u)g(v)}$$

is still a solution to (6.1).

Let Φ_λ be a solution of the zero curvature equations (3.8) with variable spectral parameter λ. To describe immersion formulas we shall use the following notational convention.

$$\Phi[\tau] := \Phi_\lambda, \ \lambda = (1-2\tau g)/(1+2\tau f), \quad \tau \in \mathbf{R}.$$

Since the zero curvature equation (3.8) is completely integrable, (3.8) has also solutions for all $\tau \in \mathbf{C}$.

Direct computations similar to those in [1], [7] and [10] show the following.

Theorem 6.5. (Immersion formulas) *Let* $\Phi[\tau] : M \times \mathbf{C} \to G^{\mathbf{C}}$ *be a complexified solution to* (3.8). *Then the following hold.*

($c = 0$) *For every* $\tau \in \mathbf{R}$, $F^{(0)}(\tau) := -\frac{\partial}{\partial \tau}\Phi[\tau] \cdot \Phi[\tau]^{-1}$, $\tau \in \mathbf{R}$ *describes a THIMC surface in* $\mathbf{E}^3_1$ *given in* Proposition 3.4.

($c = -1$) *For any* $\tau \in \mathbf{R}^*$, $F^{(-1)}(\tau) := p_H(\Phi[\tau], \Phi[-\tau])$ *is a THIMC surface in* H^3_1 *with unit normal vector field* $N = -\mu_H(\Phi[\tau], \Phi[-\tau])\mathbf{k}'$.

($c = 1$) *Let* $\Phi[\sqrt{-1}\tau]$, $\tau \in \mathbf{R}$ *be a complexified solution to* (3.8). *Then for every* $\tau \in \mathbf{R}$ $F^{(1)}(\tau) := p_S(\Phi[\sqrt{-1}\tau])$ *is a THIMC surface in* S^3_1 *with unit normal vector field* $N = -\mu_S(\Phi[\sqrt{-1}\tau])\mathbf{j}'$.

The first fundamental form of $F^{(c)}$, $c = \pm 1$ *is*

$$I^{(c)}(\tau) = \frac{4\tau^2 e^{\omega}}{(1 + 4c\tau^2 f^2)(1 + 4c\tau^2 g^2)}.$$

The mean curvature of $F^{(c)}(\tau)$, $c = \pm 1$ *is*

$$H = \frac{1 - 4c\tau^2 fg}{2\tau(f+g)}.$$

Moreover the mean curvature of $F^{(-1)}(t)$ *satisfies* $H^2 > 1$. *In particular, for* $c = \pm 1$, $F^{(\pm 1)}(1/2)$ *is the Lawson correspondent of* $F = F^{(0)}$. *The conformal deformations of THIMC surfaces in* $\mathfrak{M}^3_1(c)$ *preserve* $K/(H^2 + c)$.

Remark. The conformal deformation of HIMC surfaces in Riemannian space forms [respectively, spacelike HIMC surfaces in Lorentzian space forms] preserves $K/(H^2 + c)$ [respectively, $K/(H^2 - c)$] Note that in the case $c = 0$, the constancy of $K/(H^2 - c)$ is equivalent to the constancy of the ratio of principal curvatures.

Computing the Gaussian curvature or $K/(H^2 + c)$, we have the following theorem.

Theorem 6.7. *Let* (M, F) *be an* (ε, θ)*-isothermic THIMC surface in* $\mathfrak{M}^3_1(c)$.

(1) *If* K *is constant then* $K = 0$ *or* c.

(2) *If* $K/(H^2 + c)$ *is constant then* (M, F) *is a flat timelike Bonnet surface.*

As an application of the Lawson correspondence above, one can classify $\pm$ isothermic flat timelike Bonnet surfaces in Lorentzian space forms. In fact, since the Lawson correspondence preserves the $\pm$ isothermic property or flatness, we have obtained the following ([11, Theorem 6.2]).

Theorem 6.8. *Flat simply connected $\pm$ isothermic timelike Bonnet surfaces in one Lorentzian 3-space form correspond to those in another Lorentzian 3-space form.*

In [11], timelike Bonnet surfaces in $\mathfrak{M}^3_1(c)$ with constant Gaussian curvature are classified.

Finally we consider THIMC surfaces in H^3_1 with mean curvature $H^2 < 1$. To investigate such surfaces, we use the following invariance of (6.1) with $c = \pm 1$. Let H be a solution of (6.1) of the form:

$$H(u,v) = \frac{f(u) + g(v)}{1 - cf(u)g(v)}.$$

Then for any $\tau \in \mathbf{R}^*$, the replacements $f \longmapsto \tau f, \ \ g \longmapsto \tau^{-1} g$ produce a new solution of (6.1). Namely the function $H[\tau]$ defined by

$$H[\tau](u,v) = \frac{\tau f(u) + \tau^{-1} g(v)}{1 - cf(u)g(v)}$$

is also a solution of (6.1). Based on this deformation, we define two auxiliary functions (*variable spectral parameters*):

$$\lambda(u,\tau) := \frac{\tau(1 - cf(u)^2)}{\tau^2 - cf(u)^2}, \quad \nu(v,\tau) := \frac{\tau(1 - cg(v)^2)}{\tau^2 - cg(v)^2}.$$

Then we have the following.

Theorem 6.9. *Let $\Psi[\tau]$ be a solution to*

$$\frac{\partial}{\partial u}\Psi[\tau] = \Psi[\tau]U[\tau], \quad \frac{\partial}{\partial v}\Psi[\tau] = \Psi[\tau]V[\tau], \tag{6.2}$$

$$U[\tau] = \begin{pmatrix} -\frac{1}{4}\omega_u & -Qe^{-\omega/2} \\ \frac{1}{2}(H[\tau] + c)\lambda e^{\omega/2} & \frac{1}{4}\omega_u \end{pmatrix},$$

$$V[\tau] = \begin{pmatrix} \frac{1}{4}\omega_v & -\frac{1}{2}(H[\tau] - c)\nu e^{\omega/2} \\ Re^{-\omega/2} & -\frac{1}{4}\omega_v \end{pmatrix}.$$

Then for any $\tau \in \mathbf{R}^*$, $F^{(-1)}[\tau](u,v) := p_H(\Psi[\tau], \Psi[-\tau])$ is a THIMC surface in H_1^3 with unit normal vector field $N = -\mu_H(\Psi[\tau], \Psi[-\tau])\mathbf{k}'$ and mean curvature $H[\tau]$. The first fundamental form of $F[\tau]$ is given by

$$I^{(-1)}[\tau] = \frac{\tau^2(1-cf(u)^2)(1-cg(v)^2)e^{\omega}}{(\tau^2-cf(u)^2)(\tau^2-cg(v)^2)}dudv.$$

Since the Lax equation (6.2) with two variable spectral parameters λ and ν is completely integrable, (6.2) has solutions for all $\tau \in \mathbf{C}$. Such complexified solutions $\Psi[\tau]$ to (6.2) describe another kind of surface in S_1^3.

Theorem 6.10. *Let $\Psi[\tau] : M \times \mathbf{C} \to G^{\mathbf{C}}$ be a complexified solution to (6.2). Then for any $\tau \in \mathbf{R}^*$, $F^{(1)}[\tau](u,v) = p_S(\Psi[\sqrt{-1}\tau])$ is a timelike surface in S_1^3 with unit normal vector field $N = \mu_S(\Psi[\sqrt{-1}\tau])\mathbf{j}'$ and mean curvature $(\tau f - \tau^{-1}g)/(1-cfg)$. The first fundamental form of $F^{(1)}[\tau]$ is given by*

$$I^{(1)}[\tau] = \frac{\tau^2(1-cf(u)^2)(1-cg(v)^2)e^{\omega}}{(\tau^2+cf(u)^2)(\tau^2+cg(v)^2)}dudv.$$

The inverse mean curvature of $F^{(1)}[\tau]$ in Theorem 6.10 is a harmonic map into $I[-1]$.

References

[1] A. I. Bobenko, Constant mean curvature surfaces and integrable equations, Russian Math. Surveys, **46** (1991), 1–45.

[2] A. I. Bobenko, Surfaces in terms of 2 by 2 matrices. Old and new integrable cases, In: Harmonic Maps and Integrable Systems, (eds. A. P. Fordy and J. C. Wood), Aspects of Math., **E 23**, Vieweg, Braunschweig.

[3] A. I. Bobenko and U. Eitner, Bonnet surfaces and Painlevé equations, J. Reine Angew. Math., **499** (1998), 47–79.

[4] A. I. Bobenko and U. Eitner, Painlevé Equations in Differential Geometry of Surfaces, Lecture Notes in Math., **1753** (2000), Springer-Verlag.

[5] A. I. Bobenko, U. Eitner and A. Kitaev, Surfaces with harmonic inverse mean curvature and Painlevé equations, Geom. Dedicata, **68** (1998), 187–227.

[6] S. P. Burtsev, V. E. Zakharov and A. V. Mikhairov, Inverse scattering method with variable spectral parameter, Theoret. and Math. Phys., **70** (1987), 227–240.

[7] A. Fujioka, Surfaces with harmonic inverse mean curvature in space forms, Proc. Amer. Math. Soc., **127** (1999), 3021–3025.

[8] A. Fujioka and J. Inoguchi, Bonnet surfaces with constant curvature, Results Math., **33** (1998), 288–293.

[9] A. Fujioka and J. Inoguchi, On some generalisations of constant mean curvature surfaces, Lobachevskii J. Math., **3** (1999), 73–95, `http:ljm.ksu.ru/vol3/fujioka.htm`.

[10] A. Fujioka and J. Inoguchi, Spacelike surfaces with harmonic inverse mean curvature, J. Math. Sci. Univ. Tokyo, **7** (2000), 657–698..

[11] A. Fujioka and J. Inoguchi, Timelike Bonnet surfaces in Lorentzian space forms, Diff. Geom. Appl., **18** (2003), 103–111.

[12] L. K. Graves, Codimension one isometric immersions between Lorentz spaces, Trans. Amer. Math. Soc., **252** (1979), 367–392.

[13] J. N. Hazzidakis, Biegung mit Erhaltung der Hauptkrümmungsradien, J. Reine Angew. Math., **117** (1897), 42–56.

[14] J. Inoguchi, Timelike surfaces of constant mean curvature in Minkowski 3-space, Tokyo J. Math., **21** (1998), 141–152.

[15] D. A. Korotkin, On some integrable cases in surface theory, J. Math. Sci. (New York), **94** (1999), 1177–1217.

[16] D. A. Korotkin and V. A. Reznik, Bianchi surfaces in $\mathbf{R}^3$ and deformation of hyperelliptic curves, Math. Notes, **52** (1992), 930–937.

[17] L. McNertney, One-parameter families of surfaces with constant curvature in Lorentz 3-space, Ph.D. thesis., Brown Univ., 1980.

[18] B. O'Neill, Semi–Riemannian Geometry with Applications to Relativity, Pure and Applied Math., **130**, Academic Press, 1983.

[19] W. K. Schief, Isothermic surfaces in spaces of arbitrary dimension: integrablity, discretization, and Bäcklund transformations–a discrete Calapso equation, Stud. Appl. Math., **116** (2001), 85–137.

[20] K. Voss, Bonnet surfaces in spaces of constant curvature In: First MSJ International Research Institute on Geometry and Global Analysis, Lecture Notes, **2** (1993), Tôhoku Univ., Sendai.

[21] T. Weinstein, An Introduction to Lorentz Surfaces, de Gruyter Exp. Math., Walter de Gruyter, Berlin.

Atsushi Fujioka
Department of Mathematics
Faculty of Science
Kanazawa University
Kakuma-machi, Kanazawa
Ishikawa 920–1192
JAPAN

Current address:
Graduate School of Economics
Hitotsubashi University
2-1 Naka, Kunitachi
Tokyo 186–8601
JAPAN

E-mail address:
fujioka@math.hit-u.ac.jp

Jun-ichi Inoguchi
Department of Applied Mathematics
Fukuoka University
Fukuoka 814-0180
JAPAN

Current address:
Department of Mathematics Education
Utsunomiya University
Utsunomiya 321-8505
JAPAN

E-mail address:
inoguchi@cc.utsunomiya-u.ac.jp