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On the Castelnuovo-Severi inequality for a double covering

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§0. Introduction, motivation and the results

Let C be a smooth projective irreducible complex algebraic curve of genus $g \geq 2$. We denote g^1_d by a 1-dimensional possibly incomplete linear system of degree d on C. For any $d \geq g+1$, every curve C of genus g has a base point free g^1_d which may be taken as a general pencil of a general element in $W^{d-g}_d(C) = J(C)$. If C is a hyperelliptic curve with the hyperelliptic pencil g^1_2, it is well-known that any base point free pencil of degree $d \leq g$ is a subsystem of the complete rg^1_2 where $r = \frac{d}{2}$; cf. [1, p.109]. In particular, the only base point free and complete pencil on a hyperelliptic curve is the g^1_2. On the other hand, a non-hyperelliptic curve C has a base point free and complete pencil of degree g, by taking off $g-2$ general points from the very ample canonical linear system $|K_C|$.

Furthermore, a theorem of Harris asserts that any non-hyperelliptic curve of genus g has a base point free and complete pencil of degree $g-1$; cf. [1, p.372]. However, this seemingly simple fact requires a proof which is somewhat involved. Especially, in case C is a bi-elliptic curve, one needs to show that the variety $W^1_{g-1}(C)$ consisting of special pencils of degree $g-1$ is reducible by using enumerative methods; see also [3, Proposition 3.3],[6, Proposition 2.5] for the other proofs concerning the existence of a base point free and complete pencil g^1_{g-1} on a bi-elliptic curve. At this point, it is worthwhile to recall the following classical result known as Castelnuovo-Severi inequality.

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Proposition 0.1 (Castelnuovo-Severi inequality, [1, p.366]**).** *Let C, B_1, B_2 be curves of respective genera g, g_1, g_2. Assume that*

$$\pi_i : C \to B_i, \ i = 1, 2,$$

is a d_i-sheeted mapping such that

$$(\pi_1, \pi_2) : C \to B_1 \times B_2$$

is birational to its image. Then

$$g \leq (d_1 - 1)(d_2 - 1) + d_1 g_1 + d_2 g_2.$$

As an easy application of Proposition 0.1, we make a note of the following remarks.

Remark 0.2. (i) *Let C be a hyperelliptic curve with the 2-sheeted covering $\pi_1 : C \to \mathbb{P}^1$ induced by the unique hyperelliptic pencil g^1_2. Let g^1_d be a base point free pencil not composed with the g^1_2. In other words, g^1_d induces a covering $\pi_2 : C \to \mathbb{P}^1$ of degree d such that $(\pi_1, \pi_2) : C \to \mathbb{P}^1 \times \mathbb{P}^1$ is birational to its image. By the Castelnuovo-Severi inequality, we have $g \leq d - 1$. This recovers the fact that any base point free pencil of degree $d \leq g$ is a subsystem of a multiple of the hyperelliptic pencil, which was mentioned earlier.*
(ii) *More generally, let $\pi : C \to E$ be a double covering of a smooth curve E of genus h. Let g^1_d be a base point free pencil of degree d not composed with the involution determined by π (composed with π for short). Again by the Castelnuovo-Severi inequality, we have*

$$d \geq g - 2h + 1. \tag{0.1}$$

Therefore it follows that any base point free pencil of degree $d \leq g - 2h$ is of the the form $\pi^* g^1_e$ for some g^1_e on E.

In case $h = 1$, the theorem of Harris quoted earlier indicates that the inequality (0.1) is indeed sharp on a bi-elliptic curve. For the case $h = 2$, it only has been known that there exists a base point free and complete pencil of degree $g - 2$ not composed with the double covering under somewhat unsatisfactory genus assumption $g \geq 11$, whereas the existence of a base point free and complete g^1_{g-3} has remained open; cf. [2, Proposition 2.6]. Therefore, we would like to raise the following questions regarding the sharpness of the inequality (0.1).

Question 0.3. (i) *Let $\pi : C \to E$ be double covering of a smooth curve E of genus h. Does there exist a base point free pencil of degree $g-2h+1$ not composed with π ?*
(ii) *Let $\pi : C \to E$ be double covering of a smooth curve E of genus h. Does there exists a base point free pencil of degree d not composed with π for every $d \geq g-2h+1$?*
(iii) *What is the optimal range for the genus g of the double covering with respect to the genus h of the base curve E ensuring affirmative answers to the questions above ? Or, find examples of double coverings for which questions* (i) *or* (ii) *fail.*

We may even pose a more naive question: Given a smooth curve E of genus h, does there exist a smooth double covering $C \xrightarrow{\pi} E$ of genus g possessing a base point free pencil of degree $g-2h+1$ not composed with π ? However this turns out to be relatively easy to answer.

Example 0.4. *Given a smooth curve E of genus $h \geq 0$ and an integer $g \geq 4h$, let $C \subset \mathbb{P}^1 \times E$ be a general divisor linearly equivalent to $D := 2p \times E + \mathbb{P}^1 \times N$ with* $\deg N = g-2h+1$ *and $p \in \mathbb{P}^1$. By the condition $g \geq 4h$, D is very ample and hence C is a smooth curve of genus g by the adjunction formula. Furthermore, the two projection maps of $E \times \mathbb{P}^1$ to E and $\mathbb{P}^1$ restricted to C correspond to a degree two morphism $C \xrightarrow{\pi} E$ and a base point free and complete pencil g^1_{g-2h+1} not composed with π.*

Motivated by Example 0.4, the main result of this paper is the following theorem which provides an affirmative answer to the Question 0.3 (i).

Theorem A. *Let C be a curve of genus g which admits a double covering $\pi : C \longrightarrow E$ with $g(E) = h \geq 0$ and $g \geq 4h$. Then C has a base point free and complete g^1_{g-2h+1} not composed with π.*

By using Theorem A, we are also able to answer the Question 0.3 (ii) in the affirmative.

Theorem B. *Let C be a curve of genus g which admits a double covering $\pi : C \longrightarrow E$ with $g(E) = h$ and $g \geq 8h-4$. Then there exists a base point free pencil of degree d not composed with π for any degree d with $d \geq g-2h+1$.*

The organization of this paper is as follows. In §1, after giving a general theory between a double covering $\pi : C \to E$ and an embedding of C into a ruled surface (see Proposition 1.5), we prove necessary and sufficient conditions for the existence of base point free and complete g^1_{g-2h+1} (see Theorem 1.1). This can be done by observing the relationship between the above associated embedding of C into a ruled surface and the embedding $(\pi, g^1_{g-2h+1}) : C \hookrightarrow E \times \mathbb{P}^1$ by using elementary transformations. In §2, we prove that the necessary and sufficient condition in §1 for the existence of such g^1_{g-2h+1} holds for any smooth double covering under the numerical assumption $g \geq 4h$ by using Theorem 2.1, thereby proving Theorem A. This will be carried out by using an elementary theory of determinantal varieties. We then proceed to prove Theorem B by the excess linear series argument. In §3, we mainly deal with the Question 0.3 (iii). Specifically, we show that the numerical assumption $g \geq 4h$ in Theorem A is the best possible one by constructing an example of a double covering of $g = 4h - 1$ without a base point free and complete g^1_{g-2h+1}. We also exhibit an example of a double covering with a base point free and complete g^1_{g-2h+1} under the same numerical condition $g = 4h - 1$. Throughout we use the same notations and conventions as in [1].

§1. Curves on ruled surfaces

In this section we study double coverings on a ruled surface. In particular we collect and develop some methods realizing a double covering with a base point free pencil of particular degree as a smooth divisor on a ruled surface. The goal of this section is to prove the following result:

Theorem 1.1. *Let $C = \mathbf{Spec}(\mathcal{O}_E \oplus \mathcal{O}_E(-N)) \xrightarrow{\pi} E$ be a smooth double covering and let $\iota : C \to \mathbb{P}(\mathcal{O}_E \oplus \mathcal{O}_E(-N))$ be an embedding associated with π such that $\rho_N \iota = \pi$. Then the following four conditions are equivalent:*

1) *C has a base point free and complete g^1_{g-2h+1} which is not composed with π, and $\pi_* D \in |N|$ for $D \in g^1_{g-2h+1}$.*

2) *There is a section $H \in |T_N + \rho_N^*(N)|$ such that $H|_{\iota(C)} = D_1 + D_2$ with $\pi_* D_1, \pi_* D_2 \in |N|$ and $D_1 \sim \sigma^* D_2$.*

3) *There is a divisor $H \in |T_N + \rho_N^* N|$ such that $H \cap T_N = \emptyset$ satisfying $H|_{\iota(C)} = D_1 + D_2$ with $\pi_* D_1, \pi_* D_2 \in |N|$.*

4) *There is a divisor $A \in |\pi^* N| \setminus \{\pi^* L \mid L \in |N|\}$ such that $\pi_* A = N_1 + N_2$ and $N_1, N_2 \in |N|$.*

Let M be an effective divisor on a smooth projective curve E of genus h and let $\mathcal{O}_E(M)$ be the line bundle associated with M. Throughout this paper, we denote the structure morphism of the ruled surface $\mathbb{P}(\mathcal{O}_E \oplus \mathcal{O}_E(-M))$ by

$$\rho_M : \mathbb{P}(\mathcal{O}_E \oplus \mathcal{O}_E(-M)) \to E$$

and its minimal section by T_M; by the minimal section, we always mean the section of minimal degree on a normalized ruled surface. For $P \in \mathbb{P}(\mathcal{O}_E \oplus \mathcal{O}_E(-M))$, let F be the fibre over $p = \rho_M(P)$. In the blowing-up

$$\eta : S_P \to \mathbb{P}(\mathcal{O}_E \oplus \mathcal{O}_E(-M))$$

of the ruled surface $\mathbb{P}(\mathcal{O}_E \oplus \mathcal{O}_E(-M))$ at P, let e be the exceptional divisor of η, f the proper transform of F and

$$\tau : S_P \to S'$$

the contraction of f. We put $P' = \tau(f) \in S'$. Since S' is an elementary transformation of $\mathbb{P}(\mathcal{O}_E \oplus \mathcal{O}_E(-M))$ with center P, S' is a ruled surface over E; cf. [4, p.416]. We define ρ' as its ruling $S' \to E$.

We choose a section $H_M \in |T_M + \rho_M^* M|$ and hence $H_M \cap T_M = \emptyset$. Let $\widetilde{T_M}$ and $\widetilde{H_M}$ be the proper transforms of T_M and H_M on S_P respectively, and set $T' = \tau(\widetilde{T_M})$, $H' = \tau(\widetilde{H_M})$. Since $H_M \cap T_M = \emptyset$, we have $H' \cap T' = \emptyset$ for $P \in T_M \cup H_M$ which implies

$$S' \cong \mathbb{P}(\mathcal{O}_E \oplus \mathcal{O}_E(-M'))$$

for some $M' \in \mathrm{Div}(E)$; cf. [4, p.383]. Let C_0 be an irreducible curve on $\mathbb{P}(\mathcal{O}_E \oplus \mathcal{O}_E(-M))$ with $C_0 \sim 2T_M + \rho_M^*(Z)$ for some $Z \in \mathrm{Div}(E)$, let $\phi : C \to C_0$ be its normalization, let $\widetilde{C_0}$ be the proper transform of C_0 on S_P, let $C_0' = \tau(\widetilde{C_0})$ and let $\phi' : C \to C_0'$ be its normalization. Let $\pi = \rho_M \phi$. Note that $\pi = \rho' \phi'$ and $\pi : C \to E$ is a double covering and we denote the associated involution by σ.

From now, we assume that $P \in T_M \cup H_M$. First, we consider the case, the point $P \in T_M \cup H_M$ is a smooth point of C_0. By $(\widetilde{C_0} + e.f + e) = (C_0.F) = 2$ and $(\widetilde{C_0}.e) = 1$, we have $(\widetilde{C_0}.f) = 1$. Hence C_0' is non-singular at P'. Therefore $C_0 \cong \widetilde{C_0} \cong C_0'$, when C is non-singular.

Lemma 1.2. (i) *T' is a minimal section $T_{M'}$ on $\mathbb{P}(\mathcal{O}_E \oplus \mathcal{O}_E(-M'))$.*
(ii) *In case $P \in H_M$ and* $\deg(M - p) \geq 0$, *we have*

$$\mathbb{P}(\mathcal{O}_E \oplus \mathcal{O}_E(-M')) \cong \mathbb{P}(\mathcal{O}_E \oplus \mathcal{O}_E(-(M - p))),$$

$H' \sim T' + {\rho_{M-p}}^*(M-p), C_0' \sim 2T' + {\rho_{M-p}}^*(Z-p)$ and $\phi'^* H' = \phi^* H_M - P$.

(iii) *In case $P \in T_M$, we have*

$$\mathbb{P}(\mathcal{O}_E \oplus \mathcal{O}_E(-M')) \cong \mathbb{P}(\mathcal{O}_E \oplus \mathcal{O}_E(-(M + p))),$$

$H' \sim T' + {\rho_{M+p}}^*(M + p)$, $C_0' \sim 2T' + {\rho_{M+p}}^*(Z + p)$ and $\phi'^* H' = \phi^* H_M + \sigma^* P$.

Proof. We only give a proof for the case $P \in H_M$ and $\deg(M-p) \geq 0$; the case $P \in T_M$ is similar. Since $H_M|_{C_0} = \widetilde{H_M} + e|_{\widetilde{C_0}}$ and $(\widetilde{C_0}.e) = 1$, we have

$$H'|_{C_0'} = \widetilde{H_M}|_{\widetilde{C_0}},\ e|_{\widetilde{C_0}} = P,\ H'|_{C_0'} = H_M|_{C_0} - P.$$

Now we show that T' is a minimal section. Since $P \notin T_M$,

$$\eta^* T_M = \widetilde{T_M} \cong T_M \cong T'.$$

Since T_M is a (minimal) section, $T_M \cong E$ and $\mathcal{O}_{T_M}(T_M) \cong \mathcal{O}_E(-M)$. Therefore we have

$$\mathcal{O}_E(-M) \cong \mathcal{O}_{T_M}(T_M) \cong \mathcal{O}_{\widetilde{T_M}}(\eta^* T_M) = \mathcal{O}_{\widetilde{T_M}}(\widetilde{T_M}).$$

Since $\mathcal{O}_{T'}(T') \cong \mathcal{O}_{\widetilde{T_M}}(\tau^* T') = \mathcal{O}_{\widetilde{T_M}}(\widetilde{T_M} + f)$,

$$\mathcal{O}_{T'}(T') \cong \mathcal{O}_E(-(M-p)). \tag{1.2.1}$$

To see $\mathbb{P}(\mathcal{O} \oplus \mathcal{O}(-M')) \cong \mathbb{P}(\mathcal{O} \oplus \mathcal{O}(-(M-p)))$, we argue as follows. If $(T'.T_{M'}) < 0$, $T' = T_{M'}$ which implies $M' \sim -T'|_{T'} \sim M-p$ and we are done for this case. Therefore we may assume $(T'.T_{M'}) \geq 0$. Let $T' \sim aT_{M'} + \rho_{M'}^* B$ with $\deg B = b$ and let $(T_{M'}^2) = -n'$. By the assumption $\deg(M-p) \geq 0$, we have $(T'^2) = a(2b - an') \leq 0$. Since T' is a section, $a = 1$ and $b \geq an'$ by $(T'.T_{M'}) \geq 0$, which implies $b = 0$. On the other hand, since T' is effective

$$\{0\} \neq \Gamma(S', \mathcal{O}(T')) \cong \Gamma(E, \mathcal{O}_E(B) \oplus \mathcal{O}_E(B - M')) \tag{1.2.2}$$

by projection formula. When $M' > 0$, $\deg(B - M') = \deg(-M') = -n' < 0$ implying $B \sim 0$ and hence $T' = T_{M'}$. Therefore it follows that $M' \sim -T'|_{T'} \sim M - p$ by (1.2.1). When $M' = 0$, we have either $B \sim 0$ or $M' \sim B$ by (1.2.2). Since $M' = \deg M' = 0$, $T_{M'} + \rho_{M'}^* M'$ is linearly equivalent to a minimal section $T_{-M'} \subset \mathbb{P}(\mathcal{O}_E \oplus \mathcal{O}_E(M')) \cong \mathbb{P}(\mathcal{O}_E \oplus \mathcal{O}_E(-M'))$. Therefore we have either $T' = T_{M'}$ when $B \sim 0$ or $T' = T_{-M'}$ when $M' \sim B$. In either cases, T' is a minimal section satisfying (1.2.1). Therefore

$$\mathbb{P}(\mathcal{O}_E \oplus \mathcal{O}_E(-M')) \cong \mathbb{P}(\mathcal{O}_E \oplus \mathcal{O}_E(-(M-p))) \text{ and } T' = T_{M-p}.$$

Now we prove $H' \sim T' + \rho_{M-p}{}^*(M-p)$. Let $H' \sim T' + \rho_{M-p}{}^*(G)$ for some $G \in \operatorname{Div}E$. Since $H' \cap T' = \emptyset$, $H'|_{T'} \sim 0$ and hence $T'|_{T'} + G \sim 0$ which implies $G \sim -T'|_{T'} \sim M - p$. Hence $H' \sim T' + \rho_{M-p}{}^*(M-p)$.

Finally we prove $C_0' \sim 2T' + \rho_{M-p}{}^*(Z-p)$. Since C_0' is smooth and $\tau^* C_0' \sim \widetilde{C_0} + f$,

$$\tau^* C_0' \sim \eta^* C_0 - e + f. \tag{1.2.3}$$

Since $\eta^* C_0 \sim 2\widetilde{T_M} + \eta^* \rho_M^*(Z-p+p) \sim 2\widetilde{T_M} + \eta^* \rho_M^*(Z-p) + (e+f)$,

$$\begin{aligned} \tau^* C_0' &\sim 2(\widetilde{T_M} + f) + \tau^* \rho_{M-p}{}^*(Z-p) \\ &= 2\tau^* T' + \tau^* \rho_{M-p}^*(Z-p) = \tau^*(2T' + \rho_{M-p}^*(Z-p)) \end{aligned}$$

by (1.2.3). Hence $C_0' \sim 2T' + \rho_{M-p}{}^*(Z-p)$.

Q.E.D.

Next, we consider the case, the point $P \in T_M \cup H_M$ is a singular point of C_0. Let $F = \rho_M^* p$ where $p = \rho_M(P)$. Since $\rho_M \phi = \pi : C \to E$ is a double covering, $\phi^* F = \pi^* p$ which means P is a double point or a cusp.

Lemma 1.3. (i) *In case* $P \in H_M$ *and* $\deg(M-p) \geq 0$, *we have*

$$S' \cong \mathbb{P}(\mathcal{O}_E \oplus \mathcal{O}_E(-(M-p))),$$

$H' \sim T' + \rho_{M-p}{}^*(M-p), C_0' \sim 2T'' + \rho_{M-p}{}^*(Z)$ and $\phi'^* H' = \phi^* H_M - \pi^* p$.

(ii) *In case* $P \in T_M$, *we have*

$$S' \cong \mathbb{P}(\mathcal{O}_E \oplus \mathcal{O}_E(-(M+p))),$$

$H' \sim T' + \rho_{M+p}{}^*(M+p), C_0' \sim 2T' + \rho_{M+p}{}^*(Z)$ and $\phi^* H' = \phi' * H_M + \pi^* p$.

Proof. We only give a proof for the case $P \in H_M$ and $\deg(M-2p) \geq 0$; the case $P \in T_M$ is similar. By Lemma 1.2, $S'' \cong \mathbb{P}(\mathcal{O}_E \oplus \mathcal{O}_E(-(M-2p)))$, $H'' \sim T'' + \rho_{M-2p}{}^*(M-2p)$. We now prove $C_0'' \sim 2T'' + \rho_{M-2p}{}^*(Z-2p)$. Since $P \in C_0$ is a double point, $\eta_1^* C_0 \sim \widetilde{C_0} + 2e_1$. Therefore $2 = (C_0.F) = (\widetilde{C_0} + 2e_1.e_1 + f_1)$ which implies $(\widetilde{C_0}.e_1) = 2$. Hence $(\widetilde{C_0}.f_1) = 0$, so we have $\widetilde{C_0} = \tau_1^* C_0'$ because τ_1 is a contraction of f_1. Since $1 = (T_M.F) = (\widetilde{T_M} + e_1.e_1 + f_1)$ and $(\widetilde{T_M}.e_1) = 1$, $(\widetilde{T_M}.f_1) = 0$. Therefore $\tau_1^* T' = \widetilde{T_M}$ which implies

$$\tau_1^* C_0' \sim \eta_1^*(2T_M + \rho_M^* Z) - 2e_1 \sim 2\tau_1^* T' + \eta_1^* \rho_M^* Z.$$

Since $\eta_1^* \rho_M^* Z \sim \tau_1^* \rho_1^* Z$, $\tau_1^* C_0' \sim \tau_1^*(2T' + \rho_1^* Z)$, i.e.

$$C_0' \sim 2T' + \rho_1^* Z.$$

Q.E.D.

Finally, we consider the case, the point $P \in T_M \cup H_M$ does not lie on C_0.

Lemma 1.4. (i) *In case* $P \in H_M$ *and* $\deg(M-p) \geq 0$, *we have*

$$S' \cong \mathbb{P}(\mathcal{O}_E \oplus \mathcal{O}_E(-(M-p))),$$

$H' \sim T' + \rho_{M-p}{}^*(M-p), C_0' \sim 2T'' + \rho_{M-p}{}^*(Z-2p)$ and $\phi_1^* H' = \phi^* H_M$.

(ii) *In case* $P \in T_M$, *we have*

$$S' \cong \mathbb{P}(\mathcal{O}_E \oplus \mathcal{O}_E(-(M+p))),$$

$H' \sim T' + \rho_{M+p}{}^*(M+p), C_0' \sim 2T' + \rho_{M+p}{}^*(Z+2p)$ and $\phi^* H' = \phi'^* H_M$.

Proof. We only give a proof for the case $P \in H_M$ and $\deg(M-2p) \geq 0$; the case $P \in T_M$ is similar. By Lemma 1.2, $S'' \cong \mathbb{P}(\mathcal{O}_E \oplus \mathcal{O}_E(-(M-2p)))$, $H'' \sim T'' + \rho_{M-2p}{}^*(M-2p)$. We now prove $C_0'' \sim 2T'' + \rho_{M-2p}{}^*(Z-2p)$. Since $P \in C_0$ is a double point, $\eta_1^* C_0 \sim \widetilde{C_0} + 2e_1$. Therefore $2 = (C_0.F) = (\widetilde{C_0} + 2e_1.e_1 + f_1)$ which implies $(\widetilde{C_0}.e_1) = 2$. Hence $(\widetilde{C_0}.f_1) = 0$, so we have $\widetilde{C_0} = \tau_1^* C_0'$ because τ_1 is a contraction of f_1. Since $1 = (T_M.F) = (\widetilde{T_M} + e_1.e_1 + f_1)$ and $(\widetilde{T_M}.e_1) = 1$, $(\widetilde{T_M}.f_1) = 0$. Therefore $\tau_1^* T' = \widetilde{T_M}$ which implies

$$\tau_1^* C_0' \sim \eta_1^*(2T_M + \rho_M^* Z) - 2e_1 \sim 2\tau_1^* T' + \eta_1^* \rho_M^* Z.$$

Since $\eta_1^* \rho_M^* Z \sim \tau_1^* \rho_1^* Z$, $\tau_1^* C_0' \sim \tau_1^*(2T' + \rho_1^* Z)$, i.e.

$$C_0' \sim 2T' + \rho_1^* Z.$$

Q.E.D.

We recall some basics of a double covering of a curve E of genus h; see [5] for a full treatment. For $N \in E_{g-2h+1}$, let R be an effective divisor on E with $\mathcal{O}_E(R) \cong \mathcal{O}_E(2N)$. Given an isomorphism

$$\phi : \mathcal{O}_E(-N)^{\otimes 2} \xrightarrow{\sim} \mathcal{O}_E(-R) \subset \mathcal{O}_E,$$

one defines an $\mathcal{O}_E$-algebra structure on $\mathcal{O}_E \oplus \mathcal{O}_E(-N)$ by

$$(a,b) \cdot (c,d) = (ac + \phi(bd), ad + bc).$$

One then has a double covering $\pi : C = \mathbf{Spec}(\mathcal{O}_E \oplus \mathcal{O}_E(-N)) \to E$ with $\pi_* \mathcal{O}_C \cong \mathcal{O}_E$. The virtual genus of C is g, i.e. $\dim H^1(C, \mathcal{O}_C) = g$. Note that $(a,b) \mapsto (a,-b)$ is an $\mathcal{O}_E$-algebra isomorphism of order 2 which induces an involution $\sigma : C \to C$ over E. Conversely, every double covering over E is of this form. We also recall that a double covering $\pi : C = \mathbf{Spec}(\mathcal{O}_E \oplus \mathcal{O}_E(-N)) \to E$ is an irreducible reduced non-singular curve if and only if R is reduced. Let $\lambda : \pi^*\mathcal{E} \to \mathcal{O}_C$ be the restriction of a natural map $\lambda : \pi^*\pi_*(\mathcal{O}_C) \to \mathcal{O}_C$ to $\pi^*\mathcal{E}$. Since λ is surjective, we have a morphism

$$\iota : C \to \mathbb{P}(\mathcal{E}) = \mathbb{P}(\mathcal{O}_E \oplus \mathcal{O}_E(-N))$$

with $\rho_N \iota = \pi$.

Proposition 1.5. *ι is embedding and $\iota(C) \sim 2(T_N + \rho_N^*(N))$ on $\mathbb{P}(\mathcal{E}) = \mathbb{P}(\mathcal{O}_E \oplus \mathcal{O}_E(-N))$.*

Proof. Let $U = \mathrm{Spec}(A) \subset E$ be an affine open subset and $\mathrm{Spec}B = \pi^{-1}U$. Let $\mathcal{E}|_U = \pi_*(\mathcal{O}_C)|_U = \mathcal{O}_U e_1 \oplus \mathcal{O}_U e_2$ where $e_1 = 1_A$ and $\pi^*e_1 = (e_1, 0)$ is the identity of $B = \Gamma(U, \pi_*\mathcal{O}_C)$. Let $(\pi^{-1}U)_{\pi^*e_1} = \{P \in \pi^{-1}U \mid \pi^*e_1(P) \neq 0\}$. Note that $(\pi^{-1}U)_{\pi^*e_1} = \pi^{-1}U = \mathrm{Spec}(B)$ is affine and the homomorphism $A[e_1, e_2] \to \Gamma(\pi^{-1}(U), \mathcal{O}_C) = B$ defined by $e_i \mapsto \dfrac{\pi^*e_i}{\pi^*e_1} = \pi^*e_i$ $(i = 1, 2)$ is surjective. By [4, p.151 Proposition 7.2], $\iota|_{\pi^{-1}U}$ is embedding and hence ι is embedding. Since $\rho_{N*}\mathcal{O}(T_N) \cong \pi_*\mathcal{O}_C$,

$$\pi^*\rho_{N*}\mathcal{O}(T_N + \rho_N^* N) \cong \pi^*(\rho_{N*}\mathcal{O}(T_N) \otimes \mathcal{O}_E(N)) \cong \pi^*\pi_*(\mathcal{O}_C) \otimes \mathcal{O}_C(\pi^*(N))$$

by the projection formula. Therefore $\lambda \otimes \mathcal{O}_C(\pi^*(N)) : \pi^*\rho_{N*}\mathcal{O}(T_N + \rho_N^*(N)) \to \mathcal{O}_C(\pi^*(N))$ again defines ι. This means $\phi_{|\pi^*(N)|} = \phi_{|T_N + \rho_N^*(N)|}\iota$ where $\phi_{|\pi^*(N)|} : C \to \mathbb{P}(\Gamma(C, \mathcal{O}_C(\pi^*(N))))$ is a morphism defined by $|\pi^*(N)|$ and $\phi_{|T_N+\rho_N^*(N)|} : \mathbb{P}(\mathcal{O}_E \oplus \mathcal{O}_E(-N)) \to \mathbb{P}(\Gamma(\mathcal{O}(T_N + \rho_N^*(N))))$ is a morphism defined by $|T_N + \rho_N^*(N)|$. Since ι is an embedding and $\phi_{|T_N+\rho_N^*(N)|}$ is an birational morphism only contracting T_N, $\phi_{|\pi^*(N)|}$ is a birational morphism onto its image. Hence

$$(\iota(C).T_N + \rho_N^*(N)) = \deg\phi_{|\pi^*(N)|}(C) = 2(g - 2h + 1).$$

Since ι is an embedding and $\rho_N\iota = \pi$, $(\iota(C).\rho_N^*(N)) = \deg\pi^*(N) = 2(g-2h+1)$ which implies $(\iota(C).T_N) = 0$, i.e. $\iota(C) \cap T_N = \emptyset$ because $\iota(C)$ and T_N are irreducible. Therefore $\iota(C)|_{T_N} \sim 0$. Let $\iota(C) \sim 2T_N + \rho_N^* B$. Then

$$0 \sim \iota(C)|_{T_N} \sim 2T_N|_{T_N} + B.$$

Since $T_N|_{T_N} \sim -(N)$, $\iota(C) \sim 2(T_N + \rho_N^*(N))$.

Q.E.D.

Corollary 1.6. *Let $\pi : C \to E$ be a smooth double covering. Then C is isomorphic to $\mathbf{Spec}(\mathcal{O}_E \oplus \mathcal{O}_E(-N))$ over E if and only if it has an embedding $\iota : C \hookrightarrow \mathbb{P}(\mathcal{O}_E \oplus \mathcal{O}_E(-N))$ with $\rho_N \iota = \pi$ and $\iota(C) \sim 2(T_N + \rho_N^* N)$.*

Proof. Assume that there is an embedding $\iota : C \hookrightarrow \mathbb{P}(\mathcal{O}_E \oplus \mathcal{O}_E(-N))$ with $\rho_N \iota = \pi$ and $\iota(C) \sim 2(T_N + \rho_N^* N)$. Let R' be the branch locus of the double covering π. Then there is a divisor N' on E with an isomorphism $\phi' : \mathcal{O}_E(N')^{\otimes 2} \xrightarrow{\sim} \mathcal{O}_E(R')$ such that:

$\mathcal{O}_E \oplus \mathcal{O}_E(-N')$ is an $\mathcal{O}_E$-algebra by $(a,b)\cdot(c,d) = (ac + \phi'(bd), ad + bc)$
$C \cong \mathbf{Spec}(\mathcal{O}_E \oplus \mathcal{O}_E(-N'))$ over E.

We need to show $N \sim N'$. By the Hurwitz relation, $K_C \sim \pi^*(K_E + N')$. On the other hand, we have

$$K_{\mathbb{P}(\mathcal{O}_E \oplus \mathcal{O}_E(-N))} + \iota(C)|_{\iota(C)} \sim \rho_N^*(K_E + N)|_{\iota(C)} \sim \pi^*(K_E + N)$$

by the adjunction formula and the assumption $\iota(C) \sim 2(T_N + \rho_N^* N)$. Therefore we have $K_E + N' \sim K_E + N$ and hence $N \sim N'$. For the converse part, we denote $\iota = \iota_0$. Then the result is clear by Proposition 1.5.

Q.E.D.

Remark 1.7. *Let $\iota : C \to \mathbb{P}(\mathcal{O}_E \oplus \mathcal{O}_E(-N))$ with $\iota(C) \sim 2(T_N + \rho_N^*(N))$ be an embedding associated with a double covering $C \cong \mathbf{Spec}(\mathcal{O}_E \oplus \mathcal{O}_E(-N)) \to E$. Since*

$$\Gamma(\mathbb{P}(\mathcal{O}_E \oplus \mathcal{O}_E(-N)), \mathcal{O}(T_N + \rho_N^*(N))) \cong \Gamma(E, \mathcal{O}_E(N) \oplus \mathcal{O}_E),$$

we have

$$|\pi^*(N)| = \{H|_{\iota(C)} \mid H \in |T_N + \rho_N^*(N)|\}. \tag{1.7.1}$$

Since $T_N|_{\iota(C)} \sim 0$, we have

$$\{\pi^* L \mid L \in |N|\} = \{(T_N + \rho_N^* L)|_{\iota(C)} \mid L \in |N|\}. \tag{1.7.2}$$

Now we prove Theorem 1.1.

Proof of Theorem 1.1: 1) $\Rightarrow$ 2): Since C has a base point free g^1_{g-2h+1} not composed with π, the morphism $\phi = (g^1_{g-2h+1}, \pi) : C \to \mathbb{P}^1 \times E = \mathbb{P}(\mathcal{O}_E \oplus \mathcal{O}_E)$ is a birational morphism. Note that $\phi(C) \sim 2T_0 + \rho_0^*(J)$ where $J = \pi_* L$ and $L \in g^1_{g-2h+1}$. Take $a \neq b \in \mathbb{P}^1$ and put $T_0 = \{a\} \times E$, $H_0 = \{b\} \times E$ which implies $T_0 \cap H_0 = \emptyset$. Let $D_1' = \phi^* T_0$, $D_2' = \phi^* H_0$ and note that $D'_1, D'_2 \in g^1_{g-2h+1}$. Applying Lemma 1.2-(iii) and Lemma 1.3-(ii) to every $P \leq D'_1$, we get a ruled surface $\rho_M : \mathbb{P}(\mathcal{O}_E \oplus \mathcal{O}_E(-M)) \to E$ and a non-singular curve C' on $\mathbb{P}(\mathcal{O}_E \oplus \mathcal{O}_E(-M))$ such that $C' \cong C$ and $C' \sim 2(T_M + \rho_M^* M)$. We put $D_1 = \sigma^* D'_1$ and $D_2 = D'_2$ which implies $D_1 \sim \sigma^* D_2$. Note that $\pi_* D'_1 \sim \pi_* D'_2$ since $D'_1 \sim D'_2$. By Theorem 1.6, $M \sim N$ and we finally have $\pi_* D_1, \pi_* D_2 \in |N|$, $H' \sim T_N + \rho_N^*(N)$, $H'|_{C_2} = D_1 + D_2$.

2) $\Rightarrow$ 4): We take a section $H \in |T_N + \rho_N^* N|$, D_1, $D_2 \in \mathrm{Div}(C)$ satisfying the condition 2). Since $H \cap T_N = \emptyset$, $H \notin \{T_N + \rho^* N_0 \mid N_0 \in |N|\}$. We put $A = H|_{\iota(C)}$, $N_1 = \pi_* D_1$ and $N_2 = \pi_* D_2$. Then we have $A \in |\pi^* N| \setminus \{\pi^* N_0 \mid N_0 \in |N|\}$ by Remark 1.7 and $\pi_* A = N_1 + N_2$, $N_1, N_2 \in |N|$.

4) $\Rightarrow$ 3): We take a divisor $A \in |\pi^* N| \setminus \{\pi^* N_0 \mid N_0 \in |N|\}$ such that $\pi_* A = N_1 + N_2$ and $N_1, N_2 \in |N|$. By (1.7.1) there is a divisor $H \in |T_N + \rho_N^* N|$ such that $H|_{\iota(C)} = A$. Since $\pi_* A = N_1 + N_2$, there exist two effective divisors D_1, $D_2 \in \mathrm{Div}(C)$ such that $H|_{\iota(C)} = D_1 + D_2$ and $\pi_* D_1, \pi_* D_2 \in |N|$. By (1.7.2), $H \cap T_N$ is finite and hence $H \cap T_N = \emptyset$ by $(H.T_N) = 0$.

3) $\Rightarrow$ 1): Take a divisor $H \in |T_N + \rho_N^* N|$ such that $H \cap T_N = \emptyset$ satisfying $H|_{\iota(C)} = D_1 + D_2$ with $\pi_* D_1, \pi_* D_2 \in |N|$. We prove that H is a section. Since $(H.\rho_N^* p) = 1$, there exists an irreducible divisor $\widehat{H}$ and a divisor B such that $H = \widehat{H} + B$ with $(\widehat{H}.\rho_N^* p) = 1$ and $(B.\rho_N^* p) = 0$. Therefore $B = \rho^*(p_1 + \cdots + p_s)$ for some $p_1, \cdots p_s \in E$. By $(H.T_N) = 0$, $(\widehat{H}.T_N) + s = 0$. If $s > 0$, then $(\widehat{H}.T_N) < 0$ implying $\widehat{H} = T_N$ and hence $H \cap T_N \neq \emptyset$, a contradiction. Therefore $s = 0$ and $H = \widehat{H}$, i.e. H is a section. Applying Lemma 1.2-(ii) and Lemma 1.3-(i) to every $P \leq D_1$, we get a ruled surface $\rho_0 : \mathbb{P}(\mathcal{O}_E \oplus \mathcal{O}_E) = \mathbb{P}^1 \times E \to E$, a non-singular curve C' such that $C' \cong C$, and $C' \sim 2T_0 + \rho_0^*(N) = 2\{\mathrm{pt}\} \times E + \mathbb{P}^1 \times N$ with $\deg N = g - 2h + 1$. Therefore the second projection $\mathbb{P}(\mathcal{O}_E \oplus \mathcal{O}_E) \cong \mathbb{P}^1 \times E \to \mathbb{P}^1$ restricted on C' induces a base point free g^1_{g-2h+1} not composed with π.

Q.E.D.

§2. Proof of Theorem A

Let

$$\phi : \Gamma(E, \mathcal{O}_E(N)) \otimes \Gamma(E, \mathcal{O}_E(N)) \to \Gamma(E, \mathcal{O}_E(2N))$$

be the natural cup product map. Our eventual goal is to prove Theorem A, but for most of this paper, we prove the following theorem.

Theorem 2.1. *Let $C = \mathbf{Spec}(\mathcal{O}_E \oplus \mathcal{O}_E(-N)) \xrightarrow{\pi} E$ be a double covering of genus g over a curve E of genus h with $g \geq 4h-2$. Choose $r \in \Gamma(E, \mathcal{O}_E(2N))$ whose zero is the branch locus of π. Then C has a base point free and complete g^1_{g-2h+1} not composed with π if and only if $r \in \mathrm{im}[\phi : \Gamma(E, \mathcal{O}_E(N)) \otimes \Gamma(E, \mathcal{O}_E(N)) \to \Gamma(E, \mathcal{O}_E(2N)]$.*

We put $V = \Gamma(E, \mathcal{O}_E(N))$. We assume that $g \geq 4h-2$. Since $\deg(\mathcal{O}_E(N)) = g-2h+1 \geq 2h-1$, $\mathcal{O}_E(N)$ is non-special and hence $\dim V = g-3h+2$ by the Riemann-Roch Theorem. Let $\mathrm{M}(m)$ be the variety of $m \times m$ complex matrices. Let

$$M_k(g-3h+2) \subset M(g-3h+2)$$

be the kth determinantal variety, i.e. the subvariety of $M(g-3h+2)$ defined by the ideal generated by $(k+1) \times (k+1)$-minors of (a_{ij}). The codimension of $M_k(g-3h+2)$ is

$$\text{(2.1)} \qquad \mathrm{codim} M_k(g-3h+2) = (g-3h+2-k)^2$$

by [1, p.67 Proposition]. Let $e_1, \cdots, e_{g-3h+2}$ be a basis of V and let

$$\chi : V \otimes V \to \mathrm{M}(g-3h+2)$$

be the natural isomorphism defined by $\displaystyle\sum_{i,j=1,\cdots,g-3h+2} a_{ij} e_i \otimes e_j \mapsto (a_{ij})$.

Lemma 2.2. $\chi^{-1}(M_1(g-3h+2)) = \{u \otimes v \mid u, v \in V\}$.

Proof. Since

$$M_1(g-3h+2) = \{(a_{ij}) \mid a_{ij} = u_i v_j,\ i,j = 1, \cdots, g-3h+2\},$$

we have $\chi^{-1}(M_1(g-3h+2)) = \{u \otimes v \mid u = \sum u_i e_i, v = \sum v_j e_j\}$.

Q.E.D.

We put

$$M_1 = \chi^{-1}(M_1(g-3h+2)) \text{ and } M_0 = \{a \otimes a \mid a \in V\},$$

which are affine cones, i.e. if $c \in M_i$ and $\lambda \in \mathbb{C}$, then $\lambda c \in M_i$ for $i = 0, 1$. For an affine cone A, we denote $\mathbb{P}(A)$ by $A/\mathbb{C}^*$. For an element

$z \in A$, we denote $[z]$ by $\mathbb{C}z/\mathbb{C}^* \in \mathbb{P}(A)$. Let S^2V be the subspace of $V \otimes V$ generated by $\{a \otimes b + b \otimes a \mid a, b \in V\}$. Then S^2V is indeed the second symmetric product of V containing M_0.

Lemma 2.3. *$\mathbb{P}(M_0) \subset \mathbb{P}(\mathrm{S}^2V)$ is the image of $\mathbb{P}(V)$ under the Veronese embedding.*

Proof. Let $a = \sum_{i=1}^{g-3h+2} a_i e_i \in V$. Then

$$a \otimes a = \sum_{i=1}^{g-3h+2} a_i^2 e_i \otimes e_i + \sum_{i<j} a_i a_j (e_i \otimes e_j + e_j \otimes e_i)$$

which gives a coordinate of the Veronese embedding $\mathbb{P}(V) \hookrightarrow \mathbb{P}(\mathrm{S}^2V)$.
Q.E.D.

By (2.1) and Lemma 2.3, we have the following:

Corollary 2.4. $\dim\mathbb{P}(M_1) = 2g - 6h + 2$ *and* $\dim\mathbb{P}(M_0) = g - 3h + 1$.

Let $\widetilde{V}$ be a vector space. For affine cones $S, T \subset \widetilde{V}$, we put

$$S * T = \{\lambda\widetilde{x} + \mu\widetilde{y} \mid \widetilde{x} \in S, \widetilde{y} \in T, \lambda, \mu \in \mathbb{C}\}.$$

Note that $S * T$ is again an affine cone and we may consider $\mathbb{P}(S * T) \subset \mathbb{P}(\widetilde{V})$.

Lemma 2.5. *Let $M^* = M_0 * M_1 \subset V \otimes V$. Then* $\dim\mathbb{P}(M^*) = 3g - 9h + 4$.

Proof. We define a morphism

$$\theta : V \oplus V \oplus V \to M^*$$

by $\theta(x, y, u) = x \otimes y + u \otimes u$, which is surjective. Let $x, y, u \in V$ be general elements and let $x', y', u' \in V$ be arbitrary elements. We may assume that x, y, u are linearly independent. We put

$$x = \sum_{i=1}^{g-3h+2} x_i e_i,\ y = \sum_{i=1}^{g-3h+2} y_i e_i,\ u = \sum_{i=1}^{g-3h+2} u_i e_i$$

and

$$x' = \sum_{i=1}^{g-3h+2} x'_i e_i,\ y' = \sum_{i=1}^{g-3h+2} y'_i e_i,\ u' = \sum_{i=1}^{g-3h+2} u'_i e_i.$$

Assume that $\theta(x,y,u)=\theta(x',y',u')$. Then

$$x\otimes y+u\otimes u=x'\otimes y'+u'\otimes u'. \tag{2.5.1}$$

Since $V\otimes V$ can be decomposed as $(V\otimes e_1)\oplus\cdots\oplus(V\otimes e_{g-3h+2})$,

$$y_i x+u_i u=y_i' x'+u_i' u' \ (i=1,\cdots,g-3h+2)$$

by (2.5.1). Since $x,y,u\in V$ are general elements, we may assume that $\det\begin{pmatrix} y_i & u_i \\ y_j & u_j \end{pmatrix}\neq 0$ for any $i,j=1,\cdots g-3h+2$ with $i\neq j$, and hence x,u are linear combinations of x',u'. Note that x,y are linearly independent, we have

$$x'=\alpha x+\beta u \text{ and } u'=\gamma x+\delta u$$

for some $\alpha,\beta,\gamma,\delta\in\mathbb{C}$. Since $V\otimes V=(e_1\otimes V)\oplus\cdots\oplus(e_{g-3h+2}\otimes V)$ and $x,y,u\in V$ are general elements, we again have $x_i y+u_i u=x_i' y'+u_i' u'$ for $i=1,\cdots,g-3h+2$ and hence

$$y'=\xi y+\eta u \text{ and } u'=\lambda y+\mu u$$

for some $\xi,\eta,\lambda,\mu\in\mathbb{C}$. Especially

$$u'=\gamma x+\delta u=\lambda y+\mu u.$$

Since x,y,u are linearly independent, $\gamma=\lambda=0$ and $\delta=\mu$. Therefore $x'=\alpha x+\beta u$, $y'=\xi y+\eta u$ and $u'=\delta u$. By (2.5.1), $\alpha\xi=1,\alpha\eta=0,\beta\xi=0,\beta\eta+\delta^2=1$. Therefore $\beta=\eta=0,\alpha\xi=1,\delta^2=1$, i.e.

$$x'=\alpha x, y'=\frac{1}{\alpha}y \text{ and } u'=\pm u.$$

Therefore $\theta^{-1}(x\otimes y+u\otimes u)$ is 1-dimensional for general elements $x,y,u\in V$. Hence $\dim M^*=3\dim V-1=3g-9h+5$, i.e. $\dim\mathbb{P}(M^*)=3g-9h+4$.

Q.E.D.

We take $0\neq\kappa\in\mathrm{im}\phi\subset\Gamma(E,\mathcal{O}_E(2N))$, $\widetilde{\kappa}\in\phi^{-1}(\kappa)$ and consider a linear subspace

$$L_\kappa=\{\lambda\widetilde{\kappa}+x \mid \lambda\in\mathbb{C},x\in\ker\phi\}=\mathbb{C}\widetilde{\kappa}+\ker\phi\subset V\otimes V.$$

Note that $\mathbb{C}\widetilde{\kappa}\cap\ker\phi=\{0\}$ and hence $\dim L_\kappa=\dim\ker\phi+1\geq\dim V\otimes V-\dim\Gamma(E,\mathcal{O}_E(2N)+1=\dim V\otimes V-(2g-5h+3)+1$, therefore

$$\dim\mathbb{P}(L_\kappa)\geq\dim\mathbb{P}(V\otimes V)-(2g-5h+2). \tag{2.2}$$

We now prove Theorem 2.1.

Proof of Theorem 2.1:Let $s \in \Gamma(C, \mathcal{O}_C(\pi^*N))$ and let $(s)_0 = A$. Since $s\sigma^*s = \pi^*\lambda$ for some $\lambda \in \Gamma(E, \mathcal{O}_E(2N))$, we put $\mathrm{Nm}_{C/E}(s) = \lambda$ and call it the Norm of s for the Galois covering $\pi : C \to E$. Since $\pi_*\mathcal{O}_C(\pi^*N) \cong \mathcal{O}_E(N) \oplus \mathcal{O}_E$, there is an isomorphism

$$\Gamma(C, \mathcal{O}_E(\pi^*N)) \cong \Gamma(E, \mathcal{O}_E(N)) \oplus \Gamma(E, \mathcal{O}_E).$$

Therefore s can be written as $s = (\alpha, \beta)$ for some $\alpha \in \Gamma(E, \mathcal{O}_E(N))$, $\beta \in \Gamma(E, \mathcal{O}_E)$ and

$$A = (s)_0 \in \{\pi^*N_0 \mid N_0 \in |N|\} \text{ if and only if } \beta = 0.$$

By the $\mathcal{O}_E$-algebra structure on $\mathcal{O}_E \oplus \mathcal{O}_E(-N)$, we have

$$s\sigma^*s = (\alpha^2 - r\beta^2, 0) \in \Gamma(E, \mathcal{O}_E(2N)) \oplus \Gamma(E, \mathcal{O}_E(N)).$$

Therefore $\mathrm{Nm}_{C/E}(s) = \alpha^2 - r\beta^2$. Since π_*A is defined by $\mathrm{Nm}_{C/E}(s)$, $\pi_*A = (\alpha^2 - r\beta^2)_0$. Let

$$W = \{\mathrm{Nm}_{C/E}(s) \mid s \in \Gamma(C, \mathcal{O}_C(\pi^*N))\} \subset \Gamma(E, \mathcal{O}_E(2N)).$$

Since $M_0 = \{a \otimes a \mid a \in \Gamma(E, \mathcal{O}_E(N))\}$ and $\mathrm{Nm}_{C/E}(\pi^*a) = a^2$ for any $a \in \Gamma(E, \mathcal{O}_E(N))$,

$$\phi(M_0) = \{\mathrm{Nm}_{C/E}(\pi^*a) \mid a \in \Gamma(E, \mathcal{O}_E(N))\} \subset \Gamma(E, \mathcal{O}_E(2N)).$$

Then $W = \{\alpha^2 - r\beta^2 \mid \alpha \in \Gamma(E, \mathcal{O}_E(N)), \beta \in \Gamma(E, \mathcal{O}_E)\}$ which implies $\mathbb{P}(W) = \mathbb{P}(\phi(M_0) * \mathbb{C}r)$.

We now assume C has a base point free and complete g^1_{g-2h+1} not composed with π. By Theorem 1.1, there exist $l, m \in \Gamma(E, \mathcal{O}_E(N))$ such that $lm \in W \setminus \phi(M_0)$. Then there exists $a_0 \in \Gamma(E, \mathcal{O}_E(N))$ such that $r \in \mathbb{P}(\mathbb{C}a_0^2 + \mathbb{C}lm)$. Hence $r = \alpha a_0^2 + \beta lm = \phi(\alpha a_0 \otimes a_0 + \beta l \otimes m)$ for some $\alpha, \beta \in \mathbb{C}$ which implies $r \in \mathrm{im}\phi$.

Next we assume $r \in \mathrm{im}(\phi)$. Since $\mathbb{P}(L_r)$ is a linear subspace of $\mathbb{P}(V \otimes V)$,

$$\dim\mathbb{P}(M^*) \cap \mathbb{P}(L_r) \geq \dim\mathbb{P}(M^*) - (2g - 5h + 2) = g - 4h + 2 \geq 0$$

by (2.2) and Lemma 2.5. Hence there exists $\widetilde{x} \in M^*$ such that $\phi(\widetilde{x}) = r$. Since $\widetilde{x} = l \otimes m + a \otimes a$ for some $l, m, a \in V$, we have

$$r = lm + a^2.$$

Assume $[l] \neq [m]$. Then $lm \in W \setminus \phi(M_0)$. We now put $\alpha = \sqrt{-1}a$, $\beta = \sqrt{-1}(\neq 0)$ and $s = (\alpha, \beta)$. Let $A = (s)_0 \in |\pi^*N|$. Since $r = lm + a^2$,

$\alpha^2 - r\beta^2 = lm$. Therefore $\pi_* A = N_1 + N_2$, $N_1, N_2 \in |N|$ and $A \in |\pi^* N| \setminus \{\pi^* N_0 \mid N_0 \in |N|\}$. By Theorem 1.1, there exists a base point free g^1_{g-2h+1} not composed with π. Assume $[l] = [m]$. We may assume that $l = m$. Then $r = (l + \sqrt{-1}a)(l - \sqrt{-1}a)$. When $(l + \sqrt{-1}a)_0 = (l - \sqrt{-1}a)_0$, we have $(l)_0 = (a)_0$ which implies the branch locus $(r)_0$ is not reduced. This is a contradiction, since C is non-singular. Therefore $(l + \sqrt{-1}a)_0 \neq (l - \sqrt{-1}a)_0$. We put $s = (0, 1)$ and let $A = (s)_0 \in |\pi^* N|$. Then $\pi_* A = (r)_0$. Since $r = (l + \sqrt{-1}a)(l - \sqrt{-1}a)$, we again have $\pi_* A = N_1 + N_2$, $N_1, N_2 \in |N|$ and $A \in |\pi^* N| \setminus \{\pi^* N_0 \mid N_0 \in |N|\}$, which implies that there exists a base point free g^1_{g-2h+1} not composed with π by Theorem 1.1.

Q.E.D.

Finally we prove Theorem A:

Proof of Theorem A: Since $g \geq 4h$, $\deg(\mathcal{O}_E(N)) = g - 2h + 1 \geq 2h + 1$. Therefore $\mathcal{O}_E(N)$ is normally generated and hence ϕ is automatically surjective. Hence we have Theorem A by Theorem 2.1.

Q.E.D.

We are now ready to prove Theorem B as a corollary to Theorem A.

Proof of Theorem B

Claim. Fix an integer $e \geq 1$. Let C be a smooth curve of genus $g \geq 4e - 4$, not necessarily a double covering. Let Σ^1_d be the union of those components of $W^1_d(C)$ whose general element is base point free and complete. If $\Sigma^1_{g-e+1} \neq \emptyset$ then $\dim \Sigma^1_{g-e+1}$ has the expected dimension and $\Sigma^1_{g-e+2} \neq \emptyset$.

Proof of the Claim. Since it is assumed that $\Sigma^1_{g-e+1} \neq \emptyset$, any component of Σ^1_{g-e+1} has dimension at least $\rho(g - e + 1, g, 1) = g - 2e$. Suppose there exists a component $\Sigma \subset \Sigma^1_{g-e+1}$ such that $\dim \Sigma = n \geq g - 2e + 1$ and take a general $L \in \Sigma$. By the base point free pencil trick and the description of the tangent space to the scheme $W^r_d(C)$ in general, we have

$$\begin{aligned} h^0(C, L^2) &= 2(g - e + 1) - g + 1 + h^1(C, L^2) \\ &= 2(g - e + 1) - g + 1 + \ker \mu_0 \\ &\geq g - 2e + 3 + n - \rho(g - e + 1, g, 1) = n + 3, \end{aligned}$$

where $\mu_0 : H^0(C, L) \otimes H^0(C, KL^{-1}) \to H^0(C, K)$ is the natural map given by multiplication of sections; cf. [1, p 189]. Therefore it follows that

$$g - 2e + 1 \leq n \leq \dim W^{n+2}_{2g-2e+2}(C) \leq \dim W^{g-2e+3}_{2g-2e+2}(C) = 2e - 4,$$

contrary to the assumption $g \geq 4e - 4$. And this completes the proof of the first assertion of the Claim. Suppose now that $\Sigma^1_{g-e+2} = \emptyset$. Then, we have

$$W^1_{g-e+2}(C) = [\Sigma^1_{g-e+1} + W_1(C)] \cup [W^1_{g-e}(C) + W_2(C)].$$

Since

$$\dim[\Sigma^1_{g-e+1} + W_1(C)] = \rho(g-e+1, g, 1) + 1 < \rho(g-e+2, g, 1),$$

it follows that the closed locus $\Sigma^1_{g-e+1} + W_1(C)$ is contained in $W^1_{g-e}(C) + W_2(C)$. Note that a general element in the locus $\Sigma^1_{g-e+1} + W_1(C)$ is a complete pencil with only one base point, whereas a complete pencil in $W^1_{g-e}(C) + W_2(C)$ has at least two base points, which is an absurdity. This completes the proof of the Claim.

We now take $e = 2h$ in the Claim. By Theorem A, we have $\Sigma^1_{g-2h+1} \neq \emptyset$ and hence $\Sigma^1_{g-2h+2} \neq \emptyset$ by the Claim. By taking $e' = 2h - 1$ in the Claim, we again have $\Sigma^1_{g-e'+2} = \Sigma^1_{g-2h+3} \neq \emptyset$; note that $g \geq 8h - 4 > 4e' - 4$. We may continue this process by taking smaller $e's$ and we are done.

Q.E.D.

§3. Examples

In this final section, we exhibit two examples which show that the genus assumption $g \geq 4h$ in Theorem A is the best possible one. We first give an example of a double covering $C \xrightarrow{\pi} E$ of genus $g = 4h - 1$ without a base point free and complete g^1_{g-2h+1} not composed with π. We also give another example of a double covering $C \xrightarrow{\pi} E$ possessing a base point free and complete g^1_{g-2h+1} not composed with π under the same genus assumption $g = 4h - 1$. In these examples we shall make use of the following well-known fact regarding the normal generation of line bundles on a hyperelliptic curve.

Remark 3.1. *Let E be a hyperelliptic curve of genus h. A very ample line bundle on E of degree $2h$ is not normally generated; cf.* [1, p.221 C-3].

Example 3.2. *There exists a double covering $C \xrightarrow{\pi} E$ of genus $g = 4h-1$ which does not have a base point free and complete g^1_{g-2h+1} not composed with π.*

Proof. Let E be a hyperelliptic curve of genus $h \geq 2$. Given $N \in \mathrm{Div}(E)$, we consider the natural cup product map

$$\phi : \Gamma(E, \mathcal{O}_E(N)) \otimes \Gamma(E, \mathcal{O}_E(N)) \to \Gamma(E, \mathcal{O}_E(2N)).$$

For $h = 2$, let $\mathcal{O}_E(N)$ be a base point free line bundle of degree $2h = 4$. Note that $\mathcal{O}_E(N)$ is not very ample. Assume that ϕ is surjective. By using [1, p.222 C-4] inductively, we easily see that

$$\Gamma(E, \mathcal{O}_E(N))^{\otimes k} \to \Gamma(E, \mathcal{O}(kN))$$

is surjective for every $k \geq 1$, i.e. $\mathcal{O}_E(N)$ is normally generated. Hence $\mathcal{O}_E(N)$ is very ample which is a contradiction. Therefore we may choose $r \notin \mathrm{im}\phi$ such that $(r)_0 = R$ is reduced. For $h \geq 3$, by a well-known theorem of Halphen, we may take a very ample line bundle $\mathcal{O}_E(N)$ of degree $2h = g - 2h + 1$. By Remark 3.1, $\mathcal{O}_E(N)$ is not normally generated and hence ϕ is not surjective by [1, p.222]. Therefore we may again choose $r \notin \mathrm{im}\phi$ such that $(r)_0 = R$ is reduced.

Let $C \xrightarrow{\pi} E$ be a double covering of genus $g = 4h - 1$ with the branch locus R. By Theorem 2.1, C does not have a base point free and complete g^1_{g-2h+1} not composed with π.

Q.E.D.

For an example of a double covering of genus $g = 4h-1$ with a base point free and complete g^1_{g-2h+1}, we have implictly exhibited such one for $h \geq 3$ in the Example 0.4. We simply note that, in the Example 0.4, it is possible to take a very ample $N \in E_{g-2h+1}$ even in the range $3h + 2 \leq g \leq 4h - 1$ for any curve E of genus $h \geq 3$. One may also construct such an example by a similar method as in Example 3.2.

Example 3.3. *There is double covering $C \xrightarrow{\pi} E$ of gneus $g = 4h - 1$ which has a base point free and complete g^1_{g-2h+1}.*

Proof. Let $\mathcal{O}_E(N)$ be a line bundle of degree $2h = g - 2h + 1$. Since $|N|$ is base point free , we may take $r \in \mathrm{im}\phi$ whose zero $R = (r)_0$ is reduced and let $C \xrightarrow{\pi} E$ be a double covering of genus genus $g = 4h - 1$ with the branch locus R. Then C has a base point free and complete g^1_{g-2h+1} not composed with π by Theorem 2.1.

Q.E.D.

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