Advanced Studies in Pure Mathematics 49, 2007
Probability and Number Theory — Kanazawa 2005
pp. 323–365

Approximations to the Goldbach and twin prime problem and gaps between consecutive primes

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Abstract.

We give a survey about the topics mentioned in the title, with a more detailed description of the recent joint results of Goldston, Yıldırım and the author about small gaps between consecutive primes.

§1. Introduction

In the present work we give a short survey of the results concerning the above mentioned problems with special emphasis to the developments in the last 5 years. We will discuss in greater detail the recent (still mostly unpublished) results of the author reached partly in collaboration with R. C. Baker and G. Harman (Large gaps between primes), I. Z. Ruzsa (Goldbach–Linnik problem) and mainly with D. Goldston, C. Yıldırım, S. W. Graham and Y. Motohashi (Small gaps between primes and almost primes).

Finally we will discuss the ideas behind the proof of the basic result (p_n denotes the n^{th} prime)

$$\liminf_{n\to\infty} \frac{p_{n+1} - p_n}{\log p_n} = 0, \tag{1.1}$$

reached in a recent joint work with Goldston and Yıldırım [34].

A more detailed discussion of the earlier results connected with Goldbach's conjecture can be found in the survey paper of Pintz (2006). Further discussion of all the mentioned topics can be found in the excellent monograph of Narkiewicz (2000).

Received March 9, 2006.
Revised April 27, 2006.
2000 *Mathematics Subject Classification.* Primary 11N05; Secondary 11P32.

Sections 3–15 will be devoted to the developments reached in the twentieth century, while Section 11 will contain a description of the results of the past 5 years.

Acknowledgement. The author would like to thank the referee for many helpful comments.

§2. Origin of the problems

The time of the origin of the twin prime problem is unclear, but it is plausible to suppose that already the Greeks observed that there are neighbouring odd integers larger and larger which are both primes. In view of Euclid's proof about the infinitude of primes we may justifiably suppose that the same was believed to be true about twin primes as well. In this way there is a good chance that the twin prime problem is one of the oldest (if not *the* oldest) unsolved problems in mathematics.

What we know for sure is that de Polignac (1849) formulated in 1849 already a generalization of it, namely

De Polignac conjecture (Generalized twin prime conjecture). *Every even integer can be written in infinitely many ways as the difference of two primes.*

A weaker form of this conjecture is a complete analogue of the Goldbach conjecture.

Weak de Polignac conjecture (Weak form of the generalized twin prime conjecture). *Every even integer can be written in at least one way as the difference of two primes.*

This conjecture clearly does not follow from and does not imply the Twin Prime Conjecture. In contrary to this, the following conjecture is a weaker form of the Twin Prime Conjecture.

Conjecture about bounded prime differences. *There exists an absolute constant C such that*

$$p_{n+1} - p_1 \le C \quad \textit{infinitely often.} \tag{2.1}$$

In order to see the difference in the depth of the de Polignac conjecture and its weaker form it is sufficient to remark that the truth of the de Polignac conjecture for at least one even integer is equivalent to the conjecture (2.1) about bounded prime differences. On the other hand, the weak de Polignac conjecture is already known to be true for almost all even integers since 1937, as it follows directly from the proof of the results of van der Corput (1937) Čudakov (1938), Estermann (1937) (see Section 4).

In contrast to the immense uncertainty in the time of the origin of the twin prime conjecture, the Goldbach conjecture is usually considered to be originated in the letter of Goldbach to Euler, dated June 7^{th}, 1742 (Euler, Goldbach (1965)). But the situation is also a little bit more complicated here. Goldbach used even two different formulations in his above mentioned letter from the usual one. In his answer dated June 30^{th}, 1742 Euler essentially formulated the presently known form of the

Binary Goldbach conjecture. *Every even integer greater than* 2 *can be written as the sum of two primes.*

(In his time still the number 1 was considered as a prime, so he did not need the assumption "greater than 2".)

Euler mentions in his letter that

(i) the above quoted formulation was mentioned to him earlier in a conversation by Goldbach and

(ii) that it implies one of the more complicated original formulations of the conjecture contained in the June 7^{th} letter of Goldbach.

Another fact, which is an important aspect of the history of the Goldbach problem is the nearly forgotten short remark of Descartes (1908) which we can formulate as

Descartes conjecture. *Every even integer can be written as the sum of* 1, 2 *or* 3 *primes.*

This remark appears just in the 1908 edition of the Collected Works of Descartes (Vol. 10, p. 298), so it was probably not known to Goldbach, although the possibility cannot be excluded completely.

In the survey paper (Pintz 2006) we made a more detailed analysis of the logical connections between various forms of the Goldbach conjecture which revealed that (surprisingly) the two original versions of the (binary) Goldbach conjecture formulated by Goldbach in his letter are both equivalent with the present version, unlike the Descartes conjecture, which is equivalent to the weaker assertion that for any even N at least one of N and $N+2$ is a Goldbach number, that is, can be written as the sum of two primes.

Finally, for the sake of completeness, we may formulate the

Ternary Goldbach conjecture. *Every odd integer greater than* 5 *can be written as the sum of three primes.*

This conjecture is not contained in the correspondence of Goldbach and Euler, but this terminology is generally used nowadays. It is easy to see that it follows both from the binary Goldbach conjecture and from the Descartes conjecture.

It was a great triumph of (analytic) number theory when I. M. Vinogradov (1937) proved that the ternary Goldbach conjecture is true for all sufficiently large odd integers $N > N_0$. The best known value $N_0 = e^{3100}$ was reached very recently by Liu Ming-Chit and Wang Tianze (2002). In view of the almost complete solution of this problem we shall restrict our attention for the binary Goldbach conjecture, called further on just Goldbach conjecture.

As practically no advance was made in either the Goldbach or the twin prime conjecture in the eighteenth and nineteenth centuries, Hilbert included both problems (and a common generalization of them) into its 8^{th} problem – together with the Riemann Hypothesis – into the collection of the famous 23 problems in his celebrated lecture at the 1900 Paris International Congress (see Hilbert (1935)). After more than hundred years we can say with some justification that his 8^{th} problem turned out to be the most difficult. Although most of his problems are already completely solved, none of the three above mentioned problems about primes have been solved in the past 105 years.

No progress was made in the first 12 years. So, at the 1912 Cambridge International Congress Landau (1912) listed four basic problems about primes

(1) Goldbach conjecture
(2) Twin prime conjecture
(3) There are always primes between n^2 and $(n+1)^2$ $(n \in \mathbb{Z})$
(4) There are infinitely many primes of the form $n^2 + 1$.

He characterized in 1912 in his speech these problems as "unangreifbar beim heutigen Stand der Wissenschaft" (unattackable at the present state of science).

In fact, they are all open today, after nearly 100 years. Since we will deal with the most important approximations to Problems 1–3 in the next sections but not with Problem 4 we remark here that modern sieve methods have already reached a close approximation to Problem 4. We are referring to the deep result of Iwaniec (1978) which gave the following weaker form of Problem (4) of Landau.

Theorem (Iwaniec). $n^2 + 1 = P_2$ *infinitely often.*

Here we use the well-known notation P_r for numbers having at most r prime factors (counted with multiplicity).

§3. Approximations to the Goldbach problem. (i). Almost primes

The first significant approximation of Goldbach's problem was achieved by Vigo Brun (1920) who showed that the analogue of the binary Goldbach conjecture is true if we allow instead of primes numbers of type P_9, that is (with the notation introduced at the end of Section 2), numbers with at most 9 prime factors.

Theorem (Brun). *If $N > N_0$ is even, then N can be written in the form $N = P_9 + P_9$.*

Selberg showed in the early 1950's with the sieve invented by himself that

$$N = P_2 + P_3, \quad N > N_0, \ 2 \mid N \tag{3.1}$$

but details of the proof appeared much later (Selberg (1992)).

Rényi (1947, 1948) could combine traditional sieve methods, analytic methods and the large sieve of Linnik to show the first result when one of the summands is definitely prime in the "quasi Goldbach" decomposition of a large even integer N.

Theorem (Rényi). *There exists a fixed (large) integer K such that $N = P_1 + P_K$ if $N > N_0$, N even.*

The value of K was reduced in subsequent works of Pan Cheng Dong (1963) and Barban (1963) to $K = 4$, then to $K = 3$ by Buhštab (1965). Finally, in 1966 Chen Jing Run (1966, 1973) obtained his celebrated $\{1, 2\}$ result with an ingenious application of the weighted sieve.

Chen's Theorem. *$N = P_1 + P_2$ if $N > N_0$ is even.*

A crucial role was played in Chen's theorem by (a new version of) the famous Bombieri–Vinogradov theorem.

§4. Approximations to the Goldbach problem. (ii) Exceptional sets

The first method which was able to attack approximations to the binary and ternary Goldbach problems with primes was developed by Hardy and Littlewood (1923) soon after Brun's work. Although the results were initially just conditional, it was the circle method of Hardy and Littlewood which formed the basis of Vinogradov's famous three prime theorem. In their ground-breaking paper they showed that the ternary Goldbach conjecture holds for $N > N_0$ subject to the Hypothesis

$$L(s,\chi) \neq 0 \quad \text{for} \quad \sigma > \Theta, \tag{4.1}$$

where Θ is any fixed constant with $\Theta < 3/4$.

In their next paper (Hardy–Littlewood (1924)) they showed under the Generalized Riemann Hypothesis that almost all even numbers are Goldbach numbers with a strong estimate on the exceptional set. Let us use the notation

$$E(X) = \#\{n \leq X,\ 2 \mid n,\ n \neq p_1 + p_2\}. \tag{4.2}$$

They showed on GRH in 1924 the estimate

$$E(X) \ll_\varepsilon X^{1/2+\varepsilon}, \tag{4.3}$$

which is apart from the factor X^ε even today the best conditional result on GRH.

The first unconditional estimate of type

$$E(X) < X\left(\frac{1}{2} - c\right) \tag{4.4}$$

was shown by an entirely different elementary method by Schnirelman (1930, 1933) (see (5.2)). However, the crucial result

$$E(X) = O_A\left(\frac{X}{\log^A X}\right), \quad A > 0 \ \text{arbitrary} \tag{4.5}$$

of van der Corput (1937), Čudakov (1938) and Esterman (1937) proved simultaneously and independently by them was a suitable modification of Vinogradov's method in proving the ternary Goldbach conjecture, which was based on the circle method of Hardy and Littlewood.

The pioneering work of Montgomery and Vaughan (1975) yielded the first estimate of type

$$E(X) \ll X^{1-\delta} \tag{4.6}$$

with some (explicitly calculable) small unspecified value of δ.

Despite many efforts of Chen, to show (4.6) with a good explicit value of δ, the results $\delta = 0.05$ of Chen and Liu (1989) and $\delta = 0.079$

of Hongze Li (1999) remained far from the conditional estimate (4.3) of Hardy and Littlewood. The hitherto best published result

$$E(X) \ll X^{0.914} \Longleftrightarrow \delta = 0.086 \tag{4.7}$$

was shown one year later by Hongze Li (2000a).

§5. 5. Approximations to the Goldbach problem. (iii) Another conjecture of Landau

In the same talk at the Cambridge International Congress in 1912, Landau proposed the following weaker form of the Goldbach conjecture.

Landau's conjecture. *There exists an integer k such that every integer exceeding* 1 *can be written as the sum of at most k primes.*

We may formulate also a weaker form of this as

Weak Landau conjecture. *There exists an integer k such that every sufficiently large integer can be written as the sum of at most k primes.*

We remark that although the two conjectures are equivalent with each other as long as no bound on k is required, they are of different difficulty if we would require them to hold with the same value k. Let us denote the minimal value of k by S in the first formulation, and by S_1 in the second formulation.

It was actually Brun's work on sieves which opened the way to an unconditional treatment of Landau's conjecture. Namely, Schnirelmann (1930, 1933) succeeded to show that the set $\mathcal{G}$ of Goldbach numbers, that is the set

$$\mathcal{G} = \{n; 2 \mid n, n = p_1 + p_2\} \tag{5.1}$$

has positive lower density (see (4.4)), i.e.

$$G(X) = \#\{n \leq X;\ n \in \mathcal{G}\} \gg X \quad \text{for } X \geq 4. \tag{5.2}$$

He could show further that this property implies Landau's conjecture with the bound

$$S_1 \leq 800\,000. \tag{5.3}$$

Vinogradov's (1937) three primes theorem implied

$$S_1 \leq 4 \tag{5.4}$$

but left open the value for S, as well as the question whether

$$S_1 = 3, \tag{5.5}$$

which is equivalent to Descartes' conjecture for $N > N_0$, and so it would be a consequence of the binary Goldbach conjecture.

The best known unconditional result is due to Ramaré (1995) who showed that all even integers can be written as a sum of 6 primes, therefore

$$S_1 \le 7. \tag{5.6}$$

His proof relies both on analytic and elementary arguments, combined with computations. Schnirelman's method was developed by him to the point that

$$G(X) > \frac{X}{5} \quad \text{for} \quad X > e^{67}, \tag{5.7}$$

as an explicit form of (5.2).

Finally we mention that the best known conditional results, subject to the Riemann Hypothesis (RH) and the Generalized Riemann Hypothesis (GRH), respectively, were achieved by Kaniečki (1995), who showed

$$\text{RH} \Longrightarrow S_1 \le 6, \tag{5.8}$$

and by Deshouillers, Effinger, te Riele, Zinoviev (1997) and Saoter (1998):

$$\text{GRH} \Longrightarrow \text{TGC} \Longrightarrow S_1 \le 4, \tag{5.9}$$

where TGC means the ternary Goldbach conjecture (for every odd $N > 5$).

§6. Approximations to the Goldbach problem. (iv) Goldbach numbers in short intervals

The results (4.5) of van der Corput, Čudakov and Estermann raised the following two natural problems.

I) What are the possible longest intervals without any Goldbach numbers?

II) What are the possible longest intervals where the possible Goldbach exceptional numbers form a positive proportion of all even integers in the interval?

According to the above let

$$\mathcal{G} = \{g_i\}_{i=1}^{\infty}, \quad \mathcal{E} = \{n; 2 \mid n, n \notin \mathcal{G}\} \tag{6.1}$$

$$A(x) = \max_{g_k \le x}(g_{k+1} - g_k) \tag{6.2}$$

$$E(X,Y) = E(X+Y) - E(X) = \#(\mathcal{E} \cap (X, X+Y]), \tag{6.3}$$

where g_n denotes the sequence of all Goldbach numbers.

It turns out that in contrast to essentially all problems concerning Goldbach numbers, the most effective methods to answer question I are the ones which use no specific additive methods, just results about the distribution of prime numbers and the mere definition of the Goldbach numbers. We can formulate the simple idea of Montgomery and Vaughan (1975) in the following slightly more general form

Proposition. *Let us suppose we have four positive constants ϑ_1, ϑ_2 c_1 and $c_2 < c_1\vartheta_1$ with the following properties:*

(a) *every interval of type $[X - Y, X]$ with $X^{\vartheta_1} < Y < X/2$ contains at least $c_1 Y/\log X$ primes for any $X > X_0$,*

(b) *for all but $c_2 X/\log X$ integer values $n \in [X, 2X]$ the interval $[n - X^{\vartheta_2}, n]$ contains a prime for any $X > X_0$.*

Then

$$A(X) \ll X^{\vartheta_1\vartheta_2}. \tag{6.4}$$

According to this, the estimate

$$A(X) \ll X^{\vartheta_3} \tag{6.5}$$

follows directly from the results about ϑ_1 and ϑ_2. (We mention that results about ϑ_1 are dealt with in more detail in Section 14). The results of Huxley (1972)

$$\vartheta_1 = \frac{7}{12} + \varepsilon, \quad \vartheta_2 = \frac{1}{6} + \varepsilon \tag{6.6}$$

accordingly led Montgomery and Vaughan to the result

$$\vartheta_3 = \frac{7}{72} + \varepsilon. \tag{6.7}$$

The strongest published results before 2000, $\vartheta_1 = 0.535$ of R. C. Baker and G. Harman (1996), further the result $\vartheta = \frac{1}{20} + \varepsilon$ of Ch. Jia (1996a) implied

$$\vartheta_3 = 0.02675. \tag{6.8}$$

Based on these results we may formulate the following weaker conjecture, which is perhaps more accessible than the Goldbach problem.

Conjecture A. $g_{k+1} - g_k \ll g_k^{\varepsilon}$ *for any* $\varepsilon > 0$.

Our feeling that Conjecture A may be proved easier than the Goldbach conjecture is supported by the result of Linnik (1952), according to which the Riemann Hypothesis implies Conjecture A, with the estimate $\log^3 g_k$. The strongest conditional result is the following

Theorem (Kátai 1967). RH $\Longrightarrow g_{k+1} - g_k \ll (\log g_k)^2$.

The first answer for Problem II was furnished by the result of K. Ramachandra (1973)

$$E\big(X, X^{\vartheta_4}\big) = o\big(X^{\vartheta_4}\big) \quad \text{if } \vartheta_4 = \frac{3}{5} + \varepsilon. \tag{6.9}$$

This was improved twenty years later when simultaneously and independently Perelli–Pintz (1933) and H. Mikawa (1992) showed (6.9) with $\vartheta_4 = \frac{7}{36} + \varepsilon$ and $\vartheta_4 = \frac{7}{48} + \varepsilon$, respectively. Further significant improvements were

$\vartheta_4 = \frac{7}{81} + \varepsilon$ Hongze Li (1995),

$\vartheta_4 = \frac{11}{160} + \varepsilon$ Baker, Harman, Pintz (1995/96).

Finally we remark that we can show an analogue of the above mentioned theorems of Linnik and Kátai for the more difficult question II, if we suppose the Generalized Riemann Hypothesis in place of RH.

Theorem (Kaczorowski–Perelli–Pintz (1993)). *The GRH implies for any* $\varepsilon > 0$

$$E(X, Y) = o_\varepsilon(Y) \quad \textit{for} \quad Y = (\log X)^{6+\varepsilon}. \tag{6.10}$$

§7. Approximations to the Goldbach problem. (v) The Goldbach–Linnik problem

The following approximation of the Goldbach problem was examined by Linnik more than half a century ago, before most of the short interval problems. He imposed the question whether we can write any (sufficiently large) even integer as the sum of a Goldbach number and at most K powers of two, with a fixed number K. In two subsequent papers Linnik (1951, 1953) succeeded to prove this with an unspecified K, first under GRH, two years later unconditionally. Since the binary Goldbach conjecture (for large enough even numbers) is equivalent to $K = 0$, the proof of Linnik's result with a reasonable value of K shows how close we are to the Goldbach conjecture in the above sense. It may be interesting to note that Descartes' conjecture implies $K = 1$.

The problem can be formulated in a similar (but not equivalent) following way, too.

Problem: Suppose an arbitrary even integer is given in the binary form. How many binary digits have to be changed at most in order to obtain a Goldbach number?

In 1975 Gallagher [29] found a significantly simpler proof for Linnik's theorem. The first explicit results, obtained at the end of the 1990es, were based on Gallagher's proof. We list the results below up to 2000.

$K = 54000$, J. Y. Liu, M. C. Liu and T. Z. Wang (1998b);

$K = 25000$, Hongze Li (2000b);

$K = 2250$, T. Z. Wang (1999).

The conditional estimates were the following.

GRH $\Longrightarrow K = 770$, J. Y. Liu, M. C. Liu and T. Z. Wang (1998a);

GRH $\Longrightarrow K = 200$, J. Y. Liu, M. C. Liu and T. Z. Wang (1999);

GRH $\Longrightarrow K = 160$, T. Z. Wang (1999).

Finally at the Debrecen Number Theory meeting in 2000 the author announced the results $K = 10$ (under GRH) and $K = 12$ (unconditionally), which were later further improved in joint work with I. Z. Ruzsa (see Section 11).

§8. Approximations to the Goldbach problem. (vi) Moments of differences of Goldbach numbers

The idea of examination of moments

$$M_\alpha(X) = \sum_{g_n \le X} (g^*_{n+1} - g_n)^\alpha \quad (g^*_k = \max(g_k, X)) \tag{8.1}$$

of differences of Goldbach numbers originates from H. Mikawa (1993). He proved

$$M_\alpha(X) = 2^{\alpha-1}X + o(X) \quad \text{for } 0 < \alpha < 3 \tag{8.2}$$

and

$$M_3(X) \ll X \log^{300} X. \tag{8.3}$$

The expression (8.1) (for all $\alpha \ge 0$) tells more, in some sense, about irregularities of the distribution of Goldbach numbers than $E(X)$ in (4.2) since large blocks of Goldbach exceptional numbers obtain larger weight in (8.1) for $\alpha > 1$; in this way it tells something about the structure of the exceptional set $\mathcal{E}$ (see (6.1)), too.

Since the binary Goldbach conjecture (for all sufficiently large even numbers) is equivalent to

$$M_\alpha(X) = 2^{\alpha-1}X + O_\alpha(1) \quad \text{for } \alpha \geq 0,\ X > 0, \tag{8.4}$$

it is plausible to formulate the following weaker conjectures.

Conjecture $\mathbf{C}_1$: $M_\alpha(X) = 2^{\alpha-1}X + o(X)$ for any $\alpha \geq 0$.

Conjecture $\mathbf{C}_2$: $M_\alpha(X) = 2^{\alpha-1}X + O(X^{1-\delta})$ for any $\alpha \geq 0$ with a positive $\delta = \delta(\alpha)$.

In order to see the depth of Conjecture C_2 we remark that its assertion for $\alpha = 0$ is obviously the deep theorem (4.6) of Montgomery and Vaughan (1975).

It is interesting to note that in the above formulation (that is for all $\alpha \geq 0$) the seemingly weaker Conjecture C_1 is equivalent to C_2 (using again (4.6)). In fact, (4.6) implies the following connection

Proposition. *Conjectures C_1, C_2 and A (see Section 6) are equivalent.*

The above equivalence implies that similarly to Conjecture A, also Conjectures C_1 and C_2 follow from the Riemann Hypothesis (unlike most other problems in connection with the Goldbach conjecture, where we need assumptions about zeros of Dirichlet L-functions, like the Generalized Riemann Hypothesis, for example).

Since Mikawa's method has its natural limit at $\alpha = 3$ we used entirely different methods to improve his results (see Section 11).

§9. Approximations to the Twin Prime Conjecture and to the Weak Form of the Generalized Twin Prime Conjecture

The results about Goldbach decompositions of even integers using almost primes in place of primes (see Section 3) all have their natural analogues for the twin prime conjecture. So, we mention just the strongest result.

Chen's theorem (1966, 1973). *There are infinitely many primes p such that $p+2$ is a P_2 number (and the same holds for $p-2$, too, or with an arbitrary even integer $2d$ in place of 2).*

In case of the examination one can ask the same questions for the exceptional set in the weaker form of the generalized twin prime conjecture, that is for

$$E'(X) = \#\{n \le X;\ 2 \mid n,\ n \ne p_1 - p_2\} \quad \text{(see (4.2))} \tag{9.1}$$

or for the analogue $E'(X,Y) = E'(X+Y) - E'(X)$ (see (6.3)).

Although the results were not worked out in most cases, the applied methods would yield the same results as those of Sections 4, 6, 8 and (with perhaps slight changes) 7, if we substitute Goldbach numbers by the numbers which can be written as the difference of two primes. (We ignore trivial differences, like $E(2) = 1$ in contrast to $E'(2) = 0$, and consequently the Goldbach conjecture is equivalent to $E(X) = 1$ for $X \ge 2$, while we conjecture $E'(X) = 0$ for all X, naturally.)

Thus, the difference is only with results of Section 5, concerning Landau's conjecture, which have no natural analogue with respect to any form of the twin prime conjecture.

§10. Large Gaps Between Consecutive Primes

It is a very natural question to ask how large the gaps can be in the sequence of primes. The first published conjecture was the well-known Bertrand's postulate. Bertrand (1845) conjectured that there is always a prime in $(x/2, x-2]$ for any $x > 6$, which means that we have essentially always at least one prime between an integer and its double. The same conjecture appeared about 100 years earlier in an unpublished manuscript of Euler (cf. p. 104 of Narkiewicz (2000)). This conjecture was proved by Chebyshev (1850), ([13]), who even showed the much stronger result

$$0.92129\frac{x}{\log x} \le \pi(x) \le 1.10555\frac{x}{\log x} \quad \text{if } x > x_0, \tag{10.1}$$

where $\pi(x)$ denotes the number of primes not exceeding x. The proof of Chebyshev was completely elementary.

The proof of the prime number theorem with the classical error term

$$\pi(x) = li\,x + O\left(\frac{x}{e^{c\sqrt{\log x}}}\right) = \int_0^x \frac{dt}{\log t} + O\left(\frac{x}{e^{c\sqrt{\log x}}}\right) \tag{10.2}$$

due to de la Vallée Poussin (1899) automatically leads to an estimate for the gaps between primes. Since under the Riemann Hypothesis we have even

$$\pi(x) = li\,x + O\big(\sqrt{x}\log x\big), \tag{10.3}$$

this implies the existence of primes in intervals of the form

$$[x, x + C\sqrt{x}\log^2 x] \quad \text{on } RH. \tag{10.4}$$

This was the knowledge about the differences of consecutive primes when Landau expressed his conjecture (Problem 3 in Section 2) in 1912, which asserts (roughly) the existence of primes in intervals of type

$$[x, x + 2\sqrt{x}]. \tag{10.5}$$

On the other hand, the Prime Number Theorem (10.2) obviously implies that the average size of gaps between primes around x is much smaller, namely

$$\log x, \tag{10.6}$$

which indicates that a much stronger assertion than (10.5) might hold in reality. In fact, Cramér (1936) constructed a heuristic probabilistic model for the primes in which the integers n are chosen to be prime independently with probability $1/\log n$, based on the function $li\, x$ and the approximation (10.2), describing the average density of primes around x. This model is very effective to produce conjectures about the true behaviour of primes. Even if it spectacularly fails in some cases – like shown first by the celebrated result of Helmut Maier (1985);

$$\frac{\pi(x + (\log x)^\lambda) - \pi(x)}{(\log x)^{\lambda-1}} \not\to 1 \tag{10.7}$$

for any fixed $\lambda > 0$, as $x \to \infty$ – we have no better tools to make plausible predictions. Based on his model Cramér (1936) expressed the conjecture

$$\limsup_{n\to\infty} \frac{p_{n+1} - p_n}{\log^2 p_n} = 1. \tag{10.8}$$

This conjecture is accepted by most experts, at least apart from the value 1; definitely it is believed to be true in the weaker form

$$p_{n+1} - p_n \ll_\varepsilon (\log p_n)^{2+\varepsilon} \quad \text{for any } \varepsilon > 0; \tag{10.9}$$

and is considered to be false for any $\varepsilon < 0$.

The first result, going beyond the "trivial" gap size implied by the error term of the prime number formula was reached by Hoheisel (1930), nearly two decades after Landau's lecture in 1912.

His idea could be formulated in the following way. Let us consider the explicit formula for the well-known function

$$\psi(x) = \sum_{n \le x} \Lambda(n), \quad \Lambda(n) = \begin{cases} \log p & \text{if } n = p^m \\ 0, & \text{otherwise.} \end{cases} \tag{10.10}$$

Denoting the non-trivial zeros $\varrho = \beta + i\gamma$ of the ζ-function we have

$$\psi(x) = x - \sum_{|\gamma| \le T} \frac{x^\varrho}{\varrho} + O\left(\frac{x \log^2 Tx}{T}\right). \tag{10.11}$$

(We can set here $T = \sqrt{x}$, e.g.) The crucial observation is that although the individual error terms can make up the total error in (10.2), but their change $((x+y)^\varrho - x^\varrho)/\varrho$ once we move from x to $x+y$ with an y of the form

$$y = x^\vartheta \quad (\vartheta < 1), \tag{10.12}$$

for example, is generally much smaller than the absolute value of x^ϱ/ϱ. Therefore, if we have some information about the number of zeros

$$N(\sigma, T) = \sum_{\substack{\varrho; \zeta(\varrho)=0 \\ |\gamma| \le T, \beta \ge \sigma}} 1, \tag{10.13}$$

then we can estimate the change in the error term much better than the size of the error term itself. Since the density theorem of Carlson (1920),

$$N(\sigma, T) \ll_\varepsilon T^{(4\sigma+\varepsilon)(1-\sigma)} \quad \text{for any } \varepsilon > 0 \tag{10.14}$$

was available since 10 years in 1930, Hoheisel succeeded to show the existence of primes, even to give a prime number theorem for short intervals

$$\pi(x + x^\vartheta) - \pi(x) \sim \frac{x^\vartheta}{\log x} \quad \text{for } \vartheta = 1 - \frac{1}{33000}. \tag{10.15}$$

Later discoveries about the zero-free region of the zeta-function revealed that the crucial point is to have a good density theorem as (10.14), and in case of a bound

$$N(\sigma, T) \ll T^{A(1-\sigma)} \log^B T \tag{10.16}$$

one can prove (10.15) with

$$\vartheta = 1 - \frac{1}{A} + \varepsilon. \tag{10.17}$$

Continuing this research, after many subsequent improvements, finally Huxley (1972) succeeded to prove the above with

$$A = \frac{12}{5}, \quad \vartheta = \frac{7}{12} + \varepsilon = 0.58\dot{3} + \varepsilon, \tag{10.18}$$

which is even today essentially the best result for (10.15).

However, surprisingly Iwaniec and Jutila (1979) found a way to show the existence of primes in short intervals without requiring the asymptotic (10.15). With an ingenious combination of modern forms of sieve methods and analytic methods they succeeded to prove in place of (10.15) the weaker relation

$$\pi(x + x^{\vartheta}) - \pi(x) \gg \frac{x^{\vartheta}}{\log x}, \quad \vartheta = \frac{13}{23} = 0.5652\ldots . \tag{10.19}$$

The estimate (10.19) was improved several times before the end of the twentieth century until the distance $\vartheta - 1/2$ was finally nearly halved compared to (10.19) by the result

$$\vartheta = 0.535, \quad \text{R. C. Baker – G. Harman (1996).} \tag{10.20}$$

The present best conditional result, the improvement of (10.4) with a factor $\log N$; that is, the existence of primes in

$$\left[x, x + C\sqrt{x}\log x\right] \quad \text{on } RH \tag{10.21}$$

originates from Cramér (1936), but still fails to decide Landau's original problem.

As we mentioned earlier, according to the general belief, the actual size of the largest possible gaps is around (10.8), which is exactly the square of the average gap size $\log p$ around p.

The value $\log p$ was improved by elementary arguments to

$$\lambda = \limsup_{n\to\infty} \frac{p_{n+1} - p_n}{\log p_n} \geq 2 \tag{10.22}$$

by Backlund (1929); further by Brauer and Zeitz (1930) to

$$\lambda \geq 4. \tag{10.23}$$

The first estimate of type $\lambda = \infty$ was found one year later by Westzynthius (1931)

$$\limsup_{n\to\infty} \frac{(p_{n+1} - p_n)\log_4 p_n}{\log p_n \log_3 p_n} > 0 \tag{10.24}$$

where $\log_\nu x$ denotes the ν-fold iterated logarithm. This was improved by Erdős (1935) to

$$\limsup_{n\to\infty} \frac{(p_{n+1}-p_n)(\log_3 p_n)^2}{\log p_n \log_2 p_n} > 0; \tag{10.25}$$

further three years later by a $\log\log\log\log p$ factor by Rankin (1938) to

$$\limsup_{n\to\infty} \frac{(p_{n+1}-p_n)(\log_3 p_n)^2}{\log p_n \log_2 p_n \log_4 p_n} \geq \frac{1}{3}. \tag{10.26}$$

Apart from the constant $1/3$ this result is even today, after nearly 70 years the best known despite of the prize \$ 10 000 offered by Erdős in 1979 for the proof (or disproof) that (10.26) holds with any positive constant c in place of $1/3$. Although (10.26) was improved four times (by Schönhage, Ricci, Maier–Pomerance and the author), the best known result at present is only

$$\limsup_{n\to\infty} \frac{(p_{n+1}-p_n)(\log_3 p_n)^2}{\log p_n \log_2 p_n \log_4 p_n} \geq 2e^\gamma, \tag{10.27}$$

proved by the author [Pintz (1997)] by a combination of methods from graph theory, probability theory, analytic and elementary number theory.

§11. Small Gaps Between Consecutive Primes

The smallest possible existing gap between primes is clearly 2, apart from the exceptional distance 1, between 2 and 3. However, whether gaps of size 2 appear beyond any limit, is obviously exactly the twin prime conjecture. According to this, the detection of small gaps between primes is a natural approximation to the twin prime conjecture. The progress can be compared to that in the problem of finding large gaps between consecutive primes as described in (10.22)–(10.27).

Since the average gap size is $\log p$ by the Prime Number Theorem, one can set the goal to reach some nontrivial estimate (possibly zero) for the quantity

$$\Delta_1 = \liminf_{n\to\infty} \frac{p_{n+1}-p_n}{\log p_n}. \tag{11.1}$$

The first result was proved eighty years ago under GRH by Hardy and Littlewood (1926)

$$\Delta_1 \leq 2/3, \tag{11.2}$$

by the aid of their circle method. However, the first unconditional result was reached much later by Erdős (1940);

$$\Delta_1 \leq 1 - c, \tag{11.3}$$

with an unspecified small positive constant c. Erdős used Brun's sieve in order to show that for any particular integer k the equation

$$p_1 - p_2 = 2k \quad p_i \leq x \tag{11.4}$$

has not too many solutions. This showed that the relation $\Delta_1 = 1$, which would imply a too big concentration of the differences $p_{n+1} - p_n$ around $\log x$ if $p_n \in [x/2, x]$, is not possible.

After several improvements in sieve methods enabled an improvement of (11.3) to $c = 3/32$ in 1965, Bombieri and Davenport (1966) were able to substitute the GRH by the new Bombieri–Vinogradov theorem, which led itself to $\Delta_1 \leq 1/2$. Further, a combination with the method of Erdős yielded

$$\Delta_1 \leq \left(2 + \sqrt{3}\right)/8 = 0.466\ldots . \tag{11.5}$$

Further improvements reduced this to

$$\Delta_1 \leq 0.44254\ldots \quad \text{Huxley (1977)}, \tag{11.6}$$

when H. Maier (1985) invented his method which led to the unexpected irregular behaviour (10.7) of primes in intervals of length $(\log x)^\lambda$ for any $\lambda > 0$. Since quantitatively he showed that primes might be distributed more densely than on average by a factor e^γ in intervals of length $\omega(x) \log x$ with $\omega(x) = (\log x)^{o(1)}$, that is

$$\limsup_{n\to\infty} \frac{\pi(x + \omega(x)\log x) - \pi(x)}{\omega(x)} \geq e^\gamma, \tag{11.7}$$

this automatically yielded $\Delta_1 \leq e^{-\gamma} = 0.56145\ldots$.

However, his method could be again combined with that of Huxley (1977) in proving (11.6), so the combination of the three basically distinctive approaches of Erdős, Bombieri–Davenport and Maier finally led, in a fairly non-standard way, to the result of Maier (1988),

$$\Delta_1 \leq e^{-\gamma} \cdot 0.44254 = 0.2486\ldots . \tag{11.8}$$

The results (11.2), (11.5) and (11.8) show that regularity of distribution of primes in arithmetic progressions (GRH or Bombieri–Vinogradov theorem) implies the existence of small gaps between consecutive primes.

On the other hand, Heath-Brown [39] proved in 1983, that the existence of Siegel zeros (that is very strong irregularities in the distribution of primes in arithmetic progressions) implies even more, namely the Generalized Twin Prime Conjecture.

This was the state of the problem at the end of the twentieth century, until very recently. Thus the improvements over a period of nearly 80 years were just to gain slightly more than a factor 4 with respect to the "trivial" estimate $\Delta_1 \le 1$, which could be compared to the result $\lambda \ge 4$ of Brauer and Zeitz (1930), proved 75 years ago (see (10.23)).

Finally, we mention that the above mentioned methods were also able to estimate the quantity

$$\Delta_\nu = \liminf_{n\to\infty} \frac{p_{n+\nu} - p_n}{\log p_n}. \tag{11.9}$$

The subsequent results proved about Δ_ν were

$\Delta_\nu \le \nu - 1/2$, Bombieri–Davenport (1966);

$\Delta_\nu \le \nu - 5/8 + o(1)$ Huxley (1968/69);

and finally

$\Delta_\nu \le e^{-\gamma}(\nu - 5/8 + o(1))$ Maier (1988).

The twin prime conjecture was generalized already 100 years ago by L. E. Dickson (1904) in the following way. Let $h_1, h_2, \ldots, h_k$ be non-negative distinct integers,

$$\mathcal{H} = \{h_1, \ldots, h_k\} \tag{11.10}$$

and let $\nu_p(\mathcal{H})$ denote the number of distinct residue classes modulo p, occupied by the elements of $\mathcal{H}$.

Dickson raised the question under what condition on $\mathcal{H}$ is it plausible to conjecture that all elements

$$(n + h_1, \ldots, n + h_k) \tag{11.11}$$

can be primes for infinitely many values of n. A trivial necessary condition is the requirement

$$\nu_p(\mathcal{H}) < p \quad \text{for all primes } p. \tag{11.12}$$

Dickson (1904) conjectured that this condition is also sufficient. We will call a set $\mathcal{H}$ admissible if (11.12) holds. The conjecture was later examined in greater detail in the quantitative form by Hardy and Littlewood (1923). They introduced the so-called singular series

$$\mathfrak{S}(\mathcal{H}) = \prod_p \left(1 - \frac{\nu_p(\mathcal{H})}{p}\right)\left(1 - \frac{1}{p}\right)^{-k} \tag{11.13}$$

(which arose as a series, not a product, in their work), and expressed the following quantitative form of Dickson's conjecture. They were probably unaware of Dickson's qualitative formulation, since they do not mention his work in their paper. Since nowadays even the qualitative version of Dickson's conjecture is associated with the names of Hardy and Littlewood, due to their valuable results in connection with these type of problems we may call the qualitative conjecture Hardy–Littlewood–Dickson (HLD) conjecture in the following.

Hardy–Littlewood's Prime k-tuple Conjecture. *In case of any admissible set $\mathcal{H}$ of size k we have*

$$\sum_{\substack{n\le N\\ n+h_i\in\mathcal{P}\ (1\le i\le k)}} 1 \sim \mathfrak{S}(\mathcal{H})\frac{N}{\log^k N}. \tag{11.14}$$

The arithmetic meaning of $\mathfrak{S}(\mathcal{H})$ is that the correct "probability" that all elements of a k-tuple (11.13) are not divisible by a given prime is

$$1-\frac{\nu_p(\mathcal{H})}{p}, \tag{11.15}$$

compared by the naive (and false) "probability"

$$\left(1-\frac{1}{p}\right)^k, \tag{11.16}$$

when considering Cramér's model (see Section 10). Therefore we have to multiply the naive probability (obtained by the product rule of independent random variables)

$$\frac{1}{\log^k N} \tag{11.17}$$

for the simultaneous primality of all components $n+h_i$ with the correction factor (11.13). The factor $\mathfrak{S}(\mathcal{H})$ takes into account that the primality of the components $n+h_i$ is not independent.

We remark that the factor $\mathfrak{S}(\mathcal{H})$ is 1 in average if $\mathcal{H}$ runs through all sets of size k with elements in $[1,H]$ for a fixed k and $H\to\infty$.

Theorem (Gallagher (1976)). *If all sets $\mathcal{H}$ are counted with a multiplicity $k!$ (are regarded as ordered sets) then for fixed k and $h\to\infty$ we* *have*

$$\sum_{\substack{|\mathcal{H}|=k \\ \mathcal{H}\subset[1,H]}} \mathfrak{S}(\mathcal{H}) \sim \sum_{\substack{|\mathcal{H}|=k \\ \mathcal{H}\subset[1,H]}} 1 \sim H^k. \tag{11.18}$$

The plausibility of the quantitative conjecture (11.14) is supported by the following results:

(i) it is true for almost all k-tuples $\mathcal{H} \subset [1, N]$ if k is fixed, $N \to \infty$ as shown by Lavrik (1961);

(ii) we have for any fixed $\mathcal{H}$ as $N \to \infty$

$$\sum_{\substack{n\le N \\ n+h_i\in\mathcal{P}\ (1\le i\le k)}} 1 \le \left(2^k k! + o(1)\right)\mathfrak{S}(\mathcal{H})\frac{N}{\log^k N} \tag{11.19}$$

(see Theorem 5.3 of Halberstam and Richert (1974)).

§12. Small gaps between almost primes. Conjectures of Erdős on consecutive integers

Chen's theorem (see Section 2) showed that the analogue of the twin prime problem can be solved if we allow almost primes of type P_2, that is numbers with at most two prime factors. However, there is no way to decide which one of the equations

$$p + 2 = p' \quad \text{or} \quad p + 2 = p'p'' \tag{12.1}$$

has infinitely many solutions. Even the seemingly much easier question than Chen's result to show that both cases

$$2 \mid \Omega(p+2) \quad \text{and} \quad 2 \nmid \Omega(p+2) \tag{12.2}$$

occur infinitely often, is still open. ($\Omega(n)$ denotes the number of prime factors of n, counted with multiplicity).

The reason for this is the so-called parity problem, a heuristic principle, first formulated by Selberg, which says that sieve methods cannot differentiate between integers with an even and odd number of prime factors.

According to this, Chen's theorem leaves many questions still open about the distribution of almost primes, where the almost primes have a given number of prime divisors. Let us call E_r or E_r' numbers the integers with *exactly* r prime factors, counted with or without multiplicity, that is

$$E_r = \{n; \Omega(n) = r\}, \quad E_r' = \{n; \omega(n) = r\}, \tag{12.3}$$

where $\omega(n)$ denotes the number of distinct prime factors of n. Particular attention can be paid to E_2-numbers which are products of *exactly* two primes, since they are in this sense the closest approximations to primes.

It seemed to be until very recently that due to the parity phenomenon it is as difficult to show results for E_r (or E'_r) numbers for any specific r as for primes itself. Thus, for example, one can raise the problem, whether

$$\liminf_{n\to\infty} \frac{q_{n+1} - q_n}{\log q_n (\log\log q_n)^{-1}} < \infty \tag{12.4}$$

if $q_1 = 4$, $q_2 = 6, \dots$ denotes the sequence of E_2-numbers, or any analogue of (12.4) with E_r-numbers for an $r > 2$.

Erdős proposed some interesting problems about divisors and prime divisors of consecutive integers

Problem 1 (Erdős–Mirsky (1952). Is $d(n) = d(n+1)$ infinitely often?

Problem 2 (Erdős (1983)). Is $\Omega(n) = \Omega(n+1)$ infinitely often?

Problem 3 (Erdős (1983)). Is $\omega(n) = \omega(n+1)$ infinitely often?

There seemed to be no other way to attack these conjectures than to look for almost prime solutions, and this is the situation still today. On the other hand, due to the parity phenomenon, a solution of any of these problems seemed to be as difficult as the twin prime problem.

It turned out, however, that there is a possibility to find almost prime solutions for these problems by "side stepping" the parity problem, that is, without specifying the number of prime divisors, or even the parity of the number of prime divisors of n and $n+1$. After a first idea of C. Spiro (1981) (who showed that $d(n) = d(n+5040)$ has infinitely many solutions) Heath-Brown (1984) succeeded to solve Problems 1 and 2. However, Problem 3 was not solved before the end of the century.

If we consider the problems of Erdős when n is restricted to almost primes, these problems can be considered like further close approximations to the twin prime conjecture or generalized twin prime conjecture. We remark that in case of almost primes there is no need to restrict the distance of the two almost primes to even integers therefore we can consider pairs of type n, $n + b$ for odd values of b too, like, in particular $b = 1$. In fact, Ch. Pinner (1997) extended the method of Heath-Brown and showed that both equations

$$d(n) = d(n+b), \qquad \Omega(n) = \Omega(n+b) \tag{12.5}$$

have infinitely many solutions for any given integer value b.

Although Chen's theorem answered even the generalized twin prime conjecture affirmatively for almost primes of type P_r (with $r = 2$) the following problems remained open for any value of r. Let ν and r be given. Let $P_r = \{m_i^{(r)}\}_{i=1}^{\infty}$.

Problem. Is it true that any admissible k-tuple contains at least ν almost primes of type P_r if k is sufficiently large ($k > C(\nu, r)$)?

Problem. Is $\liminf_{n\to\infty}\bigl(m_{n+\nu}^{(r)} - m_n^{(r)}\bigr) < \infty$?

§13. Recent developments

In the present section we will give an account of the developments of the past 5 years 2001–2005 including many still unpublished results of the author.

Concerning the size of the exceptional sets $E(X)$ (see (4.2)) and $E'(X)$ (see (9.1)) for the Goldbach and the weaker form of the generalized twin prime problem we state the following unconditional estimates.

Theorem 1. $E(X) \ll X^{2/3}$.

Theorem 2. $E'(X) \ll X^{2/3}$.

If we define the analogous counting function describing the size of the exceptional set in Descartes' conjecture (see Section 2),

$$D(X) = \Bigl\{n \le X; n \ne \sum_{i=1}^{j} p_i \ (1 \le j \le 3)\Bigr\}, \tag{13.1}$$

we can show

Theorem 3. $D(X) \ll X^{3/5}\log^{10} X$.

We can see that these results are nearer to the conditional estimate $X^{1/2+\varepsilon}$ (see (4.3)) of Hardy and Littlewood (valid on GRH) than to the earlier sharpest unconditional bound $X^{0.914}$ (4.7) of Hongze Li (2000).

The best upper bound for large gaps between consecutive primes was improved in 2001, which is the best known approximation to Landau's 3rd problem.

Theorem 4 (R. C. Baker, G. Harman, J. Pintz (2001)). *We have*

$$\pi(x + x^{\vartheta}) - \pi(x) \gg \frac{x^{\vartheta}}{\log x} \quad \textit{for } \vartheta = \frac{21}{40} = 0.525. \tag{13.2}$$

The above result (which is also valid for some ϑ very close but smaller than 0.525) together with the theorem of Ch. Jia (1996a) assuring the existence of primes in almost all short intervals of size $x^{1/20+\varepsilon}$ implies by the Proposition of Section 6 the strongest known upper bound for gaps between consecutive Goldbach numbers

Theorem 5 (Baker, Harman, Jia, Pintz). *We have*

$$g_{n+1} - g_n \ll g_n^{21/800}. \tag{13.3}$$

The best known bounds for the Goldbach–Linnik problem were shown simultaneously and independently in joint works of D. R. Heath-Brown and J. C. Puchta (2002) ($K = 7$ on GRH and $K = 13$ unconditionally) and Pintz and I. Z. Ruzsa (2003, 200?) which we state as

Theorem 6 (Pintz, Ruzsa). *Every sufficiently large even number can be written as the sum of two primes and K powers of two, where $K = 7$ on GRH and $K = 8$ unconditionally.*

The methods of proof of Theorem 1 allow a considerable improvement on Mikawa's estimates (8.1)–(8.3).

Theorem 7. *Let $g_1, g_2, \ldots$ be the series of Goldbach numbers. Then*

$$M_\alpha(X) := \sum_{g_n \le X} (g_{n+1}^* - g_n)^\alpha = 2^{\alpha-1}X + O(X^{1-\delta}) \tag{13.4}$$

for any $\alpha < 341/21 = 16.238\ldots$ with a suitably chosen $\delta = \delta(\alpha) > 0$ (with $g_k^* = \max(g_k, X)$.

Concerning small gaps between primes we proved the following results (see [34])

Theorem 8 (Goldston, Pintz, Yıldırım). *We have for any $\nu \ge 1$*

$$\Delta_\nu = \liminf_{n\to\infty} \frac{p_{n+\nu} - p_n}{\log p_n} \le \left(\sqrt{\nu} - 1\right)^2, \tag{13.5}$$

in particular

$$\Delta_1 = \liminf_{n\to\infty} \frac{p_{n+1} - p_n}{\log p_n} = 0 \tag{13.6}$$

and the obtained small gaps of size $\left(\left(\sqrt{\nu} - 1\right)^2 + \varepsilon\right) \log p_n$ represent a positive proportion of all gaps $p_{n+\nu} - p_\nu$.

A simplified version of (13.6) was proved in a joint work with Goldston, Motohashi and Yıldırım ([32]).

The above Theorem 8 can be improved in two directions. First, using many further ideas we can give a significant improvement of the qualitative result (13.6) as

Theorem 9 (Goldston, Pintz, Yıldırım). *We have*

$$\liminf_{n\to\infty} \frac{p_{n+1}-p_n}{(\log p_n)^{1/2}(\log\log p_n)^2} < \infty. \tag{13.7}$$

Secondly, we can combine the method of Maier (1985) with that of Theorem 8 as to yield

Theorem 10 (Goldston, Pintz, Yıldırım). *We have*

$$\Delta_\nu = \liminf_{n\to\infty} \frac{p_{n+\nu}-p_n}{\log p_n} \le e^{-\gamma}\left(\sqrt{\nu}-1\right)^2. \tag{13.8}$$

The proofs of Theorems 8 and 10 revealed that proofs of results about small gaps between primes rely heavily on our knowledge about the regularity of distribution of primes in arithmetic progressions. We say that primes have *level of distribution* ϑ if the relation

$$\sum_{q\le X^{\vartheta-\varepsilon}} \max_{\substack{a\\(a,q)=1}} \left| \sum_{\substack{p\le x\\ p\equiv a \pmod q}} \log p - \frac{X}{\varphi(q)} \right| \ll_{A,\varepsilon} \frac{X}{\log^A X} \tag{13.9}$$

holds for any positive ε and A as $X\to\infty$.

The Bombieri–Vinogradov theorem asserts that primes have level of distribution 1/2. The Elliott–Halberstam conjecture (EH) asserts that even $\vartheta = 1$ is an admissible level for primes.

It is a very surprising fact that any improvement of the Bombieri–Vinogradov theorem to a fixed $\vartheta > 1/2$ implies already the very strong relation (2.1) and even more.

Theorem 11 (Goldston, Pintz, Yıldırım). *If primes have level of distribution $\vartheta > 1/2$, then any admissible k-tuple contains at least two primes infinitely often if $k > C_1(\vartheta)$. In particular we have then*

$$\liminf_{n\to\infty}(p_{n+1}-p_n) \le C_2(\vartheta). \tag{13.10}$$

The Elliott–Halberstam conjecture implies the existence of two primes in every admissible 6-tuple, and thereby

$$\liminf_{n\to\infty}(p_{n+1}-p_n) \le 16. \tag{13.11}$$

Remark. To show (13.11) we can use the admissible 6-tuple $\mathcal{H} = \{0,4,6,10,12,16\}$.

It is also surprising that in order to have 3 primes in a block of length $o(\log x)$, which seems to be easier than the bounded difference problem (13.10) we need to assume more, namely EH.

Theorem 12 (Goldston, Pintz, Yıldırım). *EH implies*

$$\Delta_2 = \liminf_{n\to\infty} \frac{p_{n+2} - p_n}{\log p_n} = 0. \tag{13.12}$$

The methods applied in the proofs of Theorem 8–12 turned out to be even more successful for almost primes of type E_r (or E'_r) than for primes. The further results were all obtained in collaboration with D. Goldston, S. W. Graham and C. Yıldırım, abbreviated later as GGPY (see [31] and its continuations).

First we mention that we can give an affirmative answer to the last two problems of Section 12, even if we restrict ourselves to E_r numbers for any specific $r \geq 2$.

Theorem 13 (GGPY. Weak form of the HLD conjecture for almost primes). *Let $r \geq 2$ and $\nu \geq 1$ be given. Any admissible k-tuple contains simultaneously at least $\nu + 1$ E_r (or E'_r) numbers infinitely often if*

$$k \geq C_1(\nu) = (1 + o(1))e^{\nu-\gamma} \tag{13.13}$$

as $\nu \to \infty$.

Since $\mathcal{H} = \{p_{k+1}, \ldots, p_{2k}\}$ is always an admissible k-tuple of diameter $\sim k \log k$ we obtain from (13.13)

Corollary 14 (GGPY). *If $E_r = \{q_i^{(r)}\}_{i=1}^{\infty}$, $r \geq 2$, then*

$$\Delta(\nu, r) := \liminf_{n\to\infty} \left(q_{n+\nu}^{(r)} - q_n^{(r)}\right) \leq C_2(\nu) = (1 + o(1))\nu e^{(\nu-\gamma)}. \tag{13.14}$$

The above results can be generalized for a k-tuple of linear forms $L_i(n) = a_i n + b_i$ ($a_i, b_i \in \mathbb{Z}$) which is admissible, that is the product of all forms have no fixed prime divisors. This extension is particularly important in the treatment of Erdős' problems.

Theorem 15 (GGPY). *Let $r \geq 2$ be given. Any admissible triplet of linear forms contains two forms which take simultaneously E_r (or E'_r) numbers infinitely often.*

Taking the triplet n, $n + 2$, $n + 6$, this implies

Corollary 16 (GGPY). $\liminf_{n\to\infty} \left(q_{n+1}^{(r)} - q_n^{(r)}\right) \leq 6.$

Theorems 13 and 15 already show that the parity obstacle can be overcome in some way, but they still do not show that we can obtain even neighbouring E_r numbers for some r. However, one can derive this from Theorem 15 with a slightly bigger value of r, and to give a solution to Erdős' problems when the value of ω, Ω, or d is a prescribed number. In this connection we mention that Schlage–Puchta (2003/05) succeeded to solve recently Problem 3 of Erdős. However, his result does not extend to the general case $\omega(n) = \omega(n+b)$ (even no result is known for any single $b > 1$).

Theorem 17 (GGPY). *Let $A \geq 3$, $B \geq 4$, $C \geq 1$ be arbitrary given integers. Then the equations*

$$\omega(n) = \omega(n+1) = A, \tag{13.15}$$

$$\Omega(n) = \Omega(n+1) = B, \tag{13.16}$$

$$d(n) = d(n+1) = 24C \tag{13.17}$$

have all infinitely many solutions.

Apart from a singe exceptional case for the divisor function ($b \equiv 15 (\bmod 30)$) these results can be extended for an arbitrary shift b.

Theorem 18 (GGPY). *Let $b \in \mathbb{Z}$, $b \not\equiv 15 (\bmod 30)$ in case of (13.20). There exists a constant $A(b)$ such that if $A \geq A(b)$, $B \geq 5$, $C \geq 1$, then the equations*

$$\omega(n) = \omega(n+b) = A, \tag{13.18}$$

$$\Omega(n) = \Omega(n+b) = B, \tag{13.19}$$

$$d(n) = d(n+b) = 48C \tag{13.20}$$

have all infinitely many solutions.

We can see from (13.19) that the de Polignac conjecture can be solved if primes are substituted by E_r numbers for any given $r \geq 5$.

§14. Basic ideas of the proof of $\Delta_1 = 0$

In the following we outline the basic ideas leading to the proof of (13.6) and (13.10). A long series of investigations of Goldston and Yıldırım on short divisor sums, Gallagher's work (1976) about the connection of the Hardy–Littlewood–Dickson (HLD) k-tuple conjecture with the distribution of primes in short intervals and ideas of Granville and Soundararajan led to the conclusion that the distribution of primes in short intervals could be examined through some approximation of the HLD conjecture in the form of

$$S = S(\mathcal{H}, \nu) := \sum_{N<n\le 2N} \Biggl(\sum_{h_i \in \mathcal{H}} \chi^*(n+h_i) - \nu \Biggr) a(n), \tag{14.1}$$

where $\nu \in \mathbb{Z}^+$, χ^* is the characteristic function of primes,

$$\chi^*(n) = \begin{cases} 1 & \text{if } n \text{ is prime} \\ 0 & \text{otherwise} \end{cases}, \tag{14.2}$$

and $a(n)$ are suitably chosen non-negative weights (which depend on $\mathcal{H}$) satisfying the "trivial" condition

$$A := A(N) := \sum_{N \le n \le 2N} a(n) > 0. \tag{14.3}$$

In fact, for any k the positivity of S in (14.1) for an arbitrary fixed $\nu \in \mathbb{Z}^+$ and $N > N_0(\nu, \mathcal{H})$ implies the existence of at least $\nu + 1$ primes of the form $n + h_i$ for some $n \in (N, 2N]$; in particular even for $\nu = 1$ bounded gaps between primes infinitely often, more precisely

$$\limsup_{n\to\infty} (p_{n+1} - p_n) \le C(\mathcal{H}) = h_k - h_1. \tag{14.4}$$

Although the above goal looks too ambitious, it turned out that it can be reached under the supposition that the level ϑ of distribution of primes (cf. (13.9)) is an absolute constant bigger than $1/2$ if $k > k(\vartheta)$ and this approach led finally to an unconditional solution of the weaker relation (13.6).

The crucial condition $S > 0$ can be reformulated as follows. First we normalize the weights $a(n)$ by introducing

$$w(n) = \frac{a(n)}{A} \ge 0 \qquad \Biggl(\sum_{N<n\le 2N} w(n) = 1 \Biggr), \tag{14.5}$$

then we have to show that

$$E(\mathcal{H}) := \sum_{i=1}^{k} E(\mathcal{H}; h_i) := \sum_{i=1}^{k} \sum_{N<n\le 2N} w(n)\chi^*(n+h_i) > 1. \tag{14.6}$$

The notation above suggests that we consider $w(n)$ as a probability measure and $E(\mathcal{H};\, h_i)$ stands then for the expected number of primes in the i^{th} component of the k-tuple $\mathcal{H}$. If this expected value is in all k components together bigger than 1, then we proved the existence of at least two primes in the k-tuple $\mathcal{H}$ with $n \in (N, 2N]$. So our goal can be formulated as to try to choose some weights $a(n)$, equivalently $w(n)$, which maximizes $E(\mathcal{H})$.

The best conjectured choice would be to set

$$a(n) = \prod_{i=1}^{k} \chi^*(n+h_i), \tag{14.7}$$

which would lead "in reality" most probably to the optimal result

$$E(\mathcal{H}) = k. \tag{14.8}$$

The "only" problem is that in this case the trivially necessary condition (14.3) is clearly equivalent with the Dickson conjecture about the simultaneous primality of all $n + h_i$ for at least one $n \in (N, 2N]$, an assertion which is much stronger than our present aim.

Another, also "trivial" choice, is to set with a fixed $i \in [1, k]$

$$a(n) = \chi^*(n+h_i); \tag{14.9}$$

which yields in contrast to the former choice unconditionally

$$E(\mathcal{H}) \ge E(\mathcal{H}; h_i) = 1. \tag{14.10}$$

However, similarly to (14.7)–(14.8), in this case the condition (14.6) is only a trivial reformulation of our original goal. Paradoxically, still at present, any other known choices of $a(n)$ (apart from trivial variations of (14.9)) with (14.3) yield in the unconditional case a final result inferior to (14.10), that is

$$E(\mathcal{H}) \ge E^*(\mathcal{H}), \quad E^*(\mathcal{H}) < 1; \tag{14.11}$$

so in some sense (14.9) is still the best known choice in order to maximize $E(\mathcal{H})$, although completely useless to prove any reasonable result.

Despite the mentioned obstacles these choices (or at least one of them) are still helpful as to get an idea how to create some weights

$a(n)$ which might lead to non-trivial results about small gaps between primes. Although the choice (14.7) does not yield unconditionally any positive estimate of $E(\mathcal{H})$, in contrast to (14.10), the choice (14.7) is still promising since it captures at least the simultaneous primality of $n+h_i$, whereas the weight $\chi^*(n+h_i)$ has nothing to do with other elements of $\mathcal{H}$. The idea is to approximate in some way (14.7) and hope that

(i) the approximation will make possible the asymptotic evaluation of the weighted sum

$$A := A(\mathcal{H}) = \sum_{N<n\leq 2N} a(n), \tag{14.12}$$

(as $N \to \infty$) and that of the sums, twisted by primes

$$S(i) = S(\mathcal{H}; h_i) = \sum_{N<n\leq 2N} a(n)\chi^*(n+h_i); \tag{14.13}$$

(ii) will finally lead to a possibly not small value of

$$E(\mathcal{H}) = \frac{1}{A}\sum_{i=1}^{k} S(i). \tag{14.14}$$

Since in practice we can substitute the characteristic function $\chi^*(n)$ of the primes by

$$\Lambda(n) = \sum_{d|n} \mu(d) \log \frac{n}{d}, \tag{14.15}$$

the choice (14.7) offers two possibilities.

(a) To approximate the product of all $\Lambda(n+h_i)$'s term by term with a truncated divisor sum

$$\Lambda_R(n) = \sum_{\substack{d|n \\ d\leq R}} \mu(d) \log \frac{R}{d}, \tag{14.16}$$

leading to the approximation

$$a_1(n) = \left(\prod_{i=1}^{k} \Lambda_R(n+h_i) \right)^2, \tag{14.17}$$

where the squaring is performed to ensure the condition $a(n) \geq 0$.

(b) To work with the generalized von Mangoldt function which detects numbers with at most k prime-factors,

$$\sum_{d|m} \mu(d) \log^k \frac{m}{d}, \tag{14.18}$$

and detect whether $\{n + h_i\}_{i=1}^k$ is a prime k-tuple or not by applying the above formula (14.8); then, to approximate this by

$$a_2(n) := \Lambda_R^2(n; \mathcal{H}) := \Bigg(\sum_{\substack{d|P_{\mathcal{H}}(n) \\ d \le R}} \mu(d) \log^k \frac{R}{d} \Bigg)^2, \tag{14.19}$$

where

$$P_{\mathcal{H}}(n) = \prod_{i=1}^{k} (n + h_i). \tag{14.20}$$

A big advantage of the later choice (14.19) is that we have just one parameter, d, involved; instead of k parameters d_i, implicit in (14.17).

It is interesting to note that actually both choices $a_1(n)$ and $a_2(n)$ with the "truncation parameters"

$$R = R_1 = N^{1/(4k)-\varepsilon} \quad \text{and} \quad R = R_2 = N^{1/4-\varepsilon}, \tag{14.21}$$

respectively, make possible an execution of the steps (i) and (ii) (cf. (14.12)–(14.14)), and yielded (after relatively elaborate calculations) the values

$$E(\mathcal{H}) = \frac{1}{4} - \varepsilon_1(k) \quad \text{and} \quad E(\mathcal{H}) = \frac{1}{2} - \varepsilon_2(k). \tag{14.22}$$

For the case $a_1(n)$ see Goldston and Yıldırım ([33]). These could be used to obtain further (in a non-trivial way)

$$\Delta_1 \le \frac{1}{4} \quad \text{and} \quad \Delta_1 \le 1 - \sqrt{3}/2 = 0.1339\ldots, \tag{14.23}$$

respectively, which were actually better than earlier values reached by other methods without the combination with Maier's matrix method (which would improve these estimates by a factor $e^{-\gamma} = 0.5614\ldots$). Interestingly and fortunately the simpler choice $a_2(n)$ leads to the better result. Thus, in what follows, we will continue with a suitable generalization of the weight $a_2(n)$ in (14.19).

Expressions, similar to (14.1), of the form

$$\sum_{N<n\le 2N} \big(\tau(\mathcal{P}_{\mathcal{H}}(n)) - C\big)a(n) \qquad \Big(\tau(n) = \sum_{d\mid n} 1\Big) \tag{14.24}$$

were investigated for $k = 2$, $\mathcal{H} = \{0, 2\}$ by Selberg in the early 1950's (see Selberg (1992)) and for general k and $\mathcal{H}$ by D. R. Heath-Brown (1997) to prove results about twin almost primes and k-tuples of almost primes, respectively. The choice of the weights $a_2(n)$ and $a_3(n)$ was motivated, in fact, by Heath-Brown's work (1997).

It is plausible to work with weights which give the maximal weights for the set of all prime k-tuples. In other words we can try to choose the weights in such a way that, supposing that

$$a(n) = 1 \text{ if } n + h_i \in \mathcal{P} \quad (1 \le i \le k), \tag{14.25}$$

we should have A in (14.3) as small as possible, which is in this formulation equivalent to an upper bound sieve problem.

The best known upper bounds for the number of prime k-tuples are essentially reached in this case by the use of the function $a_2(n)$ in (14.19) (see (11.19)).

However, a closer look of the expressions in (14.12)–(14.14) reveals that we have to maximize the quantity $E(\mathcal{H})$, so we need to maximize the ratio

$$E(\mathcal{H}) = \frac{\sum\limits_{i=1}^{k} S(i)}{A} \tag{14.26}$$

not just A^{-1}, that is, to minimize A. This means we have no convincing heuristic argument to choose the weights as in (14.19).

If we consider the starting heuristics that we try to choose an approximation for the detector function of prime k-tuples, the question arises: if our goal is to show the existence of at least two primes among the k components why not try to choose a function which simulates k-tuples with at least two primes, for example to approximate numbers of the form $\mathcal{P}_{\mathcal{H}}(n)$ with

$$\omega\Big(\prod_{i=1}^{k}(n + h_i)\Big) \le k + \ell \text{ with } 0 \le \ell \le k - 2, \tag{14.27}$$

which definitely contain at least two primes among $n + h_i$. This gives the idea to try the weight-function

$$(14.28)\quad a_3(\mathcal{H}, n) := a_3(n) := \Lambda_R^2(n; \mathcal{H}, \ell) := \Bigg(\sum_{\substack{d \mid P_{\mathcal{H}}(n) \\ d \le R}} \mu(d) \log^{k+\ell} \frac{R}{d} \Bigg)^2$$

in place of the simpler $a_2(n)$ in (14.9).

After the evaluation of the corresponding weighted sums it will turn out that actually any choice of ℓ with

$$(14.29)\qquad \ell = o(k), \qquad \ell \to \infty \quad \text{as} \quad k \to \infty$$

is much (exactly twice) better for $k \to \infty$ than the choice $\ell = 0$. It will lead unconditionally after performace of steps (i) and (ii) (cf. (14.12)–(14.14)) to

$$(14.30)\qquad E(\mathcal{H}; h_i) = \frac{1}{k}\big(1 - \varepsilon_0(k)\big), \quad E(\mathcal{H}) = 1 - \varepsilon_0(k)$$

with an $\varepsilon_0(k) \to 0$ as $k \to \infty$ (under the condition (14.28)), which is, however, unfortunately positive

$$(14.31)\qquad \varepsilon_0(k) > 0 \Longleftrightarrow E(\mathcal{H}) < 1.$$

To be more precise, the function $\varepsilon_0(k) = \varepsilon_0(k, \ell)$ depends on ℓ, too, and the optimal choice $\ell \sim \sqrt{k}/2$ yields

$$(14.32)\qquad \varepsilon_0(k, \ell) \sim \frac{2}{\sqrt{k}}.$$

At first sight this is still a total failure. What does it help to show in average the existence of at least $1 - 10^{-10}$ primes in a k-tuple instead of $1/2 - 10^{-10}$ primes in (14.22) once we have clearly infinitely often at least one prime in a k-tuple?

Fortunately, there are even (at least) two possibilities. The first one is that if we allow conditional results, too, then an easy analysis of the proof leading to (14.30) and (14.22) shows that the number 1 in the numerator of these expressions arises in the unconditional treatment actually as

$$(14.33)\qquad 2\vartheta,$$

where ϑ is an admissible level of distribution of primes. This observation implies that if any fixed absolute constant $\vartheta > 1/2$ is an admissible level, then we have already

$$(14.34)\qquad E(\mathcal{H}) = 2\vartheta(1 - \varepsilon_0(k)) > 1$$

primes in "average" in our k-tuple $\mathcal{H}$, that is, (13.10) holds.

The other useful, although nearly trivial observation is that we were now looking at the much more difficult problem than (13.6), namely we tried to find at least two primes in a given k-tuple. We missed just with a hair-breadth to solve it, which in general might not help naturally the solution of the easier problem. However, what we achieved, can be measured quantitatively, and that makes a big difference between the results of (14.22) and (14.30), too. We constructed a suitable weight function $a_3(n)$ (or a probability measure, in other words) in (14.28) such that the expected number of primes of the form $n + h_i$ for any $i \in [1, k]$ is $\frac{1}{k}(1 - \varepsilon_0(k))$. In other words, choosing $n \in (N, 2N]$ randomly with probability $w_n = a(n)/A$, we have already an expected number of $1 - \varepsilon_0(k)$ primes among the mere k candidates $\{n + h_i\}_{i=1}^k$. Since the probability to choose a prime of size cN randomly with uniform distribution is asymptotically just

$$\frac{1}{\log N} \tag{14.35}$$

in general, our achievement $1 - \varepsilon_0(k)$ for $E(\mathcal{H})$, even in the weaker form (14.22), is much more than the expected number

$$\frac{k}{\log N} \tag{14.36}$$

in a uniformly chosen random set $\{n + h_i\}_{i=1}^k$.

Fortunately, these $1 - \varepsilon_0(k)$ primes are not lost by our failure to prove the existence of 2 primes in $\mathcal{H}$. They are for our disposal: if we can add to them for a $H = \varepsilon \log N$ in average more than

$$\varepsilon_0(k) \tag{14.37}$$

additional primes among

$$\{n + h\}; \quad 1 \le h \le H, \quad h \ne h_i, \tag{14.38}$$

then we have proved the existence of two primes among $\{n + j\}_{j=1}^H$, so we proved (13.6).

The weight-function $a_3(n)$, depending on $\mathcal{H}$ was naturally constructed in an especially intricate way: to catch as much primes of the form $\{n + h_i\}_{i=1}^k$ as possible. Thus we cannot expect it to be so extremely sensible for primes of the form (14.38) as well. However, we can justifiably expect it to be as effective as the simplest uniform random choice, that is, to capture for any given $h \in [1, H] \setminus \mathcal{H}$ a prime value for $n + h$ with probability

$$\frac{1}{\log N}. \tag{14.39}$$

An evaluation of the corresponding weighted sum is again possible. It reveals that even if (14.39) is not exactly true, but the correct answer is near to it; using the random choice corresponding to $\{w(n)\}_{N<n\le 2N} = a(n)/A$, we obtain for $h_0 \in [1, H] \setminus \mathcal{H}$, $\mathcal{H}^0 = \mathcal{H} \cup h_0$, $k, N \to \infty$:

$$\begin{aligned} P(n + h_0 \text{ is prime}) &:= E(\mathcal{H}; h_0) \\ &:= \sum_{N<n\le 2N} w(n)\chi^*(n + h_0) \sim \frac{\mathfrak{S}(\mathcal{H}^0)}{\mathfrak{S}(\mathcal{H})} \cdot \frac{1}{\log N}, \end{aligned} \tag{14.40}$$

and it follows from the theorem of Gallagher (see (11.18)) that the fraction $\mathfrak{S}(\mathcal{H}^0)/\mathfrak{S}(\mathcal{H})$ is one if averaged for all $\mathcal{H} \subset [1, H]$, $|\mathcal{H}| = k$ and $h_0 \in [1, H] \setminus \mathcal{H}$.

In this way it is really enough to choose

$$H = (1 + c)\varepsilon_0(k) \log N \qquad (c > 0 \text{ arbitrary, fixed}) \tag{14.41}$$

in order to get more than $\varepsilon_0(k)$ primes among $\{n + h\}_{h\in[1,H]\setminus\mathcal{H}}$ in average; that is, to obtain more than 1 prime among

$$\{n + h\}_{h=1}^{H} \tag{14.42}$$

in "average". This proves (13.6), if k is chosen enough large, $N > N_0(k)$.

Remark. If instead of (14.41) we choose

$$H = \big(\nu + (1 + c)\varepsilon_0(k)\big) \log N \qquad (c > 0 \text{ arbitrary fixed}) \tag{14.43}$$

we obtain at least $\nu + 1$ primes for some $n \in (N, 2N]$ in some interval

$$(n, n + H] \tag{14.44}$$

thereby showing as an extension of $\Delta_1 = 0$,

$$\Delta_\nu \le \nu - 1. \tag{14.45}$$

§15. Ideas of the proof of $\boldsymbol{\Delta_1 = 0}$. Main Propositions

In order to see some of the technical details of proofs of (13.6) and (13.10) we formulate the following two basic propositions, which are needed for the evaluation of

$$A(\mathcal{H}) = \sum_{N<n\le 2N} a_3(n) \tag{15.1}$$

and

$$S(\mathcal{H}, h_0) = \sum_{N<n\le 2N} a_3(n)\chi^*(n+h_0) \tag{15.2}$$

for any $h_0 \in [1, H]$, with $a_3(n) = \Lambda_R^2(n; \mathcal{H}, \ell)$ defined in (14.28).

In the propositions below we suppose that k and ℓ, further $\varepsilon > 0$ are fixed constants, implicit constants may depend on k, ℓ and ε. Let

$$|\mathcal{H}| = k, \quad \mathcal{H}^0 = \mathcal{H} \cup \{h_0\}, \qquad h_0 \le H \le R \le N, \quad 0 \le \ell \le k, \quad R, N \to \infty, \tag{15.3}$$

$$B(k,\ell) := \binom{2\ell}{\ell} \frac{(\log R)^{k+2\ell}}{(k+2\ell)!} N, \quad \Theta = \frac{\log R}{\log N}. \tag{15.4}$$

Proposition 19. *If $R \ll N^{1/2}(\log N)^{-16k}$, then*

$$A := \sum_{M<n\le 2N} \Lambda_R^2(n; \mathcal{H}, \ell) = (\mathfrak{S}(\mathcal{H}) + o(1))B(k,\ell). \tag{15.5}$$

Proposition 20. *If $R \ll N^{\vartheta/2-\varepsilon}$ with an arbitrary fixed $\varepsilon > 0$, then*

(15.6)

$$\sum_{N<n\le 2N} \Lambda_R^2(n; \mathcal{H}, \ell)\chi^*(n+h_0) = \begin{cases} \frac{(\mathfrak{S}(\mathcal{H}^0)+o(1))B(k,\ell)}{\log N} & \textit{if } h_0 \notin \mathcal{H}, \\ \frac{(\mathfrak{S}(\mathcal{H})+o(1))B(k-1,\ell+1)}{\log N} & \textit{if } h_0 \in \mathcal{H}. \end{cases}$$

Remark. Although the right-hand side of (15.6) looks quite different if $h_0 \in \mathcal{H}$ or $h_0 \notin \mathcal{H}$, the two relations are actually equivalent. If $n + h_0$ is namely a prime and $h_0 \in \mathcal{H}$, then clearly

$$d \mid P_{\mathcal{H}}(n) \Longleftrightarrow d \mid P_{\mathcal{H}\setminus\{h_0\}}(n). \tag{15.7}$$

Taking into account $|\mathcal{H}| = k$, $|\mathcal{H} \setminus \{h_0\}| = k-1$, this implies by $k + \ell = k - 1 + \ell + 1$ the relation

$$\Lambda_R(n; \mathcal{H}, \ell) = \Lambda_R(n; \mathcal{H} \setminus \{h\}, \ell+1) \tag{15.8}$$

and applying the shift $k \to k-1$, $\ell \to \ell+1$ we obtain the case $h_0 \in \mathcal{H}$ from the case $h_0 \notin \mathcal{H}$ or vice versa.

Comparing the right-hand sides of the upper equality in (15.6) with (15.5) we immediately see (14.40). Further, dividing the lower equality in (15.6) by (15.5) we obtain for $h_i \in \mathcal{H}$ (cf. (14.6)) the claim (14.30):

$$E(\mathcal{H}; h_i) = \frac{\sum\limits_{N<n\le 2N} a(n)\chi^*(n+h_i)}{A} \sim \frac{B(k-1,\ell+1)}{B(k,\ell)\log N} \tag{15.9}$$

$$= \frac{\binom{2(\ell+1)}{\ell+1}}{\binom{2\ell}{\ell}} \cdot \frac{1}{k+2\ell+1} \cdot \frac{\log R}{\log N} \sim \frac{4\Theta}{k} = \frac{2\vartheta - 4\varepsilon}{k}$$

if we suppose $\ell = o(k)$, $\ell \to \infty$ (cf. (14.29)).

Remark. Since we have

$$\frac{\binom{2(\ell+1)}{\ell+1}}{\binom{2\ell}{\ell}} \cdot \frac{1}{k+2\ell+1} = \frac{2(2\ell+1)}{(\ell+1)(k+2\ell+1)}, \tag{15.10}$$

it is easy to calculate that the maximum of (15.10) is reached if

$$\frac{k}{2\ell+1} + 2\ell + 1 \tag{15.11}$$

is minimal, that is if

$$\ell = \left[\frac{\sqrt{k}-1}{2}\right] \quad \text{or} \quad \ell = \left\lceil\frac{\sqrt{k}-1}{2}\right\rceil, \tag{15.12}$$

which leads for (15.10) to the maximum value (cf. (14.32))

$$\frac{4}{k}(1-\varepsilon_0(k)), \quad \varepsilon_0(k) \sim \frac{2}{\sqrt{k}} \quad (k \to \infty). \tag{15.13}$$

The actual dependence on k has great importance if we want to show results beyond (13.6) as given in (13.7), for example. In this case we have to choose sets of cardinality $k = k(N) \to \infty$ as $N \to \infty$. The proof of (13.7) is far from being an immediate extension of the proof of (13.6) and requires many new ideas also, as well as the successful treatment of some serious technical difficulties. It might be, however, worthwhile to note that in order to show it we need to work with all possible sets $\mathcal{H}$ of size K with

$$\mathcal{H} \in [1, H] = \left[1, \frac{C \log N}{K}\right], \tag{15.14}$$

where

$$K = \frac{c\sqrt{\log N}}{(\log\log N)^2}. \tag{15.15}$$

It is also an important change in the proof that the actual weight function, used in the proof of (13.7), although similar to (14.28), was far more elaborate. The actual choice is too complicated to state here; but it is closer to

$$a_4(n) = \Big(\sum_{\substack{\mathcal{H}\subset[1,H]\\ |\mathcal{H}|=k}} \Lambda_R(n;\mathcal{H},\ell) \Big)^2 \tag{15.16}$$

than to $a_3(\mathcal{H}, n)$ or to the function

$$\sum_{\substack{\mathcal{H}\subset[1,H]\\ |\mathcal{H}|=k}} \Lambda_R^2(n;\mathcal{H},\ell) = \sum_{\substack{\mathcal{H}\subset[1,H]\\ |\mathcal{H}|=k}} a_3(\mathcal{H},n). \tag{15.17}$$

We remark here that we actually used in the works of Goldston, Pintz, Yıldırım ([34]) and Goldston, Motohashi, Pintz, Yıldırım ([32]) the above functions (15.16), (15.17) and a further one, which is of type

$$a_5(n) = \Big(\sum_{\ell=0}^{L} \sum_{\substack{\mathcal{H}\subset[1,H]\\ |\mathcal{H}|=k}} b_\ell \Lambda_R(n;\mathcal{H},\ell) \log^{-\ell} R \Big)^2 \tag{15.18}$$

with some constants b_ℓ, depending on ℓ, k and L.

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