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On a mean value of a multiplicative function of two variables

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Abstract.

We prove the existence of a mean value of arithmetical functions of two variables : $\lim_{x,y\to\infty}(xy)^{-1}\sum_{m\le x,n\le y} f(m,n)$ under some conditions, and, when f is a multiplicative function of two variables, express the mean value as an infinite product over all primes. Five examples are given which are not obtained by trivial generalizations of results on arithmetical functions of one variable.

§1. Introduction and main results.

In this paper we investigate a mean value of arithmetical functions of two variables, and give an expression of the mean value as an infinite product over all primes. This expression is not a straightforward extension of that in the case of one variable. For an arithmetical function of one variable f, the mean value $M(f)$ is defined by the limit : $\lim_{x\to\infty} x^{-1}\sum_{n\le x} f(n)$ if this limit exists. When we want to express the mean values of arithmetical functions by an infinite product over primes, it is usually necessary to require the property of multiplicativeness. So far, numerous expressions of the mean values of multiplicative functions of one variable have been known. For example, $M(\mu^2) = \prod_p (1 - 1/p^2) = 6/\pi^2$ where μ is the Möbius function. Also many other examples are given in Tenenbaum[4]. It is also well known that if f is a multiplicative function satisfying $\sum_p p^{-1}|f(p) - 1| < \infty$ and $\sum_p \sum_{k\ge 2} p^{-k}|f(p^k)| < \infty$ then the mean value $M(f)$ exists which is equal to $\prod_p(1+\sum_{k\ge 1} p^{-k}(f(p^k)-f(p^{k-1})))$ by Corollary 2.3 in Schwarz-Spilker[3] (p.51), which is a corollary of a well known theorem of Wintner. We want to extend the above theorem to the case of two variables

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and obtain some interesting examples which are not expected to be found from the results in the case of one variable.

For an arithmetical function of two variables f, the mean value $M(f)$ is defined by the limit : $\lim_{x,y\to\infty}(xy)^{-1}\sum_{m\leq x,n\leq y} f(m,n)$ if this limit exists. For an arithmetical function of two variables the definition of multiplicativeness is given in Delange[2]. In that paper, he investigated a density for sets of pairs (m,n) of positive integers satisfying certain conditions, and he studied the mean value of the function $f=1_E$ where 1_E is the characteristic function of a set E, under the assumption of the multiplicativeness of 1_E. He applied his mean value theorems to the set of irreducible fractions of the form n/m, and derived interesting formulas concerning densities. More precisely, let $\mathcal{F}_{a,b,x}$ be the set of all irreducible fractions n/m with $a\leq n/m\leq b$ and whose denominator is $\leq x$. Let $\Phi_{a,b}(x)$ be the number of elements of $\mathcal{F}_{a,b,x}$, and let $\Phi^*_{a,b}(x)$ be the number of elements n/m of $\mathcal{F}_{a,b,x}$ for which m and n are squarefree. Then he proved that $\Phi^*_{a,b}(x)/\Phi_{a,b}(x)$ tends to $\frac{6}{\pi^2}\prod\{1-1/(p+1)^2\}$ as x tends to infinity.

After Delange's paper, to my knowledge, few results with respect to a mean value of a multiplicative function of two variables have been known. In Babu[1], the distribution of a real valued additive or multiplicative function of two variables is investigated and the conditions under which the distribution is absolutely continuous or singular are described. But in this paper we concentrate our attention not to the distribution but to the mean value.

Delange treated the case $|f|\leq 1$, but we do not confine ourselves to the case when f is bounded, and our theorems below are valid even when f is unbounded. Our methods of proving mean value theorems are different from Delange's work and our infinite product representation of a mean value is simple.

Lastly, five examples are given, which are not trivial generalizations in cases of one variable.

Notations.

$\mathbb{N}=\{1,2,....\}$, the set of positive integers,
$\mathbb{C}$ the set of complex numbers,
μ the Möbius function,
δ the delta function, i.e. , $\delta_{a,b}$=1,0 according to $a=b$ or not,
$\mathcal{P}$ the set of prime numbers (p or q [in general] denotes prime),
gcd (a,b) the greatest common divisor of integers a,b; often also written as (a,b),
$f\ll g$ the Vinogradov symbol, i.e., $f\ll g$ means that there exists a constant $M>0$ such that $|f(x)|\leq Mg(x)$ holds for all $x>0$,

$f * g$ the convolution of f and g, i.e.,
$f * g(n) = \sum_{d|n} f(d)g(n/d)$ when f, g: $\mathbb{N} \to \mathbb{C}$ and
$f * g(m,n) = \sum_{k|m,\ l|n} f(k,l)g(m/k, n/l)$ when f, g: $\mathbb{N}^2 \to \mathbb{C}$.

We first generalize Wintner's theorem, which is well known in the case of a usual arithmetical function of one variable (Schwarz-Spilker[3]), to the case of an arithmetical function of two variables. Let $\mu(m,n) = \mu(m)\mu(n)$, for $n, m \in \mathbb{N}$.

Theorem 1. *Let $f : \mathbb{N}^2 \to \mathbb{C}$ be an arithmetical function of two variables. Suppose $\sum_{m,n=1}^{\infty}(mn)^{-1}|f * \mu(m,n)| < \infty$. Then, the mean value $M(f) = \lim_{x,y\to\infty}(xy)^{-1}\sum_{m\le x, n\le y} f(m,n)$ exists, and equals $\sum_{m,n=1}^{\infty}(mn)^{-1} f * \mu(m,n)$.*

In the usual case when f is an arithmetical function of one variable, we require the multiplicativeness of f to represent the mean value as an infinite product over primes. We next give a definition of a multiplicative function of two variables which appeared in Delange[2].

Definition. Let $f : \mathbb{N}^2 \to \mathbb{C}$ be an arithmetical function of two variables. We say that f is a multiplicative function if f satisfies $f(m_1 m_2, n_1 n_2) = f(m_1, n_1) f(m_2, n_2)$ for any $m_1, m_2, n_1, n_2 \in \mathbb{N}$ satisfying $(m_1 n_1, m_2 n_2) = 1$.

We also generalize Cor.2.3 of Schwarz-Spilker[3] (p.51) to the case of multiplicative functions of two variables.

Theorem 2. *For a multiplicative function of two variables f, we set for $k, l \in \mathbb{N}$ and two distinct primes p, q,*

$$\Delta_{00} f(p,q) = f(1,1) = 1,$$
$$\Delta_{k0} f(p,q) = f(p^k, 1) - f(p^{k-1}, 1),$$
$$\Delta_{0l} f(p,q) = f(1, q^l) - f(1, q^{l-1}),$$
$$\Delta_{kl} f(p,q) = f(p^k, q^l) - f(p^k, q^{l-1}) - f(p^{k-1}, q^l) + f(p^{k-1}, q^{l-1}).$$

Suppose

$$\sum_{p\in\mathcal{P}} \sum_{k,l\ge 0, k+l\ge 1} \frac{1}{p^{k+l}} |\Delta_{kl} f(p,p)| < \infty.$$

Then the mean value $M(f)$ exists, and equals

$$\prod_{p\in\mathcal{P}} \Big(\sum_{k,l\ge 0} \frac{1}{p^{k+l}} \Delta_{kl} f(p,p) \Big).$$

Oun class of multiplicative functions of two variables that satisfies the condition of Theorem 2 is given in the next theorem.

Theorem 3. *Let g be a multiplicative function of one variable satisfying*

$$\sum_{p\in\mathcal{P}}\sum_{k\geq 1}\frac{1}{p^{2k}}|g(p^k)-g(p^{k-1})|<\infty.$$

If we set $f(m,n)=g((m,n))$, then f becomes a multiplicative function of two variables and the mean value $M(f)$ exists, which is equal to

$$\prod_{p\in\mathcal{P}}(1+\sum_{k\geq 1}\frac{1}{p^{2k}}(g(p^k)-g(p^{k-1}))).$$

§2. Proofs.

Lemma 4. *For arithmetical functions of two variables f and g, we have*

$$\sum_{m\leq x,n\leq y} f*g(m,n)=\sum_{m\leq x,n\leq y} F(\frac{x}{m},\frac{y}{n})g(m,n),$$

where $F(x,y)=\sum_{m\leq x,n\leq y} f(m,n)$.

(Proof.) By the definition of the convolution we have

$$\sum_{m\leq x,n\leq y} f*g(m,n)=\sum_{k_1k_2\leq x,\ l_1l_2\leq y} f(k_1,l_1)\ g(k_2,l_2)$$

$$=\sum_{k_2\leq x,\ l_2\leq y}\ \sum_{k_1\leq[x/k_2],\ l_1\leq[y/l_2]} f(k_1,l_1)\ g(k_2,l_2)$$

$$=\sum_{k_2\leq x,\ l_2\leq y} F(\ [\frac{x}{k_2}],\ [\frac{y}{l_2}]\ )\ g(k_2,l_2)=\sum_{m\leq x,\ n\leq y} F(\ \frac{x}{m},\ \frac{y}{n}\ )\ g(m,n).$$

Lemma 5. *The identity $\mu*1=\delta$ holds, where $1(m,n)\equiv 1$,*

$$\mu(m,n)=\mu(m)\mu(n),\ and\ \delta(m,n)=\delta_{m,1}\delta_{n,1}.$$

(Proof.) The proof is the same as that in the case of one variable.

Proof of Theorem 1.

Noting that $f=f*\mu*1$ by Lemma 5, we have by Lemma 4

$$\sum_{m\leq x,n\leq y} f(m,n)=\sum_{m\leq x,n\leq y} f*\mu*1(m,n)=\sum_{m\leq x,n\leq y} f*\mu(m,n)[\frac{x}{m}][\frac{y}{n}]$$

$$= \sum_{m \le x, n \le y} f * \mu(m,n) \ \ (\frac{x}{m} + O(1))(\frac{y}{n} + O(1)) =: I_1 + I_2 + I_3 + I_4,$$

where

$$I_1 = xy \sum_{m \le x, n \le y} \frac{1}{mn} f * \mu(m,n), \quad I_2 = x \sum_{m \le x, n \le y} \frac{O(1)}{m} f * \mu(m,n),$$

$$I_3 = y \sum_{m \le x, n \le y} \frac{O(1)}{n} f * \mu(m,n), \quad I_4 = \sum_{m \le x, n \le y} O(1) f * \mu(m,n).$$

Obviously,

$$\lim_{x,y \to \infty} \frac{1}{xy} I_1 = \sum_{m,n=1}^{\infty} \frac{1}{mn} \mu * f(m,n).$$

As for the remainder terms we first see that

$$\frac{1}{xy}|I_2| \ll \frac{x}{xy} \sum_{m \le x, n \le y} \frac{1}{m} |f*\mu(m,n)| \ll \sum_{m \le x, n \le y} \frac{n}{y} \frac{1}{mn} |f*\mu(m,n)|$$

$$= \sum_{m \le x, n \le \sqrt{y}} \frac{n}{y} \frac{1}{mn} |f * \mu(m,n)| + \sum_{m \le x, \sqrt{y} < n \le y} \frac{n}{y} \frac{1}{mn} |f * \mu(m,n)|$$

$$\le \frac{1}{\sqrt{y}} \sum_{m \le x, n \le \sqrt{y}} \frac{1}{mn} |f * \mu(m,n)| + \sum_{m \le x, \sqrt{y} < n \le y} \frac{1}{mn} |f * \mu(m,n)|$$

$$\le \frac{1}{\sqrt{y}} \sum_{m \ge 1, n \ge 1} \frac{1}{mn} |f * \mu(m,n)| + \sum_{m \ge 1, n > \sqrt{y}} \frac{1}{mn} |f * \mu(m,n)|$$

$$\to 0 \ \ (\ x, y \to \infty).$$

Therefore $I_2/xy \to 0$. Similarly we also see that $I_3/xy \to 0$. As for the term I_4, we have

$$\frac{1}{xy}|I_4| \ll \frac{1}{xy} \sum_{m \le x, \ n \le y} |f * \mu(m,n)| = \sum_{m \le x, \ n \le y} \frac{mn}{xy} \frac{1}{mn} |f * \mu(m,n)|$$

$$\ll \frac{1}{\sqrt{xy}} \sum_{m \le \sqrt{x}, \ n \le \sqrt{y}} \frac{1}{mn} |f*\mu(m,n)| + \frac{1}{\sqrt{y}} \sum_{\sqrt{x} < m \le x, \ n \le \sqrt{y}} \frac{1}{mn} |f*\mu(m,n)|$$

$$+ \frac{1}{\sqrt{x}} \sum_{m \le \sqrt{x}, \ \sqrt{y} < n \le y} \frac{1}{mn} |f*\mu(m,n)| + \sum_{\sqrt{x} < m \le x, \ \sqrt{y} < n \le y} \frac{1}{mn} |f*\mu(m,n)|$$

$$\ll \frac{1}{\sqrt{xy}} \sum_{m \ge 1, \ n \ge 1} \frac{1}{mn} |f * \mu(m,n)| + \frac{1}{\sqrt{y}} \sum_{m \ge 1, \ n \ge 1} \frac{1}{mn} |f * \mu(m,n)|$$

$$+\frac{1}{\sqrt{x}}\sum_{m\geq 1,\ n\geq 1}\frac{1}{mn}|f*\mu(m,n)|+\sum_{m>\sqrt{x},\ n>\sqrt{y}}\frac{1}{mn}|f*\mu(m,n)|$$

$$\rightarrow 0\ \ (\ x,y\rightarrow\infty).$$

Therefore we have $I_4/xy \rightarrow 0$ $(\ x,\ y\ \rightarrow\infty)$, and thus Theorem 1 is proved.

Lemma 6. (Delange[2]) If $f,\ g$ are multiplicative functions of two variables, then the convolution $f*g$ is also a multiplicative function of two variables.

Proof of Theorem 2.

We see that f satisfies the assumption of Theorem 1 when f satisfies the assumption of Theorem 2. As in the proof of Schwarz-Spilker[3] p.51 Cor.2.3, noting that $f*\mu$ is mutiplicative if f is mutiplicative by Lemma 6, we have

$$\sum_{m\leq x,\ n\leq y}\frac{1}{mn}|f*\mu(m,n)|\leq\prod_{p\leq x,\ q\leq y}\Big(\sum_{k,l\geq 0}\frac{1}{p^kq^l}|f*\mu(p^k,q^l)|\Big)$$

$$=\prod_{p\leq x,\ q\leq y}\Big(\sum_{k,l\geq 0}\frac{1}{p^kq^l}|\Delta_{kl}f(p,q)|\Big)\cdots(*)$$

Since $\Delta_{kl}f(p,q)=\Delta_{k0}f(p,q)\Delta_{0l}f(p,q)=\Delta_{k0}f(p,p)\Delta_{0l}f(q,q)$ if $p\neq q$ and $k,l\geq 1$ by the multiplicativeness of f, we have

$$(*)\leq\Big(\prod_{p\in\mathcal{P}}\sum_{k,l\geq 0}\frac{1}{p^{k+l}}|\Delta_{kl}f(p,p)|\Big)^2$$

$$=\Big(\prod_{p\in\mathcal{P}}\Big(1+\sum_{k,l\geq 0,\ k+l\geq 1}\frac{1}{p^{k+l}}|\Delta_{kl}f(p,p)|\Big)\Big)^2$$

$$\leq\exp\Big(2\sum_{p\in\mathcal{P}}\sum_{k,l\geq 0,\ k+l\geq 1}\frac{1}{p^{k+l}}|\Delta_{kl}f(p,p)|\Big)<\infty,$$

by the well-known inequality $1+\beta\leq\exp(\beta)$ for $\beta\geq 0$ and the assumption of Theorem 2. Therefore by Theorem 1, the mean value $M(f)$ exists and equals $\sum_{m,n=1}^{\infty}(mn)^{-1}f*\mu(m,n)$. It is easy to see that

$$\sum_{m,n=1}^{\infty}\frac{1}{mn}f*\mu(m,n)=\prod_{p\in\mathcal{P}}\Big(\sum_{k,l\geq 0}\frac{1}{p^{k+l}}\Delta_{kl}f(p,p)\Big)$$

by Lemma 2 of Delange[2].

Proof of Theorem 3.

Let g satisfy the assumption of Theorem 3 and we set $f(m,n) = g((m,n))$. It is easy to see that f becomes a multiplicative function of two variables. Moreover we easily see that

$$\Delta_{k0} f(p,p) = \Delta_{0l} f(p,p) = 0, \text{ for } k,l \geq 1,$$

$$\Delta_{kl} f(p,p) = 0, \quad \text{for} \quad k \neq l \quad \text{and} \quad k,l \geq 1, \quad \text{and}$$

$$\Delta_{kl} f(p,p) = g(p^k) - g(p^{k-1}), \quad \text{for} \quad k = l \geq 1.$$

Therefore we obtain

$$\sum_{p \in \mathcal{P}} \sum_{k,l \geq 0, k+l \geq 1} \frac{1}{p^{k+l}} |\Delta_{kl} f(p,p)| = \sum_{p \in \mathcal{P}} \sum_{k \geq 1} \frac{1}{p^{2k}} |g(p^k) - g(p^{k-1})| < \infty.$$

By Theorem 2 the mean value $M(f)$ exists, and it is easy to see that

$$M(f) = \prod_{p \in \mathcal{P}} (1 + \sum_{k \geq 1} \frac{1}{p^{2k}} (g(p^k) - g(p^{k-1}))).$$

§3. Examples.

Example 1. (Delange[2]) If $f(m,n) = \mu^2(mn)$, then f becomes a multiplicative function of two variables, and the mean value $M(f)$ exists and

$$M(f) = \left(\frac{6}{\pi^2}\right)^2 \prod_{p \in \mathcal{P}} \{1 - \frac{1}{(p+1)^2}\}.$$

(Proof.) It is easy to see that f satisfies the assumption of Theorem 2. Since $\Delta_{20} f(p,p) = \Delta_{02} f(p,p) = \Delta_{11} f(p,p) = -1$, $\Delta_{00} f(p,p) = \Delta_{12} f(p,p) = \Delta_{21} f(p,p) = 1$, and $\Delta_{kl} f(p,p) = 0$ otherwise, we have

$$M(f) = \prod_{p \in \mathcal{P}} (1 - \frac{3}{p^2} + \frac{2}{p^3}) = \prod_{p \in \mathcal{P}} (1 - \frac{1}{p^2})^2 \{1 - \frac{1}{(p+1)^2}\}$$

$$= \left(\frac{6}{\pi^2}\right)^2 \prod_{p \in \mathcal{P}} \{1 - \frac{1}{(p+1)^2}\}.$$

Example 2. If $f(m,n) = (m,n)^\alpha$ $(\alpha < 1)$, then f becomes a multiplicative function of two variables, and the mean value $M(f)$ exists and

$$M(f) = \prod_{p \in \mathcal{P}} (1 + \frac{p^\alpha - 1}{p^2 - p^\alpha}).$$

(Proof.) This is the case when $g(n) = n^\alpha$ in Theorem 3, which satisfies the assumption of Theorem 3 because

$$\sum_{p\in\mathcal{P}}\sum_{k\geq 1}\frac{1}{p^{2k}}|g(p^k)-g(p^{k-1})| = \sum_{p\in\mathcal{P}}\sum_{k\geq 1}\frac{1}{p^{2k}}|p^{k\alpha}-p^{(k-1)\alpha}|$$

$$= \sum_{p\in\mathcal{P}}\frac{p^\alpha - 1}{p^2 - p^\alpha} < \infty.$$

The proofs of the following examples are similar.

Example 3. If $f(m,n) = \mu((m,n))$, then f becomes a multiplicative function of two variables, and the mean value $M(f)$ exists and

$$M(f) \ = \prod_{p\in\mathcal{P}}(1-\frac{1}{p^2})^2 = (\frac{6}{\pi^2})^2.$$

Example 4. If $f(m,n) = \mu^2((m,n))$, then f becomes a multiplicative function of two variables, and the mean value $M(f)$ exists and

$$M(f) \ = \prod_{p\in\mathcal{P}}(1-\frac{1}{p^4}).$$

Example 5. If $f(m,n) = \sigma_\alpha((m,n))$, $(0<\alpha<1)$, where $\sigma_\alpha(n) = \sum_{d|n} d^\alpha$, then f becomes a multiplicative function of two variables, and the mean value $M(f)$ exists and

$$M(f) \ = \prod_{p\in\mathcal{P}}(1+\frac{1}{1-p^{\alpha-2}}).$$

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