

Characterization of the rational homogeneous space associated to a long simple root by its variety of minimal rational tangents

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Abstract.

Let $S = G/G'$ be a rational homogeneous space defined by a complex simple Lie group G and a maximal parabolic subgroup G' . For a base point $s \in S$, let $C_s \subset \mathbb{P}T_s(S)$ be the variety of minimal rational tangents at s . In the study of rigidity of rational homogeneous spaces, the following question naturally arises. Let X be a Fano manifold of Picard number 1 such that the variety of minimal rational tangents at a general point $x \in X$, $C_x \subset \mathbb{P}T_x(X)$, is isomorphic to $C_s \subset \mathbb{P}T_s(S)$. Is X biholomorphic to S ? An affirmative answer has been given by Mok when S is a Hermitian symmetric space or a homogeneous contact manifold. Extending Mok's method further and combining it with the theory of differential systems on S , we will give an affirmative answer when G' is associated to a long simple root.

§1. Introduction

Let X be a Fano manifold of Picard number 1. An irreducible component \mathcal{K} of the space of rational curves on X is called a minimal dominating rational component, if for a general point $x \in X$, the subvariety \mathcal{K}_x consisting of members passing through x is non-empty and projective. The tangent directions at x of members of \mathcal{K}_x form a subvariety C_x of $\mathbb{P}T_x(X)$, called the variety of minimal rational tangents at x , and the closure of the union of C_x as x varies over general points of X gives the subvariety $C \subset \mathbb{P}T(X)$, called the variety of minimal rational tangents (associated with \mathcal{K}). It is generally believed that the projective

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geometry of $C_x \subset \mathbb{P}T_x(X)$ controls the geometry of the Fano manifold X . There are many results manifesting this philosophy which were surveyed in [Hw01]. In this paper, we will give another example of this philosophy in the study of rational homogeneous spaces, namely, the projective varieties homogeneous under the action of a complex semi-simple Lie group.

Recall that when S is a rational homogeneous space of Picard number 1, there is a unique choice of a minimal dominating rational component and the varieties of minimal rational tangents at two different points are isomorphic. In the study of rigidity of rational homogeneous spaces, the following conjecture naturally arises. This is Conjecture 2.2 in [Hw06].

Conjecture *Let S be a rational homogeneous space of Picard number 1 and $C_s \subset \mathbb{P}T_s(S)$ be the variety of minimal rational tangents at a base point $s \in S$. Let X be a Fano manifold of Picard number 1 and $C_x \subset \mathbb{P}T_x(X)$ be the variety of minimal rational tangents at a general point $x \in X$ associated to a minimal dominating rational component K . Suppose that $C_s \subset \mathbb{P}T_s(S)$ and $C_x \subset \mathbb{P}T_x(X)$ are isomorphic as projective subvarieties. Then X is biholomorphic to S .*

We can write $S = G/G'$ for a complex simple Lie group G and a maximal parabolic subgroup G' . As explained in Section 2, the subgroup G' is determined by a choice of a simple root α of G . The main goal of this paper is to prove Conjecture for $S = G/G'$ when G' is associated to a long simple root.

Main Theorem *Let $S = G/G'$ where G' is a maximal parabolic subgroup associated to a long root and let $C_s \subset \mathbb{P}T_s(S)$ be the variety of minimal rational tangents at a base point $s \in S$. Let X be a Fano manifold of Picard number 1 and $C_x \subset \mathbb{P}T_x(X)$ be the variety of minimal rational tangents at a general point $x \in X$ associated to a minimal dominating rational component K . Suppose that $C_s \subset \mathbb{P}T_s(S)$ and $C_x \subset \mathbb{P}T_x(X)$ are isomorphic as projective subvarieties. Then X is biholomorphic to S .*

There are three classes of rational homogeneous spaces associated to long simple roots. When the isotropy representation of G' on $T_s(S)$ is irreducible, S is a Hermitian symmetric space. When the isotropy representation on $T_s(S)$ has an irreducible subspace of codimension 1, S is a homogeneous contact manifold. The third class consisting of the remaining homogeneous spaces is distinguished from the first two classes from the fact that the automorphism group of the homogeneous space in

this class is determined by the basic linear differential system on them (cf. Section 2).

For Hermitian symmetric spaces and homogeneous contact manifolds, Main Theorem was proved by N. Mok in [Mo]. The essential part of Mok's argument is to show that if $\mathcal{C}_x \subset \mathbb{P}T_x(X)$ is isomorphic to $\mathcal{C}_s \subset \mathbb{P}T_s(S)$ at a general point $x \in X$, then it is true for every point x on a general member of \mathcal{K} . This is equivalent to extending the corresponding geometric structure up to codimension 1. After this is done, he used the result of [HM97] for Hermitian symmetric spaces and the result of [Ho] for homogeneous contact manifolds to conclude that the geometric structure defined by \mathcal{C} is isomorphic to that of S .

To prove Main Theorem for the third class of homogeneous spaces associated to long simple roots, we have to do two things. The first is to extend Mok's argument to the homogeneous spaces in question. The second is to establish an analog of [HM97] and [Ho] for the homogeneous spaces in the third class. In both parts, a crucial role is played by the theory of linear differential systems on homogeneous spaces. This theory plays a minor role in [Ho], [HM97] and [Mo], because for Hermitian symmetric spaces or homogeneous contact manifolds, the basic linear differential system is either trivial or very simple. But it is an essential component in the current paper, as was the case for other works on the third class of homogeneous spaces, as explained in the introduction of [HM02].

Main Theorem verifies Conjecture except for symplectic Grassmannians and two F_4 -homogeneous spaces. For these remaining cases, our method cannot be applied at all. These cases remain a challenge for future research.

We will start with reviewing the basics of the theory of linear differential systems in Section 2. It turns out that this theory plays such a decisive role in the problem that Main Theorem can be proved easily except for quadric Grassmannians of isotropic 3-spaces. We will collect some basic properties of these homogeneous spaces in Section 4, after reviewing some known results about the variety of minimal rational tangents in Section 3. The analog of Mok's argument, extending the geometric structure up to codimension 1, is proved in Section 5 and the analog of [HM97] and [Ho] is proved in Section 6.

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§2. Review of Tanaka's theory of linear differential systems

We recall the theory of linear differential systems on rational homogeneous spaces, due to Tanaka [Ta].

First, let us recall a few basic definitions. A linear differential system on a complex manifold is just a subbundle of the tangent bundle. Given a subbundle D of the tangent bundle $T(M)$ of a complex manifold M , the k -th *weak derived system* \mathcal{D}^k of D is a subsheaf of $T(M)$ defined inductively by $\mathcal{D}^{-1} = \mathcal{D}$ and $\mathcal{D}^k = \mathcal{D}^{k+1} + [D, \mathcal{D}^{k+1}]$ for $k < -1$, where \mathcal{D} is the sheaf of sections of D . We say that D is *regular at a point* $x \in M$, or equivalently, $x \in M$ is a *regular point* of D , if the k -th derived system \mathcal{D}^k of D is a subbundle D^k of $T(M)$ in a neighborhood of x for every $k \leq -1$. The *symbol algebra* of D at a regular point $x \in M$ is defined as the graded nilpotent Lie algebra

$$\text{sym}_x(D) := D_x^{-1} + D_x^{-2}/D_x^{-1} + \cdots + D_x^{-\mu}/D_x^{-\mu+1}$$

where μ is the largest integer satisfying $D^{-\mu} \neq D^{-\mu+1}$ and the Lie bracket is induced by the bracket of local vector fields in a neighborhood of x .

Now fix a complex simple Lie algebra \mathfrak{g} . Choose a Cartan subalgebra \mathfrak{h} and the root system $\Phi \subset \mathfrak{h}^*$ of \mathfrak{g} with respect to \mathfrak{h} . Fix a system of simple roots $\{\alpha_1, \dots, \alpha_l\}$ and a distinguished choice of a simple root α , say, $\alpha = \alpha_i$. Given an integer k , $-\mu \leq k \leq \mu$, we define Φ_k as the set of all roots $\sum_{j=1}^l m_j \alpha_j$ with $m_i = k$. Here μ is the largest integer such that $\Phi_\mu \neq \emptyset$. For $\beta \in \Phi$, let \mathfrak{g}_β be the corresponding root space. Define

$$\begin{aligned} \mathfrak{g}_0 &= \mathfrak{h} \oplus \bigoplus_{\beta \in \Phi_0} \mathfrak{g}_\beta \\ \mathfrak{g}_k &= \bigoplus_{\beta \in \Phi_k} \mathfrak{g}_\beta, \quad k \neq 0. \end{aligned}$$

The decomposition $\mathfrak{g} = \bigoplus_{k=-\mu}^{\mu} \mathfrak{g}_k$ gives a graded Lie algebra structure on \mathfrak{g} . Define

$$\begin{aligned} \mathfrak{g}' &= \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_\mu \\ \mathfrak{m} &= \mathfrak{g}_{-1} \oplus \cdots \oplus \mathfrak{g}_{-\mu}. \end{aligned}$$

We say that \mathfrak{g}' is the maximal parabolic subalgebra associated to the simple root α and \mathfrak{m} is the nilpotent graded Lie algebra of type (\mathfrak{g}, α) . Here we follow the convention of [Ya] for the sign of roots in the definition of the parabolic subalgebra. This is opposite to the choice in [HM02].

Now let G be a complex Lie group with Lie algebra \mathfrak{g} and G' be the complex Lie subgroup with Lie algebra \mathfrak{g}' . The quotient space G/G' is called the rational homogeneous space of type (\mathfrak{g}, α) . The quotient map $G \rightarrow G/G'$ defines a G' -principal bundle on G/G' . The vector bundle

associated to this principal bundle by the natural G' -representation on \mathfrak{m}

$$\rho : G' \longrightarrow \text{GL}(\mathfrak{m}), \quad \rho(a)v := \text{Ad}(a)v \pmod{\mathfrak{g}' \text{ for } v \in \mathfrak{m}}$$

is the tangent bundle $T(G/G')$. Let $E \subset T(G/G')$ be the subbundle corresponding to the G' -invariant subspace \mathfrak{g}_{-1} of \mathfrak{m} . This E will be called the *standard differential system* on G/G' . The symbol algebra $\text{sym}_s(E)$ of E at each point $s \in G/G'$ is isomorphic to the nilpotent graded Lie algebra \mathfrak{m} .

When $\mu = 1$, namely, $E = T(G/G')$, the homogeneous space G/G' is a Hermitian symmetric space. When $\mu = 2$ and $\dim \mathfrak{g}_{-2} = 1$, namely, when $E \subset T(G/G')$ is of corank 1, G/G' is a homogeneous contact manifold. We will exclude these two cases. We will say that (\mathfrak{g}, α) is neither of symmetric type nor of contact type.

The fundamental question regarding the basic differential system E is the following equivalence problem. Let D be a linear differential system on a complex manifold M and $x \in M$ be a regular point of D . We say that D is a *regular differential system of type \mathfrak{m}* at x if $\text{sym}_y(D)$ is isomorphic to \mathfrak{m} for each point y in a neighborhood of x . The equivalence problem asks for a regular differential system D of type \mathfrak{m} on a manifold M , when we can find a biholomorphic map $\varphi : U \rightarrow U'$ between a neighborhood U of $x \in M$ and a neighborhood U' of $\varphi(x)$ in G/G' such that its differential $d\varphi : T(U) \rightarrow T(U')$ sends D onto E . An answer to this question was given in [Ta]. We will review this result.

Before going into details, let us remark that the assumption that (\mathfrak{g}, α) is neither of symmetric type nor of contact type guarantees that \mathfrak{g} is the prolongation of \mathfrak{m} by Theorem 5.2 of [Ya] and thus a $G_0^\#$ -structure of type \mathfrak{m} that Tanaka considered in [Ta] is nothing but a regular differential system of type \mathfrak{m} .

A *Cartan connection* of type G/G' on a manifold M of dimension $= \dim G/G'$ is a principal G' -bundle P on M with a \mathfrak{g} -valued 1-form ω on P satisfying the following conditions:

- (1) For a tangent vector v of P , $\omega(v) = 0$ implies $v = 0$.
- (2) $\omega(A^*) = A$ for all $A \in \mathfrak{g}'$ where A^* denotes the natural vector field on P induced by A via the G' -action on P .
- (3) $R_a^* \omega = \text{Ad}(a^{-1})\omega$ for $a \in G'$ where R_a denotes the right G' -action on P .

For example, the principal G' -bundle given by the canonical projection $G \rightarrow G/G'$ with the Maurer-Cartan form ω_G of the Lie group G is a Cartan connection on the homogeneous space G/G' .

The \mathfrak{g} -valued 2-form $\Omega = d\omega + \frac{1}{2}[\omega, \omega]$ is called the *curvature form* of (P, ω) . If the curvature form Ω vanishes then (P, ω) is locally isomorphic to (G, ω_G) (Theorem 5.1 of [Sh]). The curvature form Ω can be considered as a function K on P , to be called the *curvature function*, in the following way. This proposition is just Lemma 2.2 and Lemma 2.3 in [Ta].

Proposition 2.1. *Let (P, ω) be a Cartan connection of type G/G' .*

(1) *There is a unique function $K : P \rightarrow \mathfrak{g} \otimes \wedge^2 \mathfrak{m}^*$ such that*

$$\Omega = \frac{1}{2}K(\omega_- \wedge \omega_-)$$

where ω_- is the \mathfrak{m} -component of ω with respect to the decomposition $\mathfrak{g} = \mathfrak{g}' \oplus \mathfrak{m}$.

(2) *For $z \in P$, $a \in G'$ and $v_1, v_2 \in \mathfrak{m}$,*

$$K(za)(v_1, v_2) = Ad(a^{-1})K(z)(\rho(a)v_1, \rho(a)v_2),$$

where $\rho : G' \rightarrow GL(\mathfrak{m})$ is the adjoint representation modulo \mathfrak{g}' .

To get a more refined condition for the vanishing of K , we define the harmonic space $H(\mathfrak{m}, \mathfrak{g})$. The coboundary operator ∂ of the complex

$$\rightarrow \mathfrak{g} \otimes \wedge^q \mathfrak{m}^* \rightarrow \mathfrak{g} \otimes \wedge^{q+1} \mathfrak{m}^* \rightarrow$$

is defined by

$$\begin{aligned} \partial c(v_1 \wedge \cdots \wedge v_{q+1}) &= \sum_{1 \leq i \leq q+1} (-1)^{i+1} [v_i, c(v_1 \wedge \cdots \wedge \hat{v}_i \wedge \cdots \wedge v_{q+1})] \\ &+ \sum_{1 \leq i < j \leq q+1} (-1)^{i+j} c([v_i, v_j] \wedge v_1 \wedge \cdots \wedge \hat{v}_i \wedge \cdots \wedge \hat{v}_j \wedge \cdots \wedge v_{q+1}) \end{aligned}$$

for $c \in \mathfrak{g} \otimes \wedge^q \mathfrak{m}^*$ and $v_i \in \mathfrak{m}$. Let ∂^* be the adjoint operator with respect to the inner product induced by the Killing form. Then we have the orthogonal decomposition

$$\begin{aligned} \mathfrak{g} \otimes \wedge^q \mathfrak{m}^* &= \text{Ker}(\partial\partial^* + \partial^*\partial) \oplus \text{Im}(\partial\partial^* + \partial^*\partial) \\ &= (\text{Ker}\partial \cap \text{Ker}\partial^*) \oplus \text{Im}\partial \oplus \text{Im}\partial^* \end{aligned}$$

so that the cohomology space $H^q(\mathfrak{m}, \mathfrak{g}) = \text{Ker}\partial/\text{Im}\partial$ is isomorphic to the harmonic space $\text{Ker}(\partial\partial^* + \partial^*\partial) = \text{Ker}\partial \cap \text{Ker}\partial^*$. From now on we will identify them and consider the cohomology space $H^q(\mathfrak{m}, \mathfrak{g})$ as

a subspace of $\mathfrak{g} \otimes \wedge^q \mathfrak{m}^*$. Under this identification, we will denote by $H(K)$, the $H^2(\mathfrak{m}, \mathfrak{g})$ -component of the curvature function K .

Decompose the complex $\mathfrak{g} \otimes \mathfrak{m}^* \rightarrow \mathfrak{g} \otimes \wedge^2 \mathfrak{m}^* \rightarrow \mathfrak{g} \otimes \wedge^3 \mathfrak{m}^*$ as the direct sum of the subcomplexes

$$\begin{aligned}
 C^{r+1,1} &= \bigoplus_{i < 0} \mathfrak{g}_{r+i+1} \otimes \mathfrak{g}_i^* \rightarrow C^{r,2} = \bigoplus_{i,j < 0} \mathfrak{g}_{r+i+j+1} \otimes (\mathfrak{g}_i \wedge \mathfrak{g}_j)^* \\
 &\rightarrow C^{r-1,3} = \bigoplus_{i,j,k < 0} \mathfrak{g}_{r+i+j+k+1} \otimes (\mathfrak{g}_i \wedge \mathfrak{g}_j \wedge \mathfrak{g}_k)^*.
 \end{aligned}$$

We denote by $H^{r,2}(\mathfrak{m}, \mathfrak{g})$ the cohomology of this subcomplex. Then

$$H^2(\mathfrak{m}, \mathfrak{g}) = \bigoplus_r H^{r,2}(\mathfrak{m}, \mathfrak{g}).$$

Let K^r be the $C^{r,2}$ -component of K . A normal Cartan connection of type G/G' is a Cartan connection (P, ω) of type G/G' on M such that the curvature function K satisfies

- (1) $K^r = 0$ for $r < 0$ (admissibility)
- (2) $\partial^* K^r = 0$ for $r \geq 0$ (normality).

The existence of a normal Cartan connection which governs the geometric structure associated with a regular differential system of type \mathfrak{m} is guaranteed by the following, which is in Theorem 2.7 of [Ta].

Proposition 2.2. *Let (M, D) be a regular differential system of type \mathfrak{m} . Then there is a normal Cartan connection (P, ω) of type G/G' such that*

- (1) $D^{-k}/D^{-(k-1)}$ is the vector bundle associated to P with the fiber \mathfrak{g}_{-k} where the unipotent part of G' acts on \mathfrak{g}_{-k} trivially and
- (2) if the curvature function K vanishes then (M, D) is locally isomorphic to the standard differential system $(G/G', E)$.

The existence of a normal Cartan connection simplifies the curvature computation greatly as follows, which is just Corollary of Theorem 2.9 in [Ta]

Proposition 2.3. *For a normal Cartan connection (P, ω) , the curvature function K vanishes if and only if its harmonic part $H(K)$, which is a $\bigoplus_{r \geq 0} H^{r,2}(\mathfrak{m}, \mathfrak{g})$ -valued function on P , vanishes.*

Regarding the cohomology group $H^{r,2}(\mathfrak{m}, \mathfrak{g})$, we have the following result.

Proposition 2.4. *Let (\mathfrak{g}, α) be a complex simple Lie algebra with a choice of a long simple root, which is neither of symmetric type nor of contact type. Then*

(1) $H^{r,2}(\mathfrak{m}, \mathfrak{g}) = 0$ for all $r \geq 0$ except when (\mathfrak{g}, α) is either (B_ℓ, α_3) with $\ell \geq 4$ or (D_ℓ, α_3) with $\ell \geq 5$ and

(2) when (\mathfrak{g}, α) is either (B_ℓ, α_3) with $\ell \geq 4$ or (D_ℓ, α_3) with $\ell \geq 5$, $H^{r,2}(\mathfrak{m}, \mathfrak{g}) = 0$ for all $r > 0$ and $H^{0,2}(\mathfrak{m}, \mathfrak{g})$ is contained in $\mathfrak{g}_{-1} \otimes \wedge^2 \mathfrak{g}_{-1}^* \subset C^{0,2}$.

Proof. (1) follows from the vanishing results in Proposition 5.5 of [Ya]. In fact, Yamaguchi listed all non-vanishing $H^{r,2}(\mathfrak{m}, \mathfrak{g})$ with $r \geq 0$ for any parabolic subalgebra of a simple Lie algebra. In his list, when \mathfrak{g} is an exceptional Lie algebra, the cases with a single root α are either of symmetric type or of contact type. When \mathfrak{g} is of type A or C , and α is a long root, (\mathfrak{g}, α) is of symmetric type. Thus we are left with \mathfrak{g} of B or D type only. From Yamaguchi’s list, we can easily check that α must be α_3 to have non-zero $H^{r,2}(\mathfrak{m}, \mathfrak{g})$ for some $r \geq 0$.

For (2), we need to recall some terms. Let θ be the maximal root of \mathfrak{g} . Let x_β denote a root vector of a root β . Given a collection $\Psi = \{\beta_1, \dots, \beta_q\}$ of positive roots, define

$$x_\Psi := x_{\beta_1} \wedge \cdots \wedge x_{\beta_q}.$$

Let σ_j be the element in the Weyl group associated to the simple root α_j . By Kostant’s Theorem on the Lie algebra cohomology (Theorem 5.14 of [Ko] or p.471 of [Ya]), there is a collection, say Σ , of elements of the Weyl group and a collection Φ_σ of positive roots such that $H^q(\mathfrak{m}, \mathfrak{g})$ is a direct sum of irreducible \mathfrak{g}_0 -modules with the highest weight vector $x_{-\sigma(\theta)} \otimes x_{\Phi_\sigma}$ for $\sigma \in \Sigma$. Proposition 5.5 of [Ya] says that if

$$x_{-\sigma(\theta)} \otimes x_{\Phi_\sigma} \in H^{r,2}(\mathfrak{m}, \mathfrak{g})$$

for some $r \geq 0$, then $\sigma = \sigma_3\sigma_2$ in the cases of (B_ℓ, α_3) with $\ell \geq 4$ and (D_ℓ, α_3) with $\ell \geq 5$. In this case, $\Phi_{\sigma_3\sigma_2} = \{\alpha_3, \alpha_2 + \alpha_3\}$ from the description of $\Phi_{\sigma_i\sigma_j}$ in p.475 of [Ya]. The generator $x_{-\sigma_3\sigma_2(\theta)} \otimes (x_{\alpha_3} \wedge x_{\alpha_2+\alpha_3})$ is contained in $\mathfrak{g}_{-1} \otimes (\wedge^2 \mathfrak{g}_{-1}^*) \subset C^{0,2}$ because the coefficient of $\sigma_3\sigma_2(\theta)$ in α_3 is 1. Therefore $H^{r,2}(\mathfrak{m}, \mathfrak{g}) = 0$ for all $r > 0$ and $H^{0,2}(\mathfrak{m}, \mathfrak{g})$ is contained in $\mathfrak{g}_{-1} \otimes (\wedge^2 \mathfrak{g}_{-1}^*)$. Q.E.D.

Combining 2.3 and 2.4, we have the following.

Proposition 2.5. *Let (\mathfrak{g}, α) be a complex simple Lie algebra with a choice of a long simple root which is neither of symmetric type nor of contact type and \mathfrak{m} be the corresponding nilpotent graded algebra. Then a regular differential system of type \mathfrak{m} is locally isomorphic to the standard one on G/G' , except when (\mathfrak{g}, α) is either (B_ℓ, α_3) with $\ell \geq 4$ or (D_ℓ, α_3) with $\ell \geq 5$.*

§3. Review on the theory of the variety of minimal rational tangents

Here we will collect some general facts about the variety of minimal rational tangents on Fano manifolds of Picard number 1. See [Hw01] and [Mo] for details.

We will start with recalling some definitions in projective differential geometry. Let V be a complex vector space and $\mathbb{P}V$ be its projectivization. Given a (not necessarily closed) complex submanifold $Z \subset \mathbb{P}V$ and a point $z \in Z$, its affine tangent space \hat{T}_z is the subspace of V whose projectivization is the projective tangent space of Z in $\mathbb{P}V$ at z . The tangent space of Z at z is naturally isomorphic to

$$T_z(Z) \cong \text{Hom}(\hat{z}, \hat{T}_z/\hat{z})$$

where \hat{z} denotes the 1-dimensional subspace of V corresponding to z . The second fundamental form II_z is a homomorphism

$$\text{II}_z : S^2T_z(Z) \rightarrow T_z(\mathbb{P}V)/T_z(Z)$$

defined as the derivative of the Gauss map. The image of II_z is called the *first normal space* of Z at z and is denoted by $N_z^1(Z)$. The third fundamental form III_z is a homomorphism

$$\text{III}_z : S^3T_z(Z) \rightarrow T_z(\mathbb{P}V)/(T_z(Z) + N_z^1(Z))$$

defined as the derivative of the second fundamental form. The fourth fundamental form is defined similarly.

Now let \mathcal{V} be a vector bundle on a complex manifold X and $\pi : \mathbb{P}\mathcal{V} \rightarrow X$ be its projectivization. For a complex analytic subvariety $\mathcal{Z} \subset \mathbb{P}\mathcal{V}$ and a smooth curve $R \subset \mathcal{Z}$ we say that \mathcal{Z} is *relatively immersed along R* if for each point $x \in \pi(R)$, $Z_x := \mathcal{Z} \cap \pi^{-1}(x)$ is immersed at $R \cap \pi^{-1}(x)$. In this case, the collection of the affine tangent space \hat{T}_z as z varies on R give a vector bundle on R , called *the bundle of the relative affine tangent spaces* of \mathcal{Z} along R . Similarly, *the relative tangent bundle, the relative fundamental forms, the relative first normal spaces* of \mathcal{Z} along R can be defined.

Let X be a Fano manifold of dimension n and Picard number 1. Fix a minimal dominating rational component \mathcal{K} and its variety of minimal rational tangents $\mathcal{C} \subset \mathbb{P}T(X)$. If a member C of \mathcal{K} is immersed, its tangent directions form a smooth rational curve in $\mathbb{P}T(X)$, which we denote by C^\sharp and call the *tangential lift* of C . By definition, C^\sharp lies on the variety of minimal rational tangents $\mathcal{C} \subset \mathbb{P}T(X)$. It is well-known that a general member C of \mathcal{K} is standard (e.g. Theorem 1.2 in [Hw01]).

This means that C is immersed and the pull-back of $T(X)$ to C^\sharp splits as

$$\mathcal{O}(2) \oplus \mathcal{O}(1)^p \oplus \mathcal{O}^{n-1-p}$$

where $p + 2 = C \cdot K_C^{-1}$. When C is standard, C is relatively immersed along C^\sharp in the sense of the previous paragraph (e.g. Proposition 1.4 in [Hw01]). Thus we can consider the relative tangent bundle, the bundle of the relative affine tangent spaces, the relative fundamental forms and the relative first normal spaces of C along C^\sharp . It is well-known that the bundle of the relative affine tangent spaces of C along C^\sharp , which is a vector bundle on C^\sharp , corresponds to the $\mathcal{O}(2) \oplus \mathcal{O}(1)^p$ -part of the splitting of $T(X)$ pulled back to C^\sharp . The next result is essentially contained in the proof of Proposition 2.2 and Proposition 3.1 in [Mo]. Since it was stated there only for the special cases considered in [Mo], we will give a general proof here.

Proposition 3.1. *Let C be a standard rational curve belonging to \mathcal{K} and C^\sharp be its tangential lift. Then the relative tangent spaces of C along C^\sharp is a vector bundle T^π on C^\sharp of the splitting type $\mathcal{O}(-1)^p$ and the relative first normal spaces of C along C^\sharp is a vector bundle N^π on C^\sharp of splitting type $\mathcal{O}(-2)^m$ for some integer $m > 0$. In particular, the relative second fundamental forms of C along C^\sharp , forms a constant section of $\text{Hom}(S^2T^\pi, N^\pi)$.*

Proof. The relative affine tangent bundle of C along C^\sharp has splitting type $\mathcal{O}(2) \oplus \mathcal{O}(1)^p$. In this splitting, the $\mathcal{O}(2)$ part is the relative tautological bundle of $\mathbb{P}T(X)$ along C^\sharp . Thus the relative tangent bundle of C along C^\sharp has the splitting type $\text{Hom}(\mathcal{O}(2), \mathcal{O}(1)^p) = \mathcal{O}(-1)^p$. The relative second fundamental forms define a homomorphism

$$\text{II} : S^2(\mathcal{O}(-1)^p) \rightarrow \text{Hom}(\mathcal{O}(2), \mathcal{O}^{n-1-p}) = \mathcal{O}(-2)^{n-1-p}.$$

Thus the image must be of the form $\mathcal{O}(-2)^m$ and II must be a constant section. Q.E.D.

Now let us assume that C_x is irreducible for a general $x \in X$. Let $D_x \subset T_x(X)$ be the linear span of C_x , namely, $\mathbb{P}D_x$ is the smallest linear subspace of $\mathbb{P}T_x(X)$ containing C_x . The collection of D_x as x varies over general points of X defines a subsheaf of $T(X)$. By taking double dual of D , we can assume that there exists a subvariety $\text{Exc}(D) \subset X$ of codimension ≥ 2 such that $D|_{X \setminus \text{Exc}(D)}$ is a subbundle of $T(X)|_{X \setminus \text{Exc}(D)}$. We will call D the *linear differential system spanned by C* . If the rank of D is smaller than $n = \dim X$, it is well-known that D is not integrable, namely, the Frobenius bracket at a general point $x \in X$,

$$[\cdot, \cdot] : \wedge^2 D_x \rightarrow T_x(X)/D_x$$

is not zero. In fact, if it is integrable, the leaves define a holomorphic fibration outside a set of codimension ≥ 2 , which is a contradiction to the assumption that X has Picard number 1 (cf. Proposition 2.2 in [Hw01]). Given any subvariety of codimension ≥ 2 , we can choose a member of \mathcal{K} disjoint from it (e.g. Lemma 2.1 in [Hw01]). Thus we can assume that a general member of \mathcal{K} is disjoint from $\text{Exc}(D)$.

Proposition 3.2. *Let D be the linear differential system spanned by \mathcal{C} . Assume that the rank of D is smaller than n . Then for a general member $C \subset X \setminus \text{Exc}(D)$ of \mathcal{K} , the pull-back of D to C^\sharp is of the form*

$$\mathcal{O}(2) \oplus \mathcal{O}(1)^p \oplus \mathcal{O}^q \oplus \bigoplus_{i=1}^r \mathcal{O}(a_i)$$

where $q \geq m$, m being the rank of the relative first normal spaces of C along C^\sharp , $r > 0$ and $a_i < 0$.

Proof. Since the pull-back of D to C^\sharp is a vector subbundle of $\mathcal{O}(2) \oplus \mathcal{O}(1)^p \oplus \mathcal{O}^{n-1-p}$ and $\mathcal{C}_x \subset \mathbb{P}D_x$ for each $x \in C$, it must be of the form

$$\mathcal{O}(2) \oplus \mathcal{O}(1)^p \oplus \bigoplus_{j=1}^{d-1-p} \mathcal{O}(b_j)$$

where $d = \text{rank}(D)$ and $b_j \leq 0$. Moreover, from 3.1, at least m of b_j 's must be 0. Thus it suffices to exclude the possibility that $b_j = 0$ for all j . Suppose this is the case. Then the pull-back of $T(X)/D$ to C^\sharp must have degree 0. Since it is the quotient of $\mathcal{O}(2) \oplus \mathcal{O}(1)^p \oplus \mathcal{O}^{n-1-p}$, we see that $T(X)/D$ has splitting type \mathcal{O}^{n-d} where d is the rank of D . Then the Frobenius bracket $[\cdot, \cdot] : \wedge^2 D_x \rightarrow T_x(X)/D_x$ must satisfy $[\alpha, D_x] = 0$ for α corresponding to the tangent direction $\mathcal{O}(2)$ of C at x . Since this is true for general $[\alpha] \in \mathcal{C}_x$ while \mathcal{C}_x spans D_x , we conclude that D is integrable, a contradiction. Q.E.D.

§4. Variety of minimal rational tangents on the rational homogeneous space associated to a long simple root

Let $S = G/G'$ be a rational homogeneous space of type (\mathfrak{g}, α) as explained in Section 2. There is a unique minimal dominating rational component. For a base point $s \in S$, let $\mathcal{C}_s \subset \mathbb{P}T_s(S)$ be the variety of minimal rational tangents. When α is a long simple root, or equivalently, when \mathcal{C}_s is homogeneous, the following fact was proved in Proposition 1 and Proposition 7 of [HM02].

Proposition 4.1. *Let S be a rational homogeneous space of type (\mathfrak{g}, α) with α a long root. Let \mathfrak{m} be the associated graded nilpotent Lie algebra. Let $\hat{C}_s \subset E_s$ be the homogeneous cone of the variety of minimal rational tangents C_s at a base point $s \in S$. Define a graded Lie algebra $\mathfrak{n} = \mathfrak{n}_{-1} + \mathfrak{n}_{-2} + \cdots$ as the quotient of the graded free Lie algebra generated by $\mathfrak{n}_{-1} = E_s$ modulo the relations given by $[v_1, v_2] = 0$ for $v_1, v_2 \in \mathfrak{n}_{-1}$ if $v_1 \in \hat{C}_s$ and $\langle v_1, v_2 \rangle$ is tangent to \hat{C}_s . Then \mathfrak{n} is isomorphic to \mathfrak{m} as graded nilpotent Lie algebras.*

This has several consequences.

Proposition 4.2. *In the setting of 4.1, suppose (\mathfrak{g}, α) is neither of symmetric type nor of contact type. Then the group of the graded Lie algebra automorphisms of \mathfrak{n} acts irreducibly on $\mathfrak{n}_{-1} = E_s$ and C_s is the highest weight orbit of this action.*

In Proposition 1 of [HM02], it was proved that the variety of minimal rational tangents C_s is the highest weight orbit of the isotropy representation of G' on E_s . This does not imply 4.2 immediately. It is a result about the differential system E , while 4.2 is a result about the symbol algebra of E .

Proof. The Lie algebra $\text{aut}(\mathfrak{m})$ of the automorphism group of \mathfrak{m} is the 0-th prolongation of \mathfrak{m} in the sense of p.430 of [Ya], which is exactly \mathfrak{g}_0 by Theorem 5.2 of [Ya]. By 4.1, the Lie subalgebra $\text{aut}(\hat{C}_s) \subset \mathfrak{gl}(T_s(S))$ of infinitesimal linear automorphisms of the cone is contained in $\text{aut}(\mathfrak{m})$. From the list of C_s in p. 176 of [HM02], $\text{aut}(\hat{C}_s)$ is isomorphic to \mathfrak{g}_0 and C_s is the highest weight orbit of the irreducible representation of \mathfrak{g}_0 on \mathfrak{g}_{-1} . It follows that

$$\text{aut}(\mathfrak{m}) = \text{aut}(\hat{C}_s) = \mathfrak{g}_0$$

and C_s is the highest weight orbit of the $\text{aut}(\mathfrak{m})$ -representation on $T_s(S)$.
Q.E.D.

Proposition 4.3. *Let S be a rational homogeneous space of type (\mathfrak{g}, α) , with α a long root, neither of symmetric type nor of contact type. Let X be a Fano manifold of Picard number 1 with a choice of a minimal dominating rational component \mathcal{K} such that the variety of minimal rational tangents at a general point $x \in X$ is isomorphic to that of S . Let D be the differential system spanned by the variety of minimal rational tangents. Then at a general point x of X ,*

(1) *the symbol algebra $\text{sym}_x(D)$ is isomorphic to \mathfrak{m} , the symbol algebra of the standard differential system on S , and*

(2) the variety of minimal rational tangents C_x is the highest weight orbit of the representation of the automorphisms of the symbol algebra $\text{sym}_x(D)$ on $\mathbb{P}D_x$.

Proof. (1) is just Proposition 5 of [HM02]. (2) follows from 4.2. Q.E.D.

Proposition 4.4. *In the setting of 4.3, suppose the linear differential system D is equivalent to the standard system E on S in an analytic neighborhood of x . Then X is biholomorphic to S .*

Proof. By 4.3 (2), there exists a neighborhood U of x and a biholomorphic map $\varphi : U \rightarrow U'$ into an open subset U' of S such that $\varphi(C|_U)$ agrees with the variety of minimal rational tangents of S restricted to U' . By the main theorem in p. 564 of [HM01], this implies that X is biholomorphic to S . Q.E.D.

Corollary 4.5. *Unless S is of type (B_ℓ, α_3) with $\ell \geq 4$ or (D_ℓ, α_3) with $\ell \geq 5$, Main Theorem follows from 2.5.*

Thus from now on till the end of the paper, we will assume that S is of type (B_ℓ, α_3) with $\ell \geq 4$ or (D_ℓ, α_3) with $\ell \geq 5$.

The homogeneous space S is the quadric Grassmannian of 3-dimensional isotropic subspaces in an orthogonal vector space. Let $p + 2 > 0$ be the degree of minimal rational curves of S with respect to K_S^{-1} .

Proposition 4.6. *Let S and p be as defined above. Let C be a minimal rational curve on S . Then*

- (1) $\dim S = 3p + 3$ and $\text{rank}(E) = 3p$,
- (2) $E|_C = \mathcal{O}(2) \oplus \mathcal{O}(1)^p \oplus \mathcal{O}^{2p-3} \oplus \mathcal{O}(-1)^2$,
- (3) $(T(S)/E)|_C = \mathcal{O}(1)^2 \oplus \mathcal{O}$ and
- (4) $\dim[\mathfrak{g}_{\alpha_3}, \mathfrak{g}_1] = 2$.

Proof. Let $\alpha_1, \dots, \alpha_\ell$ be a set of simple roots of Lie algebras of type B_ℓ or D_ℓ . For B_ℓ , we define

$$\beta_i := \sum_{k=i}^{\ell} \alpha_k \text{ for } 1 \leq i \leq \ell \text{ and } \gamma_{ij} := \sum_{k=i}^{j-1} \alpha_k \text{ for } 1 \leq i < j \leq \ell.$$

For D_ℓ , we define

$$\beta_i := \sum_{k=i}^{\ell-2} \alpha_k + \alpha_{\ell-1}, \beta'_i := \sum_{k=i}^{\ell-2} \alpha_k + \alpha_\ell \text{ for } 1 \leq i \leq \ell - 2$$

$$\text{and } \gamma_{ij} := \sum_{k=i}^{j-1} \alpha_k \text{ for } 1 \leq i < j \leq \ell - 1.$$

Now from p.30 and p.35 of [Ti], we can explicitly write down the subsets of roots of \mathfrak{g} defined in Section 2 as follows. For (B_ℓ, α_3) ,

$$\Phi_1 = \{\beta_1, \beta_2, \beta_3, \gamma_{ij}, \beta_i + \beta_j \mid 1 \leq i \leq 3 < j \leq \ell\}$$

$$\Phi_2 = \{\beta_1 + \beta_2, \beta_1 + \beta_3, \beta_2 + \beta_3\}.$$

For (D_ℓ, α_3) ,

$$\Phi_1 = \{\beta_1, \beta_2, \beta_3, \beta'_1, \beta'_2, \beta'_3, \gamma_{ij}, \beta_i + \beta'_j \mid 1 \leq i \leq 3 < j \leq \ell - 1\}$$

$$\Phi_2 = \{\beta_1 + \beta'_2, \beta_1 + \beta'_3, \beta_2 + \beta'_3\}.$$

By the argument in the proof of Proposition 1 of [HM02], the splitting type of E on C is $\bigoplus_{\beta \in \Phi_1} \mathcal{O}(\beta(H_{\alpha_3}))$ where H_{α_3} denotes the coroot of α_3 . This can be computed easily from the above list for Φ_1 to prove (2). In the same way, the splitting type of $T(S)/E$ on C is $\bigoplus_{\beta \in \Phi_2} \mathcal{O}(\beta(H_{\alpha_3}))$ and (3) can be checked from the above list for Φ_2 . (1) is immediate from (2) and (3). (4) can be checked directly from the above list of roots. Q.E.D.

Proposition 4.7. *The variety of minimal rational tangents at a base point $s \in S$ satisfies the following.*

(1) $C_s \subset \mathbb{P}E_s$ is isomorphic to the Segre embedding of $\mathbb{P}_2 \times \mathbf{Q}_{p-2}$ where \mathbf{Q}_{p-2} denotes the hyperquadric of dimension $p - 2$ in \mathbb{P}_{p-1} .

(2) The first normal space at each point of C_s has rank $2p - 3$ and the fourth fundamental form of C_s is zero.

(3) Regard a tangential line to C_s as a one-dimensional subspace of $\wedge^2 E_s$. Then the kernel of the bracket $\wedge^2 E_s \rightarrow T_s(S)/E_s$, which is isomorphic to the Lie bracket $\wedge^2 \mathfrak{g}_{-1} \rightarrow \mathfrak{g}_{-2}$, is spanned by tangential lines to C_s .

Proof. (1) follows from the description of the highest weight orbit of G_0 on $\mathbb{P}E_s$, as given by p.176 of [HM02]. From (1), there is no hyperplane section of C_s with multiplicity ≥ 4 and hyperplane sections with multiplicity 3 forms a pencil. It follows that the fourth fundamental form is 0 and the sum of the tangent space and the first normal space has codimension 2 in $\mathbb{P}E_s$. This implies (2). Noting that $\mu = 2$ in this case, (3) is a direct consequence of 4.1. Q.E.D.

Proposition 4.8. *Let $s \in S$ be a base point and $C_s \subset \mathbb{P}E_s$ be the variety of minimal rational tangent. There exists a vector space W of dimension 3 with $\wedge^2 W \cong T_s(S)/E_s$ and a vector space F of dimension p equipped with a non-degenerate quadratic form $\omega : S^2 F \rightarrow \mathbb{C}$ such that*

$E_s \cong W \otimes F$. Furthermore, under the decomposition $S^2F = \mathbb{C} \oplus \omega^\perp$, the Lie bracket $\wedge^2 E_s \rightarrow T_s(S)/E$ is just the projection to $\wedge^2 W$ in

$$\begin{aligned} \wedge^2(W \otimes F) &= (\wedge^2 W \otimes S^2 F) \oplus (S^2 W \otimes \wedge^2 F) \\ &= \wedge^2 W \oplus (\wedge^2 W \otimes \omega^\perp) \oplus (S^2 W \otimes \wedge^2 F). \end{aligned}$$

Proof. This is the consequence of the following which can be checked easily. The semi-simple part of \mathfrak{g}' under the Levi-decomposition is isomorphic to $\mathfrak{sl}_3 \oplus \mathfrak{so}_p$. Its representation on \mathfrak{g}_{-1} is just the tensor product of the fundamental representations of \mathfrak{sl}_3 and \mathfrak{so}_p . Its representation on \mathfrak{g}_{-2} is just the tensor product of the dual of the fundamental representation of \mathfrak{sl}_3 and the trivial representation of \mathfrak{so}_p . Q.E.D.

§5. Extension of the structure to a neighborhood of a standard rational curve

From now on we assume that X is a Fano manifold of Picard number 1 with a minimal dominating rational component \mathcal{K} such that, for a general point $x \in X$, the projective subvariety $\mathcal{C}_x \subset \mathbb{P}T_x(X)$ is isomorphic to $\mathcal{C}_s \subset \mathbb{P}T_s(S)$.

Let $D \subset T(X)$ be the linear differential system on X spanned by \mathcal{C} at a general point. As in Section 3, D is a subbundle of $T(X)$ on $X \setminus \text{Exc}(D)$ and a general member of \mathcal{K} is disjoint from $\text{Exc}(D)$. By 4.3, the symbol algebra $\text{sym}_x(D)$ of D at a general point x is isomorphic to the nilpotent algebra $\mathfrak{m} = \mathfrak{g}_{-1} + \mathfrak{g}_{-2}$.

Proposition 5.1. *In the above setting, let C be a general member of \mathcal{K} disjoint from $\text{Exc}(D)$ and let $C^\sharp \subset C$ be the tangential lift of C . Then the pull-back of D to C^\sharp splits as*

$$\mathcal{O}(2) \oplus \mathcal{O}(1)^p \oplus \mathcal{O}^{2p-3} \oplus \mathcal{O}(-1)^2$$

and the pull-back of $T(X)/D$ to C^\sharp splits as $\mathcal{O}(1)^2 \oplus \mathcal{O}$. Moreover, the relative second fundamental forms and the relative third fundamental forms of \mathcal{C} along C^\sharp are all isomorphic to those of $\mathcal{C}_s \subset \mathbb{P}E_s$.

Proof. From 3.2 and 4.7 (2), the splitting type of D on C^\sharp is of the form

$$\mathcal{O}(2) \oplus \mathcal{O}(1)^p \oplus \mathcal{O}^{2p-3} \oplus \mathcal{O}(b_1) \oplus \mathcal{O}(b_2)$$

for some $0 \geq b_1 \geq b_2, b_2 < 0$.

The relative third fundamental forms of \mathcal{C} along C^\sharp defines a homomorphism

$$\text{III} : S^3(\mathcal{O}(-1)^p) \rightarrow \text{Hom}(\mathcal{O}(2), \mathcal{O}(b_1) \oplus \mathcal{O}(b_2)) = \mathcal{O}(b_1 - 2) \oplus \mathcal{O}(b_2 - 2).$$

By 4.7 (2), III must be surjective over a general point $x \in C$. Thus $b_2 = -1$.

Suppose that $b_1 = 0$. Then by Chern number consideration, the splitting type of $T(X)/D$ on C^\sharp must be $\mathcal{O}(1) \oplus \mathcal{O}^2$. Then the Frobenius bracket $[\cdot, \cdot] : \wedge^2 D \rightarrow T(X)/D$ must satisfy

$$[\mathcal{O}(2), D] = [\mathcal{O}(2), \mathcal{O}(2) \oplus \mathcal{O}(1)^p \oplus \mathcal{O}^{2p-2} \oplus \mathcal{O}(-1)] \subset \mathcal{O}(1).$$

Thus $\dim[\alpha, D_x] = 1$. But by 4.3, we know that $\mathbf{sym}_x(D)$ is isomorphic to \mathfrak{m} . This is a contradiction to 4.6 (4).

We conclude that $b_1 = b_2 = -1$ and III must be constant along C^\sharp . Moreover, $\dim[\alpha, D_x] = 2$. This implies that $T(X)/D$ has the splitting type of $\mathcal{O}(1)^2 \oplus \mathcal{O}$. Q.E.D.

Proposition 5.2. *Let \mathcal{V} be a vector bundle on the unit disc $\Delta := \{t \in \mathbb{C}, |t| < 1\}$ and $\pi : \mathbb{P}\mathcal{V} \rightarrow \Delta$ be its projectivization. Let \mathcal{Z} be a subvariety of $\mathbb{P}\mathcal{V}$ flat over Δ such that for each $t \neq 0$, $Z_t := \pi^{-1}(t) \cap \mathcal{Z}$ is isomorphic to the Segre embedding of $\mathbb{P}_m \times \mathbb{Q}_k$ for some positive integers m and k . Assume that there exists a section $R \subset \mathcal{Z}$ of π such that the relative second fundamental forms and the relative third fundamental forms of \mathcal{Z} along R are constant. Then $Z_0 := \pi^{-1}(0) \cap \mathcal{Z}$ is also isomorphic to the Segre embedding of $\mathbb{P}_m \times \mathbb{Q}_k$.*

Proof. This is proved in the proof of Proposition 3.2 Case (E) in [Mo] when $m = 1$. The proof there works verbatim for arbitrary m . Q.E.D.

Proposition 5.3. *Let C be a general standard rational curve through x . Then for each $y \in C$, $\mathcal{C}_y \subset \mathbb{P}T_y(X)$ is isomorphic to $\mathcal{C}_s \subset \mathbb{P}T_s(S)$ and the symbol algebra $\mathbf{sym}_y(D)$ is isomorphic to \mathfrak{m} .*

Proof. The first statement is an immediate consequence of 5.1 and 5.2. From the first statement, as in 4.8, there exist a line bundle L , a vector bundle W' of rank 3 and a vector bundle F' of rank p equipped with a non-degenerate quadratic form $\omega' : S^2 F' \rightarrow L$ in a neighborhood of C such that $D \cong W' \otimes F'$ and, at the point x , the Frobenius bracket annihilates $(\wedge^2 W'_x \otimes (\omega')^\perp_x) \oplus (S^2 W'_x \otimes \wedge^2 F'_x)$ -part of the decomposition

$$\wedge^2 D \cong \wedge^2 (W' \otimes F') = (\wedge^2 W' \otimes L) \oplus (\wedge^2 W' \otimes (\omega')^\perp) \oplus (S^2 W' \otimes \wedge^2 F').$$

From 5.1, we may assume that the splitting type of W' along C^\sharp is $\mathcal{O}(1) \oplus \mathcal{O}^2$, the splitting type of F' along C^\sharp is $\mathcal{O}(1) \oplus \mathcal{O}^{p-2} \oplus \mathcal{O}(-1)$ and L is the trivial line bundle along C^\sharp . It follows that $\wedge^2 W' \otimes L$ and $T(X)/D$ have the same splitting type along C^\sharp . Since the homomorphism $\wedge^2 W' \otimes$

$L \rightarrow T(X)/D$ induced by the bracket of D in a neighborhood of C is an isomorphism at a general point of C by 4.3, it must be an isomorphism at every point of C from the splitting type. This implies that $\text{sym}_y(D)$ is isomorphic to \mathfrak{m} at every $y \in C$. \square

§6. Vanishing of the curvature

We keep the notation from the previous section. To prove Main Theorem for X , it suffices to show, by 2.5 and 4.4, the vanishing of the curvature function of the associated normal Cartan connection at a general point of X . For this, we need to interpret the curvature function as a holomorphic section of a vector bundle as follows. This is an analog of Proposition 2.5 in [Ho].

Proposition 6.1. *Let (M, D) be a regular differential system of type \mathfrak{m} . If there is no nonzero holomorphic section of the bundle $D \otimes \wedge^2 D^*$, then (M, D) is locally isomorphic to the standard differential system E on S .*

Proof. Let (P, ω) be a normal Cartan connection given by Proposition 2.2. Then the vector bundle $D \otimes \wedge^2 D^*$ is the associated vector bundle of the principal G' -bundle P with fiber $\mathfrak{g}_{-1} \otimes \wedge^2 \mathfrak{g}_{-1}^*$ on which G' acts as follows: for $c \in \mathfrak{g}_{-1} \otimes \wedge^2 \mathfrak{g}_{-1}^*$, $a \in G'$ and $v_1, v_2 \in \mathfrak{g}_{-1}$,

$$a \cdot c(X, Y) = Ad(b^{-1})c(\rho(b)v_1, \rho(b)v_2)$$

where $a = b \exp(A_1) \exp(A_2)$, $b \in G_0$, $A_i \in \mathfrak{g}_i$ in the decomposition $G = G_0 \cdot \exp(\mathfrak{g}_1) \exp(\mathfrak{g}_2)$ given in Lemma 1.7 of [Ta] and $\rho : G' \rightarrow GL(\mathfrak{m})$ is the isotropy representation.

Let K' be the $\mathfrak{g}_{-1} \otimes \wedge^2 \mathfrak{g}_{-1}^*$ -component of K . The behavior of K under the G' -action is described in 2.1 (2). Here, we need to study the behavior of K' . We will use the following equivariance property of K from Lemma 2.4 of [Ta].

Lemma *Let m be any integer. If $K^p = 0$ for all $p < m$, then for any $z \in P$, $a \in G'$ and $v_1, v_2 \in \mathfrak{g}_{-1}$,*

$$K^m(za)(v_1, v_2) = Ad(b^{-1})(K^m(z)(\rho(a)v_1, \rho(a)v_2))$$

where b is the G_0 -component of a in the decomposition

$$G = G_0 \cdot \exp(\mathfrak{g}_1) \exp(\mathfrak{g}_2)$$

given in Lemma 1.7 of [Ta].

Noting that $C^{0,2} = \mathfrak{g}_{-1} \otimes \wedge^2 \mathfrak{g}_{-1}^* + \mathfrak{g}_{-2} \otimes (\mathfrak{g}_{-1} \wedge \mathfrak{g}_{-2})^*$ and applying Lemma to K^0 , we have

$$K'(za)(v_1, v_2) = Ad(b^{-1})K'(z)(\rho(a)v_1, \rho(a)v_2)$$

for $z \in P$, $a \in G'$ and $v_1, v_2 \in \mathfrak{g}_{-1}$. Since $\exp(\mathfrak{g}_1 + \mathfrak{g}_2)$ acts trivially on \mathfrak{g}_{-1} , K' is an equivariant function on P under the action of G' . So K' can be thought of as a section of $D \otimes \wedge^2 D^*$ on M .

If there is no nonzero section of $D \otimes \wedge^2 D^*$, then K' vanishes and thus the harmonic part $H(K)$ of the curvature function K of the Cartan connection (P, ω) is zero by 2.4. By Proposition 2.3 the whole curvature K vanishes and thus the differential system (M, D) is locally isomorphic to (S, E) . Q.E.D.

In the setting of Main Theorem with S of type $(B_\ell, \{\alpha_3\})$ ($\ell \geq 4$), or $(D_\ell, \{\alpha_3\})$ ($\ell \geq 5$), 5.3 implies that in a neighborhood M of a general member C of \mathcal{K} , we are given a regular differential system of type \mathfrak{m} . By 4.4 and 6.1, to finish the proof of Main Theorem, it suffices to prove the following.

Proposition 6.2. *Let S, X and \mathcal{K} be as before. Let D be the linear differential system of type \mathfrak{m} in a neighborhood M of a general member of \mathcal{K} given by 5.3. Then there is no nonzero holomorphic section of the vector bundle $D \otimes \wedge^2 D^*$ in that neighborhood.*

Proof. A holomorphic section of $D \otimes \wedge^2 D^*$ gives a homomorphism $\varphi : \wedge^2 D \rightarrow D$. We claim that φ factors through the Lie bracket $\wedge^2 D \rightarrow T(M)/D$ to a homomorphism $\varphi' : T(M)/D \rightarrow D$. By 4.7 (3), it suffices to show that for a general point $x \in M$, a general vector $v \in D_x$ with $[v] \in \mathcal{C}_x$ and a vector $w \in D_x$ such that $\langle v, w \rangle$ is tangent to \mathcal{C}_x at $[v]$, $\varphi(v \wedge w) = 0$. Let C be a member of \mathcal{K} passing through x with $[v] = [T_x(C)]$. Since the bundle of the relative affine tangent spaces of C along C^\sharp corresponds to the $(\mathcal{O}(2) \oplus \mathcal{O}(1)^p)$ -part of the splitting of the pull-back of D to C^\sharp given in 5.1, we have a holomorphic section \tilde{v} (resp. \tilde{w}) of D pulled back to C^\sharp such that $\tilde{v}(x) = v$ (resp. $\tilde{w}(x) = w$) and \tilde{v} (resp. \tilde{w}) has two (resp. one) zeros. Then $\varphi(\tilde{v} \wedge \tilde{w})$ is a section of the pull-back of D to C^\sharp with three zeroes. From the splitting type of D , we conclude that $\varphi(\tilde{v} \wedge \tilde{w}) = 0$, which implies that $\varphi(v \wedge w) = 0$. This proves the claim.

Now it remains to show that the induced homomorphism φ' from $T(M)/D$ to D is zero. Let us choose C through x with the tangent direction $[v] \in \mathcal{C}_x$ as above. Since the pull-back of $T(M)/D$ to C^\sharp splits as $\mathcal{O}(1)^2 \oplus \mathcal{O}$, the image $\text{Im}(\varphi')$ is contained in the $(\mathcal{O}(2) \oplus \mathcal{O}(1)^p \oplus \mathcal{O}^{2p-3})$ -part of the splitting of the pull-back of D to C^\sharp . From 4.7 (2),

at the point x , $(\mathcal{O}(2) \oplus \mathcal{O}(1)^p \oplus \mathcal{O}^{2p-3})_x$ is precisely the span of the tangent space and the first normal space of \mathcal{C}_x at $[v]$. It follows that $[\text{Im}(\varphi')_x] \subset \mathbb{P}D_x$ is contained in the span of the tangent space and the first normal space of \mathcal{C}_x at $[v]$ for each general choice of $[v] \in \mathcal{C}_x$. Thus if $[\text{Im}(\varphi')_x]$ is non-empty, it is contained in the common intersection, say, $Z \subset \mathbb{P}D_x$, of the spans of the tangent space and the first normal space of \mathcal{C}_x at $[v]$ as $[v]$ varies over \mathcal{C}_x . But such Z must be a G_0 -invariant subvariety of $\mathbb{P}D_x$ under the natural representation of G_0 on D_x . This is a contradiction because Z is linearly degenerate by its definition, while the representation of G_0 on D_x is irreducible. It follows that $\varphi' = 0$ at a general point x , and consequently $\varphi' = 0$. Q.E.D.

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