Advanced Studies in Pure Mathematics 49, 2007
Probability and Number Theory — Kanazawa 2005
pp. 263–288

Renormalized Rauzy inductions

Takehiko Morita [1]

Abstract.

The Rauzy induction is a dynamical system acting on the space of interval exchange transformations which is introduced by Rauzy in [20] and used by Veech to give an affirmative answer to the Keane Conjecture in [24] and [25]. The results in [25] enable us to construct induced transformations and jump transformations to the various sets. In this article the dynamical systems obtained by composing some of those transformations are called renormalized Rauzy inductions. Note that the two fold iteration of continued fraction transformation can be regarded as a classical example of renormalized Rauzy inductions via appropriate conjugacy. Our present goal is to establish the same kinds of central limit theorems as obtained in [17] for a class of renormalized Rauzy inductions.

§1. Introduction.

For a positive integer $d \geq 2$, we denote by Λ_d the positive cone $\{\lambda = (\lambda_1, \ldots, \lambda_d)^t \in \mathbb{R}^d : \lambda_j > 0, j = 1, 2, \ldots, d\}$ and by $\mathfrak{S}_d$ the symmetric group of degree d. An element $(\lambda, \pi) \in \Lambda_d \times \mathfrak{S}_d$ is naturally identified with an interval exchange transformation $T_{(\lambda,\pi)} : [0, |\lambda|) \to [0, |\lambda|)$, where $|\lambda|_1 = \sum_{i=1}^d \lambda_i$. If $\lambda_{\pi^{-1}d} \neq \lambda_d$, we can obtain a new interval exchange transformation by taking an induced transformation of $T_{(\lambda,\pi)}$ to the subinterval $[0, |\lambda|_1 - \min(\lambda_{\pi^{-1}d}, \lambda_d))$. Thus we obtain an almost everywhere defined dynamical system $\mathcal{T}_0 : \Lambda_d \times \mathfrak{S}_d \to \Lambda_d \times \mathfrak{S}_d$ on the space of interval exchange transformations. $\mathcal{T}_0$ determines also a dynamical system $\mathcal{T} : \Delta_{d-1} \times \mathfrak{S}_d \to \Delta_{d-1} \times \mathfrak{S}_d$, where Δ_{d-1} is the projective space consisting of elements $\lambda \in \Lambda_d$ with $|\lambda|_1 = 1$.

Received February 17, 2006.
Revised February 19, 2007.
2000 *Mathematics Subject Classification.* Primary 11K55; Secondary 60F05.
[1] Partially supported by the Grant-in-Aid for Scientific Research (B) 16340048, Japan Society for the Promotion of Science.

We call both dynamical systems $\mathcal{T}_0$ and $\mathcal{T}$ Rauzy inductions. For each $(\lambda, \pi) \in \Lambda_d \times \mathfrak{S}_d$ with $\lambda_{\pi^{-1}d} \neq \lambda_d$, there exist a $d \times d$, 0-1 regular matrix $A(\lambda, \pi)$ and a map $D(\lambda) : \mathfrak{S}_d \to \mathfrak{S}_d$ such that $\mathcal{T}_0(\lambda, \pi) = (A(\lambda, \pi)^{-1}\lambda, D(\lambda)\pi)$ and $\mathcal{T}(\lambda, \pi) = (\frac{A(\lambda,\pi)^{-1}\lambda}{|A(\lambda,\pi)^{-1}\lambda|_1}, D(\lambda)\pi)$. We choose an irreducible permutation $\pi^{(0)}$ and we pay attention to its Rauzy class $\mathfrak{R}(\pi^{(0)})$ i.e. the totality of elements $\pi' \in \mathfrak{S}_d$ such that $\mathcal{T}^n(\lambda, \pi^{(0)}) = (\lambda', \pi')$ holds for some λ, λ', and n. We consider the restricted dynamical system $\mathcal{T} : \Delta_{d-1} \times \mathfrak{R}(\pi^{(0)}) \to \Delta_{d-1} \times \mathfrak{R}(\pi^{(0)})$ in what follows.

Consider the simplest case $d = 2$. In this case π with $\pi 1 = 2$ and $\pi 2 = 1$ is the unique irreducible permutation and $\mathcal{R}(\pi) = \{\pi\}$ and the Rauzy induction $\mathcal{T}$ and $\mathcal{T}_0$ are the Euclidean algorithms acting on Λ_2 and Δ_1, respectively. Put $\Delta(L) = \{\lambda \in \Lambda_2 : \lambda_2 > \lambda_1\}$ and $\Delta(R) = \{\lambda \in \Lambda_2 : \lambda_2 < \lambda_1\}$. Then $\Delta_1 = \Delta(L) \cup \Delta(R)$ a.e. and we can define the jump transformations $\mathcal{T}_{\Delta(L),\Delta(R)} : \Delta(L) \to \Delta(R)$ and $\mathcal{T}_{\Delta(R),\Delta(L)} : \Delta(R) \to \Delta(L)$. $\mathcal{S} = \mathcal{T}_{\Delta(R),\Delta(L)} \circ \mathcal{T}_{\Delta(L),\Delta(R)} : \Delta(L) \to \Delta(L)$ is a typical example of the renormalized Rauzy induction in the title of this article. We see that $\mathcal{S}$ is an expanding map with respect the Hilbert projective metric Θ on Δ_1 restricted to $\Delta(L)$. Moreover, It is easy to see that $\mathcal{S}$ and the two-fold iteration T_G^2 of the so called continued fraction transformation $T_G : (0,1) \to (0,1), : x \mapsto \frac{1}{x} - \left[\frac{1}{x}\right]$ are conjugate to each other via the projection $\Delta(L) \ni (\lambda_1, \lambda_2) \mapsto \lambda_1/\lambda_2 \in (0,1)$.

It is well known that the ergodic theory of T_G, the metric theory of continued fractions, and the dynamical theory of the geodesic flow on the modular surface $M_1 = \mathbb{H}/PSL(2,\mathbb{Z})$ are closely related, where $\mathbb{H}$ is the upper half-plane in $\mathbb{C}$ (see [15], [16], [19], [23]). We notice that the modular surface has at least two different faces. First, it has a face of cofinite Riemann surface. Secondly, it has the face of the moduli space of complex structures of surface of genus 1. So there are two possibilities to generalize the results on M_1 according to which face we look at. If we regard M_1 as one of cofinite Riemann surface, we expect that we can construct a Markov map playing the role of T_G^2 for a general cofinite Fuchsian group. In fact, for any cofinite Fuchsian group Γ Bowen and Series [3] construct a one-dimensional Markov map whose action on an appropriately chosen subset in $\mathbb{R}$ is orbit equivalent to that of Γ on $\mathbb{R} \cup \{\infty\}$. One finds that T_G^2 is a typical example of Bowen-Series Markov map. The result concerning a determinant representation of the Selberg zeta function in [16] is generalized to the case of any cofinite Fuchsian group in [18] by making use of Bowen-Series Markov maps. Next in the case when we regard M_1 as the moduli space of genus 1, we consider the moduli space M_g of genus $g \geq 2$ instead of M_1. It is known that M_g has a similar structure to M_1. For example, the Teichmüller space of genus g, the Teichmüller modular group, and Teichmüller geodesic flow play the roles of $\mathbb{H}$, $PSL(2,\mathbb{Z})$, and the geodesic flow of M_1. Note that Masur [14] solved the Keane Conjecture independently by showing the ergodicity of the Teichmüller geodesic flow with respect to a canonical invariant measure. To the question "what plays the role of T_G^2 in the case of M_g?" we do not have sufficient answer at present. But the Rauzy induction and its renormalizations expected to play the role of T_G^2 for Teichmüller modular group. We have to emphasize that the study of the transfer operator approach to the central limit problems for Markov maps with infinite Markov partition in [17] became a cornerstone of the subsequent work [18]. Apart from such a background, it will be interesting to study the dynamical central limit problems for the renormalized Rauzy inductions.

From now on we concentrate on the central limit theorems for a class of renormalized Rauzy inductions. Veech constructed an σ finite invariant measure μ of $\mathcal{T}$ which is equivalent to the product measure $\omega_{d-1} \times \sharp_{\mathfrak{R}(\pi^{(0)})}$, where ω_{d-1} is the volume measure of the projective space Δ_{d-1} and $\sharp_{\mathfrak{R}(\pi^{(0)})}$ is the counting measure on $\mathfrak{R}(\pi^{(0)})$. Such a measure is unique in the sense that ν is another $\mathcal{T}$-invariant measure absolutely continuous with respect to $\omega_{d-1} \times \sharp_{\mathfrak{R}(\pi^{(0)})}$, then there exists a positive constant c such that $\nu = c\mu$. It is not necessarily finite but the measure-theoretic dynamical system $(\mathcal{T},\mu)$ is conservative (i.e. any Borel measurable set E with $T^{-1}E \subset E$ satisfies $\mu(E\Delta T^{-1}E) = 0$) and ergodic (i.e. any Borel measurable set E with $T^{-1}E = E$ satisfies $\mu(E)\mu(E^c) = 0$). Therefore we can see that for μ-almost every $(\lambda^{(0)},\pi^{(0)})$, there exists N such that $A_N(\lambda^{(0)},\pi^{(0)}) = A(\lambda^{(0)},\pi^{(0)})A(\mathcal{T}(\lambda^{(0)},\pi^{(0)})) \cdot \cdots \cdot A(\mathcal{T}^{N-1}(\lambda^{(0)},\pi^{(0)})) > 0$ and the induced transformation $\mathcal{T}_{A_N((\lambda^{(0)},\pi^{(0)}))}$ of $\mathcal{T}$ to the set $\Delta_{A_N(\lambda^{(0)},\pi^{(0)})} \times \{\pi_0\}$ is defined, where $\Delta_{A_N(\lambda^{(0)},\pi^{(0)})} = \Delta_{d-1} \cap A_N(\lambda^{(0)},\pi^{(0)})\Lambda_d$. Our main concern is such a renormalized Rauzy induction as $\mathcal{T}_{A_N((\lambda^{(0)},\pi^{(0)}))}$.

We fix $(\lambda^{(0)},\pi^{(0)})$ as above and put $B = A_N(\lambda^{(0)},\pi^{(0)})$ for the sake of simplicity. We can identify $\Delta_B \times \{\pi^{(0)}\}$ with Δ_B in a natural way. Let $\omega_B = \omega_{d-1}(\Delta_B)^{-1}\omega_{d-1}|_{\Delta_B}$ and $\mu_B = \mu(\Delta_B \times \{\pi^{(0)}\})^{-1} \cdot \mu|_{\Delta_B}$. Clearly, μ_B is a unique invariant Borel probability measure for $\mathcal{T}_B : \Delta_B \to \Delta_B$ equivalent to ω_B. Moreover we show that $\mathcal{T}_B$ is an expanding map with respect to the Hilbert projective metric Θ on Δ_{d-1} restricted to Δ_B having an infinite Markov partition. Let $F_\Theta(\Delta_B)$ (resp. $F_\Theta(\Delta_B \to \mathbb{R})$) be the totality of complex valued (resp. real valued) Lipschitz continuous functions on Δ_B with respect to Θ. The goal of this article is to establish the same kinds of central limit theorems as in [17] for the renormalized rauzy inductions $\mathcal{T}_B$. Precisely we show the following two theorems.

Theorem 1.1. *Let f be an element in $F_\Theta(\Delta_B \to \mathbb{R})$ satisfying $\int_{\Delta_B} f\, d\mu_B = 0$ which is not identically 0. Then there exists the limit*

$$v(f) = \lim_{n\to\infty} \frac{1}{n} \int_{\Delta_B} (S_{\mathcal{T}_B,n} f)^2\, d\mu_B > 0 \tag{1.1}$$

and there exists a positive number C_1 such that for any $g \in F_\Theta(\Delta_B)$, we have

$$\begin{aligned} \sup_{a,\, b\in\mathbb{R}\,:a<b} &|\omega_{B,g}\left(a \le \frac{S_{\mathcal{T}_B,n} f}{\sqrt{v(f) n}} \le b\right) \\ &- \left(\int_{\Delta_B} g\, d\omega_B\right) \frac{1}{\sqrt{2\pi}} \int_a^b e^{-\frac{x^2}{2}}\, dx| \le \frac{C_1 \|g\|_\Theta}{\sqrt{n}}, \end{aligned} \tag{1.2}$$

where $S_{\mathcal{T}_B,n} f = \sum_{k=0}^{n-1} f \circ \mathcal{T}_B^n$, $\omega_{B,g} = g\,\omega_B$, and $\|g\|_\Theta$ denotes the usual Banach norm of Lipschitz continuous function g with respect to the Hilbert projective metric Θ.

Theorem 1.2. *Let f be an element in $F_\Theta(\Delta_B \to \mathbb{R})$ satisfying $\int_{\Delta_B} f\, d\mu_B = 0$ which is not identically 0. Then for any $g \in F_\Theta(\Delta_B)$ and for any rapidly decreasing function u on $\mathbb{R}$, we have*

$$\begin{aligned} \lim_{n\to\infty} \sup_{\alpha\in\mathbb{R}} &|\sqrt{n} \int_{\Delta_B} u(\alpha + S_{\mathcal{T}_B,n} f(x))\, \omega_{B,g}(dx) \\ &- \left(\int_{\Delta_B} g\, d\omega_B\right) \left(\int_{\mathbb{R}} u(t)\, dt\right) \frac{1}{\sqrt{2\pi v(f)}} \exp\left(-\frac{\alpha^2}{2nv(f)}\right) | = 0. \end{aligned} \tag{1.3}$$

We should note that the exponential decay of correlations for the Rauzy-Veech-Zorich induction and the central limit theorem for the Teichmüller flow on the moduli space of abelian differentials are established recently by Bufetov [4] (see also Avila and Bufetov [1]). Moreover, Avila, Gouëzel, and Yoccoz prove the exponential mixing of the Teichmüller flow in [2]. From these results we can verify the exponential decay of correlations for a wider class of renormalized Rauzy inductions and a wider class of observables than those we consider. But the problems concerning the positivity of the limiting variance and the rate of convergence of the central limit theorem are not discussed in these papers because these problems are not their main concern. We emphasize that Theorem 1.1 asserts that we obtain both the positivity of the limiting variance and the good rate of convergence of the central limit theorem although the class of renormalized Rauzy inductions and observables are restricted. In addition, Theorem 1.2 asserts that the rate of convergence obtained in Theorem 1.1 is good enough to enable us to establish a weak version of local central limit theorem.

In Section 2, we explain about interval exchange transformations and Rauzy induction. The Keane Conjecture is also treated in Section 2. We introduce the class of renormalized Rauzy inductions that we study in Section 3. Section 4 is devoted to the transfer operator approach to the central limit theorems and proofs of Theorem 1.1 and Theorem 1.2.

Acknowledgements I am deeply grateful to the referee for informing me of the recent results [1], [2], and [4].

§2. Interval exchange transformations and Rauzy inductions.

We note that the definition of our renormalized Rauzy induction is based on the ergodic property of Rauzy induction established by Veech [25] and Masur [14] in their way to solve the Keane Conjecture on interval exchange transformations. Therefore we start with some historical topics on the metric theory of interval exchange transformations and Rauzy inductions.

For $(\lambda, \pi) \in \Lambda_d \times \mathfrak{S}_d$, we define $\beta(\lambda) \in \{0\} \times \Lambda_d$ so that $\beta_j(\lambda) = \sum_{i=1}^{j} \lambda_j$ for $0 \leq j \leq d$. Consider a partition $\alpha(\lambda)$ of the interval $X(\lambda) = [0, |\lambda|_1)$ into subintervals $X_j(\lambda) = [\beta_{j-1}(\lambda), \beta_j(\lambda))$ $(1 \leq j \leq d)$. Let $\lambda^\pi = (\lambda_{\pi^{-1}1}, \dots, \lambda_{\pi^{-1}d})^t$. Then the interval exchange transformation $T_{(\lambda.\pi)} : X(\lambda) \to X(\lambda)$ is defined by

$$T_{(\lambda,\pi)}x = x + \sum_{j=1}^{d} (\beta_{\pi j-1}(\lambda^\pi) - \beta_{j-1}(\lambda)) I_{X_j(\lambda)}(x). \tag{2.1}$$

By definition $T_{(\lambda.\pi)}$ maps the j-th interval $X_j(\lambda)$ in $\alpha(\lambda)$ onto πj-th interval $X_{\pi j}(\lambda^\pi)$ in $\alpha(\lambda^\pi)$ isometrically preserving the orientation. Thus the Lebesgue measure m restricted to $X(\lambda)$ is an invariant measure for $T_{(\lambda.\pi)}$. Consider the simplest case when $d = 2$, $\pi 1 = 2$, $\pi 2 = 1$, and $\lambda = (1 - a, a)$ with $0 < a < 1$. Then $T_a = T_{(\lambda,\pi)} : [0, 1) \to [0, 1)$ is the so called Weyl automorphism and it is conjugate to $2\pi a$-rotation $R_{2\pi a} : S^1 \to S^1$ on the unit circle. The following is well known.

Theorem 2.1. *The following are equivalent:*

(1) *a is irrational.*

(2) *For any $x \in [0, 1)$, the T_a-orbit of x is dense in $[0, 1)$.*

(3) *The Lebesgue measure m on $[0, 1)$ is the unique invariant Borel probability measure of T_a.*

Recall the definitions of minimality and unique ergodicity for a homeomorphism H on a compact metric space (X, ρ). H is called minimal if and only if there is no nonempty closed set E satisfying $TE = E$ and $E \neq X$. H is called uniquely ergodic if and only if it has a unique invariant Borel probability measure. We summarize important facts on these notions as the following theorem:

Theorem 2.2. *Let H be a homeomorphism on a compact metric space X. Then we have the following*

(1) *H is minimal if and only if for any $x \in X$, H-orbit of x is dense in X.*

(2) *H is uniquely ergodic if and only if there exists a Borel probability measure m on X such that $\frac{1}{n}\sum_{k=0}^{n-1} f(H^k x)$ converges to $\int_X f\,dm$ uniformly in x for any continuous function f on X.*

(3) *Assume H is uniquely ergodic. Then H is minimal if and only if the unique invariant Borel probability measure has positive value for any non empty open set.*

Keane [11] introduced the notion of minimality to general interval exchange transformations and proved the following.

Theorem 2.3. *For an interval exchange transformation $T_{(\lambda,\pi)}$, the following are equivalent.*

(1) *$T_{(\lambda,\pi)}$ satisfies the following two conditions:*

(M.1) *There is no periodic point.*

(M.2) *If an non empty subset F of $X(\lambda)$ satisfies $T_{(\lambda.\pi)}F = F$ and if F can be expressed as a finite union of left closed and right open intervals whose endpoints are elements in $(\bigcup_{j=1}^{d-1}\bigcup_{n\in\mathbb{Z}}\{T^n_{(\lambda,\pi)}\beta_j(\lambda)\}) \cup \{1\}$, then $F = X(\lambda)$.*

(2) *For any $x \in X(\lambda)$, $T_{(\lambda,\pi)}$-orbit of x is dense in $X(\lambda)$.*

More concrete sufficient condition for an interval exchange transformation to be minimal are also obtaineds in [11]. We give one of them below. $\lambda \in \Lambda_d$ is called irrational if its components are linearly independent over $\mathbb{Q}$ and $\pi \in \mathfrak{S}_d$ is irreducible if $\pi\{1, 2, \ldots, k\} = \{1, 2, \ldots, k\}$ implies $k = d$.

Theorem 2.4. *If λ is irrational and π is irreducible, then $T_{(\lambda,\pi)}$ is minimal.*

From this theorem one may expect that a similar assertion to Theorem 2.1 holds. But it is shown in [10] (see also [12]) that there exists a minimal interval exchange transformation which is not uniquely ergodic. Keane conjectured that for fixed irreducible $\pi \in \mathfrak{S}_d$, $T_{(\lambda.\pi)}$ is uniquely ergodic Lebesgue almost every $\lambda \in \Lambda_d$. The outline and the strategy of solving the Keane Conjecture can be found in Veech [25]. Veech introduced a sort of 'renormalization group approach' to the problem and solving the conjecture by showing that the so called Rauzy induction acting on the space of interval exchange transformation has a natural invariant measure. Namely, the unique ergodicity can be regarded as 'almost universal property' of interval exchange transformations and the Rauzy induction plays the role of renormalization group in the procedure to establish such an 'almost universal property'.

Next we recall the definition of Rauzy induction $\mathcal{T}_0 : \Lambda_d \times \mathfrak{S}_d \to \Lambda_d \times \mathfrak{S}_d$ for our convenience. Consider the following $d \times d$ matrices $L(\pi)$ and $R(\pi)$

(2.2)
$$L(\pi) = \begin{pmatrix} I_{d-1} & \mathbf{0}_{d-1} \\ \mathbf{e}_{d-1}(\pi^{-1}j)^t & 1 \end{pmatrix}, \quad R(\pi) = \begin{pmatrix} I_{\pi^{-1}d} & K_{\pi^{-1}d,d-\pi^{-1}d} \\ O_{d-\pi^{-1}d,\pi^{-1}d} & J_{d-\pi^{-1}d} \end{pmatrix},$$

where I_k is the $k \times k$ identity matrix, $\mathbf{0}_{d-1}$ is $d-1$-dimensional zero column vector, $\mathbf{e}_{d-1}(\pi^{-1}j)$ is the $d-1$-dimensional unit vector whose $\pi^{-1}j$-th component is 1, $O_{k,l}$ is the $k \times l$ zero matrix and $K_{\pi^{-1}d,d-\pi^{-1}d}$ and $J_{d-\pi^{-1}d}$ are $\pi^{-1}d \times (d - \pi^{-1}d)$ matrix and $(d-\pi^{-1}d) \times (d-\pi^{-1}d)$ matrix given by

$$K_{\pi^{-1}d,d-\pi^{-1}d} = \begin{pmatrix} 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \\ 1 & 0 & \dots & 0 \end{pmatrix}, \quad J_{d-\pi^{-1}d} = \begin{pmatrix} 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \\ 0 & 0 & \dots & 0 \end{pmatrix},$$

respectively. In addition we consider two transformations L, $R : \mathfrak{S}_d \to \mathfrak{S}_d$ defined by

(2.3)
$$(L\sigma)j = \begin{cases} \sigma j & (\sigma j \le \sigma d) \\ \sigma d + 1 & (\sigma j = d) \\ \sigma j + 1 & \text{otherwise} \end{cases}, \quad (R\sigma)j = \begin{cases} \sigma j & (j \le \sigma^{-1}d) \\ \sigma d & (j = \sigma^{-1}d + 1) \\ \sigma(j-1) & \text{otherwise} \end{cases}$$

For $(\lambda, \pi) \in \Lambda_d \times \mathfrak{S}_d$ with $\lambda_{\pi^{-1}d} \neq \lambda_d$, we put

(2.4)
$$A(\lambda, \pi) = \begin{cases} L(\pi) & (\text{ if } \lambda_d > \lambda_{\pi^{-1}d}) \\ R(\pi) & (\text{ if } \lambda_d < \lambda_{\pi^{-1}d}) \end{cases},$$
$$D(\lambda) = \begin{cases} L & (\text{ if } \lambda_d > \lambda_{\pi^{-1}d}) \\ R & (\text{ if } \lambda_d < \lambda_{\pi^{-1}d}). \end{cases}$$

Then the Rauzy inductions $\mathcal{T}_0 : \Lambda_d \times \mathfrak{S}_d \to \Lambda_d \times \mathfrak{S}_d$ and $\mathcal{T} : \Delta_{d-1} \times \mathfrak{S}_d \to \Delta_{d-1} \times \mathfrak{S}_d$ are defined for (λ, π) with $\lambda_{\pi^{-1}d} \neq \lambda_d$ by

(2.5)
$$\mathcal{T}_0(\lambda, \pi) =(A(\lambda, \pi)^{-1}\lambda, D(\lambda)\pi),$$
$$\mathcal{T}(\lambda, \pi) =(\frac{A(\lambda, \pi)^{-1}\lambda}{|A(\lambda, \pi)^{-1}\lambda|_1}, D(\lambda)\pi).$$

We are only interested in irreducible permutations. Fix a irreducible element $\sigma \in \mathfrak{S}_d$. Consider the Rauzy class $\mathfrak{R} = \mathfrak{R}(\sigma)$ introduced in [20]. ω_{d-1} and $\sharp_{\mathfrak{R}}$ below denote the volume measure on Δ_{d-1} and the counting measure on $\mathfrak{R}$, respectively in the below. The result about $\mathcal{T} : \Delta_{d-1} \times \mathfrak{R} \to \Delta_{d-1} \times \mathfrak{R}$ that we need is the following.

Theorem 2.5 (Veech [25] (see also [26])). *There exists a $\mathcal{T}$ invariant measure μ equivalent to $\omega_{d-1} \times \sharp_{\mathfrak{R}}$ on $\Delta_{d-1} \times \mathfrak{R}$ which makes $\mathcal{T}$ both conservative and ergodic. For each $\pi \in \mathfrak{R}$, the density μ on $\Delta_{d-1}(= \Delta_{d-1} \times \{\pi\})$ with respect to ω_{d-1} is given by the restriction of a function on Λ_d which is rational, positive, and homogeneous of degree $-d$.*

We shall explain how to use Theorem 2.5 to solve the Keane Conjecture. In what follows we only consider the irrational element in Λ_d and permutations in a fixed Rauzy class $\mathfrak{R}$. In this case $T_{(\lambda,\pi)}$ is minimal and $\mathcal{T}^n(\lambda, \pi)$ can be defined for any $n \in \mathbb{N}$. Let $M(\lambda, \pi)$ denote the totality of $T_{(\lambda,\pi)}$ invariant Borel measures on $X(\lambda)$. Consider the map $\Phi_{(\lambda,\pi)} : M(\lambda, \pi) \to \Lambda_d$ defined by $\Phi_{(\lambda,\pi)}(\nu) = (\nu(X_1(\lambda)), \dots, \nu(X_d(\lambda)))^t$. Then we can show that $\Phi_{(\lambda,\pi)}$ is affine, continuous, and injective. Thus $M(\lambda, \pi)$ is identified with its image $S(\lambda, \pi)$. Moreover it is not hard to see by definition that

$$S(\lambda, \pi) = A_n(\lambda, \pi)S(\lambda^{(n)}, \pi^{(n)}) \quad (\subset A_n(\lambda, \pi)\Lambda_d) \tag{2.6}$$

holds, where $(\lambda^{(n)}, \pi^{(n)}) = \mathcal{T}_0^n(\lambda, \pi)$ and $A_n(\lambda, \pi) = A(\lambda^{(0)}, \pi^{(0)}) \cdot \dots \cdot A(\lambda^{(n-1)}, \pi^{(n-1)})$ for $n \geq 0$. Note that if $T_{(\lambda,\pi)}$ is minimal then $m(X(\lambda^{(n)})) \to 0$ $(n \to \infty)$. Therefore there exists $N = N(\lambda, \pi)$ such that $A_N(\lambda, \pi) > 0$, i.e. all entries of the matrix $A_N(\lambda, \pi)$ are positive and moreover there exists a neighborhood $U(\lambda, \pi) \subset \Delta_{d-1}$ of λ such that $A_N(\lambda', \pi) = A_N(\lambda, \pi)$ for any $\lambda' \in U(\lambda, \pi)$. Now we are in a position to make use of Theorem 2.5. Theorem 2.5 implies that the Poincaré Recurrence Theorem holds for $(\mathcal{T}, \mu)$. Since μ is equivalent to ω_{d-1} on $\Delta_{d-1} \times \{\pi\}$ for each $\pi \in \mathfrak{R}$, so ω_{d-1}-almost every λ is a recurrent point i.e. having the property that not only $T_{(\lambda,\pi)}$ is minimal for any $\pi \in \mathfrak{R}$ but also for any neighborhood $U \subset \Delta_{d-1}$ of λ, $\mathcal{T}^n(\lambda, \pi)$ visits U for infinitely many number of n. Choose λ satisfying such a property and for $\pi \in \mathfrak{R}$, choose N such that $B = A_N(\lambda, \pi) > 0$. Then we see that both $\pi^{(n)} = \pi$ and $A_N(\lambda^{(n)}\pi^{(n)}) = B$ hold for infinitely many n. Combining this with the fact (2.6), we find a infinite sequence $C_n = C_n(\lambda, \pi)$ of $d \times d$ nonnegative matrices such that

$$S(\lambda, \pi) \subset BC_1BC_2 \cdot \cdots \cdot BC_n\Lambda_d \tag{2.7}$$

for any n. The positivity of $B > 0$ implies that $\dim \bigcap_{n=1}^{\infty} BC_1BC_2 \cdot \cdots \cdot BC_n\Lambda_d = 1$. Hence we arrived at the affirmative answer to the Keane Conjecture.

Theorem 2.6. *Let $\pi \in \mathfrak{S}_d$ be irreducible. Then $T_{(\lambda,\pi)}$ is uniquely ergodic for Lebesgue almost every $\lambda \in \Lambda_d$.*

We finish this section with the following remark.

Remark Choose any $\pi \in \mathfrak{R}$ and fix it. For any irrational $\lambda \in \Delta_{d-1}$, we can assign an infinite sequence $(D_n) \in \{L, R\}^{\mathbb{Z}_+}$ so that $D(\lambda^{(n)}) = D_n$, equivalently, $A(\lambda^{(n)}, \pi^{(n)}) = D_n(\pi^{(n)})$ for each $n \geq 0$ (see (2.4)). In virtue of Theorem 2.5, there exists a Borel set $\Omega(\pi) \subset \Delta_{d-1}$ such that $\omega_{d-1}(\Omega(\pi)) = \omega_{d-1}(\Delta_{d-1})$ and the map $\Xi : \Omega(\pi) \to \{L, R\}^{\mathbb{Z}_+}$ defined by

$$\Xi(\lambda) = (D(\lambda^{(n)})) \tag{2.8}$$

is injective and for any finite sequence $(D_k)_{k=0}^{n-1} \in \{L, R\}^n$, the subset of the elements λ for which $(D_k)_{k=0}^{n-1}$ occurs in $\Xi(\lambda)$ infinitely many times has the total measure.

The assertions except for the injectivity of Ξ are easy consequences of Theorem 2.5. But it is also easy to see that $A_n(\lambda, \pi) = A_{n'}(\lambda', \pi)$ if and only if $n = n'$ and $A(\mathcal{T}^k(\lambda, \pi)) = A(\mathcal{T}^k(\lambda', \pi))$ for every k with $0 \leq k \leq n-1$ since from the definition of $\mathcal{T}_0$ and $\mathcal{T}$ we have for each $n \geq 0$

$$\lambda' \in A_n(\lambda, \pi)\Lambda_d \text{ if and only if } A_n(\lambda', \pi) = A_n(\lambda, , \pi) \tag{2.9}$$

§3. Renormalization of Rauzy inductions.

Before introducing renormalized Rauzy inductions, we recall the definition of jump transformations and induced transformations. Let $(X, \mathcal{B}, \mu)$ be a σ finite measure space and $T : X \to X$ a μ-nonsingular transformation such that μ almost every $x \in X$ has the property that for any $E \in \mathcal{B}$ with $\mu(E) > 0$, $T^n x \in E$ holds for infinitely many $n \geq 0$. Then for any E, $F \in \mathcal{B}$ with $\mu(E) > 0$ and $\mu(F) > 0$, we put for $x \in E$

$$n(E, F; x) = \inf\{n \geq 1 : T^n x \in F\}. \tag{3.1}$$

In the case when $E = F$ we just write as $n(E; x) = n(E, E : x)$. From our assumption $n(E, F; x) < \infty$ μ-a.e. Thus we obtain almost everywhere defined transformation $T_{E,F} : E \to F$ called the jump transformation of T from E to F by

$$T_{E,F}x = T^{n(E,F;x)}x. \tag{3.2}$$

In the case $E = F$, $T_{E,F}$ is denoted by T_E and called the induced transformation of T to E or the first return map of T to E. Roughly speaking, 'renormalization of the transformation T' means the procedure constructing a new transformation by producing jump transformations and their composition.

From the fact mentioned in the previous section we can consider the renormalization of the Rauzy induction $\mathcal{T} : \Delta_{d-1} \times \mathfrak{R} \to \Delta_{d-1} \times \mathfrak{R}$. Set

$$\begin{aligned}
\Delta(L, \pi) &= (L(\pi)\Lambda_{d-1} \cap \Delta_{d-1}) \times \{\pi\} \\
&= \{\lambda \in \Delta_{d-1} : \lambda_d > \lambda_{\pi^{-1}d}\} \times \{\pi\} \\
\Delta(R, \pi) &= (R(\pi)\Lambda_{d-1} \cap \Delta_{d-1}) \times \{\pi\} \\
&= \{\lambda \in \Delta_{d-1} : \lambda_d < \lambda_{\pi^{-1}d}\} \times \{\pi\} \\
\Delta(L) &= \bigcup_{\pi \in \mathfrak{R}} \Delta(L, \pi), \quad \Delta(R) = \bigcup_{\pi \in \mathfrak{R}} \Delta(R, \pi).
\end{aligned} \tag{3.3}$$

Consider a jump transformation $\mathcal{T}_{\Delta(L),\Delta(R)} : \Delta(L) \to \Delta(R)$ and $\mathcal{T}_{\Delta(R),\Delta(L)} : \Delta(R) \to \Delta(L)$. Then we can define a transformation $\mathcal{S} = \mathcal{T}_{\Delta(L),\Delta(R)} \circ \mathcal{T}_{\Delta(R),\Delta(L)} : \Delta(L) \to \Delta_(L)$. $\mathcal{S}$ is a typical example of the renormalized Rauzy induction. Note that Zorich [27] shows that $\mathcal{S}$ has a finite invariant measure equivalent to the restriction of $\omega_{d-1} \times \sharp_{\mathfrak{R}}$ to $\Delta(L)$. In particular, in the case of $d = 2$ it has a classical meaning. In this case there exists a unique Rauzy class $\{\pi : \pi(1) = 2, \pi(2) = 1\}$. If one notices that $\mathcal{T}$ acts as the Euclidean algorithm on the projective space $\Delta_{d-1} = \Delta(R) \cup \Delta(L)$, one obtains the following commutative diagram.

$$\begin{array}{ccc} \Delta(L) & \xrightarrow{\mathcal{S}} & \Delta(L) \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\varphi} \\ (0,1) & \xrightarrow[T_G^2]{} & (0,1) \end{array} \tag{3.4}$$

where $\Delta(L)\times\{\pi\}$ is identified with $\Delta(L)$, $\varphi:\Delta(L)\to(0,1)$ is the homeomorphism given by $\varphi(\lambda)=\lambda_1/\lambda_2$, and $T_G:(0,1)\to(0,1)$ is the continued fraction transformation given by $T_Gx=\frac{1}{x}-[\frac{1}{x}]$.

The rest of the section is devoted to the study of a special class of renormalized Rauzy inductions whose members are not generalization of T_G^2 but it is useful to explain our idea. Let $(\lambda,\pi)\in\Delta(L,\pi)$ (resp. $\Delta(R,\pi)$) be such that λ is irrational. Then we can find $N\geq 2$ such that

$$\pi^{(N)}=\pi,\quad A_N(\lambda,\pi)>0. \tag{3.5}$$

We denote $A_N(\lambda,\pi)$ by B for the sake of simplicity. Consider the set $\Delta_B=B\Lambda_d\cap\Delta_{d-1}$ and $\Delta(B,\pi)=\Delta_B\times\{\pi\}$. We are interested in the induced transformation $\mathcal{T}_B$ of the transformation $\mathcal{T}$ to the set $\Delta(B,\pi)$. We regard $\mathcal{T}_B$ as a transformation on Δ_B in a natural way.

For nonnegative invertible matrix A, let $\Delta_A=A\Lambda_d\cap\Delta_{d-1}$ and $\bar{A}$ denotes the map $\bar{A}:\Delta_{d-1}\to\Delta_{d-1}$ given by $\bar{A}x=\frac{Ax}{|Ax|_1}$ for $x\in\Delta_{d-1}$.

Lemma 3.1. *Let $\mathcal{T}_B$ be as above. There exist sequences of distinct nonnegative integral matrices $\mathcal{A}=\{A^{(k)}\}$ and $\mathcal{C}=\{C^{(k)}\}$ satisfying the following:*

(1) $A^{(k)}B=BC^{(k)}$ *and* $\det A^{(k)}=\det C^{(k)}=\pm 1$.

(2) $\mathcal{T}_B|_{\Delta_{AB}}=\overline{A^{-1}}$, *i.e.* $\mathcal{T}_Bx=\dfrac{A^{-1}x}{|A^{-1}x|_1}$ *for $A\in\mathcal{A}$. In particular,* $\mathcal{T}_B\Delta_{AB}=\Delta_B$ *for each $A\in\mathcal{A}$.*

(3) *The family of the set $\mathcal{P}=\{\Delta_{AB}:A\in\mathcal{A}\}$ forms a measurable partition of Δ_B, i.e. $\omega_B(\Delta_{AB}\cap\Delta_{A'B})=0$ for A, $A'\in\mathcal{A}$ with $A\neq A'$ and $\omega_B(\Delta_B\setminus\bigcup_{A\in\mathcal{A}}\Delta_{AB})=0$, where $\omega_B=\omega_{d-1}(\Delta_B)^{-1}\omega_{d-1}|_{\Delta_B}$*

Proof. All assertions in the lemma are easy consequences of Remark in the end of Section 2. For example, we can find the family $\mathcal{A}$ as follows. Let $x\in\Delta_B$ be a irrational recurrent point for $\mathcal{T}$. Let n be the first return time of x for $\mathcal{T}$. $\mathcal{T}_Bx=\mathcal{T}^nx$ holds. Therefore we see that $A_n(x,\pi)^{-1}x\in B\Lambda_d$. Thus $x\in\Delta_{A_n(x,\pi)B}$. In virtue of (2.9) we have $A(\mathcal{T}^{n+k}(x,\pi))=A(\mathcal{T}^k(x,\pi))$ for each k with $(0\leq k\leq N-1)$ and $A_N(x,\pi)=B$. Put $A(x)=A_n(x,\pi)$ and $C(x)=B^{-1}A_n(x,\pi)B$.

There are countably many possibilities of $A_(x)$ and $C(x)$ even if x varies. Hence there exists a countable set $\{x_k\}$ such that $\mathcal{A} = \{A(x_k)\}$ and $\mathcal{C} = \{C(x_k)\}$ are desired sequences. Q.E.D.

Next we introduce the Hilbert projective metric on Δ_{d-1}. Note that the results on the Hilbert projective metrics that we need as well as their application to the study of ergodic behavior of dynamical systems are summarized in [13].

For x, $y \in \Lambda_d$, we write $x \le y$ if each entry of $y - x$ is nonnegative. Put

$$\begin{aligned} \alpha(x,y) &= \sup\{a \ge 0 \, : \, ax \le y\}, \\ \beta(x,y) &= \inf\{b \ge 0 \, : \, y \le bx\}, \\ \Theta(x,y) &= \log \frac{\beta(x,y)}{\alpha(x,y)}. \end{aligned} \tag{3.6}$$

Θ is called the Hilbert projective metric on Λ_d. Θ is a pseudo-metric on Λ_d such that $\Theta(x,y) = 0$ if and only if $x = cy$ holds for some $c > 0$. Thus Θ is a metric on the projective space Δ_{d-1}. The following two lemmas are well known facts and their more general forms can be found in [13].

Lemma 3.2. *Let A be a nonnegative matrix. Then we have*

$$\Theta(Ax, Ay) \le \tanh\left(\frac{\operatorname{diam}(\Delta_A)}{4}\right)\Theta(x,y) \tag{3.7}$$

for any x, $y \in \Lambda_d$, where $\operatorname{diam}(\Delta_A) = \sup\{\Theta(Ax, Ay) \, : \, x, \, y \in \Lambda_d\}$

Lemma 3.3. *For any x, $y \in \Lambda_d$ with $|x|_1 = |y|_1$, we have*

$$|x - y|_1 \le (e^{\Theta(x,y)} - 1)|x|_1. \tag{3.8}$$

For $n \ge 1$, put

$$\begin{aligned} \mathcal{A}_n &= \{A_1 A_2 \cdots A_n \, : \, A_1, A_2, \, \ldots, \, A_n \in \mathcal{A}\} \\ \mathcal{P}_n &= \{\Delta_{AB} \, : \, A \in \mathcal{A}_n\}. \end{aligned} \tag{3.9}$$

We summarize the basic properties of the renormalized Rauzy induction $\mathcal{T}_B$ as the following lemma.

Lemma 3.4. *Let $\mathcal{T}_B$ be as above. Then we have the following.*

(1) (Markov property) *For any $n \ge 1$ we have*

$$\mathcal{P}_n = \bigvee_{k=0}^{n-1} \mathcal{T}_B^{-k}\mathcal{P} \text{ and } \mathcal{T}_B^n \Delta_{AB} = \Delta_B \tag{3.10}$$

for any $\Delta_{AB} \in \mathcal{P}_n$. In particular $\mathcal{T}_B^n : \Delta_{AB} \to \Delta_B$ is a homeomorphsim.

(2) (expanding) *There exist $C_2 > 0$ and $\theta \in (0,1)$ such that for any $n \geq 1$*

$$\Theta(\mathcal{T}_B^n x, \mathcal{T}_B^n y) \geq C_2^{-1}\theta^{-n}\Theta(x,y) \tag{3.11}$$

holds for any x, $y \in \Delta_{AB} \in \mathcal{P}_n$.

(3) (finite distortion) *There exists $C_3 > 0$ such that for any $n \geq 1$*

$$\left|\log \frac{J(\mathcal{T}_B^n)(x)}{J(\mathcal{T}_B^n)(y)}\right| \leq C_3\Theta(\mathcal{T}_B^n x, \mathcal{T}_B^n y) \tag{3.12}$$

holds for any x, $y \in \Delta_{AB} \in \mathcal{P}_n$, where $J\mathcal{T}_B^n)$ denotes the Jacobian of $\mathcal{T}_B^n$ with respect to ω_B.

Proof. (1) follows from Lemma 3.1. Since $\operatorname{diam}(\Delta_B) < \infty$, (2) is an easy consequence of Lemma 3.2. It remains to prove (3). To this end we need the following fact which is proved in [24]. For nonnegative matrix with $|\det A| = 1$, the Jacobian $J(\bar{A})$ of the map $\bar{A} : \Delta_{d-1} \to \Delta_{d-1}$ with respect to ω_{d-1} is given by

$$J(\bar{A})(x) = \frac{1}{|Ax|_1^d} \quad \text{for } x \in \Delta_{d-1}. \tag{3.13}$$

Note that if $x \in \Delta_{AB}$, then $x = \bar{A}\mathcal{T}_B^n x$ by definition. Therefore we have

$$J(\mathcal{T}_B^n)(x) = \frac{1}{|A^{-1}x|_1^d} = |A\mathcal{T}_B^n x|_1^d.$$

Putting $x' = \mathcal{T}_B^n x$ and $y' = \mathcal{T}_B^n y$ we see that

$$\left|\log \frac{J(\mathcal{T}_B^n)(x)}{J(\mathcal{T}_B^n)(y)}\right| = d\left|\log \frac{|Ay'|_1}{|Ax'|_1}\right| = d\left|\frac{|Ay'|_1 - |Ax'|_1}{|Az'|_1}\right|$$

holds for some $z' \in \Delta_B$.

$$\left|\frac{|Ay'|_1 - |Ax'|_1}{|Az'|_1}\right| \leq \frac{\sum_i \sum_j A_{ij}|x'_j - y'_j|}{\sum_i \sum_j A_{ij} z'_j} \leq \frac{|x' - y'|_1}{\min_j z'_j}.$$

Since $z' \in \Delta_B$, $\min_j z'_j$ is bounded from blow by a positive constant depending only on B. Combining these estimate with Lemma 3.3, we arrive at the desired inequality. Q.E.D.

Remark In fact since $B > 0$, there exists $C_4 \geq 1$ depending only on B such that $C_4^{-1}\Theta(x,y) \leq |x-y|_1 \leq C_4\Theta(x,y)$ for any x, $y \in \Delta_B$. Thus combining this with the trivial fact $|x-y|_2 \leq |x-y|_1 \leq \sqrt{d}|x-y|_2$, we can obtain the similar result to Lemma 3.4 using the usual Euclidean metric on Δ_B instead of Θ. We employ the projective metric because the expanding property (2) in Lemma 3.4 follows easily from well known fact Lemma 3.3.

§4. Central limit theorems

This section is devoted to the proofs of Theorem 1.1 and Theorem 1.2. We employ the transfer operator technique as used in [8], [17], and [21]. Let $\mathcal{T}_B$ be the renormalized Rauzy induction as in the previous section. Consider the Perron-Frobenius operator $\mathcal{L}$ of $\mathcal{T}_B$ with respect to ω_B which is characterized by the identity.

$$\int_{\Delta_B} \mathcal{L}f(x)g(x)\,d\omega_B = \int_{\Delta_B} f(x)g(\mathcal{T}_B x)\,d\omega_B \tag{4.1}$$

for any $f \in L^1(\omega_B)$ and for any $g \in L^\infty(\omega_B)$. In virtue of Lemma 3.1 and the formula (3.13), it is easy to see that

$$\mathcal{L}^n f(x) = \sum_{A \in \mathcal{A}^n} \frac{1}{|Ax|_1^d} f(\bar{A}x) \tag{4.2}$$

for any $f \in L^1(\omega_B)$ and for any $n \geq 1$, where $\bar{A}x = Ax/|Ax|_1^d$ as before. Thus $\mathcal{L}$ can be thought as a bounded operator on $C(\Delta_B)$ as well as $L^1(\omega_B)$. Let $F_\Theta(\Delta_B)$ be the totality of Lipschitz continuous functions on Δ_B with respect to Θ endowed with the norm

$$\|g\|_\Theta = [g]_\Theta + \|g\|_\infty, \tag{4.3}$$

where $\|g\|_\infty = \sup_{x \in \Delta_B} |g(x)|$ and $[g]_\Theta = \sup_{x,y \in \Delta_B : x \neq y} \frac{|g(x)-g(y)|}{\Theta(x,y)}$ i.e. the Lipschitz constant of g with respect to Θ. $F_\Theta(\Delta_B \to \mathbb{R})$ denotes the totality of real valued elements of $F_\Theta(\Delta_B)$. For $f \in F_\Theta(\Delta_B \to \mathbb{R})$ and $t \in \mathbb{C}$, we define a perturbed Perron-Frobenius operator of $\mathcal{L}(t)$: $L^1(\omega_B) \to L^1(\omega_B)$ by

$$\mathcal{L}(t)g = \mathcal{L}(e^{\sqrt{-1}tf}g). \tag{4.4}$$

Then it is easy to see that

$$\mathcal{L}(t)^n g = \mathcal{L}^n(e^{\sqrt{-1}tS_{\mathcal{T}_B,n}f}g). \tag{4.5}$$

holds for any $n \geq 1$. The following estimates play important roles in our argument.

Lemma 4.1. *There exist positive constant C_5, C_6, C_7, and C_8 independent of n and f such that the following estimates hold for any $g \in F_\Theta(\Delta_B)$ and for any $t \in \mathbb{C}$.*

$$\|\mathcal{L}(t)^n g\|_\infty \le C_5 e^{n|\mathrm{Im}t|\|f\|_\infty}\|g\|_\infty. \tag{4.6}$$

$$\begin{aligned}[\mathcal{L}(t)^n g]_\Theta \le\ & C_6(e^{|\mathrm{Im}t|\|f\|_\infty}\theta)^n[g]_\Theta \\ & + C_7 e^{n|\mathrm{Im}t|\|f\|_\infty}(1+e^{C_8[f]_\Theta|t|}[f]_\Theta|t|)\|g\|_\infty.\end{aligned} \tag{4.7}$$

Proof. First we note that there exists $C_9 \ge 1$ such that $1/|Ax|_1^d \in [C_9^{-1}, C_9]\omega_B(\Delta_{AB})$ holds for any $x \in \Delta_{AB}$, $A \in \mathcal{A}_n$ and $n \ge 1$, where $a \in [c^{-1}, c]b$ means $c^{-1}b \le a \le cb$. This follows from (3) in Lemma 3.4 and the fact that each entry of $x \in \Delta_B$ is bounded below by some constant depending only on B.

In what follows we write $f_{(n)} = S_{T_B,n}f$ for convenience.

$$\begin{aligned}|\mathcal{L}(t)^n g(x)| \le & \sum_{A\in\mathcal{A}_n} \frac{|e^{\sqrt{-1}tf_{(n)}(\bar{A}x)}|}{|Ax|_1^d}|g(\bar{A}x)| \\ \le & e^{n(\mathrm{Im}t)\|f\|_\infty}\|g\|_\infty \sum_{A\in\mathcal{A}_n}\frac{1}{|Ax|_1^d}\end{aligned}$$

Thus the first estimate is obtained by choosing $C_5 = C_9$

Next we show the second estimate.

$$\begin{aligned}&\mathcal{L}(t)^n g(x) - \mathcal{L}(t)^n g(y) \\ =&\sum_{A\in\mathcal{A}_n}\frac{e^{\sqrt{-1}tf_{(n)}(\bar{A}x)}}{|Ax|_1^d}g(\bar{A}x) - \sum_{A\in\mathcal{A}_n}\frac{e^{\sqrt{-1}tf_{(n)}(\bar{A}y)}}{|Ay|_1^d}g(\bar{A}y) \\ =&\sum_{A\in\mathcal{A}_n}\left(\frac{1}{|Ax|_1^d}-\frac{1}{|Ay|_1^d}\right)e^{\sqrt{-1}tf_{(n)}(\bar{A}x)}g(\bar{A}x) \\ =&\sum_{A\in\mathcal{A}_n}\frac{1}{|Ay|_1^d}\left(e^{\sqrt{-1}tf_{(n)}(\bar{A}x)} - e^{\sqrt{-1}tf_{(n)}(\bar{A}y)}\right)g(\bar{A}x) \\ =&\sum_{A\in\mathcal{A}_n}\frac{1}{|Ay|_1^d}e^{\sqrt{-1}tf_{(n)}(\bar{A}y)}(g(\bar{A}x)-g(\bar{A}y)) \\ =&I + II + III.\end{aligned}$$

Then we have

$$
(4.8)\qquad
\begin{aligned}
|I| &\leq \sum_{A\in\mathcal{A}_n} \frac{1}{|Ay|_1^d}\left|\frac{|Ax|_1^d}{|Ay|_1^d} - 1\right| e^{n|\mathrm{Im}t|\|f\|_\infty}\|g\|_\infty \\
&\leq \sum_{A\in\mathcal{A}_n} \frac{1}{|Ay|_1^d}\left|\exp(dC_3\Theta(x,y)) - 1\right| e^{n|\mathrm{Im}t|\|f\|_\infty}\|g\|_\infty \\
&\leq C_{10}e^{n|\mathrm{Im}t|\|f\|_\infty}\|g\|_\infty\Theta(x,y)
\end{aligned}
$$

in virtue of (3.12) and (4.6), where C_{10} is a positive constant independent of n and g. Next we have

$$
(4.9)\qquad
\begin{aligned}
|II| &\leq \sum_{A\in\mathcal{A}_n} \frac{1}{|Ay|_1^d}\left|e^{(\sqrt{-1}tf_{(n)}(\bar{A}x)-\sqrt{-1}tf_{(n)}(\bar{A}y))} - 1\right| \cdot \\
&\qquad\qquad \cdot e^{n|\mathrm{Im}t|\|f\|_\infty}\|g\|_\infty \\
&\leq \sum_{A\in\mathcal{A}_n} \frac{1}{|Ay|_1^d} e^{C_{11}[f]_\theta|t|\Theta(x,y)}\cdot \\
&\qquad\qquad \cdot C_{11}[f]_\theta|t|\Theta(x,y)e^{n|\mathrm{Im}t|\|f\|_\infty}\|g\|_\infty \\
&\leq C_5C_{11}e^{C_8[f]_\theta|t|}[f]_\theta|t|e^{n|\mathrm{Im}t|\|f\|_\infty}\|g\|_\infty\Theta(x,y),
\end{aligned}
$$

where C_{11} is a positive constant independent of n and g, and $C_8 = C_{11}\mathrm{diam}(\Delta_B)$. Finally we have

$$
(4.10)\qquad
\begin{aligned}
|III| &\leq \sum_{A\in\mathcal{A}_n} \frac{1}{|Ay|_1^d} e^{n|\mathrm{Im}t|\|f\|_\infty}[g]_\Theta C_2\theta^n\Theta(x,y) \\
&\leq C_2C_5(e^{|\mathrm{Im}t|\|f\|_\infty}\theta)^n[g]_\Theta\Theta(x,y)
\end{aligned}
$$

in virtue of (3.11). Thus if we put $C_6 = C_2C_5$ and $C_7 = C_9 + C_5C_{11}$, we see that the inequalities (4.8), (4.9), and (4.10) yield the estimate (4.7). Q.E.D.

In virtue of the estimates in Lemma 4.1 we can apply the Ionescu-Tulcea and Marinescu Theorem in [7] to the perturbed Perron-Frobenius operators $\mathcal{L}(t)$ with $t \in R$. More precisely, for each $t \in \mathbb{R}$, as a bounded linear operator on $F_\Theta(\Delta_B)$, $\mathcal{L}(t)$ has at most a finite number of eigenvalues of modulus 1 whose eigenspace are finite-dimensional and the other spectrum of $\mathcal{L}(t)$ is contained in the disc with radius less than 1. In addition the spectrum on the unit circle of $\mathcal{L}(t)$ as a bounded operator on $L^1(\omega_B)$ also consists of eigenvalues and their eigenspaces are identical with those of $\mathcal{L}(t)$ as an operator on $F_\Theta(\Delta_B)$. Further we can show the following.

Lemma 4.2. *Let f be an element in $F_\Theta(\Delta_B \to \mathbb{R})$ which is not a constant function. Assume that $\mathcal{L}(t)g = \lambda g$ holds for $t \in \mathbb{R}$, λ with modulus 1, and $g \in L^1(\omega_B)$ with $\|g\|_{L^1(\omega_B)} = 1$. Then it turns out that $t = 0$, $\lambda = 1$, and $g = ch_B$ for some constant c with $|c| = 1$, where $h_B \in F_\Theta(\Delta_B)$ is the smooth density function of the unique absolutely continuous invariant probability measure $\mu_B = \mu(\Delta_B)^{-1}\mu$ of $\mathcal{T}_B$ with respect to ω_B.*

Proof. Let t, λ, g be as above. First we show that $|g| = h_B$. In fact $|g| = |\mathcal{L}(t)g| \leq \mathcal{L}|g|$ since $\mathcal{L}$ is positive operator. In addition, (4.1) yields $\int_{\Delta_B} \mathcal{L}|g|\, d\omega_B = \int_{\Delta_B} |g|\, d\omega_B$. Thus we have $\mathcal{L}|g| = |g|$ in $L^1(\omega_B)$. In virtue of the Ionescu-Tulcea and Marinescu Theorem we see that $|g|$ has a Lipschitz continuous version as noted in the above. Then we have

$$\begin{aligned}(h_B^{-1}|g|)(x) =& h_B^{-1}(x)\mathcal{L}^n(h_B h_B^{-1}|g|)(x) \\ =& \sum_{A \in \mathcal{A}_n} h_B^{-1}(x)|Ax|_1^{-d} h_B(\bar{A}x)(h_B^{-1}|g|)(\bar{A}x)\end{aligned} \tag{4.11}$$

holds for any $x \in \Delta_B$ and for any $n \in \mathbb{N}$. On the other hand $\mathcal{L}h_B = h_B$ implies that

$$\sum_{A \in \mathcal{A}_n} h_B^{-1}(x)|Ax|_1^{-d} h_B(\bar{A}x) = 1 \tag{4.12}$$

holds for any $x \in \Delta_B$ and for any $n \in \mathbb{N}$. Thus if $x_0 \in \Delta_B$ satisfies $(h_B^{-1}|g|)(x_0) = \max_{x \in \Delta_B}(h_B^{-1}|g|)(x)$, (4.11) yields that $(h_B^{-1}|g|)(\bar{A}x_0) = (h_B^{-1}|g|)(x_0)$ holds for any $A \in \mathcal{A}_n$ and for any $n \in \mathbb{N}$. Note that we can easily see that $\{\bar{A}x : A \in \mathcal{A}_n, n \in \mathbb{N}\}$ is dense in Δ_B for any $x \in \Delta_B$ from the assertions (1) and (2) in Lemma 3.4. Hence we conclude that $h_B^{-1}|g|$ is a constant function.

Next we have

$$\begin{aligned}\lambda(h_B^{-1}g)(x) =& h_B^{-1}(x)\mathcal{L}(t)(h_B h_B^{-1}g)(x) \\ =& \sum_{A \in \mathcal{A}_1} \frac{e^{\sqrt{-1}tf(\bar{A}x)}}{h_B(x)|Ax|_1^d} h_B(\bar{A}x)(h_B^{-1}g)(\bar{A}x)\end{aligned} \tag{4.13}$$

for any $x \in \Delta_B$. Since $h_B^{-1}|g| = 1$, (4.12) and (4.13) imply that $(h_B^{-1}g)(x) = \bar{\lambda} e^{\sqrt{-1}tf(\bar{A}x)}(h_B^{-1}g)(\bar{A}x)$ holds for any $x \in \Delta_B$ and for any $A \in \mathcal{A}_1$. Consequently we have

$$\begin{aligned}|(h_B^{-1}g)(x) - (h_B^{-1}g)(y)| \leq& |e^{\sqrt{-1}tf(\bar{A}x)} - e^{\sqrt{-1}tf(\bar{A}x)}| \\ &+ |(h_B^{-1}g)(\bar{A}x) - (h_B^{-1}g)(\bar{A}y)|\end{aligned} \tag{4.14}$$

for any x, $y \in \Delta_B$ and for any $A \in \mathcal{A}_1$. Since we can choose a sequence $\{A_k\} \subset \mathcal{A}$ such that $\operatorname{diam}(\Delta_{A_k B}) \to 0$ $(k \to \infty)$ and f and $(h_B^{-1} g)$ are Lipschitz continuous with respect to Θ, the inequality (4.14) implies that $(h_B^{-1} g)$ is a constant function. Thus we conclude that $g = h_B$ and $\bar{\lambda} e^{tf} = 1$. Since f is continuous non constant function, $f(\Delta_B)$ is an interval with nonempty interior. Hence we can conclude that $\lambda = 1$ and $t = 0$. Q.E.D.

Clearly $\{\mathcal{L}(t) : t \in \mathbb{C}\}$ is an analytic family of bounded linear operators on $F_\Theta(\Delta_B)$ as well as $L^1(\omega_B)$. Moreover we see the following in virtue of Lemma 4.2.

Proposition 4.3. *There exist a neighborhood U of the real line $\mathbb{R}$ and $r_0 > 0$ such that the open disc $D(0, r_0)$ of radius r_0 centered at the origin is contained in U and the analytic family $\{\mathcal{L}(t) : t \in U\}$ of bounded linear operators on $F_\Theta(\Delta_B)$ satisfies the following:*

(1) *For $t \in D(0, r_0)$, $\mathcal{L}(t)$ has the spectral decomposition*

$$\mathcal{L}(t)^n = \lambda(t)^n E(t) + R(t)^n \tag{4.15}$$

for each $n \in \mathbb{N}$, where $\lambda(t)$ is a simple eigenvalue of $\mathcal{L}(t)$ with maximal modulus, $E(t)$ is the projection onto the one-dimensional eigenspace corresponding to $\lambda(t)$, and $R(t)$ is a bounded linear operator with spectral radius less than r_1 for some $r_1 \in (0,1)$ independent of $t \in D(0, r_0)$.

(2) *For $t \in U$ with $|Re\ t| \geq r_0$, the spectral radius of $\mathcal{L}(t)$ is less than 1.*

(3) *$\lambda(t)$ in the assertion (1) is a analytic function on $D(0, r_0)$ such that*

$$\begin{aligned} &\lambda(0) = 1, \quad \lambda'(0) = 0, \quad \text{and} \\ &\lambda''(0) = -\lim_{n\to\infty} \frac{1}{n} \int_{\Delta_B} f_{(n)}^2 \, d\mu_B = -v(f) < 0. \end{aligned} \tag{4.16}$$

(4) *$E(t)$ and $R(t)$ in the assertion (1) are analytic functions on $D(0, r_0)$ with values in bounded linear operators on $F_\Theta(\Delta_B)$ given by the Dunford integrals*

$$\begin{aligned} E(t) &= \frac{1}{2\pi\sqrt{-1}} \int_{|z-1|=r_2} R(\mathcal{L}(t), z)\, dz, \\ R(t)^n &= \frac{1}{2\pi\sqrt{-1}} \int_{|z|=r_1} z^n R(\mathcal{L}(t), z)\, dz \end{aligned} \tag{4.17}$$

for each $n \in \mathbb{N}$, where $0 < r_1, r_2 < 1$ are independent of $t \in D(0, r_0)$ satisfying $r_1 + r_2 < 1$ and $R(\mathcal{L}(t), z) = (zI - \mathcal{L}(t))^{-1}$ denotes the resolvent operator of $\mathcal{L}(t)$.

Proof. Combining Lemma 4.2 and Ionescu-Tulcea and Marinescu Theorem, we can see that if $t \in \mathbb{R} \setminus \{0\}$, the spectral radius of $\mathcal{L}(t)$ is less than 1 and $\mathcal{L}(0) = \mathcal{L}$ has an eigenvalue 1 and the other spectrum of $\mathcal{L}$ is contained in $D(0, r)$ for some $0 < r < 1$. Thus all the assertions are easy consequences of the general perturbation theory for bounded linear operators (see Section VII-6 in [5] and [9]) except for the equations on $\lambda'(0)$ and $\lambda''(0)$ in (4.16).

Let g be an element in $F_\Theta(\Delta_B \to \mathbb{R})$. We consider the characteristic function $\varphi_{g,n}(t) = \int_{\Delta_B} e^{\sqrt{-1}tf_{(n)}} g\, d\omega_B$ of the distribution of $f_{(n)}$ with respect to the signed measure $g\omega_B$. Then by (4.1) and (4.5) for $t \in D(0, r_0)$ we have

$$\varphi_{g,n}(t) = \int_{\Delta_B} \mathcal{L}(t)^n g\, d\omega_B = \lambda(t)^n e_g(t) + r_{g,n}(t), \tag{4.18}$$

where

$$e_g(t) = \int_{\Delta_B} E(t) g\, d\omega_B \quad \text{and} \quad r_{g,n}(t) = \int_{\Delta_B} R(t)^n g\, d\omega_B.$$

Note that $e_g(t)$ and $r_{g,n}(t)$ are analytic functions on $D(0, r_0)$ satisfying

$$e_g(0) = \int_{\Delta_B} g\, d\omega_B, \ r_{g,n}(0) = 0, \ \text{and} \ \left| \frac{d^k r_{g,n}}{dr^k}(t) \right| \leq c_k r_1^n \|g\|_\Theta \tag{4.19}$$

for each $k \geq 0$ where c_k is a constant independent of $t \in D(0, r_0)$ and $k \geq 0$ by (4.17). First we differentiate the functions in (4.18) after substituting t/n for t and h_B for g. Next substitute 0 for t. Then we have

$$\sqrt{-1} \int_{\Delta_B} \frac{f_{(n)}}{n}\, d\mu_B = \lambda'(0) + \frac{1}{n} e'_{h_B}(0) + \frac{1}{n} r'_{h_B,n}(0).$$

Letting $n \to \infty$, we obtain $\lambda'(0) = 0$ in virtue of the Birkhoff Ergodic Theorem. Similarly, first we differentiate twice the functions in (4.18) after substituting $t/\sqrt{n}$ for t and next substitute 0 for t. Since $\lambda'(0) = 0$, we have

$$-\int_{\Delta_B} \frac{f_{(n)}^2}{n}\, d\mu_B = \lambda''(0) + \frac{1}{n} e''_{h_B}(0) + \frac{1}{n} r''_{h_B,n}(0).$$

Letting $n \to \infty$, we obtain $\lambda''(0) = -v(f)$. It remains to prove the positivity of $v(f)$. Since the autocorrelation of f decays exponentially fast by the spectral decomposition (4.15), we can apply the Leonov's result (see Chapter 18 in [6]). Therefore we can find a real valued element $u \in L^2(\mu_B)$ such that $f = u \circ \mathcal{T}_B - u$. Then we have $e^{\sqrt{-1}f} e^{\sqrt{-1}u} h_B = (e^{\sqrt{-1}u} \circ \mathcal{T}_B) h_B$. Operate $\mathcal{L}$ on both sides of this equation, we see that $\mathcal{L}(1)(e^{\sqrt{-1}u} h_B) = e^{\sqrt{-1}u} h_B$. This contradicts the result in Lemma 4.2. Hence we have seen that $v(f) > 0$. Q.E.D.

Consider $\varphi_{g,n}(t) = \int_{\Delta_B} e^{\sqrt{-1}tf_{(n)}} g \, d\omega_B$ of the distribution of $f_{(n)}$ with respect to the signed measure $g\omega_B$ as above.. Now we can show the following.

Theorem 4.4. *There exists $t_0 > 0$ such that if $|t| \le t_0\sqrt{n}$, we have*

$$\left|\varphi_{g,n}(\frac{t}{\sqrt{n}}) - \left(\int_{\Delta_B} g \, d\omega_B\right) \exp\left(-\frac{v(f)t^2}{2}\right)\right| \le \left(\left(C_{12}\frac{|t|^3}{\sqrt{n}} + C_{13}\frac{|t|}{\sqrt{n}}\right) \exp\left(-\frac{v(f)t^2}{4}\right) + C_{14}\frac{|t|}{\sqrt{n}} r_1^n\right) \|g\|_\Theta, \tag{4.20}$$

where $r_1 \in (0,1)$ are the same constants as in Proposition 4.3 and C_{12}, C_{13}, and C_{14} are positive constants independent of g and n.

Proof. By Proposition 4.3, there exists an analytic function w in $D(0, r_0)$ such that

$$\lambda(t) = 1 - \frac{v(f)t^2}{2} + w(t)t^3.$$

Thus $0 < t_0 < r_0$ is small enough, there exists $C_{15} > 0$ such that

$$\left|\lambda(\frac{t}{\sqrt{n}})^n - \exp\left(-\frac{v(f)t^2}{2}\right)\right| \le C_{15} \exp\left(-\frac{v(f)t^2}{4}\right). \tag{4.21}$$

On the other hand by using (4.18) and the fact $r_{g,n}(0) = 0$, we have

$$\begin{aligned}\varphi_{g,n}(\frac{t}{\sqrt{n}}) =& \lambda(\frac{t}{\sqrt{n}})^n e_g(0) + \lambda(\frac{t}{\sqrt{n}})^n (e_g(\frac{t}{\sqrt{n}}) - e_g(0)) \\ &+ r_{g,n}(\frac{t}{\sqrt{n}}) - r_{g,n}(0).\end{aligned} \tag{4.22}$$

Applying the inequality in (4.19) and the inequality (4.21) to (4.22), it is not hard to show the desired result. Q.E.D.

Now we are in a position to prove Theorem 1.1 and Theorem 1.2.

Proof of Theorem 1.1. Recall the Berry-Esseen inequality (see [6]). Let F and G be distribution functions of probability measures on $\mathbb{R}$. Assume that G is differentiable. Then we have

$$\sup_{x\in\mathbb{R}} |F(x) - G(x)| \leq \frac{2}{\pi}\int_0^t \frac{h(u)}{u}\, du + \frac{24}{\pi t}\sup_{x\in\mathbb{R}} |G'(x)|, \tag{4.23}$$

where $h(u) = \int_{-\infty}^{\infty} e^{\sqrt{-1}ux}\, dF(x) - \int_{-\infty}^{\infty} e^{\sqrt{-1}ux}\, dG(x)$.

If we consider the special case when g in Theorem 4.4 is a probability density, then Theorem 4.4 gives an estimate of the function which plays the role of h in (4.23). Thus we can obtain the desired result by applying the Berry-Esseen inequality to the distribution function F of $f_{(n)}$ with respect to $g\omega_B$ and the distribution function G of the normal distribution $N(0, v(f))$. The result for general g is its easy consequence.

Proof of Theorem 1.2. Let g be a element in $F_\Theta(\Delta_B \to \mathbb{R})$. Since $g^\pm$ satisfies $[g^\pm]_\Theta \leq [g]_\Theta$, we may assume that $g \geq 0$ and $\int_{\Delta_B} g\, d\omega_B = 1$.

First we consider the case when u satisfies $\hat{u} \in \mathcal{D}((-k,k))$ for some $k > 0$, where $\mathcal{D}(K)$ denotes the space of test function with support in a set $K \subset \mathbb{R}$. Let α be any real number. Then we have

$$\sqrt{n}\int_{\Delta_B} u(\alpha + f_{(n)}(x))g(x)\, d\omega_B = \frac{\sqrt{n}}{2\pi}\int_{\mathbb{R}} \hat{u}(t)\varphi_{g,n}(t)e^{\sqrt{-1}\alpha t}\, dt.$$

Therefore we see that

$$\begin{aligned}
&\sqrt{n}\int_{\Delta_B} u(\alpha + f_{(n)})g\, d\omega_B - \left(\int_{\mathbb{R}} u(\xi)\, d\xi\right)\frac{1}{\sqrt{2\pi v(f)}}e^{-\frac{\alpha^2}{2n v(f)}} \\
=&\frac{\sqrt{n}}{2\pi}\int_{\mathbb{R}} \hat{u}(t)\varphi_{g,n}(t)e^{\sqrt{-1}\alpha t}\, dt - \hat{u}(0)\frac{1}{\sqrt{2\pi v(f)}}e^{-\frac{\alpha^2}{2n v(f)}} \\
=&R_1(n) + R_2(n) + R_3(n) + R_4(n),
\end{aligned} \tag{4.24}$$

where

$$\begin{aligned}
R_1(n) =&\frac{\sqrt{n}}{2\pi}\int_{\epsilon_n \leq |t| \leq k} \hat{u}(t)\varphi_{g,n}(t)e^{\sqrt{-1}\alpha t}, \\
R_2(n) =&\frac{1}{2\pi}\int_{|t|<\epsilon_n\sqrt{n}} (\hat{u}(\frac{t}{\sqrt{n}}) - \hat{u}(0))\varphi_{g,n}(\frac{t}{\sqrt{n}})e^{\sqrt{-1}\alpha\frac{t}{\sqrt{n}}}, \\
R_3(n) =&\left(\frac{1}{2\pi}\int_{|t|<\epsilon_n\sqrt{n}} (\varphi_{g,n}(\frac{t}{\sqrt{n}}) - e^{-\frac{v(f)t^2}{2}})e^{\sqrt{-1}\alpha\frac{t}{\sqrt{n}}}\right)\hat{u}(0), \\
R_4(n) =&-\left(\frac{1}{2\pi}\int_{|t|>\epsilon_n\sqrt{n}} e^{-\frac{v(f)t^2}{2}+\sqrt{-1}\alpha\frac{t}{\sqrt{n}}}\, dt\right)\hat{u}(0).
\end{aligned}$$

The sequence ϵ_n in the above is chosen so that

$$\epsilon_n \downarrow 0, \quad \epsilon_n\sqrt{n} \uparrow \infty, \quad \text{and} \quad \epsilon_n^4 n^{\frac{3}{2}} \downarrow 0 \quad (n \to \infty). \tag{4.25}$$

Now we have

$$\begin{aligned} |R_1(n)| \leq & \frac{\sqrt{n}}{2\pi} \int_{\epsilon_n \leq |t| \leq k} \|\mathcal{L}(t)^n\|_\Theta \, dt \|g\|_\Theta \|\hat{u}\|_\infty \\ = & \frac{\sqrt{n}}{2\pi} \left(\int_{\epsilon_n \leq |t| \leq t_0} + \int_{t_0 < |t| \leq k} \right) \|\mathcal{L}(t)^n\|_\Theta \, dt \|g\|_\Theta \|\hat{u}\|_\infty \end{aligned}$$

In virtue of (1) in Proposition 4.3 and the choice of t_0 in Theorem 4.4, we see that

$$\|\mathcal{L}(t)^n\|_\Theta \leq \begin{cases} C_{16}((1 - \frac{v(f)t^2}{4})^n + r_1^n) & \text{if } |t| \leq t_0 \\ C_1(k) r(k)^n & \text{if } t_0 < |t| \leq k, \end{cases}$$

where C_{16} is a positive constant depending only on $\mathcal{T}_B$ and f, $C_1(k)$ a positive constant independent of g but dependent on k, and $r(k) < 1$ denotes the supremum of the spectral radius of $\mathcal{L}(t)$ with t running over the set $[-k, -t_0) \cup (t_0, k]$. Thus we conclude that there exists a positive number $C_2(k)$ such that

$$|R_1(n)| \leq C_2(k)\sqrt{n} \left(1 - \frac{v(f)\epsilon_n^2}{4}\right)^n \|g\|_\Theta \|\hat{u}\|_\infty \tag{4.26}$$

It is easy to see that

$$|R_2(n)| \leq \frac{1}{2\pi} \epsilon_n^2 \sqrt{n} \|g\|_\Theta \|\hat{u}'\|_\infty. \tag{4.27}$$

Next in virtue of Theorem 4.4, we obtain

$$|R_3(n)| \leq C_{17} \left(\epsilon_n^4 n^{\frac{3}{2}} + \epsilon_n^2 n^{\frac{1}{2}} \right) \|g\|_\Theta \|\hat{u}\|_\infty \tag{4.28}$$

for some C_{17} depending only on $\mathcal{T}_B$ and f. Finally we have

$$|R_4(n)| \leq \left(\frac{1}{2\pi} \int_{|t| > \epsilon_n \sqrt{n}} e^{-\frac{v(f)t^2}{2}} \, dt \right) \|g\|_\Theta \|\hat{u}\|_\infty. \tag{4.29}$$

Combining (4.26), (4.27), (4.28), and (4.29), we see that there exist $C_3(k) > 0$ depending only on $\mathcal{T}_B$, f, and k, and γ_n depending only on $\mathcal{T}_B$, f with $\gamma_n \to 0$ $(n \to \infty)$ such that

$$\begin{aligned} & |R_1(n) + R_2(n) + R_3(n) + R_4(n)| \\ \leq & \; C_3(k)\gamma_n \left(\|\hat{u}\|_\infty + \|\hat{u}'\|_\infty\right) \|g\|_\Theta. \end{aligned} \tag{4.30}$$

Combining (4.24) with (4.30), we obtain

$$\left|\sqrt{n}\int_{\mathbb{R}} \hat{u}(t)\varphi_{g,n}(t)e^{\sqrt{-1}\alpha t}\,dt\right| \leq C_4(k)\left(\|\hat{u}\|_\infty + \|\hat{u}'\|_\infty\right)\|g\|_\Theta \tag{4.31}$$

for some $C_4(k)$ depending only on $\mathcal{T}_B$, f, and k. Hence we conclude that $\{\sqrt{n}\varphi_{g,n}(\cdot)e^{\sqrt{-1}\alpha(\cdot)} : n \in \mathbb{N}, \alpha \in \mathbb{R}\}$ is a bounded family in the distribution space $\mathcal{D}((-k,k))'$. Since each $\sqrt{n}\varphi_{g,n}(\cdot)e^{\sqrt{-1}\alpha(\cdot)}$ is a distribution of positive type, the family turns out to be a bounded family in the space of $\mathcal{B}(\mathbb{R})'$ of the bounded distributions (see Chapter VII in [22]).

Next we choose any $u \in \mathcal{S}(\mathbb{R})$ and fix it. Let $\mu_{g,n,\alpha}$ denotes the Radon measure such that $\hat{\mu}_{g,n,\alpha} = \sqrt{n}\varphi_{g,n}(\cdot)e^{\sqrt{-1}\alpha(\cdot)}$. Since $\{\hat{\mu}_{g,n,\alpha}\}$ is a bounded set in $\mathcal{B}(\mathbb{R})'$ and $\hat{u} \in \mathcal{S}(\mathbb{R}) \subset \mathcal{B}(\mathbb{R})$, we have

$$\begin{aligned}&\sup_{s\in\mathbb{R}}\left|\int_{\mathbb{R}} u(t+s)\,\mu_{g,n,\alpha}(dt)\right| \\ &= \sup_{s\in\mathbb{R}}\left|\frac{1}{2\pi}\int_{\mathbb{R}} \hat{u}(t)\sqrt{n}\varphi_{g,n}(t)e^{\sqrt{-1}(\alpha+s)t}(dt)\right| \leq C_1(u),\end{aligned} \tag{4.32}$$

where $C_1(u) > 0$ is a constant depending on u but not on n. On the other hand $\{u(s,t) = s^{-1}(u(\cdot+s) - u(\cdot)) : 0 < |s| \leq 1\}$ is a bounded set in $\mathcal{S}(\mathbb{R})$, and so a bounded set in $\mathcal{B}(\mathbb{R})$. Therefore we have

$$\sup_{\alpha\in\mathbb{R}}\sup_{s:0<|s|\leq 1}\left|\int_{\mathbb{R}} u(s,t)\,\mu_{g,n,\alpha}(dt)\right| \leq C_2(u), \tag{4.33}$$

where $C_2(u) > 0$ is also a constant depending on u but not on α and n.

Let $\{\rho_j\}$ be a sequence of probability measures on $\mathbb{R}$ which converges to $\delta(0)$ (the unit mass at the origin) weakly as $j \to \infty$ such that $\hat{\rho}_j \in \mathcal{D}(\mathbb{R})$ for each j. Choose any $\delta \in (0,1)$. We consider

$$\begin{aligned}&\left|\int_{\mathbb{R}} u(t)(\rho_j * \mu_{g,n,\alpha})(dt) - \int_{\mathbb{R}} u(t)\mu_{g,n,\alpha}(dt)\right| \\ &\leq \int_{|s|<\delta}\rho_j(ds)\left|\int_{\mathbb{R}}(u(t+s)-u(t))\,\mu_{g,n,\alpha}(dt)\right| \\ &\qquad + \int_{|s|\geq\delta}\rho_j(ds)\left|\int_{\mathbb{R}}(u(t+s)-u(t))\,\mu_{g,n,\alpha}(dt)\right| \\ &= I_n(\delta) + II_n(\delta).\end{aligned} \tag{4.34}$$

By (4.32) we have

$$|I_n(\delta)| \leq \int_{|s|<\delta}\rho_j(ds)|s|\left|\int_{\mathbb{R}} u(s,t)\,\mu_{g,n,\alpha}(dt)\right| \leq C_2(u)\delta.$$

Next by (4.33) we have

$$|II_n(\delta)| \le \rho_j(|s| \ge \delta) 2C_1(u).$$

Thus we have

$$|I_n(\delta) + II_n(\delta)| \le C_2(u)\delta + 2C_1(u)\rho_j(|s| \ge \delta). \tag{4.35}$$

On the other hand (4.24) and (4.30) yield that for fixed j, we obtain

$$\sup_{\alpha\in\mathbb{R}} \left| \int_{\mathbb{R}} u(t)(\rho_j * \mu_{g,n,\alpha})(dt) - \hat{u}(0)\hat{\rho}_j(0)\frac{1}{\sqrt{2\pi v(f)}} e^{-\frac{\alpha^2}{2\pi v(f)}} \right| \to 0 \tag{4.36}$$

Hence in virtue of (4.35), (4.36), we conclude that

$$\limsup_{n\to\infty} \sup_{\alpha\in\mathbb{R}} \left| \sqrt{n} \int_{\Delta_B} u(\alpha + f_{(n)}) g \, d\omega_B - \hat{u}(0)\frac{1}{\sqrt{2\pi v(f)}} e^{-\frac{\alpha^2}{2\pi v(f)}} \right|$$
$$\le C_2(u)\delta + 2C_1(u)\rho_j(|s| \ge \delta).$$

Letting $j \to \infty$, we see that the left hand side in the above is bounded by $C_2(u)\delta$. Since $\delta \in (0, 1)$ is arbitrary, the proof of Theorem 1.2 is now complete.

Remark Since the metrics $\Theta(x, y)$ and $|x - y|_1$ are equivalent on Δ_B by Remark at the end of the previous section, the totality $Lip(\Delta_B)$ of Lipschitz continuous functions with repect to the Euclidean metric on Δ_B coincides with $F_\Theta(\Delta_B)$ and its Banach norm $\|g\|_{Lip} = Lip(g) + \|g\|_\infty$ is equivalent to $\|g\|_\Theta$. Thus all the results in this section are valid if the space $(F_\Theta(\Delta_B), \|\cdot\|_\Theta)$ is replaced by $(Lip(\Delta_B), \|\cdot\|_{Lip})$. We employ $F_\Theta(\Delta_B)$ just for the sake of convenience.

References

[1] A. Avila and A. Bufetov, Exponential decay of correlations for the Rauzy-Veech-Zorich induction map, arXiv:math.DS/0610628.

[2] A. Avila, S. Gouëzel and J.-P. Yoccoz, Exponential mixing of the Teichmüller flow, arXiv:math.DS/0511614.

[3] R. Bowen and C. Series, Markov maps associated to Fuchsian groups, Publ. Math. Inst. Hautes Études Sci., **50** (1980), 401–418.

[4] A. Bufetov Decay of correlations for the Rauzy-Veech-Zorich induction map on the space of interval exchange transformations and the central limit theorem for the Teichmüller flow on the moduli space of abelian differentials, J. Amer. Math. Soc., **19** (2006), 579–623.

[5] N. Dunford and J. T. Schwartz, Linear operators. I, Interscience, 1955.
[6] I. A. Ibragimov and Y. V. Linnik, Independent and stationary sequences of random variables, Wolter-Norolhoff, 1971.
[7] C. Ionescu-Tulcea and G. Marinescu, Théorie ergodique pour des classes d'operations non coplètement continues, Ann. of Math. (2), **52** (1950), 141–147.
[8] H. Ishitani, A central limit theorem of mixed type for a class of 1-dimensional transformations, Hiroshima Math. J., **16** (1986), 161–188.
[9] T. Kato, Perturbation theory for linear operators, Springer-Verlag, 1976.
[10] H. Keynes and D. Newton, A minimal and non-uniquely ergodic interval exchange transformation, Math. Z., **148** (1976), 101–105.
[11] M. Keane, Interval exchange transformations, Math. Z., **141** (1975), 25–31.
[12] M. Keane, Non-ergodic interval exchange transformations, Israel J. Math., **26** (1977), 188–196.
[13] C. Liverani, Decay of correlations, Ann. of Math. (2), **142** (1995), 239–301.
[14] H. Masur, Interval exchange transformations and measured foliations, Ann. of Math. (2), **1315** (1982), 169–200.
[15] D. Mayer On the thermodynamic formalism for the Gauss map, Comm. Math. Phys., **130** (1990), 311–333.
[16] D. Mayer The thermodynamic formalism approach to Selberg's zeta functions for $PSL(2,\mathbb{Z})$, Bull. Amer. Math. Soc., **25** (1991), 55–60.
[17] T. Morita, Local limit theorem and distribution of periodic orbits of Lasota-Yorke transformations with infinite Markov partition, J. Math. Soc. Japan, **46** (1994), 309–343.
[18] T. Morita, Markov systems and transfer operators associated with cofinite Fuchsian groups, Ergodic Theory Dynam. Systems, **17** (1997), 1147–1181.
[19] M. Pollicott, Distributions of closed geodesics on the modular surface and quadratic irrationals, Bull. Soc. Math. France, **114** (1986), 431–446.
[20] G. Rauzy, Exchange d'intervales et transformations induites, Acta. Arith., **34** (1979), 315–328.
[21] J. Rousseau-Egele, Un thèoréme de la limite locale pour une classes de transformations dilatantes et monotones par maruceaux, Ann. Prob., **11** (1983), 772–788.
[22] L. Schwartz, Théorie de distributions, Hermann, 1966.
[23] C. Sries, The modular surface and continued fractions, J. London. Math. Soc., **31** (1985), 69–80.
[24] W. Veech, Interval exchange transformations, J. Analyse Math., **33** (1978), 222–278.
[25] W. Veech, Gauss measures for transformations on the space of interval exchange maps, Ann. of Math. (2), **115** (1982), 201–242.
[26] W. Veech, The metric theory of interval exchange transformations I. Generic spectral properties, Amer. J. Math., **106** (1984), 1331–1359.
[27] A. Zorich, Finite Gauss measure on the space of interval exchange transformations. Lyapunov exponents, Ann. Inst. Fourier, **46** (1996), 325–370.

Takehiko Morita
Department of Mathematics
Graduate School of Science
Hiroshima University
Higashi-Hiroshima 739–8526
Japan