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The diaphony of a class of infinite sequences

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Abstract.

Let α be an irrational number with diophantine approximation properties and let $\ell(z)$ be a logarithmic-like function. We study the diaphony F_N of the sequence $(\alpha n+\ell(n))_{n\geq 1}$. As an example of our result, we show that if α has the bounded partial quotients of the continued fraction expansion and β is non-zero real, then the sequence $(x_n)_{n\geq 1} = (\alpha[(n+1)/2]+(-1)^{n+1}\beta\log([(n+1)/2]))_{n\geq 1}$ satisfies $N^{-\frac{3}{4}-\varepsilon} \ll F_N(x_n) \ll N^{-\frac{2}{3}}$ for any $0 < \varepsilon < 1/4$. In our proof, Atkinson's saddle-point lemma is very usefull.

§1. Introduction

The fractional part $\{x\}$ of a real number x is defined by $\{x\} = x-[x]$, where $[x]$ denotes the integral part of x. Let $(x_n)_{n\geq 1}$ be an infinite sequence of real numbers. For an interval $I \subset [0,1)$, let $A(I,N,x_n)$ be the number of terms x_n, $1 \leq n \leq N$, for which $\{x_n\} \in I$. The sequence $(x_n)_{n\geq 1}$ is called uniformly distributed mod 1 if for any interval $I \subset [0,1)$, $\lim_{N\to\infty} N^{-1}A(I,N,x_n) = |I|$, where $|I|$ denotes the length of I .

The discrepancy $D_N(x_n)$ of the sequence $(x_n)_{n\geq 1}$ is defined by

$$D_N(x_n) = \sup_{I\subset[0,1)} \left| \frac{A(I,N,x_n)}{N} - |I| \right|$$

(see [8] and [2]).

The diaphony $F_N(x_n)$ of the sequence $(x_n)_{n\geq 1}$ was defined by Zinterhof [13] as

$$F_N(x_n) = \left(2\sum_{h=1}^{\infty} \frac{1}{h^2} \left| \frac{1}{N} \sum_{n=1}^{N} e(hx_n) \right|^2 \right)^{1/2},$$

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where $e(\theta) = e^{2\pi i\theta}$.

It is known that for any real sequence $(x_n)_{n\geq 1}$

$$3^{-1/2}\pi D_N(x_n)^{3/2} \leq F_N(x_n) \leq 11^{1/2} D_N(x_n)^{1/2}$$

(see [10], [8], and [2]).

The sequence $(x_n)_{n\geq 1}$ is uniformly distributed mod 1 if and only if $\lim_{n\to\infty} D_N(x_n) = 0$, or, equivalently, $\lim_{n\to\infty} F_N(x_n) = 0$ (see [8]).

In [11], [3], and[4], we studied the discrepancy of the sequence $(\alpha n + f(n))_{n\geq 1}$ for the irrational number α with diophantine approximation properties and for the logarithmic-like function $f(x)$. We gave the upper bounds and the lower bounds for the discrepancy, and showed that there exists the gap between the uppler bound and lower bound for the discrepancy. For example, if α has the bounded partial quotients of the continued fraction expansion and β is non-zero real number, then the sequence $(\alpha n + \beta \log n)_{n\geq 1}$ satisfies

$$\frac{1}{N^{\frac{3}{4}}} \ll D_N(\alpha n + \beta \log n) \ll \frac{\log N}{N^{\frac{2}{3}}}.$$

In this paper, we study estimates for the diaphony of similar sequences. For example, if α has the bounded partial quotients of the continued fraction expansion and β is non-zero real, then the sequence $(x_n)_{n\geq 1} = (\alpha[(n+1)/2] + (-1)^{n+1}\beta\log([(n+1)/2]))_{n\geq 1}$ satisfies

$$\frac{1}{N^{\frac{3}{4}+\varepsilon}} \ll F_N(x_n) \ll \frac{1}{N^{\frac{2}{3}}}$$

for any $0 < \varepsilon < 1/4$. This result is deduced from Theorem 4.

§2. Results

We need some estimates for exponential sums. The following formulas are sharper than that in Theorem 1.1 of [4].

Theorem 1. *Suppose that $\ell(z)$ is a function of complex variable z that satisfies the following conditions:*

(i) *$\ell(x)$ is real and differentiable for $x > 0$,*
(ii) *there exists an integer n_0 such that $1 \leq n_0 < N$ and $\ell''(x) < 0$ for $x \geq n_0$,*
(iii) $\lim_{x\to\infty} \ell'(x) = 0$,
(iv) *$\ell(z)$ is analytic for $x > 0$, $|z - x| \leq x/2$,*
(v) *$|\ell'(z)| \ll \ell'(x)$ and $|\ell''(z)| \gg \ell'(x)/x$ for* $\mathrm{Re}\, z = x > 0$,

(vi) *for any* $\delta \geq 0$, $|\ell''(x)|^{1/2+\delta} \ll \ell'(x) \ll |\ell''(x)|^{1/2}$ *for* $x \geq n_0$, $|z-x| \leq x/2$,
(vii) *there exists* $\rho < 0$ *such that* $\ell'(2x) \leq 2^\rho \ell'(x)$ *for* $x \geq n_0$.

Let α *be an irrational number. For a positive integer* h *let us define* c_h *by* $\ell'(c_h) = (k-\alpha h)/h$, *where* $k = [\alpha h] + 1$.
If $h\ell'(n_0) > 1$ *and* $n_0 \leq c_h \leq N$, *then for any* $\varepsilon > 0$

$$\sum_{n_0 \leq n \leq N} e(h(\alpha n + \ell(n))) = h^{-1/2}|\ell''(c_h)|^{-1/2} e(\theta_h)$$

$$+ O\left(\frac{1}{k-\alpha h}\right) + O\left(\frac{h^{-1/2+\varepsilon}}{(k-\alpha h)^{1+\varepsilon}}\right) + O\left(h^{1/2}\log(\ell'(n_0)h)\right),$$

where $\theta_h = h\ell(c_h) + (\alpha h - k)c_h - 1/8$.

Let us define c'_h *by* $\ell'(c'_h) = (\alpha h - k')/h$, *where* $k' = [\alpha h]$.
If $h\ell'(n_0) > 1$ *and* $n_0 \leq c'_h \leq N$, *then for any* $\varepsilon > 0$

$$\sum_{n_0 \leq n \leq N} e(h(\alpha n - \ell(n))) = h^{-1/2}|\ell''(c'_h)|^{-1/2} e(\theta'_h)$$

$$+ O\left(\frac{1}{\alpha h - k'}\right) + O\left(\frac{h^{-1/2+\varepsilon}}{(\alpha h - k')^{1+\varepsilon}}\right) + O\left(h^{1/2}\log(\ell'(n_0)h)\right),$$

where $\theta'_h = -h\ell(c'_h) + (\alpha h - k')c'_h + 1/8$.

If $h\ell'(n_0) > 1$ *and either* $N < c_h$ *or* $N < c'_h$, *then*

$$\sum_{n_0 \leq n \leq N} e(h(\alpha n \pm \ell(n))) = O\left(h^{1/2}\log(\ell'(n_0)h)\right).$$

Theorem 2. *Suppose that the irrational number* α *has the bounded partial quotients of the continued fraction expansion, that* $\ell(z)$ *is a function satisfying the conditions* (i)–(vii) *of Theorem 1, and that* $N^{-\frac{6}{5}} < \ell'(N) < 1$. *Then*

$$F_N(\alpha n \pm \ell(n)) \ll \frac{1}{N\ell'(N)^{\frac{1}{3}}}.$$

By the same reasoning as in the proof of Theorem 2, we obtain the following theorem.

Theorem 3. *Suppose that the irrational* α *is of type* η , *that* $\ell(z)$ *is a function satisfying the conditons* (i)–(vii) *of Theorem 1, and that* $N^{-\frac{4\eta+2}{6\eta-1}} < \ell'(N) < 1$. *Then for any* $\varepsilon > 0$

$$F_N(\alpha n + \ell(n)) \ll \frac{1}{N\ell'(N)^{1-\frac{2}{2\eta+1}+\varepsilon}}.$$

Theorem 4. *Suppose that the irrational number α has the bounded partial quotients of the continued fraction expansion, and that $\ell(z)$ satisfies the conditons* (i)–(vii) *of Theorem 1. Let $x_n = \alpha[(n+1)/2] + (-1)^{n+1}\ell([(n+1)/2])$, $n = 1, 2, \ldots$. Then for any $0 < \varepsilon < 1/4$*

$$\frac{1}{N\ell'(N/2)^{\frac{1}{4}-\varepsilon}} \ll F_N(x_n) \ll \frac{1}{N\ell'(N/2)^{\frac{1}{3}}}. \tag{1}$$

§3. Lemmas

We use the following saddle-point lemma of Atkinson [1].

Lemma 1 ([1, Lemma 1]). *Let $f(z)$, $\varphi(z)$ be the two functions of the complex variable z and $[a, b]$ a real interval such that*

(i) *for $a \le x \le b$, the function $f(x)$ is real and $f''(x) < 0$,*
(ii) *for a certain positive differentiable function $\mu(x)$, defined on $a \le x \le b$, $f(z)$ and $\varphi(z)$ are analytic for $a \le x \le b$, $|z - x| \le \mu(x)$,*
(iii) *there exist positive functions $F(x)$, $\Phi(x)$ defined on $[a, b]$ such that*

$$\varphi(z) \ll \Phi(x), \quad f'(z) \ll F(x)\mu^{-1}(x), \quad (f''(z))^{-1} \ll \mu^2(x)F^{-1}(x)$$

for $a \le x \le b$, $|z - x| \le \mu(x)$.

Let r be any real number, and if $f'(x) + r$ has a zero in $[a, b]$ denote it x_0. Let the values of $f(x)$, $\varphi(x)$, and so on, at a, x_0, and b be characterized by the suffixes a, 0, and b respectively. Then

$$\begin{aligned}\int_a^b \varphi(x)e(f(x) + rx)dx &= \varphi_0(|f_0''|)^{-1/2}e(f_0 + rx_0 - 1/8)\\ &+ O\left(\int_a^b \Phi(x)\exp[-C|r|\mu(x) - CF(x)](dx + |d\mu(x)|)\right)\\ &+ O\left(\Phi_0\mu_0F_0^{-3/2}\right)\\ &+ O\left(\Phi_a(|f_a' + r| + (|f_a''|)^{1/2})^{-1}\right) + O\left(\Phi_b(|f_b' + r| + (|f_b''|)^{1/2})^{-1}\right).\end{aligned}$$

If $f'(x) + r$ has no zero for $a \le x \le b$, then the terms involving x_0 are to be omitted.

The following lemma was given by Hardy and Littlewood (see [6] and [7]). The lower bounds was stated without proof by them. Proofs for the lower bounds can be found in Haber and Osgood [5].

Lemma 2. *If α is an irrational number, and the partial quotients of the continued fraction expansion are bounded by a fixed number M, then*

$$C_1 K \log K < \sum_{n=1}^{K} \|n\alpha\|^{-1} < C_2 K \log K$$

and for $t > 1$

$$C_3 K^t < \sum_{n=1}^{K} \|n\alpha\|^{-t} < C_4 K^t,$$

where $\|x\|$ denotes the distance between x and the nearest integer and the C's depend only on M and on t. The left-hand inequalities holds without any hypothesis on the partial quotients of α.

§4. Proof of Theorem 1

We consider $\sum_{1\le n\le N} e(h(\alpha n + \ell(n)))$. In the same way, the results for $\sum_{1\le n\le N} e(h(\alpha n - \ell(n)))$ is obtained.

Suppose that $h\ell'(n_0) > 1$ and $n_0 \le c_h \le N$. Let $k = [\alpha h] + 1$. We write

$$\begin{aligned} &\sum_{1\le n\le N} e(h(\alpha n + \ell(n))) \\ &= \sum_{1\le n< n_0} e(h(\alpha n + \ell(n))) + \sum_{n_0\le n\le 2c_h} e(h(\alpha n + \ell(n))) + E \\ &= O(1) + S + E, \end{aligned} \tag{2}$$

where $S = \sum_{n_0\le n\le 2c_h} e(h(\alpha n + \ell(n)))$, and $E = \sum_{1\le n\le N} e(h(\alpha n + \ell(n))) - \sum_{1\le n\le 2c_h} e(h(\alpha n + \ell(n)))$.

From (ii) and (iii), it follows that

$$0 < \ell'(2c_h) \le \ell'(c_h) = \frac{k - \alpha h}{h} < 1/h,$$

so that

$$0 < h\ell'(2c_h) < 1.$$

To S, we apply Lemma 4.7 of [12] with $f(x) = h\alpha x + h\ell(x)$, $a = n_0$, $b = 2c_h$, $A = \alpha h + h\ell'(2c_h)$, $B = \alpha h + h\ell'(n_0)$, and $\eta = h\ell'(2c_h)$. Since

$k-1 < A-\eta = \alpha h < k$ and $(B+\eta)-(A-\eta) > h\ell'(n_0) > 1$, we have

$$\begin{aligned} S = & \int_{n_0}^{2c_h} e(h\ell(x) + (\alpha h - k)x)dx \\ & + \sum_{k+1 \le \nu \le \alpha h + h\ell'(n_0)} \int_{n_0}^{2c_h} e(h\ell(x) + (\alpha h - \nu)x)dx \\ & + \sum_{\alpha h + h\ell'(n_0) < \nu \le L} \int_{n_0}^{2c_h} e(h\ell(x) + (\alpha h - \nu)x)dx \\ & + O(\log(\ell'(n_0)h + 2)), \end{aligned} \tag{3}$$

where $L = B + \eta = \alpha h + h\ell'(n_0) + h\ell'(2c_h)$. Let

$$S_1 = \sum_{k+1 \le \nu \le \alpha h + h\ell'(n_0)} \int_{n_0}^{2c_h} e(h\ell(x) + (\alpha h - \nu)x)dx,$$

$$S_2 = \sum_{\alpha h + h\ell'(n_0) < \nu \le L} \int_{n_0}^{2c_h} e(h\ell(x) + (\alpha h - \nu)x)dx.$$

We define b_ν by $\ell'(b_\nu) = (\nu - \alpha h)/h$ for $\alpha h < \nu \le \alpha h + h\ell'(n_0)$. We note that the zero b_ν of $h\ell'(x) + \alpha h - \nu$ satisfies $n_0 \le b_\nu \le c_h$. To $\int_{n_0}^{2b_\nu} e(h\ell(x) + (\alpha h - \nu)x)dx$, applying Lemma 1 with $a = n_0$, $b = 2b_\nu$, $f(z) = h\ell(z)$, $r = \alpha h - \nu$, $\mu(x) = x/2$, $\varphi(z) = \Phi(x) = 1$, $F(x) = \frac{1}{2}hx\ell'(x)$, we find that for $\alpha h < \nu \le \alpha h + h\ell'(n_0)$,

$$\begin{aligned} & \int_{n_0}^{2b_\nu} e(h\ell(x) + (\alpha h - \nu)x)dx \\ = & h^{-1/2}|\ell''(b_\nu)|^{-1/2} e\left(h\ell(b_\nu) + (\alpha h - \nu)b_\nu - 1/8\right) \\ & + O\left(\int_{n_0}^{2b_\nu} \exp\left(\frac{C}{2}(\alpha h - \nu)x - \frac{C}{2}hx\ell'(x)\right) dx\right) \\ & + O\left(h^{-3/2} b_\nu^{-1/2} \ell'(b_\nu)^{-3/2}\right) \\ & + O\left(\frac{1}{|h\ell'(n_0) + \alpha h - \nu|}\right) + O\left(\frac{1}{|h\ell'(2b_\nu) + \alpha h - \nu|}\right). \end{aligned} \tag{4}$$

For $\alpha h < \nu \le \alpha h + h\ell'(n_0)$, we obtain

$$\int_{n_0}^{2b_\nu} \exp\left(\frac{C}{2}(\alpha h - \nu)x - \frac{C}{2}hx\ell'(x)\right) dx \ll \frac{1}{\nu - \alpha h}. \tag{5}$$

From (vii), It follows that $|h\ell'(2b_\nu)+\alpha h-\nu|^{-1} \le (1-2^\rho)^{-1}(\nu-\alpha h)^{-1}$. Let $\varepsilon > 0$ be fixed. We define δ by $\varepsilon = 2\delta/(1+2\delta)$. Since $|\ell''(x)|^{1/2+\delta} \ll \ell'(x)$ for $x \ge n_0$ by (vi), from (v) we derive that $b_\nu \gg \ell'(b_\nu)/|\ell''(b_\nu)| \gg \ell'(b_\nu)^{1-\frac{2}{1+2\delta}} = \ell'(b_\nu)^{-1+2\varepsilon}$ and so $h^{-3/2}b_\nu^{-1/2}\ell'(b_\nu)^{-3/2} \ll h^{-1/2+\varepsilon}(\nu-\alpha h)^{-1-\varepsilon}$. Hence from (4) and (5), it follows that for $\alpha h < \nu \le \alpha h + h\ell'(n_0) - 2$, (note that this range is void if $h\ell'(n_0) \le 2$)

$$\begin{aligned}&\int_{n_0}^{2b_\nu} e(h\ell(x) + (\alpha h - \nu)x)dx\\ &= h^{-1/2}|\ell''(b_\nu)|^{-1/2} e\left(h\ell(b_\nu) + (\alpha h - \nu)b_\nu - 1/8\right)\\ &+ O\left(\frac{1}{\nu - \alpha h}\right) + O\left(\frac{h^{-1/2+\varepsilon}}{(\nu-\alpha h)^{1+\varepsilon}}\right) + O\left(\frac{1}{h\ell'(n_0) + (\alpha h - \nu)}\right). \end{aligned} \tag{6}$$

For $\alpha h + h\ell'(n_0) - 2 < \nu \le \alpha h + h\ell'(n_0)$, we obtain the formula (6) with the last term replaced by $O(h^{-1/2})$.

In order to estimate $\int_{2b_\nu}^{2c_h} e(h\ell(x) + (\alpha h - \nu)x)dx$ for $\alpha h < \nu \le \alpha h + h\ell'(n_0)$, we also use Lemma 1. We note that $b_\nu \notin [2b_\nu, 2c_h]$. Since $h\ell'(2c_h) + \alpha h - \nu < h\ell'(2b_\nu) + \alpha h - \nu \le h2^\rho\ell'(b_\nu) + \alpha h - \nu = (2^\rho - 1)(\nu - \alpha h) < 0$, we have

$$\begin{aligned}&\int_{2b_\nu}^{2c_h} e(h\ell(x) + (\alpha h - \nu)x)dx\\ &\ll \int_{2b_\nu}^{2c_h} \exp\left(-\frac{C}{2}(\nu - \alpha h)x - \frac{C}{2}hx\ell'(x)\right)dx\\ &\quad + |h\ell'(2b_\nu) + \alpha h - \nu|^{-1} + |h\ell'(2c_h) + \alpha h - \nu|^{-1}\\ &\ll \frac{1}{\nu - \alpha h}. \end{aligned} \tag{7}$$

By (6) and (7), for $\alpha h < \nu \le \alpha h + h\ell'(n_0) - 2$ we have

$$\begin{aligned}&\int_{n_0}^{2c_h} e(h\ell(x) + (\alpha h - \nu)x)dx\\ &= h^{-1/2}|\ell''(b_\nu)|^{-1/2} e\left(h\ell(b_\nu) + (\alpha h - \nu)b_\nu - 1/8\right)\\ &+ O\left(\frac{1}{\nu - \alpha h}\right) + O\left(\frac{h^{-1/2+\varepsilon}}{(\nu-\alpha h)^{1+\varepsilon}}\right) + O\left(\frac{1}{h\ell'(n_0) + \alpha h - \nu}\right), \end{aligned} \tag{8}$$

and we obtain the formula (8) with the last term replaced by $O(h^{-1/2})$ for $\alpha h + h\ell'(n_0) - 2 < \nu \le \alpha h + h\ell'(n_0)$. Since $|\ell''(b_\nu)|^{-1/2} \ll \ell'(b_\nu)^{-1} =$

$\frac{h}{\nu-\alpha h}$ by (v), from (8) we deduce that

$$\begin{aligned}
S_1 =& h^{-1/2} \sum_{k+1\le\nu\le\alpha h+h\ell'(n_0)} |\ell''(b_\nu)|^{-1/2} e(h\ell(b_\nu)+(\alpha h-\nu)b_\nu+1/8) \\
&+O\left(\sum_{k+1\le\nu\le\alpha h+h\ell'(n_0)} \frac{1}{\nu-\alpha h}\right) \\
&+O\left(\sum_{k+1\le\nu<\alpha h+h\ell'(n_0)-2} \frac{1}{h\ell'(n_0)+\alpha h-\nu}\right) \\
&+O\left(\sum_{k+1\le\nu\le\alpha h+h\ell'(n_0)} \frac{h^{-1/2+\varepsilon}}{(k-\alpha h)^{1+\varepsilon}}\right)+O(h^{-1/2}) \\
\ll& h^{1/2} \sum_{k+1\le\nu\le\alpha h+h\ell'(n_0)} \frac{1}{\nu-\alpha h}+\log(\ell'(n_0)h)+O\left(h^{-1/2}\right) \\
=& O\left(h^{1/2}\log(\ell'(n_0)h)\right). \qquad (9)
\end{aligned}$$

Next, we estimate S_2. Since $0 < L-\alpha h-\ell'(n_0)h = h\ell'(2c_h) < h\ell'(c_h) = k-\alpha h < 1$, S_2 is the sum of at most 1 term. If there exists ν such that $\alpha h+\ell'(n_0)h < \nu \le L$, then $\ell'(b_\nu) = \frac{\nu-\alpha h}{h} > \ell'(n_0)$. Since $\ell'(x)$ is decreasing, $b_\nu < n_0$, and so the zero b_ν of $h\ell'(x)+(\alpha h-\nu)$ is not in $[n_0, 2c_h]$. Applying Lemma 1, we have

$$\begin{aligned}
S_2 =& \int_{n_0}^{2c_h} e(h\ell(x)+(\alpha h-\nu)x)dx \\
\ll& \int_{n_0}^{2c_h} \exp\left(-\frac{C}{2}(\alpha h-\nu)x-Ch\right)dx \\
&+|h\ell''(n_0)|^{-1/2}+|h\ell'(2c_h)+\alpha h-\nu|^{-1} \\
\ll& \frac{1}{\nu-\alpha h}+h^{-1/2}+h^{-1} \\
\ll& \frac{1}{h\ell'(n_0)}+h^{-1/2} \ll h^{-1/2} \qquad (10)
\end{aligned}$$

From (3), (9) and (10), it follows that

$$\begin{aligned}
S =& h^{-1/2}|\ell''(c_h)|^{-1/2} e\left(h\ell(c_h)+(\alpha h-k)c_h-1/8\right) \\
&+O\left(\frac{1}{k-\alpha h}\right)+O\left(\frac{h^{-1/2+\varepsilon}}{(k-\alpha h)^{1+\varepsilon}}\right)+O\left(h^{1/2}\log(\ell'(n_0)h)\right). \qquad (11)
\end{aligned}$$

Lastly we estimate the term E. Suppose that $2c_h \leq N$. Applying Lemma 4.7 of [12], we obtain

$$E = \sum_{2c_h < n \leq N} e(h(\alpha n + \ell(n)))$$
$$= \sum_{A-\eta<\nu<B+\eta} \int_{2c_h}^{N} e(h\ell(x) + (\alpha h - \nu)x)dx + O(\log(B - A + 2)),$$

where $A = h\alpha + h\ell'(N)$, $B = h\alpha + h\ell'(2c_h)$, and $\eta = h\ell'(N)/2 < 1/2$. Since $(B+\eta)-(A-\eta) = h\ell'(2c_h) < h\ell'(c_h) < 1$, the number of terms in the sum is at most one. Since $A-\eta = \alpha h + h\ell'(N)/2 < \alpha h + (k-\alpha h)/2 < k$ and $k - 1 \leq A - \eta$, if there exists the term, then $\nu = k$. Since $g'(x) = h\ell'(x) + (\alpha h - k)$ has no zero in $[2c_h, N]$ and $|h\ell'(N) + \alpha h - k| > |h\ell'(2c_h) + \alpha h - k| = -h\ell'(2c_h) - \alpha h + k > (1 - 2^\rho)(k - \alpha h)$, applying Lemma 1 with $f(x) = h\ell(x)$, $r = \alpha h - k$, $\mu(x) = x/2$, $\varphi(x) = \Phi(x) = 1$, $F(x) = \frac{1}{2}hx\ell'(x)$, we have

$$\int_{2c_h}^{N} e(h\ell(x) + (\alpha h - k)x)dx$$
$$\ll \int_{2c_h}^{N} \exp\left(-\frac{C}{2}(k - \alpha h)x\right) dx$$
$$+ |h\ell'(2c_h) + (\alpha h - k)|^{-1} + |h\ell'(N) + (\alpha h - k)|^{-1}$$
$$\ll \frac{1}{k - \alpha h}.$$

Hence

$$E = O\left(\frac{1}{k - \alpha h}\right). \tag{12}$$

On the other hand, if $c_h \leq N < 2c_h$, then

$$E = -\sum_{N<n\leq 2c_h} e(h(\alpha n + \ell(n)) = -\sum_{N+1/2<n\leq 2c_h} e(h(\alpha n + \ell(n)).$$

Similarly, we have the same estimate as (12). Combining this with (11), we get the conclusion.

In the case $N < c_h$, the conclusion is obtained in the same way as estimate of S_1.

§5. Proof of Theorem 2

We consider $F_N(\alpha n + \ell(n))$. In the same way, the estimate for $F_N(\alpha n - \ell(n))$ is obtained. Put $x_n = \alpha n + \ell(n)$. We write

$$\begin{aligned}\sum_{n=1}^{N} e(hx_n) = &\sum_{1\le h\le \ell'(n_0)^{-1}} + \sum_{\ell'(n_0)^{-1}<h\le \ell'(N)^{-\frac{2}{3}}} + \sum_{\ell'(N)^{-\frac{2}{3}}<h<\ell'(N)^{-1}} \\ &+ \sum_{\ell'(N)^{-1}\le h\le A\ell'(N)^{-1}} + \sum_{A\ell'(N)^{-1}<h} \\ =&S_1 + S_2 + S_3 + S_4 + S_5,\end{aligned}$$

say, $A > 1$ will be suitably determined. We get the trivial estimate

$$S_1 = O(1). \tag{13}$$

Let $h > \ell'(n_0)^{-1}$, and let $k = [\alpha h] + 1$. Since $(k - \alpha h)/h = \ell'(c_h) \ll |\ell''(c_h)|^{1/2}$, Theorem 1 with $\varepsilon = 1/2$ implies that

$$\sum_{n=1}^{N} e(hx_n) \ll \frac{h^{1/2}}{k-\alpha h} + \frac{1}{(k-\alpha h)^{3/2}} + h^{1/2}\log(\ell'(n_0)h). \tag{14}$$

Since α has the bounded partial quotients of the continued fraction expansion if and only if α is of constant type, $h^{-1} \ll \|\alpha h\|$ for all integers h (see [9]). Hence $h^{-1} \ll k - \alpha h$, and so $\frac{h^{1/2}}{k-\alpha h} \gg \frac{1}{(k-\alpha h)^{3/2}}$. Therefore from (14)

(15)

$$\sum_{n=1}^{N} e(hx_n) \ll \frac{h^{1/2}}{k-\alpha h} + h^{1/2}\log(\ell'(n_0)h) \le \frac{h^{1/2}}{\|\alpha h\|} + h^{1/2}\log(\ell'(n_0)h).$$

From (15), we derive

$$\begin{aligned} S_2 = &\sum_{\ell'(n_0)^{-1}<h\le \ell'(N)^{-\frac{2}{3}}} \frac{1}{h^2}\left|\sum_{n=1}^{N} e(hx_n)\right|^2 \\ \ll &\sum_{\ell'(n_0)^{-1}<h\le \ell'(N)^{-\frac{2}{3}}} \frac{1}{h^2}\left(\frac{h^{1/2}}{\|\alpha h\|} + h^{1/2}\log(\ell'(n_0)h)\right)^2 \\ \ll &\sum_{\ell'(n_0)^{-1}<h\le \ell'(N)^{-\frac{2}{3}}} \frac{1}{h\|\alpha h\|^2} + \sum_{\ell'(n_0)^{-1}<h\le \ell'(N)^{-\frac{2}{3}}} \frac{\log^2(\ell'(n_0)h)}{h}. \end{aligned} \tag{16}$$

Since $\sum_{j=1}^{h} \|\alpha j\|^{-2} \ll h^2$ by Lemma 2, applying Abel's summation formula, we get

$$\sum_{\ell'(n_0)^{-1}<h\le\ell'(N)^{-\frac{2}{3}}} \frac{1}{h\|\alpha h\|^2} \ll \sum_{h=1}^{[\ell'(N)^{-\frac{2}{3}}]} 1 + \ell'(N)^{\frac{2}{3}}\ell'(N)^{-\frac{4}{3}} \ll \ell'(N)^{-\frac{2}{3}}.$$

Hence from (16) it follows that

$$S_2 \ll \ell'(N)^{-\frac{2}{3}} + \log^3(\ell'(N)^{-\frac{2}{3}}) \ll \ell'(N)^{-\frac{2}{3}}. \tag{17}$$

If $h/(k-\alpha h) \le 1/\ell'(N)$, then by (15)

$$\left|\sum_{h=1}^{N} e(hx_n)\right| \ll \frac{h^{1/2}}{k-\alpha h} + h^{1/2}\log(\ell'(n_0)h) \ll \frac{h^{-1/2}}{\ell'(N)} + h^{1/2}\log(\ell'(n_0)h).$$

On the other hand, if $1/\ell'(N) < h/(k-\alpha h)$, then $N < c_h$, and so Theorem 1 implies

$$\left|\sum_{h=1}^{N} e(hx_n)\right| \ll h^{1/2}\log \ell'(n_0)h.$$

Hence

$$\left|\sum_{h=1}^{N} e(hx_n)\right| \ll \frac{h^{-1/2}}{\ell'(N)} + h^{1/2}\log \ell'(n_0)h.$$

Therefore

$$S_3 = \sum_{\ell'(N)^{-\frac{2}{3}}<h<\ell'(N)^{-1}} \frac{1}{h^2}\left|\sum_{n=1}^{N} e(hx_n)\right|^2 \ll \sum_{\ell'(N)^{-\frac{2}{3}}<h<\ell'(N)^{-1}} \frac{1}{h^2}\left(\frac{1}{h\ell'(N)^2} + h\log^2(\ell'(n_0)h)\right)$$

$$\ll \ell'(N)^{-\frac{2}{3}} + \log^3\left(\frac{1}{\ell'(N)}\right). \tag{18}$$

We have

$$S_5 = \sum_{A\ell'(N)^{-1}<h} \frac{1}{h^2}\left|\sum_{n=1}^{N} e(hx_n)\right|^2 \ll \sum_{A\ell'(N)^{-1}<h} \frac{N^2}{h^2} \ll \frac{N^2\ell'(N)}{A}. \tag{19}$$

We put $A = N^2\ell'(N)^{\frac{5}{3}}$. Since $N^{-\frac{6}{5}} < \ell'(N)$ by the assumption, $A > 1$ is satisfied. Hence by (19) we have

$$S_5 = \sum_{N^2\ell'(N)^{\frac{2}{3}}<h} \frac{1}{h^2}\left|\sum_{n=1}^{N} e(hx_n)\right|^2 \ll \ell'(N)^{-\frac{2}{3}}. \tag{20}$$

If $\ell'(N)^{-1} \le h$, then $\ell'(c_h) = \frac{k-\alpha h}{h} < \ell'(N)$, or $c_h > N$, and so Theorem 1 yields

$$\sum_{n=1}^{N} e(hx_n) \ll h^{1/2}\log(\ell'(n_0)h).$$

Therefore

$$\begin{aligned} S_4 &= \sum_{\ell'(N)^{-1}\le h\le N^2\ell'(N)^{\frac{2}{3}}} \frac{1}{h^2}\left|\sum_{n=1}^{N} e(hx_n)\right|^2 \\ &\ll \sum_{\ell'(N)^{-1}\le h\le N^2\ell'(N)^{\frac{2}{3}}} \frac{1}{h}\log^2(\ell'(n_0)h) \ll \log^3(N^2\ell'(N)^{\frac{2}{3}}). \end{aligned} \tag{21}$$

From (13), (17), (18) , (20) and (21), it follows that

$$\sum_{h=1}^{\infty} \frac{1}{h^2}\left|\sum_{n=1}^{N} e(hx_n)\right|^2 \ll \ell'(N)^{-\frac{2}{3}} + \log^3\left(\frac{1}{\ell'(N)}\right) + \log^3\left(N^2\ell'(N)^{\frac{2}{3}}\right). \tag{22}$$

Since $N^{-\frac{6}{5}} < \ell'(N)$, we obtain $\frac{1}{\ell'(N)} \ll N^2\ell'(N)^{\frac{2}{3}}$. Hence from (22), we obtain

$$\sum_{h=1}^{\infty} \frac{1}{h^2}\left|\sum_{n=1}^{N} e(hx_n)\right|^2 \ll \ell'(N)^{-\frac{2}{3}} + \log^3 N.$$

Therefore

$$F_N(x_n) \ll \frac{1}{N}\left(\sum_{h=1}^{\infty} \frac{1}{h^2}\left|\sum_{n=1}^{N} e(hx_n)\right|^2\right)^{1/2} \ll \frac{1}{N\ell'(N)^{\frac{1}{3}}} + \frac{\log^{3/2} N}{N}. \tag{23}$$

Since $\ell'(x) \ll (-\ell''(x))^{1/2}$, we get $\frac{-\ell''(x)}{\ell'(x)^2} \gg 1$, and so $\ell'(N) \ll 1/N$. Hence $\ell'(N) \ll \log^{-\frac{9}{2}} N$, and so $\ell'(N)^{-\frac{1}{3}} \gg \log^{3/2} N$. Combining this with (23), we have

$$F_N(x_n) \ll \frac{1}{N\ell'(N)^{\frac{1}{3}}}.$$

§6. Proof of Theorem 4

The upper bound of (1) is got from Theorem 2.

We write $N = 2m$ or $N = 2m+1$. Since α is of constant type, there exists $c > 0$ such that $\|\alpha h\| \geq c/h$ for all positive integer h. Put $A_m = [\sqrt{c/\ell'(m)}]$. Suppose that $1 \leq h \leq A_m$. Let $k = [\alpha h] + 1$ and let $k' = [\alpha h]$. We derive $\ell'(c_h) = \frac{k - \alpha h}{h} \geq \frac{\|\alpha h\|}{h} \geq \frac{c}{h^2} \geq \ell'(m)$, and so $c_h \leq m$. Similarly we have $c_h' \leq m$. Hence Theorem 1 yields

$$\begin{aligned}
\sum_{n=1}^{N} e(hx_n) =& \sum_{j=1}^{m'} e(h(\alpha j + \ell(j))) + \sum_{j=1}^{m} e(h(\alpha j - \ell(j))) \\
=& \frac{1}{h^{1/2}|\ell''(c_h)|^{1/2}} e(h\ell(c_h) + (\alpha h - k) - 1/8) \\
&+ \frac{1}{h^{1/2}|\ell''(c_h')|^{1/2}} e(-h\ell(c_h') + (k' - \alpha h) + 1/8) \\
&+ O\left(\frac{1}{k - \alpha h}\right) + O\left(\frac{1}{\alpha h - k'}\right) \\
&+ O\left(\frac{h^{-1/2+\delta}}{(k-\alpha h)^{1+\delta}}\right) + O\left(\frac{h^{-1/2+\delta}}{(\alpha h - k')^{1+\delta}}\right) \\
&+ O\left(h^{1/2} \log(\ell'(n_0)h)\right), \qquad (24)
\end{aligned}$$

where $m' = m$ if $N = 2m$, $m' = m+1$ if $N = 2m+1$, and δ will be determined later. We assume that $\|\alpha h\| = k - \alpha h$. Since $\alpha h - k' \geq 1/2$, we have

$$\frac{1}{h^{1/2}(-\ell''(c_h'))^{1/2}} \ll \frac{1}{h^{1/2}\ell'(c_h')} = \frac{h^{1/2}}{\alpha h - k'} \ll h^{1/2}.$$

Hence (24) implies

$$\begin{aligned}\sum_{n=1}^{N} e(hx_n) =& \frac{1}{h^{1/2}|\ell''(c_h)|^{1/2}} e(h\ell(c_h) + (\alpha h - k) - 1/8)\\ &+ O\left(\frac{1}{k-\alpha h}\right) + O\left(\frac{h^{-1/2+\delta}}{(k-\alpha h)^{1+\delta}}\right)\\ &+ O\left(h^{1/2}\log(\ell'(n_0)h)\right). \end{aligned} \tag{25}$$

Let $0 < \varepsilon < 1/4$ be fixed. From the condition (vi), it follows that

$$\frac{1}{h^{1/2}|\ell''(c_h)|^{1/2}} \gg h^{-1/2}\ell'(c_h)^{-\frac{1}{1+2\delta}} \gg h^{-1/2}\ell'(c_h)^{-1+\varepsilon} \gg \frac{h^{1/2-2\varepsilon}}{\|\alpha h\|},$$

where δ is defined by $\varepsilon = \frac{2\delta}{1+2\delta}$. Since $\varepsilon < 1/4$, we have

$$\frac{h^{-1/2+\delta}}{(k-\alpha h)^{1+\delta}} = \frac{h^{-1/2+\delta}}{\|\alpha h\|^{1+\delta}} = \frac{h^{-1/2+\delta}}{\|\alpha h\|}\frac{1}{\|\alpha h\|^{\delta}} \ll \frac{h^{-1/2+2\delta}}{\|\alpha h\|} < \frac{h^{1/2-2\varepsilon}}{\|\alpha h\|}.$$

Hence from (25), we derive

$$\left|\sum_{n=1}^{N} e(hx_n)\right| \geq C_1\frac{h^{1/2-2\varepsilon}}{\|\alpha h\|} - C_2 h^{1/2}\log(\ell'(n_0)h).$$

Therefore

$$\left|\sum_{n=1}^{N} e(hx_n)\right|^2 \geq C_3\frac{h^{1-4\varepsilon}}{\|\alpha h\|^2} - C_4\frac{h^{1-\varepsilon}}{\|\alpha h\|} + C_5\left(h\log^2(\ell'(n_0)h)\right). \tag{26}$$

In the same way, we obtain (26) when $\|\alpha h\| = \alpha h - k'$. Hence it follows that

$$\begin{aligned} &N^2 F_N(x_n)^2\\ &\geq C_6\left(\sum_{h=1}^{A_m}\frac{1}{h^2}\left|\sum_{n=1}^{N} e(hx_n)\right|^2\right)\\ &\geq C_7\sum_{h=1}^{A_m}\frac{h^{-1-4\varepsilon}}{\|\alpha h\|^2} - C_8\left(\sum_{h=1}^{A_m}\frac{h^{-1-\varepsilon}}{\|\alpha h\|}\right) + C_9\left(\sum_{h=1}^{A_m}\frac{\log^2(\ell'(n_0)h)}{h}\right). \end{aligned} \tag{27}$$

Applying Abel's summation formula and Lemma 2, we have

$$\sum_{h=1}^{A_m} \frac{h^{-1-4\varepsilon}}{\|\alpha h\|^2} = \sum_{h=1}^{A_m-1} \left(h^{-1-4\varepsilon} - (h+1)^{-1-4\varepsilon}\right) \sum_{n=1}^{h} \frac{1}{\|\alpha n\|^2}$$

$$+ A_m^{-1-4\varepsilon} \sum_{n=1}^{A_m} \frac{1}{\|\alpha n\|^2}$$

$$\gg A_m^{1-4\varepsilon}. \tag{28}$$

Similarly, we obtain

$$\sum_{h=1}^{A_m} \frac{h^{-1-\varepsilon}}{\|\alpha h\|} = O(1). \tag{29}$$

Since $\sum_{h=1}^{A_m} \frac{\log^2(\ell'(n_0)h)}{h} = O(\log^3 A_m)$, from (27), (28), and (29), it follows that

$$F_N(x_n) \gg \frac{A_m^{\frac{1}{2}-2\varepsilon}}{N}. \tag{30}$$

Using $A_m \gg \ell'(N/2)^{-1/2}$ with (30), we arrive at

$$F_N(x_n) \gg \frac{1}{N\ell'(N/2)^{\frac{1}{4}-\varepsilon}}.$$

References

[1] F. V. Atkinson, The mean-value of the Riemann zeta function, Acta Math., **81** (1949), 353–376.

[2] M. Drmota and R. F. Tichy, Sequences, Discrepancies and applications, Springer-Verlag, Berlin, 1997.

[3] K. Goto and Y. Ohkubo, Discrepancy of some special sequences, Analytic Number Theory, (eds. C. Jia and K. Matsumoto), Kluwer Acad. Publ. Dordrecht, 2002, pp. 143–155.

[4] K. Goto and Y. Ohkubo, Lower bounds for the discrepancy of some sequences, Math. Slovaca, **54** (2004), 487–502.

[5] S. Haber and C. F. Osgood, On the sum $\sigma\langle n\alpha\rangle^{-t}$ and numerical integration, Pacific J. Math., **31** (1969), 383–394.

[6] G. H. Hardy and J. E. Littlewood, Some problems of Diophantine approximation: The lattice points of a right-angled triangle, Abhandl. Math. Seminar. Hamburg Univ., **1** (1922), 212–249.

[7] G. H. Hardy and J. E. Littlewood, Some problems of Diophantine approximation: A series of cosecants, Bull. Calcutta Math. Soc., **20** (1930), 251–266.

[8] L. Kuipers and H. Niederreiter, Uniform Distribution of Sequences, John Wiley and Sons, New York, 1974.

[9] S. Lang, Introduction to diophantine approximations, Springer-Verlag, New York, 1995.

[10] W. J. LeVeque, An inequality connected with Weyl's criterion for uniform distribution, Proc. Symp. Pure Math. Vol VIII, Amer. Math. Soc., Providence, R. I., **31** (1965), 22–30.

[11] Y. Ohkubo, Notes on Erdös-Turán inequality, J. Austral. Math. Soc., **67** (1999), 51–57.

[12] E. C. Titchmarsh, The Theory of the Riemann Zeta-function, 2nd ed., revised by D. R. Heath-Brown, Clarendon Press, Oxford, 1986.

[13] P. Zinterhof, Über einige abschätzungen bei der approximation von funktionen mit gleichverteilungsmethoden, Österreich. Akad. Wiss. Math.-Naturwiss. Kl. S.-B. II, **185** (1976), 121–132.

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