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Gaps between consecutive zeros of the zeta-function on the critical line and conjectures from random matrix theory

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Abstract.

Assuming the Riemann hypothesis and two conjectures from random matrix theory, we prove that

$$\lambda = \limsup_{n\to\infty}(\gamma_{n+1}-\gamma_n)\frac{1}{2\pi}\log\frac{\gamma_n}{2\pi} = \infty.$$

§1. The Riemann zeta-function

The Riemann zeta-function is given by

$$\zeta(s) = \sum_{n=1}^{\infty}\frac{1}{n^s} = \prod_p \left(1-\frac{1}{p^s}\right)^{-1},$$

where the product is taken over all prime numbers. Both, the Dirichlet series and the Euler product converge absolutely for $\mathrm{Re}\, s > 1$ and uniformly in each compact subset of this half-plane. The identity between the Dirichlet series and the Euler product gives a first glance on the intimate connection between the zeta-function and the distribution of prime numbers. $\zeta(s)$ has an analytic continuation to the whole complex plane except for a simple pole at $s = 1$ with residue 1. Riemann was the first to investigate $\zeta(s)$ as a function of a complex variable. He discovered

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that the zeta-function satisfies the functional equation

$$\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s)=\pi^{-(1-s)/2}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s). \tag{1}$$

In view of the Euler product $\zeta(s)$ has no zeros in the half-plane $\mathrm{Re}\, s>1$. It follows from the functional equation and from basic properties of the Gamma-function that $\zeta(s)$ vanishes in $\mathrm{Re}\, s<0$ exactly at the so-called trivial zeros $s=-2n$ with $n\in\mathbb{N}$. All other zeros of $\zeta(s)$ are said to be nontrivial, and we denote them by $\rho=\beta+i\gamma$. The nontrivial zeros lie inside the so-called critical strip $0\leq \mathrm{Re}\, s\leq 1$, and there is none on the real axis. By the functional equation (1) in addition with the reflection principle $\zeta(\overline{s})=\overline{\zeta(s)}$ the nontrivial zeros of $\zeta(s)$ are symmetrically distributed with respect to the real axis and the so-called critical line $\mathrm{Re}\, s=\frac{1}{2}$. The number $N(T)$ of nontrivial zeros $\rho=\beta+i\gamma$ with $0<\gamma\leq T$ (counting multiplicities) is asymptotically given by the Riemann-von Mangoldt formula

$$N(T)=\frac{T}{2\pi}\log\frac{T}{2\pi e}+O(\log T). \tag{2}$$

It should be noticed that the frequency of the appearance of the nontrivial zeros is increasing as $T\to\infty$.

Riemann conjectured that all nontrivial zeros lie on the critical line $\mathrm{Re}\, s=\frac{1}{2}$. This is the famous, yet unproved Riemann hypothesis which we rewrite equivalently as

Riemann's hypothesis (RH). $\zeta(s)\neq 0$ *for* $\mathrm{Re}\, s>\frac{1}{2}$.

Van de Lune, te Riele & Winter [14] localized the first 1 500 000 001 zeros, all lying without exception on the critical line; moreover they all are simple. By observations like this it is conjectured that all or at least almost all zeros of the zeta-function are simple.

Assuming the truth of the Riemann hypothesis, Montgomery [16] studied the distribution of zeros $\frac{1}{2}+i\gamma, \frac{1}{2}+i\gamma'$ of the zeta-function and conjectured

Montgomery's pair correlation conjecture. *For fixed* α,β *satisfying* $0<\alpha<\beta$,

$$\begin{aligned}\lim_{T\to\infty}\frac{1}{N(T)}\sharp\left\{0<\gamma,\gamma'<T\,:\,\alpha\leq\frac{(\gamma-\gamma')\log T}{2\pi}\leq\beta\right\}\\ =\int_\alpha^\beta\left(1-\left(\frac{\sin\pi u}{\pi u}\right)^2\right)\,du.\end{aligned} \tag{3}$$

This conjecture plays a complementary role to the Riemann hypothesis: vertical vs. horizontal distribution of the nontrivial zeros of $\zeta(s)$. There are plenty of important consequences of this far-reaching claim. For instance, the pair correlation conjecture implies that almost all zeros of the zeta-function are simple. Conrey [1] proved unconditionally that more than two fifths of the zeros are simple and lie on the critical line.

By the Riemann-von Mangoldt formula (2) the average spacing between consecutive ordinates of nontrivial zeros is $2\pi/\log T$, which appears as scaling factor in (3). The deviations from this average value have been intensively studied for almost sixty years. In this note we shall study large gaps subject to the truth of RH and recent conjectures for the Riemann zeta-function originating from random matrix theory.

§2. Conjectures from random matrix theory

Dyson pointed out that the Gaussian unitary ensemble (GUE) has the same pair correlation function as the conjectured one for the Riemann zeta-function (3). The GUE has been an object of intensive studies in mathematical physics with respect to the distribution of energy levels in manyparticle systems; it consists of $n \times n$ complex Hermitian matrices of the form $A = (a_{jk})$, where $a_{jj} = \sqrt{2}\sigma_{jj}$, $a_{jk} = \sigma_{jk} + i\eta_{jk}$ for $j < k$, and $a_{jk} = \overline{a}_{kj} = \sigma_{kj} - i\eta_{kj}$ for $j > k$, where the σ_{jk} and η_{jk} are independent standard normal variables. After a suitable normalization the pair correlation of the eigenvalues of the matrices of the GUE becomes $1 - ((\sin \pi u)/(\pi u))^2$, as $n \to \infty$. For more information about the GUE and other ensembles we refer to Mehta [15]. By the computations of Odlyzko [19] it turned out that the pair correlation and the nearest neighbour spacing for the zeros of $\zeta(s)$ were amazingly close to those for the GUE.

However, there is more evidence for the pair correlation conjecture (3) than numerical data. Many results from random matrix theory were found which perfectly fit to certain results on the value-distribution of the Riemann zeta-function. For example, Keating & Snaith [13] showed that the characteristic polynomials associated with certain random matrix ensembles have in a sense the same value-distribution as the zeta-function on the critical line predicted by Selberg's limit law.

Recently, random matrix theory has been used for doing *good* predictions. It is a long standing conjecture that for fixed $k \geq 0$, there exists a constant $C(k)$ such that

$$\int_0^T \left|\zeta\left(\frac{1}{2} + it\right)\right|^{2k} \mathrm{d}t \sim C(k)\, T(\log T)^{k^2}, \tag{4}$$

as $T \to \infty$. The asymptotic formula (4) is known to be true only in the trivial case $k = 0$, and in the cases $k = 1$ with $C(1) = 1$ and $k = 2$ with $C(2) = (2\pi^2)^{-1}$ by the classical results of Hardy & Littlewood [8] and Ingham [12], respectively. The asymptotic formula (4) is in some applications unsatisfying as long as we do not know the value of $C(k)$. Some new insights were found by random matrix theory.

Let

$$a(k) = \prod_p \left(1 - \frac{1}{p^2}\right)^{k^2} \sum_{m=0}^{\infty} \left(\frac{\Gamma(m+k)}{m!\Gamma(k)}\right)^2 p^{-m} \tag{5}$$

and denote by $G(z)$ Barnes' G-function, defined by

$$\begin{aligned} G(z+1) &= (2\pi)^{z/2} \exp\left(-\frac{1}{2}(z(z+1) + \gamma z^2)\right) \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right)^n \\ &\quad \times \exp\left(-z + \frac{z^2}{n}\right), \end{aligned}$$

where γ is the Euler-Mascheroni constant. Extending a conjecture of Conrey & Ghosh [2], Keating & Snaith [13] claimed

Conjecture 1. *The asymptotic formula (4) holds with*

$$C(k) = a(k) \frac{G^2(k+1)}{G(2k+1)}.$$

Note that in the above definition of the numbers $C(k)$, one must take an appropriate limit if $k = 0$.

Another conjecture on the basis of the random matrix model is due to Hughes [10]. To state his conjecture let $L = \frac{1}{2\pi} \log \frac{T}{2\pi}$ and define

$$F_k(2x) = x^2 j_k(x)^2 - 2kx j_k(x) j_{k-1}(x) + x^2 j_{k-1}(x)^2, \tag{6}$$

where $j_k(x)$ is the k-th spherical Bessel function of the first kind; note that for integers k

$$j_k(x) = (-1)^k x^k \left(\frac{1}{x}\frac{\mathrm{d}}{\mathrm{d}x}\right)^k \frac{\sin x}{x}.$$

Hughes' moment conjecture yields an asymptotic formula for certain discrete moments:

Conjecture 2. *Assume the truth of the Riemann hypothesis. Then, for fixed* $k > -\frac{1}{2}$,

$$\frac{1}{N(T)} \sum_{0<\gamma\leq T} \left| \zeta\left(\frac{1}{2} + i\left(\gamma + \frac{\alpha}{L}\right)\right)\right|^{2k} \sim F_k(2\pi\alpha)a(k)\frac{G(k+1)^2}{G(2k+1)}(\log T)^{k^2}$$

as $T \to \infty$, *uniformly in* α *for* $|\alpha| \leq L$.

This conjecture is known to be true only in a few particular cases. Namely, the trivial case $k = 0$ but also in the case $k = 1$. Assuming the Riemann hypothesis, Gonek [4] proved

$$(7) \quad \frac{1}{N(T)} \sum_{0<\gamma\leq T} \left| \zeta\left(\frac{1}{2} + i\left(\gamma + \frac{\alpha}{L}\right)\right)\right|^{2} \sim \left(1 - \left(\frac{\sin(\pi\alpha)}{\pi\alpha}\right)^2\right) \log T$$

uniformly in α for $|\alpha| \leq \frac{1}{2}L$. It is easy to check that this is the quantity predicted by Conjecture 2.

§3. Gaps between consecutive zeros

Only little is known about the spacing of consecutive zeros of the zeta-function. Denote by γ_n the positive ordinates of the nontrivial zeros of the zeta-function in ascending order. Define

$$(8) \qquad \lambda = \limsup_{n\to\infty} (\gamma_{n+1} - \gamma_n)\frac{1}{2\pi}\log\frac{\gamma_n}{2\pi}.$$

It is conjectured that $\lambda = \infty$. This is also what the random matrix model predicts since the spacing distribution does not have compact support (cf. Hughes [9]). However, the best known results in this direction are far away from this. Selberg [21] was the first to show that $\lambda > 1$. Mueller [17] obtained $\lambda > 1.9$ under assumption of the Riemann hypothesis, Conrey, Ghosh & Gonek [3] proved $\lambda > 2.68$ subject to the truth of the Generalized Riemann hypothesis (for Dirichlet L-functions), and Hughes [9] succeeded in showing $\lambda > 2.7$ if his Conjecture 2 is true. Recently, Ng [18] improved all these bounds by showing that $\lambda > 2.91$ holds under assumption of the Generalized Riemann hypothesis. Now let Λ denote the quantity in (8) where only zeros $\frac{1}{2} + i\gamma_n$ on the critical line are considered. Of course, $\Lambda \geq \lambda$ and equality holds if the Riemann hypothesis is true. The best unconditional result is $\Lambda > 2.345\ldots$ due to Hall [5]; this improves Mueller's bound under assumption of RH. Furthermore, Hall used a conjectural asymptotic formula for mixed powers of Hardy's Z-function and its first derivative in order to derive larger bounds for λ.

Hardy's Z-function $Z(t)$ is a function of a real variable, given by

$$Z(t)=\pi^{-it/2}\frac{\Gamma\left(\frac{1}{4}+\frac{it}{2}\right)}{\left|\Gamma\left(\frac{1}{4}+\frac{it}{2}\right)\right|}\zeta\left(\frac{1}{2}+it\right).$$

By the functional equation for the zeta-function, $Z(t)$ is an infinitely often differentiable function which is real for real t. Moreover, $|\zeta(\frac{1}{2}+it)|=|Z(t)|$ and thus zeta-zeros on the critical line correspond bijectively to real zeros of $Z(t)$. Hall [7] claimed

Conjecture 3. *For any given pair of non-negative integers $h\le k$, there exists a constant $b(h,k)$ such that*

$$\frac{1}{T}\int_0^T Z(t)^{2k-2h}Z'(t)^{2h}\,\mathrm{d}t\sim a(k)b(h,k)(\log T)^{k^2+2h}$$

as $T\to\infty$, where $a(k)$ is defined by (5) and $b(h,k)$ is a rational number predicted by random matrix theory.

It should be noted that for $0\le h,k\le 2$ the values of $a(k)$ and $b(h,k)$ are all known (see [6]) and Conjecture 3 holds unconditionally. Further, for $h=0$ the asymptotics of Conjecture 3 simply follows from Conjecture 1. For $k\le 6$ the values of $b(h,k)$ are explicitly known (see [6, 7]).

Let $\Lambda(k)$ denote the lower bound for Λ which Hall obtained by his method using Conjecture 3 with fixed k. Then Hall's records are:

k	1	2	3	4	5	6
$\Lambda(k)\ge$	1.732...	2.345...	2.891...	3.392...	3.858...	4.298...

(see [5, 6, 7]); it should be noted that these bounds are unconditional for $k\le 2$. Hall's method relies on a sophisticated variation problem together with so-called Wirtinger type-inequalities and is designed exclusively for this problem. Note that Hall's bounds improve all bounds mentioned so far if all nontrivial zeros lie on the critical line.

We shall improve Hall's lower bounds assuming RH, and Conjectures 1 and 2 from the random matrix model. Let $\lambda(k)$ be the lower bound for λ which we obtain by applying the asymptotics of Conjectures 1 and 2 for fixed k. Our argument follows Mueller's proof [17] of such bounds (resp. Hughes' proof of his bound).

Assume that $\eta>\lambda$, then

$$(9)\qquad \int_0^T\left|\zeta\left(\frac{1}{2}+it\right)\right|^{2k}\,\mathrm{d}t<\sum_{0<\gamma\le T}\int_{\gamma-\frac{\eta}{2L}}^{\gamma+\frac{\eta}{2L}}\left|\zeta\left(\frac{1}{2}+it\right)\right|^{2k}\,\mathrm{d}t,$$

as $T \to \infty$. For the right-hand side we can write

$$\frac{2\pi}{\log T} \int_{-\frac{\eta}{2}}^{\frac{\eta}{2}} \sum_{0<\gamma\leq T} \left|\zeta\left(\frac{1}{2} + i\left(\gamma + \frac{\alpha}{L}\right)\right)\right|^{2k} d\alpha.$$

By Conjecture 2 this is asymptotically equal to

$$a(k)\frac{G(k+1)^2}{G(2k+1)}\frac{2\pi}{\log T}N(T)(\log T)^{k^2} \int_{-\frac{\delta}{2}}^{\frac{\eta}{2}} F_k(2\pi\alpha)\, d\alpha.$$

In view of Conjecture 1 the left-hand side of (9) is asymptotically equal to

$$a(k)\frac{G(k+1)^2}{G(2k+1)}T(\log T)^{k^2}.$$

Combining these bounds and taking into account (2) we get

$$1 < 2\int_0^{\frac{\eta}{2}} F_k(2\pi\alpha)\, d\alpha. \tag{10}$$

If the integral on the right-hand side is equal to $\frac{1}{2}$ for some value of η, then we obtain a contradiction to our assumption $\eta > \lambda$; we denote the infimum of all η for which the integral is $\geq \frac{1}{2}$ by $\eta(k)$. We shall prove lower bounds for $\eta(k)$ and hence for $\lambda(k)$. Using MATHEMATICA we find

k	$\lambda(k) \geq$	k	$\lambda(k) \geq$	k	$\lambda(k) \geq$	k	$\lambda(k) \geq$
1	1.902...	5	4.949...	9	7.791...	13	10.562...
2	2.706...	6	5.670...	10	8.489...	14	11.248...
3	3.473...	7	6.383...	11	9.183...	15	11.932...
4	4.218...	8	7.090...	12	9.874...	16	12.614...

The bounds obtained by this method increase with k in the computed range $1 \leq k \leq 16$; this is illustrated in Figure 3. In view of the graphs of $F_k(2x)$ in Figure 3, one may hope to prove $\lambda = \infty$ conditional to RH, and the random matrix conjectures 1 and 2.

Theorem 1. *Assume the Riemann hypothesis and Conjectures 1 and 2 for fixed $k \in \mathbb{N}$. Then*

$$\lambda(k) \geq \frac{4}{\pi}\sqrt{k}.$$

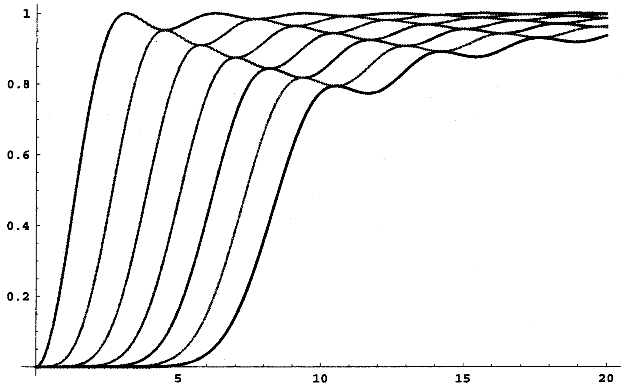

Fig. 1. $F_k(2x)$ for $k = 1, 2, 3, 4, 5, 6, 7$ and $x \in [0, 10]$.

In particular, we find that $\lambda(k)$ tends to infinity as k tends to infinity, and thus we deduce from

$$\lambda = \limsup_{n\to\infty} (\gamma_{n+1} - \gamma_n)\frac{1}{2\pi}\log\frac{\gamma_n}{2\pi} \geq \lambda(k),$$

that $\lambda = \infty$ subject to the truth of the Riemann hypothesis and of the Conjectures 1 and 2 for all $k \in \mathbb{N}$.

Proof. We assume that $k \geq 9$; for values $k < 9$ we may use the computed values from the tabular above. Put

$$a_k(m) = \frac{(k+m-1)!(k+m)!}{m!(2k+m)!(2k+2m+1)!}, \quad b_k(m) = \frac{k}{2k+2m+1}.$$

Then

$$F_k(y) = k\sum_{m=0}^{\infty}(-1)^m a_k(m) y^{2k+2m};$$

this is Formula (5) from Hughes [10]. Moreover,

$$I_k(\delta) := \int_0^\delta F_k(2\pi\alpha)\,\mathrm{d}\alpha = \frac{1}{2\pi}\sum_{m=0}^{\infty}(-1)^m a_k(m) b_k(m)(2\pi\delta)^{2k+2m+1}.$$

We write

$$I_k(\delta) = \frac{1}{2\pi}\left\{\sum_{m=0}^{2k} + \sum_{m=2k+1}^{\infty}\right\}(-1)^m a_k(m) b_k(m)(2\pi\delta)^{2k+2m+1}$$

$$= \frac{1}{2\pi}\left(\sum_1 + \sum_2\right), \tag{11}$$

say. In order to estimate these sums we shall use Stirling's formula

$$n! = \Gamma(n-1) = \sqrt{2\pi}(n-1)^{n-\frac{3}{2}} e^{1-n+\mu(n-1)},$$

where

$$|\mu(n-1)| \leq \frac{1}{12(n-1)},$$

valid for integers $n \geq 2$. The proof of this relation is based on properties of Gudermann's series and can be found, for example, in Remmert [20]. Since $e^z \leq 1 + 2z$ for $0 \leq z \leq 1$ and $e^z \geq 1 - 2z$ for $-1 \leq z \leq 0$, we obtain, for $n \geq 2$,

$$\frac{5}{6}\sqrt{2\pi}(n-1)^{n-\frac{3}{2}} e^{1-n} \leq n! \leq \frac{7}{6}\sqrt{2\pi}(n-1)^{n-\frac{3}{2}} e^{1-n}. \tag{12}$$

We start with the first sum in (11). We have, for $1 \leq \ell \leq k$,

$$\begin{aligned} &(-1)^{2\ell-1} a_k(2\ell-1) b_k(2\ell-1)(2\pi\delta)^{2k+4\ell-1} \\ &+ (-1)^{2\ell} a_k(2\ell) b_k(2\ell)(2\pi\delta)^{2k+4\ell+1} \leq 0 \end{aligned} \tag{13}$$

provided that

$$\delta \leq \frac{2k+4\ell+1}{\pi}\sqrt{\frac{k+\ell}{(k+2\ell-1)(2k+4\ell-1)}}.$$

A short computation shows that (13) holds for

$$\delta \leq \frac{2}{\pi}\sqrt{k}. \tag{14}$$

Then, by (13) and (12), we get the estimate

$$\left|\sum_1\right| \leq a_k(0) b_k(0)(2\pi\delta)^{2k+1} \leq \frac{392}{25}\frac{e}{k}\left(\frac{\pi e \delta}{2k}\right)^{2k+1}, \tag{15}$$

which tends to zero with k tending to infinity if $\delta \leq \frac{2}{\pi e}k$.

Now we estimate the second sum in (11). We have

$$\left|\sum_2\right| \leq a_k(2k+1)b_k(2k+1)(2\pi\delta)^{6k+3}$$

if

$$\frac{a_k(m+1)b_k(m+1)(2\pi\delta)^{2k+2m+3}}{a_k(m)b_k(m)(2\pi\delta)^{2k+2m+1}} \leq \frac{1}{2}$$

for all $m \geq 2k+1$ (by the geometric series expansion). A short computation in addition with (12) show that this condition is fulfilled for

$$\delta \leq \frac{\sqrt{2}}{\pi}\sqrt{(k+1)(2k+1)}, \tag{16}$$

and that in this case

$$\left|\sum_2\right| < \frac{748}{1125e^3}\sqrt{\frac{3k}{2\pi}}2^{-4k}\left(\frac{\pi e\delta}{2k}\right)^{6k+3}.$$

In view of (14) and (16) we may take $\delta = \frac{2}{\pi}\sqrt{k}$ in (15) and the latter inequality. This leads via (11) to the estimate

$$I_k(\delta) < \frac{1}{2\pi}\left(\frac{392}{25}\frac{e}{k}\left(\frac{e}{\sqrt{k}}\right)^{2k+1} + \frac{748}{1125e^3}\sqrt{\frac{3k}{2\pi}}2^{-4k}\left(\frac{e}{\sqrt{k}}\right)^{6k+3}\right).$$

The right hand-side is less than $\frac{1}{2}$ for $k \geq 9$. Now taking $\eta = 2\delta$ in (10) we obtain

$$\lambda(k) \geq \eta(k) \geq 2\delta = \frac{4}{\pi}\sqrt{k}.$$

The theorem is proved.

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