Advanced Studies in Pure Mathematics 50, 2008
Algebraic Geometry in East Asia — Hanoi 2005
pp. 289–297

Poincaré polynomial of a class of signed complete graphic arrangements

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Abstract.

We compute the Poincaré polynomial of hyperplane arrangements associated with a class of signed complete graphs. We also make a factorization of the Poincaré polynomial over the integers.

§1. Introduction

Let $\mathbf{V}$ be an n-dimensional vector space over a field $\mathbb{K}$. Let S be the symmetric algebra over the dual space $\mathbf{V}^* := \mathrm{Hom}_{\mathbb{K}}(\mathbf{V}, \mathbb{K})$. If $x_1, \ldots, x_n$ is a basis of $\mathbf{V}^*$, then there are identifications $S = \mathbb{K}[x_1, \ldots, x_n]$ and $\mathbf{V} = \mathbb{K}^n$. A hyperplane H in $\mathbb{K}^n$ is by definition the zero set of a degree one polynomial α_H in the variables $x_1, \cdots, x_n$. An arrangement of hyperplanes $\mathcal{A}$ in $\mathbb{K}^n$ is a finite collection of hyperplanes.

Let $L(\mathcal{A})$ be the collection of all non-empty intersections of hyperplanes from $\mathcal{A}$, which is a partial ordered set with the order defined by the inverse inclusion. The rank of an element $X \in L(\mathcal{A})$ is defined by $r(X) = \mathrm{codim}(X)$. Let μ be the Möbius function of $L(\mathcal{A})$, and denote $\mu(X) = \mu(\mathbf{V}, X)$. The Poincaré polynomial of L is defined by $\pi(L, t) = \sum_{X \in L} \mu(X)(-t)^{\mathrm{r}(X)}$.

If $\mathbb{K}$ is the field $\mathbb{C}$ of complex numbers, the complement M of $\mathcal{A}$ is of interest from topological point of view. One of the central topics in studying hyperplane arrangements is to describe the topology of M by

Received March 17, 2006.

Revised June 30, 2006.

2000 *Mathematics Subject Classification.* Primary 52C35; Secondary 06C05, 05C22.

Key words and phrases. signed graph, hyperplane arrangement, free arrangement.

[1]Supported by NSF of China under grant 10671009, and SRF for ROCS, SEM..

[2]Supported by NSF of China under grant 10671009.

the data from $L(\mathcal{A})$. For example, it is well known that the cohomology $H^*(M;\mathbb{C})$ of M is isomorphic to the Orlik-Solomon algebra $\mathcal{OS}(\mathcal{A})$ [7] and the Poincaré polynomial of $\mathcal{OS}(\mathcal{A})$ equals to $\pi(L,t)$.

Graphic arrangements have been studied by a number of authors [5, 4, 3, 12], since they are related to the arrangements associated with classical groups [1, 2, 9, 10]. We study a special class of arrangements associated with signed complete graphs [13] and prove the following factorization formula.

Theorem 1. *If a signed complete graph Σ_n with n vertices is switching equivalent to the signed complete graph $\Sigma_n^{(3)}$ with negative part a triangle, the Poincaré polynomial of the corresponding arrangement $\mathcal{A}(\Sigma_n)$ is*

$$\pi(\mathcal{A}(\Sigma_n),t) = (t+1)(2t+1)\cdots[(n-3)t+1]Q(t), \tag{1}$$

where

$$Q(t) := [3(n-3)(n-2)+1]\,t^3 + (n^2-6)t^2 + (2n-3)t + 1.$$

§2. Preliminaries

2.1. Intersection lattice

An arrangement $\mathcal{A}$ is called central if the intersection of all the hyperplanes in $\mathcal{A}$ is not empty. If $\mathcal{A}$ is central, $L = L(\mathcal{A})$ is a geometric lattice. The minimal element of L is denoted by $\hat{0}$ and the maximal element is denoted by $\hat{1}$.

The meet of $X, Y \in L$ is defined by $X \wedge Y = \cap\{Z \in L \mid Z \supseteq X \cup Y\}$, and if $X \cap Y \neq \emptyset$, their join is defined by $X \vee Y = X \cap Y$. A pair $(X,Y) \in L \times L$ is called a modular pair if for all $Z \leq Y$ one has $Z \vee (X \wedge Y) = (Z \vee X) \wedge Y$. A pair $(X,Y) \in L \times L$ is modular if and only if $r(X) + r(Y) = r(X \vee Y) + r(X \wedge Y)$. An element X is called a modular element if it forms a modular pair with each $Y \in L$.

An element in a geometric lattice L is called an atom if it covers the minimal element of L. If L is the intersection lattice of an arrangement, each hyperplane is an atom of L. Let $A(L)$ be the collection of the atoms in L. For any $X \in L$, let $A(L)_X = \{Y \in A(L) \mid Y \leq X\}$.

Lemma 2. *For $X, Y \in L$,*

$$X \wedge Y = \hat{0} \Longleftrightarrow A(L)_X \cap A(L)_Y = \emptyset.$$

Proof. Obviously, the following formula holds.

$$A(L)_X \cap A(L)_Y = A(L)_{X \wedge Y}.$$

The lemma follows from this formula. ◇

2.2. Stanley Theorem

Let L be a geometric lattice, $\mathcal{OS}(L)$ the Orlik-Solomon algebra generated by the atoms of L. By [7], $\mathcal{OS}(L(\mathcal{A}))$ is the same as $\mathcal{OS}(\mathcal{A})$ for a hyperplane arrangement $\mathcal{A}$. Note that, for $Y \in L$, $L_Y = \{Z \in L \mid Z \leq Y\}$ is also a geometric lattice.

Stanley Theorem [8, 11]. Let L be a geometric lattice, and $X \in L$ be a modular element. Then

$$\pi(\mathcal{OS}(L),t) = \pi(\mathcal{OS}(L_X),t) \sum_{Z \in L, Z \wedge X = \hat{0}} \mu(Z)(-t)^{r(Z)}, \tag{2}$$

where μ is the Möbius function of L.

2.3. Signed complete graph

A signed complete graph $\Sigma_n = (K_n, \sigma)$ consists of an ordinary complete graph K_n with n vertices, and an arc labelling mapping $\sigma : E \longrightarrow \{\pm\}$, where E is the edge set of K_n. Let $E_+ = \sigma^{-1}(+)$ and $E_- = \sigma^{-1}(-)$ denote the sets of the positive and negative edges respectively. An edge $\{ij\} \in E_+$ is denoted by $\{ij\}^+$ and is pictured as a line segment connecting the vertices i and j. An edge $\{ij\} \in E_-$ is denoted by $\{ij\}^-$ and is pictured as a dashed line segment connecting the vertices i and j. For general study of singed graphs we refer the reader to Zaslavsky [13].

Given a signed complete graph $\Sigma_n = (K_n, \sigma)$, define an arrangement $\mathcal{A}(\Sigma_n)$ in $\mathbb{K}^n$ as follows:

$$\{x_i - x_j = 0\} \in \mathcal{A}(\Sigma_n) \text{ if } \{ij\} \in E_+$$

and

$$\{x_i + x_j = 0\} \in \mathcal{A}(\Sigma_n) \text{ if } \{ij\} \in E_-$$

2.4. Switching equivalence

For a signed complete graph $\Sigma_n = (K_n, \sigma)$, we consider the following operations on Σ_n.

1) A permutation of the labels on the vertices of Σ_n;
2) Switching a vertex $i_0 \in [n] = \{1, 2, \ldots, n\}$ is to switch the sign of the edge $\{i_0 i\}$ for each $i \in N_{i_0} := \{i \in [n] \mid \{i_0 i\} \in E\}$, the neighborhood of i_0. We call this *the vertex switching*, or, *switching* i_0.

The first operation is essentially a permutation on the coordinates, which allows one to consider unlabeled graphs. For a vertex v, switching v corresponds to switching the sign of the coordinate x_v. Obviously,

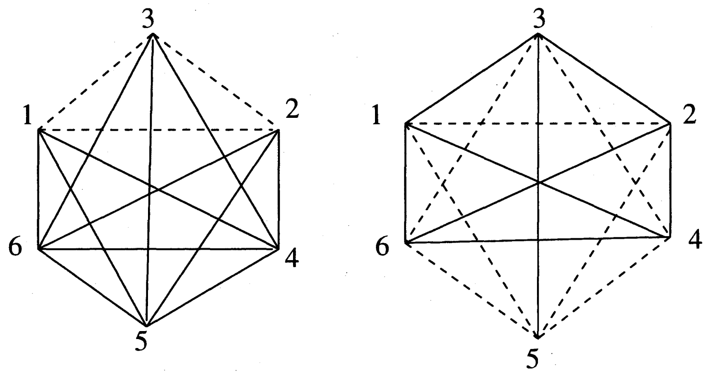

Fig. 1

the vertex switching operations form a group, which acts on the set of signed graphs with fixed number of vertices. Since a coordinate transformation on the vector space $\mathbf{V}$ does not affect the intersection lattice of an arrangement, the first two operations preserve the modular elements.

Two signed graphs Σ_n' and Σ_n'' are *switching equivalent* if there exists a series of vertex switchings such that Σ_n' can be transformed into Σ_n'' up to a permutation of the labels on the vertices. In this case, we also say that the corresponding arrangements are *switching equivalent.* For example, the two signed complete graphs with 6 vertices in figure 1 are switching equivalent by first switching the vertex 5, then the vertex 3 of the graph on the right hand side, one gets the graph on the left hand side.

§3. Modular elements

Signed complete graphs with at most 6 vertices were classified under the switching equivalence in [6]. One class of signed complete graphs with n vertices is denoted by $\Sigma_n^{(3)}$ in which the negative part is a triangle. We denote the vertices of the triangle by 1,2,3 and label the other vertices by 4, ..., n. The arrangement associated with $\Sigma_n^{(3)}$ consists of the following hyperplanes.

$$\begin{array}{llll} H_{12}: \ x_1 + x_2 = 0, & H_{13}: \ x_1 + x_3 = 0, & H_{23}: \ x_2 + x_3 = 0, \\ H_{1m}: \ x_1 - x_m = 0, & H_{2m}: \ x_2 - x_m = 0, & 4 \le m \le n, \\ H_{ij}: \ x_i - x_j = 0, & 3 \le i < j \le n. & \end{array}$$

For the case of $n = 6$ see figure 1.

To simplify notation, in the following we set $L = L(\mathcal{A}(\Sigma_n^{(3)}))$.

Lemma 3. *The element* $X = \bigcap_{3\le i<j\le n} H_{ij} = \{x \in \mathbb{K}^n \mid x_3 = \cdots = x_n\} \in L$ *is modular.*

Proof. It is enough to prove that for each $Y \in L$ with $X \wedge Y = \hat{0}$, (X, Y) is modular pair. This is equivalent to $r(X) + r(Y) = r(X \vee Y)$ since $r(X \wedge Y) = r(\hat{0}) = 0$. By lemma 2 and the definition of rank function, it is sufficient to prove that for each $Y \in L$ with $A(L)_X \cap A(L)_Y = \emptyset$, the equation

$$\dim Y - \dim(X \cap Y) = n - 3 \tag{3}$$

holds.

Let

$$A_1 = \{H_{1j} \mid j = 4, \dots n\}, A_2 = \{H_{2j} \mid j = 4, \dots n\}, A_3 = \{H_{12}, H_{13}, H_{23}\}.$$

Then

$$A(L)_X \cap A(L)_Y = \emptyset \implies A(L)_Y \subset A_1 \sqcup A_2 \sqcup A_3$$

with

$$|A(L)_Y \cap A_m| \le 1, m = 1, 2, \quad |A(L)_Y \cap A_3| \le 3.$$

Hence, there are 16 possibilities for $(|A(L)_Y \cap A_1|, |A(L)_Y \cap A_2|, |A(L)_Y \cap A_3|)$:

$$\begin{array}{cccc} (0,0,0) & (0,1,0) & (1,0,0) & (0,0,1) \\ (1,1,0) & (0,1,1) & (1,0,1) & (0,0,2) \\ (1,1,1) & (0,1,2) & (1,0,2) & (0,0,3) \\ (1,1,2) & (0,1,3) & (1,0,3) & (1,1,3) \end{array} \tag{4}$$

The case $(0, 0, 0)$ can not appear. If $(|A(L)_Y \cap A_1|, |A(L)_Y \cap A_2|, |A(L)_Y \cap A_3|) = (0, 0, 1)$, $A(Y)$ consists of one of the hyperplanes from A_3. Then $\dim Y = n - 1$, $\dim(X \cap Y) = 2$, and equation (3) holds.

If $(|A(L)_Y \cap A_1|, |A(L)_Y \cap A_2|, |A(L)_Y \cap A_3|) = (0, 0, 2)$, $A(Y)$ consists of two of the hyperplanes from A_3. Then $\dim Y = n - 2$, $\dim(X \cap Y) = 1$, and equation (3) holds.

If $(|A(L)_Y \cap A_1|, |A(L)_Y \cap A_2|, |A(L)_Y \cap A_3|) = (0, 0, 3)$, $A(Y) = A_3$. Then $\dim Y = n - 3$, $\dim(X \cap Y) = 0$, and equation (3) holds. Similar treatment will prove the cases $(0, 1, 0), (1, 0, 0), (0, 1, 1), (1, 0, 1)$ and $(1, 1, 0)$.

For the case $(0, 1, 2)$, if $A(Y) \cap A_3 = \{H_{12}, H_{13}\}$ and $A(Y) \cap A_2 = \{H_{2j}\}$ for some $j \ge 4$, then Y is the intersection of $x_1 + x_2 = 0, x_1 +$

$x_3 = 0$, and $x_2 - x_j = 0$. This implies that Y is contained in the hyperplane $H_{3j} : x_3 - x_j = 0$, which is impossible since $X \subset H_{3j}$. If $A(Y) \cap A_3 = \{H_{12}, H_{23}\}$, and $A(Y) \cap A_2 = \{H_{2j}\}$ for some $j \geq 4$, then $\dim Y = n - 3$, $\dim(X \cap Y) = 0$, and equation (3) holds. Similar treatment works for $A(Y) \cap A_3 = \{H_{13}, H_{23}\}$.

One can treat the case $(1,0,2)$ in a similar way.

The cases $(0,1,3)$ and $(1,0,3)$ do not appear, since otherwise there would be $Y \subset H_{3k}$ for some $k > 3$.

If $(|A(L)_Y \cap A_1|, |A(L)_Y \cap A_2|, |A(L)_Y \cap A_3|) = (1,1,1)$, $A(Y)$ consists of $H_{1k} : x_1 - x_k = 0$, $H_{2j} : x_2 - x_j = 0$ and one of the hyperplanes from A_3. It is easy to see that $\dim Y = n - 3$ and $\dim(X \cap Y) = 0$.

Let $(|A(L)_Y \cap A_1|, |A(L)_Y \cap A_2|, |A(L)_Y \cap A_3|) = (1,1,2)$, there are four cases for $A(Y)$:

1) $A(Y) = \{H_{1k}, H_{2j}, H_{12}, H_{13}\}$ which implies that $Y \subset H_{3j}$, a contradiction;

2) $A(Y) = \{H_{1k}, H_{2j}, H_{12}, H_{23}\}$ which implies that $Y \subset H_{3k}$, a contradiction;

3) $A(Y) = \{H_{1k}, H_{2j}, H_{13}, H_{23}\}$ with $k \neq j$ which implies that $Y \subset H_{kj}$, a contradiction;

4) $A(Y) = \{H_{1k}, H_{2k}, H_{13}, H_{23}\}$ with $\dim Y = n - 3, X \cap Y = 0$, hence equation (3) holds.

The case $(|A(L)_Y \cap A_1|, |A(L)_Y \cap A_2|, |A(L)_Y \cap A_3|) = (1,1,3)$ does not appear since otherwise we would have $Y \subset H_{jk}$. ◇

Let

$$\mathcal{A}_0 = \emptyset,\ \mathcal{A}_1 = \{H_{34}\},\ \mathcal{A}_2 = \{H_{ij} \mid 3 \leq i < j \leq 5\},\\ \ldots,\ \mathcal{A}_{n-3} = \{H_{ij} \mid 3 \leq i < j \leq n\}.$$

and

$$X_k = \bigcap_{H \in \mathcal{A}_k} H, \quad k = 0,1,2,\ldots n-3.$$

Note that $X_0 = \hat{0}$ and $X_{n-3} = X$. By [4], we have the following

Lemma 4. *There is a chain of modular elements*

$$\hat{0} < X_1 < X_2 < \cdots < X_{n-4} < X_{n-3} = X.$$

◇

For X defined in lemma 3, it follows from lemma 4 that $L_X = L(\mathcal{A}(\Sigma_n^{(3)}))_X$ is a modular lattice. By [4], we have

$$(5) \qquad \pi(\mathcal{OS}(L_X), t) = (t+1)(2t+1)\cdots((n-3)t+1).$$

§4. Proof of the main result.

It remains to calculate

$$Q = \sum_{Z \in L, Z \wedge X = \hat{0}} \mu(Z)(-t)^{r(Z)}.$$

Note that in our case X is modular with $r(X) = n - 3$. For $Z \in L, Z \wedge X = \hat{0}$, we have

$$\begin{aligned} n \geq r(X \vee Z) &= r(X \vee Z) + r(X \wedge Z) = r(X) + r(Z) \\ &= n - 3 + r(Z) \Rightarrow r(Z) \leq 3. \end{aligned}$$

Hence

(6) $$Q = \alpha(-t)^3 + \beta t^2 + \gamma(-t) + 1,$$

where

$$\alpha = \sum_{\substack{Z \in L, Z \wedge X = \hat{0} \\ r(Z)=3}} \mu(Z), \quad \beta = \sum_{\substack{Z \in L, Z \wedge X = \hat{0} \\ r(Z)=2}} \mu(Z), \quad \gamma = \sum_{\substack{Z \in L, Z \wedge X = \hat{0} \\ r(Z)=1}} \mu(Z).$$

By lemma 2,

(7) $$\gamma = -|A(L) \setminus A(L)_X| = -\left[\binom{n}{2} - \binom{n-2}{2}\right] = -(2n-3).$$

Next we compute β. Since $A(L)_Z \cap A(L)_X = \emptyset$ and $r(Z) = 2$, for each $k = 1, 2$, $A(L)_Z$ may contain H_{ki} or H_{kj} $(i \neq j)$, but does not contain both at the same time for, otherwise, there would be $H_{ki} \cap H_{kj} \supset X$. Hence, $A(L)_Z$ contains two hyperplanes, and $\mu(Z) = 1$.

There are three cases to be considered.

1) one of the two hyperplanes in $A(L)_Z$ comes from $\{H_{1i} \mid i = 4, \ldots, n\}$, and the other one comes from $\{H_{2i} \mid i = 4, \ldots, n\}$. Hence there are as many as $(n-3)^2$ possibilities;

2) the two hyperplanes in $A(L)_Z$ comes from H_{12}, H_{23}, H_{13}, there are 3 possibilities;

3) one of the two hyperplanes in $A(L)_Z$ comes from $\{H_{ki} \mid k = 1, 2, i = 4, \ldots, n\}$ and the other one comes from A_3. There are $3 \times 2(n-3)$ possibilities.

Hence

(8) $$\beta = (n-3)^2 + 3 + 6(n-3) = n^2 - 6.$$

Now we compute α. The point is that in some cases, $A(L)_Z$ contains more atoms than the the minimal possible.

Since $r(Z) = 3$, $(|A(L)_Z \cap A_1|, |A(L)_Z \cap A_2|, |A(L)_Z \cap A_3|)$ has only the following possibilities.

$$(0,0,3), (0,1,2), (1,0,2), (1,1,1), (1,1,2), (1,0,3), (0,1,3), (1,1,3).$$

The cases $(1,0,3)$, $(0,1,3)$, $(1,1,3)$ are excluded by the condition $Z \wedge X = \hat{0}$.

For the case $(0,0,3)$, $Z = H_{12} \cap H_{13} \cap H_{23}$ and $\mu(Z) = -1$, This contributes -1 to α.

In case $(0,1,2)$, for $4 \le k \le n$, we have

$$Z_1^{(k)} = H_{12} \cap H_{23} \cap H_{2k}, \ Z_2^{(k)} = H_{12} \cap H_{13} \cap H_{2k}, \ Z_3^{(k)} = H_{13} \cap H_{23} \cap H_{2k}.$$

Since $Z_2^{(k)} \subset H_{3k}$, $Z_2^{(k)}$ should be excluded. It is obvious that

$$A(L)_{Z_1^{(k)}} = \{H_{12}, H_{23}, H_{2k}\}, \quad A(L)_{Z_3^{(k)}} = \{H_{13}, H_{23}, H_{1k}, H_{2k}\}.$$

So $Z_3^{(k)}$ should belong to the case $(1,1,2)$, which will be considered later. Since $\mu(Z_1^{(k)}) = -1$, this case contributes $-(n-3)$ to α.

The case $(1,0,2)$ is similar to the case $(0,1,2)$. For $4 \le k \le n$, we have

$$\tilde{Z}_1^{(k)} = H_{12} \cap H_{23} \cap H_{1k}, \ \tilde{Z}_2^{(k)} = H_{12} \cap H_{13} \cap H_{1k}, \ \tilde{Z}_3^{(k)} = H_{13} \cap H_{23} \cap H_{1k}.$$

Since $\tilde{Z}_1^{(k)} \subset H_{3k}$, $\tilde{Z}_1^{(k)}$ should be excluded. It is obvious that $\tilde{Z}_3^{(k)} = Z_3^{(k)}$. Since $\mu(\tilde{Z}_2^{(k)}) = -1$. This case contributes $-(n-3)$ to α.

In case $(1,1,1)$, there are $3(n-3)^2$ possibilities all together. Since $H_{13} \cap H_{1k} \cap H_{2k} = H_{23} \cap H_{1k} \cap H_{2k} = Z_3^{(k)}$ which should be in the case $(1,1,2)$, this case contributes $-(3(n-3)^2 - 2(n-3))$ to α.

We consider the case $(1,1,2)$. For $4 \le j, k \le n$, we have

$$Z_4^{(jk)} = H_{12} \cap H_{13} \cap H_{1j} \cap H_{2k}, \quad Z_5^{(jk)} = H_{12} \cap H_{23} \cap H_{1j} \cap H_{2k},$$
$$Z_6^{(jk)} = H_{13} \cap H_{23} \cap H_{1j} \cap H_{2k}.$$

It is obvious that $Z_4^{(jk)} \subset H_{3k}$, $Z_5^{(jk)} \subset H_{3j}$, and for $j \neq k$, $Z_6^{(jk)} \subset H_{jk}$, which are excluded by the condition $Z \wedge X = \hat{0}$. For $j = k$, we have $Z_6^{(jk)} = Z_6^{(kk)} = Z_3^{(k)}$, and $\mu(Z_3^{(k)}) = -3$. Hence, this case contributes $-3(n-3)$ to α.

Hence

$$\alpha = -(1 + 3(n-2)(n-3)). \tag{9}$$

Combine formulae (2), (7), (8), and (9), we obtain the formula (1) in Theorem 1.

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