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Two non-conjugate embeddings of $S_3 \times \mathbb{Z}_2$ into the Cremona group II

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Abstract.

We give a detailed proof of the following main result of [Isk1]. There are two non-conjugate embeddings of $S_3 \times \mathbb{Z}_2$ into the Cremona group given by the linear action of this group on the plane and the action on the two-dimensional torus respectively.

§1. Introduction and the main result

1.1.

Let us first recall our main result which is inspired by [LPR]. Consider the following two actions of the group $G \simeq S_3 \times \mathbb{Z}_2$ on the rational surfaces P and T (where S_3 denotes the symmetric group and $\mathbb{Z}_2$ denotes the cyclic group of order two).

(I) P **- the plane** $x + y + z = 0$**:** S_3 acts by the permutations on the coordinates and $\mathbb{Z}_2$ is given by $(x, y, z) \to (-x, -y, -z)$;

(II) T **- the two-dimensional torus** $xyz = 1$**:** S_3 acts by the permutations on the coordinates and $\mathbb{Z}_2$ is given by $(x, y, z) \to (x^{-1}, y^{-1}, z^{-1})$.

In both cases we obtain two embeddings of this group into the two-dimensional Cremona group $Cr_2(\mathbb{C})$. A natural question arising here is whether the images of G under these embeddings are conjugate in $Cr_2(\mathbb{C})$ or not?

Our answer is the following proposition.

1.1.1. **Proposition** *The images of G under the above embeddings are not conjugate in $Cr_2(\mathbb{C})$. In the other words, there exists no G-equivariant birational map between G-surfaces P and T.*

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1.2. Commentary

Together with [LPR] this result means that the simple algebraic group of type $\mathbb{G}_2$ is not a Cayley group in the following sense. Let G as in [LPR] be a connected algebraic group (we assume it only in this commentary). Let $\mathfrak{g}$ be its Lie algebra. Consider G as an algebraic variety X with the action of G on itself by the conjugation gxg^{-1}, $g \in G$, $x \in X$, and the Lie algebra $\mathfrak{g}$ as an algebraic variety Y with adjoint action $Ad_G g(y) = gyg^{-1}$, $g \in G$, $y \in Y$. The group G is called *a Cayley group* if there exist G-equivariant birational map $\lambda \colon X \dashrightarrow Y$, *i. e.* $\lambda(gxg^{-1}) = Ad_G g(\lambda(x))$. The question about the Cayley property is reduced to some property of character lattice $\hat{T}$ of the maximal torus $T \in G$ with the natural action of the Weyl group W on $\hat{T}$. Many canonical linear algebraic groups are proved to be non-Cayley ones. However, there are also a lot of exceptions.

The maximal torus of the group of type $\mathbb{G}_2$ is two-dimensional, and its Weyl group is $W \simeq S_3 \times \mathbb{Z}_2$. The question about the Cayley property is exactly problem 1.1. Our result says that $\mathbb{G}_2$ is not Cayley. Surprisingly it was proved in [LPR] that the group $\mathbb{G}_2 \times G_m^2$ (where G_m is the multiplicative group of the ground field with the trivial action of the Weyl group) is a Cayley group. Thus, $\mathbb{G}_2$ is stable Cayley in this sense. It would be interesting to ask whether the group $\mathbb{G}_2 \times G_m$ is Cayley or not?

In our work [Isk1] we gave a sketchy proof of Proposition 1.1.1 based on the general method of factorization of G-equivariant birational maps between rational G-surfaces (see [Ma1], [Isk3]) with G denoting a finite group. In this paper we shall give a detailed proof of it since after the appearance of [Isk1] a great interest surrounded this fact.

1.3.

As in [Isk1] we shall use the concept of a rational G-surface in our proof. Here $G \subset Cr_2(\mathbb{C})$ denotes a finite subgroup of the Cremona group (see, for instance, [Ma1], [Isk2]). In our case $G \simeq S_3 \times \mathbb{Z}_2$ and it acts on the compactifications of Y and X of surfaces P and T corresponding respectively to the cases (I) and (II) above. Let us describe these actions more precisely.

The case (I): $Y = \mathbb{P}^2$. Let (u_0, u_1, u_2) be the homogeneous coordinates on $\mathbb{P}^2$ and $x = \frac{u_1}{u_0}$, $y = \frac{u_2}{u_0}$ and $z = -\frac{u_1+u_2}{u_0}$. Then the action of G on Y is given in the matrix form as follows (we use the homogeneous coordinates).

$$\text{The involution } \sigma_{xy} = (xy) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix};$$

$$\text{the 3-cycle } \sigma_{xyz} = (xyz) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & -1 \end{pmatrix};$$

$$\text{the involution } \tau = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \text{ is the generator of } \mathbb{Z}_2.$$

Fixed elements: the point $(1, 0, 0)$.

Invariant elements: the line $L_0 = (u_0 = 0)$. The group $\mathbb{Z}_2$ acts on L_0 trivially and S_3 acts as the standard irreducible two-dimensional linear representation. A unique G-invariant 0-orbit of length 2 is $\{(0, 1, \lambda), (0, \lambda, 1)\}$, where $\lambda = e^{\frac{2\pi i}{3}}$ is the cubic root of 1. There are also two G-invariant 0-orbits of length 3 given by $\{(0, 0, 1), (0, 1, -1), (0, 1, 0)\}$ and $\{(0, 1, 1), (0, 1, -2), (0, -1, 1)\}$.

There is a G-equivariant pencil of lines with G-fixed point $(1, 0, 0)$ and G-invariant section (the line L_0). The group S_3 acts on the base of this pencil and on the section L_0. The group $\mathbb{Z}_2$ acts on the fibers by $t \to -t$.

The case (II): X is the most natural smooth compactification of the two-dimensional torus T: $(xyz = 1)$. Consider the threefold $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ with homogeneous coordinates $(x_1, x_0) \times (y_1, y_0) \times (z_1, z_0)$ and coordinates $(x = \frac{x_1}{x_0}, y = \frac{y_1}{y_0}, z = \frac{z_1}{z_0})$. This compactification is given by

$$x_1 y_1 z_1 = x_0 y_0 z_0. \tag{1}$$

Under the Segre map $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \hookrightarrow \mathbb{P}^7$, the equation (1) is the G-invariant equation of the hyperplane $\mathbb{P}^6 \supset X$. The surface X is given there by the quadratic equations that are restrictions on $\mathbb{P}^6$ of the equations which give the Segre map. Under this embedding X is a smooth projective del Pezzo surface of degree 6 in $\mathbb{P}^6$.

G acts in (1) as follows. S_3 acts by the coordinate permutations induced by the permutations of factors of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ (for example σ_{xy}: $(x_1, x_0) \longleftrightarrow (y_1, y_0)$). $\mathbb{Z}_2$ acts as follows.

$$(x_1, y_1, z_1) \longleftrightarrow (x_0, y_0, z_0).$$

These actions are linear under the Segre map.

According to the classification of G-minimal rational G-surfaces (see [Isk2], [Mo]), $X \subset \mathbb{P}^6$ is the minimal del Pezzo G-surface of degree 6 with G-invariant Picard group $\operatorname{Pic}^G(X) = \mathbb{Z}(-K_X)$, where K_X denotes the canonical class. The configuration of (-1)-curves on X is a unique G-orbit and has the form of a regular hexagon. Its equation is the infinite hyperplane section

$$x_0 y_0 z_0 = 0. \tag{2}$$

These lines are one-dimensional fibers of three birational projections $X \to \mathbb{P}^1 \times \mathbb{P}^1$, two one-dimensional fibers in each of them.

Further, we need other G-birational equivalent to X models X_0, X_1, X_2 and the classification of some 0-dimensional (of small length) and 1-dimensional orbits on them.

1.4.

Start from the surface X given by (1). $\operatorname{Pic} X$ is generated by (-1)-curves and all (-1)-curves form a unique G-orbit, which is equivalent to $-K_X$, so $\operatorname{Pic}^G_X = \mathbb{Z}(-K_X)$. We may consider an affine model $T\colon (xyz = 1)$ for studying of G-orbits, because the orbit at infinity is known (it is all (-1)-curves) and there is no 0-orbit of length less than 6.

1.4.1. **Lemma** *Let $\mathcal{A} \subset T$ be a G-equivariant 0-orbit of length $d < 6$. Then $\mathcal{A}$ is one of the following.*

- $d = 1$: *a unique fixed point $P = (1,1,1)$;*
- $d = 2$: *a unique orbit $\{P_1 = (\lambda,\lambda,\lambda), P_{-1} = (\lambda^{-1},\lambda^{-1},\lambda^{-1})\}$, where $\lambda = e^{\frac{2\pi i}{3}}$ is a cubic root of 1 (notice that $\lambda^{-1} = \lambda^2$);*
- $d = 3$: *a unique orbit $\{Q_1 = (1,-1,-1), Q_2 = (-1,1,-1), Q_3 = (-1,-1,1)\}$;*
- $d = 4$: *there are no such orbits;*
- $d = 5$: *there are no such orbits, because $5 \nmid 12 = |G|$.*

Indeed, one can see this from the equation $xyz = 1$ and the action of G on T. In the case $d = 4$ the stabilizer of the point is $\langle \sigma_{xyz} \rangle = \mathbb{Z}_3$ acting by $(x,y,z) \to (y,z,x)$, which means that $x = y = z$.

1.4.2. We now find some G-orbits consisting of rational curves.

$\mathcal{T} = \Gamma_x + \Gamma_y + \Gamma_z \sim -K_X$. This G-orbit consists of the curves of genus 0 which contain the fixed point $P = (1,1,1)$ and is given by $x = 1$, $y = 1$, $z = 1$. In the projective form these equations on $\mathbb{P}^1 \times \mathbb{P}^1$ are as follows.

$$\Gamma_x\colon y_1 z_1 - y_0 z_0 = 0,$$

$$\Gamma_y\colon x_1 z_1 - x_0 z_0 = 0, \tag{3}$$

$$\Gamma_z\colon x_1y_1 - x_0y_0 = 0.$$

Each of these curves Γ is smooth of genus 0 with $-K_X\Gamma = 2$, $\Gamma^2 = 0$, and any two of them intersect by a unique point P. Blow up this point. Then after a G-equivariant contraction of these curves we have three G-conjugate points on the image of the exceptional (-1)-curve of the blowing-up on the model $X_2 \subset \mathbb{P}^3$. This is an example of G-equivariant link $\Phi_{6,1}$, *cf.* Section 2 below.

This birational G-map $\gamma\colon X \dashrightarrow X_2$ is a projection from the tangent plane to the fixed point P with the embedding $X \subset \mathbb{P}^6$ described above. Under this projection the point P is blown up, and three conics passing through P are contracted.

$\mathcal{D} = \Delta_x + \Delta_y + \Delta_z \sim -2K_X$. This is a triple of smooth curves of genus 0 with equations $y = z$, $z = x$, and $x = y$ respectively. In the projective form we have

$$\Delta_x\colon x_1y_1^2 - x_0y_0^2 = 0,$$

$$\Delta_y\colon y_1z_1^2 - y_0z_0^2 = 0, \tag{4}$$

$$\Delta_z\colon z_1x_1^2 - z_0x_0^2 = 0.$$

All these curves Δ intersect in three points P, P_1 and P_{-1} with $-K_X\Delta = 4$ and $\Delta^2 = 2$.

1.4.3. Curves Γ and Δ are components of the fibers of the G-invariant pencil of rational curves $\Pi = |-K_X - 2P - P_1 - P_{-1}|$ (*i. e.* the pencil of curves from the linear system $|-K_X|$ which contain points P_1, P_{-1} and twice point P). Moreover $\Gamma_x + \Delta_x \sim \Gamma_y + \Delta_y \sim \Gamma_z + \Delta_z$ are the fibers of this pencil. Equations (3) and (4) show that $\Gamma_x \cap \Delta_x = \{P, Q_1\}$, $\Gamma_y \cap \Delta_y = \{P, Q_2\}$, $\Gamma_z \cap \Delta_z = \{P, Q_3\}$.

$\mathcal{E} = E_x + E_y + E_z \sim -K_X$. This triple of smooth curves of genus 0 is given by $x = -1$, $y = -1$, $z = -1$ accordingly. One can see from these equations that $E_x \ni Q_2, Q_3$, $E_y \ni Q_1, Q_3$, $E_z \ni Q_1, Q_2$, and Q_i are the only intersection points for the components of $\mathcal{E}$. As above, $-K_XE = 2$, $E^2 = 0$, so after blowing up Q_1, Q_2 and Q_3 strict transforms E'_x, E'_y, E'_z form a G-conjugate triple of (-2)-curves which do not intersect each other. They may be G-equivariant contracted to three ordinary double points on a cubic surface in $\mathbb{P}^3$.

1.5.

Consider now a G-birational model X_0 of G-surface X, namely, the standard projectivisation of torus $T \subset \mathbb{A}^3 \subset \mathbb{P}^3$

$$X_0\colon xyz = w^3, \tag{5}$$

where (x, y, z, w) are homogeneous coordinates on $\mathbb{P}^3$. This model differs from X only at infinity $w = 0$. Three lines given by $x = w = 0$, $y = w = 0$ and $z = w = 0$ are on X_0 unlike on X there are 6 lines. It has also 3 ordinary double points $(0,0,1,0)$, $(0,1,0,0)$, and $(1,0,0,0)$. G acts on X_0 not linearly (as on $X \subset \mathbb{P}^6$ it does). More precisely S_3 acts linearly by permutations of x, y, z and $\mathbb{Z}_2$ acts via the birational involution $(x, y, z, w) \rightarrowtail (x^{-1}, y^{-1}, z^{-1}, w^{-1})$.

However there is another cubic model $X_1 \subset \mathbb{P}^3$ such that G acts linearly on it. As X_0 it has three ordinary double points. This model is given by G-equivariant birational projection

$$p_1 \colon X \subset \mathbb{P}^6 \dashrightarrow \mathbb{P}^3 \supset X_1$$

from the plane $\langle Q_1, Q_2, Q_3 \rangle$, which is a linear span of $Q_1, Q_2, Q_3 \in X$ in $\mathbb{P}^6$. As noted in 1.4 conics E_x, E_y, E_z are contracted to the singular points.

Choose homogeneous coordinates (x, y, z, w) in $\mathbb{P}^3$ such that the image of P is the point $(0,0,0,1)$ and $\mathbb{Z}_2$ acts via the involution $(x, y, z, w) \rightarrowtail (-x, -y, -z, w)$. Then S_3 acts by coordinate permutations as on X_0. Under the projection $p_1 \colon X \dashrightarrow X_1$ three singular points (the images of E_x, E_y, and E_z) lie on the plane at infinity $w = 0$. The images of conics Γ_x, Γ_y, and Γ_z are three lines passing through $P_0 = p_1(P)$. Thus P_0 is the Eckardt point whose tangent plane is G-invariant, *i. e.* it is given by the linear equation $x + y + z = 0$.

So the equation of $X_1 \subset \mathbb{P}^3$ is given by

$$X_1 \colon xyz - aw^2(x + y + z) = 0, \quad a \in \mathbb{C}. \tag{6}$$

For convenience one may assume $a = \frac{1}{3}$. Then the singular points are $(1,0,0,0)$, $(0,1,0,0)$, $(0,0,1,0)$, and the images of P_1 and P_{-1} are $(1,1,1,1)$ and $(-1,-1,-1,-1)$ respectively. Three lines $x = 0$, $y = 0$, $z = 0$ lie on the tangent plane $x + y + z = 0$.

1.6. Remark

As on each cubic surface, there is Geiser's birational involution concerned with P_0, *i. e.* the projection from this point to the plane composed with the automorphism of the double covering. Obviously this birational involution is G-equivariant and in our case it is biregular because P_0 is the Eckardt point. G-equivariant birational Bertini involution is concerned with two G-conjugate points $P_1, P_{-1} \in X$. Its action on $\operatorname{Pic} X_1$ differs from the standard one because X_1 is singular and the points P_1, P_{-1} are not in the general position. This means that the third intersection point of the line $\langle P_1, P_{-1} \rangle \subset \mathbb{P}^3$ and X_1 (i.e. P_0) lies on the (-1)-curves.

1.7.

Another G-equivariant birational model of X is the quadric $X_2 \subset \mathbb{P}^3$ of type

$$X_2\colon xy + yz + zx = 3w^2. \tag{7}$$

It can be obtained from X_1 by G-equivariant birational transform $(x, y, z, w) \rightarrowtail (x^{-1}, y^{-1}, z^{-1}, w^{-1})$. Indeed this coordinate change with the multiplication to the common factor transform the equation (6) with $a = \frac{1}{3}$ to the equation (7). The action of G on (x, y, z, w) is still linear. Moreover X_2 is nothing but the image of $X \subset \mathbb{P}^6$ under the linear projection

$$p_2\colon X \subset \mathbb{P}^6 \dashrightarrow \mathbb{P}^3 \supset X_2$$

from the tangent plane to X at the G-fixed point $P \in X$. The projection p_2 blows up P to the conic $C_0\colon (w = 0) \subset X_2$ and contracts three conics $\Gamma_x, \Gamma_y, \Gamma_z \subset X$ to the G-invariant 0-orbit $\mathcal{A} = \{(1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1, 0)\} \subset C_0$.

The curve C_0 is the image of the point $P_0 \in X_1$ and the 0-orbit $\mathcal{A}$ is the image of the lines $x = 0, y = 0, z = 0$ under our map $X_1 \dashrightarrow X_2$.

1.8. Lemma

(i): *The surface X_2 is smooth and there are no G-fixed points on it;*

(ii): *The conics $C_0 = (w = 0)$ and $C_1 = (x + y + z = 0)$ are G-invariant and on C_0 acts only S_3 by a unique two-dimensional irreducible representation and G acts effective on C_1;*

(iii): *The G-invariant 0-orbits of length $d < 6$ on X_2 are:*

$d = 2$: $\{P_1 = (1, 1, 1, 1), P_{-1} = (-1, -1, -1, 1)\}$, *the images on X_2 of P_1 and P_{-1} on X (or X_1); $\{R_1 = (1, \lambda, \lambda^{-1}), R_2 = (\lambda, 1, \lambda^{-1})\} = C_0 \cap C_1$ - the base points of the conic pencil on X_2 generated by C_0 and C_1, $\lambda = e^{\frac{2\pi i}{3}}$*

$d = 3$: $\mathcal{A} = \{(1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1, 0)\}$,
$\mathcal{B} = \{(-1, 2, 2, 0), (2, -1, 2, 0), (2, 2, -1, 0)\}$, *$\mathcal{A}, \mathcal{B} \subset C_0$;*

$d = 4$: *there are no such orbits;*

$d = 5$: *there are no such orbits;*

(iv): *Let*

$$\Pi_0 = (t_0(x + y + z) + t_1 w = 0), \quad (t_0, t_1) \in \mathbb{P}^1,$$

be the pencil of conics generated by C_0 and C_1; then on the base of this pencil $\mathbb{P}^1$ acts only $\mathbb{Z}_2 = G/S_3\colon (t_0, t_1) \rightarrowtail (-t_0, t_1)$ with two fixed points $(1, 0)$ and $(0, 1)$; there are two reducible *fibers F_1 and F_{-1} on Π_0 with intersection points P_1 and P_{-1} of the components with coordinates* $(1,-3)$ *and* $(1,3)$ *accordingly; on the irreducible fibers G acts without fixed points;*

(v): *there is another pencil of conics*

$$\Pi_1 = \{u_0(x-y)+u_1(y-z)+u_2(z-x) = 0;\ u_0+u_1+u_2 = 0,\ (u_0,u_1,u_2) \in \mathbb{P}^2\}$$

passing through $P_1, P_{-1} \in X_2$; it is an image of Π from 1.4.3 under the projection $p_2\colon X \dashrightarrow X_2$; on the base of this pencil $\mathbb{P}^1 = (u_0+u_1+u_2 = 0)$ as usual acts S_3 by a two-dimensional irreducible representation, $\mathbb{Z}_2$ acts on each fiber with fixed points $\Pi_1 \cap C_0$ and invariant 2-section $\Pi_1 \cap C_1$.

There are two reducible fibers G_1 and G_2 on Π_1 with intersection points of the components R_1 and R_2 with coordinates $(\lambda^{-1},1,\lambda)$ *and* $(\lambda^{-1},\lambda,1)$ *on the base accordingly.*

All these statements are straightforward by analyzing the equations and the action of G.

§2. Proof of the main result

2.1.

The idea of the proof of Proposition 1.1.1 is the same as in [Isk1]. We have to show that there is no G-equivariant birational map between the G-surfaces X and $Y = \mathbb{P}^2$ defined in section 1.3. According to the general theory (see [Isk3]), each G-equivariant birational map can be decomposed into a sequence of G-equivariant elementary birational maps—links, via the G-equivariant Sarkisov program. The classification of such links was obtained in the algebraic case for G a finite Galois group acting on smooth k-minimal rational surfaces over a perfect field k (see [Isk3]).

However according to the general concept of rational G-surfaces (see [Ma1], [Isk2]) this classification may be applied to the geometrical case, *i. e.* to the case of action of a finite group G on smooth projective minimal rational G-surfaces. Their classification is also known (see [Isk2] and G-equivariant 2-dimensional Mori theory [Mo]).

Some differences between geometrical and algebraical cases are concerned with the description of G-orbits. For example, in the geometrical case (with non-trivial G-action) G-orbits are closed proper subset. In the algebraical case for del Pezzo surfaces, say, of large degrees, the existence of one k-point (orbit of degree 1) implies the density everywhere of such k-points in the Zariski topology.

The decomposition theorem for a birational G-map to a sequence of links is known (see [Isk3], 2.5) and all links are determined by their centers, *i. e.* G-orbits of dimension 0, all points of which are in the general position on the corresponding G-surface. So for the proof of our main result we need the following

a) Choose from the general classification (see [Isk3], 2.6) the links which are concerned with our case. Indicate minimal G-surfaces on which these links act (as G-equivariant birational maps) starting from X.

b) Classify all zero-dimensional G-orbits of length $d < \deg X_i = K_{X_i}^2$ on all of such surfaces X_i, and choose those whose points are in the general position (this is necessary condition for the existence of links).

Recall that the points $x_1, x_2, \ldots, x_d \in X$ are in the general position, if

1) X is del Pezzo surface, $\sigma\colon X' \to X$ is the simultaneous blow up of all $x_1, \ldots, x_d$ and X' is also del Pezzo surface, or

2) $\pi\colon X \to \mathbb{P}^1$ is a conic bundle, then these points do not lie on the degenerated fibers and any two of $x_1, \ldots, x_d$ do not lie on the same fiber; this is necessary and sufficient condition for the existence of link with center in $x_1, \ldots, x_d$, *i. e.* the birational transform

$$\begin{tikzcd} X \arrow[rr, dashed] \arrow[dr] & & X', \arrow[dl] \\ & \mathbb{P}^1 & \end{tikzcd}$$

which is a blow up of $x_1, \ldots, x_d$ and a contraction of strict transforms of the fibers on which they lie.

2.2.

Suppose that there exists a G-equivariant birational map $\chi\colon X \dashrightarrow Y$. Then χ can be decomposed into a composition of links. According to the particular algorithm of decomposition and considering all possibilities for links we will show that there is no link whose image is $\mathbb{P}^2$, *i. e.* a contradiction with the decomposition theorem. Thus such a χ can not exist. This means that two embeddings of G to the Cremona group, defined in (I) and (II), section 1.1, are not conjugate in $Cr_2(\mathbb{C})$.

By the mentioned algorithm, choose a very ample G-invariant linear system $\mathcal{H}' = \mathcal{H}_Y$ on Y, for example, $\mathcal{H}' = |-K_Y|$. Let $\mathcal{H} = \mathcal{H}_X = \chi_*^{-1}\mathcal{H}'$ be its strict transform on X. By the definition of strict transform, the linear system $\mathcal{H}$ has no fixed components and $\dim \mathcal{H} = \dim \mathcal{H}'$ ($\mathcal{H}$ has base points if χ is not morphism). $\operatorname{Pic}^G(X) = \mathbb{Z}(-K_X)$, so $\mathcal{H} \sim -aK_X$, $a \in \mathbb{Z}_{>0}$.

The map $\chi\colon X \dashrightarrow Y$ is not isomorphism. So by the G-equivariant Noether inequality (see [Isk3], 2.4) the linear system $\mathcal{H}$ has a maximal singularity which is a zero-dimensional G-orbit $x \subset X$ of length $d = d(x)$ and multiplicity $mult_x\mathcal{H} = r = r(x) > a$ at all points $x_i \in x$. We have $\mathcal{H}^2 = a^2K_X^2 > r^2d$ ($\mathcal{H}$ is a linear system without fixed components passing through all points of G-orbits x with multiplicity r and determining a birational map) so $d < K_X^2$. In our case $d < K_X^2 = 6$. The length of the orbit is divided by the order of the group. So the maximal singularities giving rise to our links in the decomposition can be only G-orbits of length $d = 1$, 2, 3 or 4. Such orbits may be found in Lemma 1.4.1.

2.3.

Now for every such G-orbit we have to verify whether its points are in the general position or not and if yes, then choose from the classification ([Isk3], 2.6) appropriate link $\Phi\colon X \dashrightarrow X_1$ (do not confuse with X_1 from paragraph 1.5). The decomposition algorithm is going as follows. The map $\chi_1\colon X_1 \dashrightarrow Y$ in the composition $\chi = \chi_1 \circ \Phi$ is given by the linear system $\mathcal{H}_1 = \Phi_*(\mathcal{H}) \sim -a_1K_{X_1}$ with $a_1 < a$. Since $a \in \mathbb{Z}_{>0}$ the decomposition process must be terminated on an isomorphism after a finite number of steps. We begin to find links. There is no G-orbit of length 4 on X (see 1.4.1). So let's start from $d = 3$.

The case $d = 3$**.** There is one such orbit (Q_1, Q_2, Q_3). It can be shown that in this case there is no such an "untwisting" link Φ. Indeed the points Q_1, Q_2, Q_3 on X are not in the general position. On the blow up of these three points $\sigma\colon X' \to X$ strict transforms E'_x, E'_y, E'_z of conics E_x, E_y, E_z (see 1.4.2) are G-invariant triple of (-2)-curves, so $-K_X$ is not ample ($-K_X \cdot E'_x = 0$).

(As we have seen in paragraph 1.5, the projection $p_1\colon X \dashrightarrow X_1$ from the plane $\langle Q_1, Q_2, Q_3\rangle \subset \mathbb{P}^6 \dashrightarrow \mathbb{P}^3$ blows up these three points and contracts E_x, E_y, E_z to the ordinary double points on X_1.)

Notice that in [Isk1] there was an inaccuracy claiming that there is such a link $\Phi_{6,3}$. The reason was that we did not check the condition of generality (although this does not affect to the final result).

2.3.1. **Remark** The fact that the link $\Phi_{6,3}$ does not exists can be seen easily. For the existence of a link with center in the maximal singularity $x \subset X$ it is necessary and sufficient that under the blow up $\sigma\colon X' \to X$ of the cycle x on the surface X' there are two extremal rays, one of which is an exceptional divisor. Since $\operatorname{Pic}^G(X') = \mathbb{Z} \oplus \mathbb{Z}$, the Mori cone is generated by these extremal rays and $-K_{X'}$ is ample by

Kleiman's criterion. The link Φ is nothing but the blow up σ and the extremal contraction of the second ray.

In the classical terms, the non-general position condition for Q_1, Q_2 and Q_3 means that this G-orbit could not be a maximal singularity in the linear system $\mathcal{H}$, because if $r > a$, then E_x, E_y and E_z are fixed components of $\mathcal{H}$. Indeed $\mathcal{H} \subset |-aK_X - rQ_1 - rQ_2 - rQ_3|$, the cycles $E_x - Q_2 - Q_3$, $E_y - Q_1 - Q_3$, $E_z - Q_1 - Q_2$ are effective and must have non-negative intersection with $\mathcal{H}$. But $\mathcal{H} \cdot (E_x - Q_2 - Q_3) = (-aK_x - rQ_1 - rQ_2 - rQ_3) \cdot (E_x - Q_2 - Q_3) = 2a - 2r < 0$. So we would get a contradiction.

2.3.2. **The case $d = 2$.** There is such a unique orbit $x = \{P_1, P_{-1}\}$ (see 1.4.1). The pair P_1, P_{-1} is in the general position on X. Indeed, each of these points does not lie on the (-1)-curves (which lie on the infinity $x_0 y_0 z_0 = 0$), and there is no curve of genus 0 and degree 2 (with respect to $-K_X$) on which both of these points lie.

There is corresponding link $\Phi = \Phi_{6,2}\colon X \dashrightarrow X_1$ (see [Isk3], 2.6). Find out what is X_1. Let $\sigma\colon X' \to X$ be the blow up of $x = \{P_1, P_{-1}\}$. Then $-K_{X'}$ is ample and $K_{X'}^2 = 4$. The linear system $|-K_{X'}|$ gives a G-equivariant embedding $X' \subset \mathbb{P}^4$ into the del Pezzo surface of degree 4 (*i. e.* into an intersection of two quadrics). The image of the exceptional (-1)-curves are two skew lines whose linear span is a hyperplane $\mathbb{P}^3$. This image is G-invariant and intersects $X' \subset \mathbb{P}^4$ by another G-invariant pair of skew lines which is reducible curve of genus 1 together with the pair that we blow up.

Contracting this residual pair of lines we get a del Pezzo surface $X_1 \subset \mathbb{P}^6$ of degree 6.

In the general algebraic case the surface X_1 may be not isomorphic to X, *i. e.* $\Phi_{6,2}$ is not always a birational involution (but I do not know such an example). However, in our particular case $\Phi_{6,2}$ is a birational involution of Bertini type. Indeed, if $\sigma\colon X'' \to X' \to X$ is a blow up of G-orbit $\{Q_1, Q_2, Q_3\}$ together with $\{P_1, P_{-1}\}$, then $K_{X''}^2 = 1$. But $-K_{X''}$ is not ample (because X'' is a degenerated del Pezzo surface of degree 1 with three (-2)-curves, or, after their contraction, three ordinary double points). Nevertheless, the linear system $|-2K_{X''}|$ gives a contraction of (-2)-curves composed with double covering $X'' \to Q^* \subset \mathbb{P}^3$ of the quadratic cone branched over a fixed point and a curve on Q^* with three ordinary double points.

The involution of this double covering $X'' \to X$ induces the birational involution $\Phi_{6,2}$. It differs from the standard Bertini involution given by the Picard group, because 5 blown up points are not in the general position. The point P and ordinary double points are still fixed under this involution. It also induces the involution on the cubic surface X_1 considered in Remark 1.6.

The birational involution $\Phi_{6,2}$ maps the pencil of rational curves $\Pi = |-K_X - 2P - P_1 - P_{-1}|$ onto itself and interchanges the components of its three fibers: $\Gamma_x \leftrightarrow \Delta_x$, $\Gamma_y \leftrightarrow \Delta_y$, $\Gamma_z \leftrightarrow \Delta_z$ (see 1.4.3).

Apparently one can write the particular equation for the residual G-invariant pair of the curves of genus 0 on X (although we will not need this in the sequel). This 3-linear 3-homogeneous equation must be as follows.

$$H\colon \sum_{g\in G} g((x_1-\lambda x_0)(y_1-\lambda y_0)(z_1-\lambda^2 z_0)+(x-\lambda x_0)(y_1-\lambda^2 y_0)(z_1-\lambda^2 z_0)) = 0.$$

The curve $H \sim -K_X$ is G-invariant. It passes through P_1, P_{-1}, and has them as ordinary double singularities. This means that the curve is reducible.

The link $\Phi_{6,2}\colon X \dashrightarrow X$ maps the linear system $\mathcal{H} \sim -aK_X$ with the maximal singularity $x = \{P_1, P_{-1}\}$ of multiplicity $r = r(x)$ to the linear system $\mathcal{H}_1 \sim -a_1K_X$ with the base cycle $x_1 = x = \{P_1, P_{-1}\}$ with the multiplicity $r_1 < a_1$ (this is not maximal singularity for $\mathcal{H}_1$). The formulas for the coefficients are given in [Isk3], Theorem 2.6, case $K_X^2 = 6$, case d):

$$(8) \qquad a_1 = 2a - r(x) < a, \qquad r_1 = 3a - 2r(x) < a_1.$$

So, $x = \{P_1, P_{-1}\}$ already is not a maximal singularity for $\mathcal{H}_1$. The map $\chi_1 = \chi \circ \Phi^{-1}\colon X \dashrightarrow Y$ defined by $\mathcal{H}_1$ is not isomorphism, so, by Lemma – Noether's inequality, $\mathcal{H}_1$ must have a maximal singularity. Now it can be only the point P of multiplicity $r_1 = r(P) > a_1$.

2.3.3. **The case $d = 1$.** The only fixed point $x_1 = P$. It does not lie on the (-1)-curves, so it is in a general position. The corresponding link $\Phi_{6,1}\colon X \dashrightarrow X_2$, where $X_2 \subset \mathbb{P}^3$ is a smooth quadric with $\operatorname{Pic}^G(X_2) = \frac{1}{2}\mathbb{Z}(-K_X)$ considered in section 1.7, is the birational projection $p_2\colon X \dashrightarrow X_2 \subset \mathbb{P}^3$ from the tangent plane in $\mathbb{P}^6$ to X at P. This link blows up the point P and contracts three conics Γ_x, Γ_y, Γ_z to the G-orbit $\mathcal{A}$ of length 3 on the conic $C_0 \subset X_2$ which is the image of blowing up P. The stabilizator of $P \in X$ is the whole G which acts on the projectivisation of tangent plane to C_0 as S_3 by the standard irreducible 2-dimensional linear representation. This situation was completely investigated in the Lemma 1.8.

There is a G-invariant curve $C_0 \sim -\frac{1}{2}K_{X_2}$, which generates $\operatorname{Pic}^G(X_2)$ on X_2. In the notations from 1.8 one sees that the formulas for the action of $\Phi_{6,1}$ can be given as follows.

$$-K_X \rightarrowtail -\frac{3}{2}K_{X_2} - 2\mathcal{A} \sim 3C_0 - 2\mathcal{A},$$

(9)
$$P \rightarrowtail -\frac{1}{2}K_{X_2} - \mathcal{A} \sim C_0 - \mathcal{A},$$

$$\mathcal{H}_1 \subset |-a_1K_X - r_1P| \rightarrowtail \mathcal{H}_2 \subset |-a_2K_{X_2} - r_2\mathcal{A}|, \qquad \text{where}$$

$$a_2 = \frac{3}{2}a_1 - \frac{1}{2}r_1, \qquad r_2 = 2a_1 - r_1.$$

Under our maximality hypothesis of the singularity at P we have $r_0 = mult_P\mathcal{H}_1 > a_1$, so $a_2 < a_1$ and $r_2 < a_2$. This means that the G-orbit $\mathcal{A}$ is not a maximal singularity for $\mathcal{H}_2$.

2.4.

Now we are on the quadric $X_2 \subset \mathbb{P}^3$ with linear system $\mathcal{H}_2 \sim -a_2K_{X_2}$. Denote this system by $\mathcal{H}$ and put $a = a_2$ for simplicity. The map $\chi_2\colon X_2 \dashrightarrow Y = \mathbb{P}^2$ is not isomorphism, so, by Noether inequality, there is a maximal singularity, G-orbit x of multiplicity $r = r(x) > a$ and of length $d < K_{X_2}^2 = 8$. $d|12 = |G|$ so the only possibilities for d are $d = 1, 2, 3, 4, 6$. We can exclude the cases $d = 1$ and $d = 4$ by Lemma 1.8. It remains to consider the cases $d = 2, 3, 6$.

2.4.1. **The case** $d = 2$. There are two G-orbits $\{R_1, R_2\}$ and $\{P_1, P_{-1}\}$ (see 1.8). First we start from $x = \{R_1, R_2\} = C_0 \cap C_1$ of multiplicity $r = r(x) > a$. Such R_1, R_2 do not lie on one line on the quadric X_2, because each line intersects C_0 at a unique point (C_0 is the hyperplane section). So R_1 and R_2 are in the general position. There exists a link $\Phi_{8,2}$ (see [Isk3], 2.6, the case $K_X^2 = 8$, a), $d = 2$), which is a blow up $X_2 \dashrightarrow X_1$ of the points $\{R_1, R_2\}$. There is a G-invariant structure of conic bundle $\pi\colon X_1 \to \mathbb{P}^1$ on X_1, which is given by the pencil Π_0 from 1.8 (iv). Let $\mathcal{H}_1$ be the strict transform on X_1 of linear system $\mathcal{H}$. Then $\mathcal{H}_1 \sim -a_1K_{X_1} + b_1f_1$, where $f_1 \cong C_0'$ is a G-invariant fiber. We have

$$K_{X_1}^2 = 6, \qquad -K_{X_1} = E + C_0' + C_1' \sim E + 2f_1,$$

where E is an exceptional divisor on $X_1 \to X_2$, *i. e.* a G-invariant pair of (-1)-curves, and C_0', C_1' are strict transforms of C_0, C_1. By [Isk3], 2.6

(10)
$$a_1 = 2a - r(x) < a,$$

$$b_1 = 2(r(x) - a).$$

From (10) one can see that $a_1 < a$ and $b_1 > 0$. By the Noether inequality (because $X_1 \not\cong Y$) there is a maximal singularity in $\mathcal{H}_1$.

2.4.2. **Lemma** *There is no zero-dimensional G-orbit x_1 of length d_1 on X_1, whose points are in the general position in the sense of* 2.1 2), *i. e. whose points do not lie on the degenerated fibers and no two points among them lie on the same fiber of morphism $\pi\colon X_1 \to \mathbb{P}^1$ (this is a condition of existence of "untwisting" link for $\mathcal{H}_1$ on the conic bundle with $b_1 \geq 0$).*

Proof. By Lemma 1.8 (iv), a cyclic group of order 2 acts on $\mathbb{P}^1$, so for the generality condition the length d_1 must be less or equal than 2. There are no G-fixed points on X_1 (as on X_2). The orbit $\{P_1, P_{-1}\}$ of length 2 by Lemma 1.8 lies on the degenerated fibers. Also on the non-degenerated fibers S_3 acts without fixed points. Q.E.D.

So the decomposition algorithm in this situation is impossible. This means that $\{R_1, R_2\}$ is not maximal, and we need to come back to the quadric X_2. Consider the other possibilities.

2.4.3. **The case** $d = 2$ with maximal singularity $x = \{P_1, P_{-1}\}$, $r(x) > a$. As in 2.4.1 the pair $\{P_1, P_{-1}\}$ is in the general position and there exists a link $\Phi_{8,2}$ (see [Isk3], Theorem 2.6, the case $K_X^2 = 8$, c), $d = 2$), which is the blow up $X \dashrightarrow X_1$ of P_1, P_{-1}. There is a structure of conic bundle $\pi\colon X_1 \to \mathbb{P}^1$ on X_1 that is given by the pencil Π_1 from 1.8 (v).

The difference from the previous case is the following. The action of $G = S_3 \times \mathbb{Z}_2$ on $\pi\colon X_1 \to \mathbb{P}^1$ is given by two commuting actions. S_3 acts on the base $\mathbb{P}^1$ without fixed points and this action can be shifted to X_1 by changing the fibers. $\mathbb{Z}_2$ acts on $X_1 \to \mathbb{P}^1$ only on the fibers. The fixed points in this case are the intersection points of the fibers with the curve $C_0' \subset X_1$, which is the pre-image of the curve $C_0 \subset X_2$. So $\mathbb{Z}_2$ acts as the classical De Jonquieres involution. The fixed curve C_0' determines $\pi\colon X_1 \to \mathbb{P}^1$. Indeed the factor by this involution is ruled surface $\mathbb{F}_1 \to \mathbb{P}^1$ branched over C_0'. The pair of the (-1)-curves E covers the exceptional section $S_1 \subset \mathbb{F}_1$.

Notice that there are no fixed points on the base $\mathbb{P}^1$ (so there is no G-invariant fiber as in the previous case). But the class of the fiber $[f_1]$ in $\operatorname{Pic}^G(X)$ is G-invariant, so the action of this link is the same as in (10).

By the Noether inequality there must be a maximal singularity in the linear system $\mathcal{H} \sim -a_1K_{X_1} + b_1 f_1$ which is G-orbit $x_1 \subset X_1$ of multiplicity $r_1 = r(x_1) > a_1$, whose points are in the general position in the sense of 2.4.2.

There are no G-fixed points on X_1 and the only length 2 G-invariant orbit $\{R_1, R_2\}$ lies in the degenerated fibers (see 1.8 (v)).

Two G-orbits of length 3, $\mathcal{A}$ and $\mathcal{B}$, satisfy the generality condition. Both of them lie on C_0' and they are interchanged under the involution of double covering $\pi|_{C_0'}\colon C_0' \to \mathbb{P}^1$ (do not confuse with the involution of the action of $\mathbb{Z}_2$).

There are also G-orbits of length 6. By the generality condition they lie on C_0', one point on the fiber. The other intersection points of C_0' with fibers are complementary G-orbit of length 6.

There is an "untwisting" link concerned with every maximal singularity $x_1 \subset X_1$ of length 3 or 6. That is, the elementary transform

(11)
$$\begin{tikzcd} \Phi\colon X_1 \arrow[rr, dashed, "\mathcal{E}_{x_1}"] \arrow[dr] & & X_1'. \arrow[dl] \\ & \mathbb{P}^1 & \end{tikzcd}$$

Recall that the elementary transform blows up the points of orbit $x_1 \subset X_1$ and contracts the strict transforms of the fibers on which these points lie. In the general case an elementary transform is not a birational involution. But in our case it is so. Indeed the transform $\mathcal{E}_{X_1}$ induces the isomorphism on the curve C_0' and, as noted above, C_0' completely determines $\pi\colon X_1 \to \mathbb{P}^1$. So $X_1' = X_1$ in (11).

The birational involution $\Phi = \mathcal{E}_x$ acts by the following formulas from [Isk3], 2.6.

$$\begin{aligned} -K_{X_1} &\rightarrowtail -K_{X_1} + d_1 f_1 - 2x_1, & a_1' &= a_1, \\ f_1 &\rightarrowtail f_1, & b_1' &= b_1 + d_1(a_1 - r_1), \\ x_1 &\rightarrowtail d_1 f_1 - x_1, & r_1' &= 2(r_1 - a_1), \end{aligned}$$

where d_1 is the length of the orbit x_1. As the last formula shows, the multiplicities of maximal singularities decrease. After a finite number of such transforms we can "untwist" all maximal singularities (which means that the multiplicities of the base points in the modified linear system will be not greater then a). By the Noether inequality the coefficient at b_1 at the fiber will be less than 0.

In this situation the link $\Phi_{8,2}^{-1}\colon X_1 \to X_2$ (the inverse to $\Phi_{8,2}$) contracts G-invariant pair of (-1)-curves E (the sections of the pencil $\pi\colon X_1 \to \mathbb{P}^1$) to G-orbit of length two $x = \{P_1, P_{-1}\}$. It acts by the following formulas.

(12)
$$-K_{X_1} \rightarrowtail -K_{X_2} - x, \qquad a = a_1 + \frac{2}{3} b_1,$$

$$2f_1 \rightarrowtail -K_{X_2} - 2x, \qquad r(x) = a_1 + b_1.$$

These formulas show that $a < a_1$ and $r(x) < a$, because $b_1 < 0$. It means that the link is untwisted and $x = \{P_1, P_{-1}\}$ already is not a maximal singularity.

2.5.

The last cases are those whose singularities of $\mathcal{H}$ are G-orbits of length 3 or 6. There are only two orbits of length 3: $\mathcal{A}$ and $\mathcal{B}$ (see 1.8). Let us first consider

2.5.1. **The case** $d = 6$. Suppose that there is a maximal singularity, G-orbit x of length 6, all whose points are in the general position and the multiplicity $r = r(x) > a$, where $\mathcal{H} \sim -aK_{X_2}$.

Then there is an untwisting link $\Phi_{8,6}$, the Geiser's involution (see [Isk3], 2.6). Indeed, blowing up x we get a del Pezzo surface of degree 2, on which the classical Geiser's involution acts biregularly. Now apply this involution and contract the blown up divisor. The link $\Phi_{8,6}$ acts by the following formulas.

$$a_1 = 7a - 6r(x) < a,$$

$$r(x_1) = 8a - 7r(x) < a_1.$$

2.5.2. **The case** $d = 3$. If the maximal singularity is $x = \mathcal{A}$, then the link $\Phi_{8,3}$ with center in this singularity is the inverse map $\Phi_{6,3}^{-1}\colon X_2 \dashrightarrow X$ and we are in the situation from which we started. If the maximal singularity is $x = \mathcal{B}$ (we missed this case in [Isk1]), then the link $\Phi'_{8,3}\colon X_2 \dashrightarrow X'$ acts from X_2 to del Pezzo surface of degree 6 X'. But $X' \cong X$, because the birational involution on X with the center in $\{P_1, P_{-1}\}$ considered in 2.3.2 interchanges $\{\Gamma_2, \Gamma_y, \Gamma_z\} \leftrightarrow \{\Delta_x, \Delta_y, \Delta_z\}$. Thus it interchanges the cycles $\mathcal{A}$ and $\mathcal{B}$ on C_0 that correspond to these directions in the tangent plane to $P \in X$.

So we completed our consideration of all possibilities but have not found any G-equivariant G-map $X \dashrightarrow Y = \mathbb{P}^2$. So there is no such a map, which prove the Proposition 1.1.1.

2.6. Commentary

In the proof we determined that the smooth relative minimal G-models on which there exists G-equivariant map from X are only the quadric X_2 or the conic bundle with two different actions of G. It is interesting to find by this method all G-surfaces on which $Y = \mathbb{P}^2$ may be G-equivariant mapped.

The following fact is also of interest. If we consider only S_3 instead of G with the similar actions on X and Y, then these surfaces are S_3-birational equivalent. Indeed in this case there are three fixed points

P, P_1 and P_{-1} on X. Therefore there are two fixed points P_1, P_{-1} on the quadric X_2. The stereographic projection from one of them is the required G-equivariant birational isomorphism, so both of these embeddings of S_3 in $Cr_2(\mathbb{C})$ are conjugate. On the other hand, the involution $(x, y) \to (x^{-1}, y^{-1})$ of course is conjugate to the linear one.

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