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Low discrepancy sequences generated by dynamical systems

Makoto Mori

Abstract.

This is a survey of papers([3],[7],[8],[10],[11]). Uniformly distributed sequences generated by dynamical systems are considered. Relations between discrepancies and the spectra of the associated Perron-Frobenius operator are discussed. Examples of two and three dimensional low discrepancy sequences are also given.

§1. One dimensional cases

First, we will consider one dimensional cases. Let F be a piecewise linear transformation from $[0,1]$ into itself. One of the main tool to study the ergodic properties of the dynamical system is the spectra of the Perron–Frobenius operator P associated with F defined by

$$\int f(x)g(F(x))\,dx = \int Pf(x)g(x)\,dx \quad (f \in L^1, g \in L^\infty),$$

or in other words,

$$Pf(x) = \sum_{y:\ F(y)=x} f(y)|F'(y)|^{-1}.$$

For example,

(1) the eigenfunction $\rho \geq 0$ associated with eigenvalue 1 of the Perron–Frobenius operator is a density function of an invariant measure,
(2) assume that 1 is a simple eigenvalue and there exists no other eigenvalue modulus 1, then the dynamical system is mixing,

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that is,

$$\int f(x)g(F^n(x))\,dx \to \int f\,dx \times \int g\,d\mu,$$

where μ is the unique invariant probability measure.

Originally the Perron–Frobenius operator is defined on L^1, the set of integrable functions. But, as is well known(cf. [4]), all the eigenfunctions associated with eigenvalue 1 in modulus belongs to BV (in particular, the density function of invariant measure belongs to BV). So we restrict its domain to BV, the set of functions with bounded variation. Here, we consider BV as a subset of L^1 and on it we introduce a norm by

$$||f|| = ||f||_1 + V(f),$$

where $||\cdot||_1$ is the L^1 norm and $V(\cdot)$ is the total variation. Then, the second greatest eigenvalue in modulus determines the decay rate of correlation.

In these results, what we need in this article are the following:

Theorem 1. *Assume that the absolute value of the slope of F is constant β and greater than 1. Then 1 is the greatest eigenvalue of the Perron-Frobenius operator and the essential spectral radius equals β^{-1}. Moreover, if the dynamical system generated by F is topologically transitive, then it becomes mixing, that is, 1 is a simple eigenvalue and no other eigenvalue exists on the unit circle. Let η be the second greatest eigenvalue in modulus, then, roughly speaking, the decay rate of correlation satisfies*

$$\int f(x)g(F^n(x))\,dx - \int f\,dx \times \int g\,d\mu = O(|\eta|^n).$$

where μ is the unique invariant probability measure.

Any z ($|z| < \beta{-}1$) is an eigenvalue of P. Hence we call β^{-1} the essential spectrum radius.

Hereafter, we assume that F is topologically transitive and the absolute value of the slope β is greater than 1. Then there exists a unique invariant probability measure μ with its density function ρ which satisfies $P\rho = \rho$, and the dynamical system is mixing.

However, the Perron–Frobenius operator is not compact, so it is not easy to find its spectra. One of the tool is the dynamical zeta function defined by

$$\zeta(z) = \exp\left[\sum_{n=1}^{\infty} \frac{z^n}{n} \sum_{p:\ F^n(p)=p} |F^{n\prime}(p)|^{-1}\right].$$

Then its singularities in $|z| < \beta$ is proved to be the reciprocals of the spectra of the Perron–Frobenius operator. These results are proved by many authors such as Baladi, Hofbauer, Keller, Ruelle etc.(cf. [1, 2, 12]) in a different way. Another way developed by the author is to construct a renewal equation described in Lemma 2(cf. [5, 6]). Using this renewal equation, we can prove the solution of $\det(I-\Phi(z)) = 0$ is the reciprocals of the eigenvalue of the Perron–Frobenius operator in $|z| < \beta$. This looks like the Fredholm determinant of nuclear operators, so we call $\Phi(z)$ the Fredholm matrix and $\det(I - \Phi(z))$ the Fredholm determinant. The Fredholm determinant for nuclear operators is an entire function, but our Fredholm determinant is analytic only in $|z| < \beta$ when F is not Markov. On the other hand, we can prove

$$\zeta(z) = \frac{1}{\det(I - \Phi(z))}.$$

This shows that the zeros of the Fredholm determinant coincide with the singularities of the zeta function.

Now using F, we will construct uniformly distributed sequences. Let $\mathcal{A}$ be an alphabet associated with F:

(1) $\langle a \rangle$ is a subinterval corresponding to a $(a \in \mathcal{A})$,
(2) F is monotone on $\langle a \rangle$,
(3) $\{\langle a \rangle\}_{a \in \mathcal{A}}$ becomes a partition of $[0, 1]$,
(4) $\operatorname{sgn} a = \begin{cases} +1 & \text{if } F' > 0 \text{ on } \langle a \rangle, \\ -1 & \text{if } F' < 0 \text{ on } \langle a \rangle. \end{cases}$

We call a finite sequence of symbols $a_1 \cdots a_n$ a word $(a_i \in \mathcal{A},\ 1 \leq i \leq n)$, and

(1) $|w| = n$,
(2) $\langle w \rangle = \bigcap_{i=1}^{n} F^{-i+1}(\langle a_i \rangle)$.

For a finite or infinite sequence of symbols $s = a_1 a_2 \cdots$, we use notations $s[m, n] = a_m \cdots a_n$ and $s[m] = a_m$. For convenience, we consider the empty word ϵ, and set $|\epsilon| = 0$, $\langle \epsilon \rangle = [0, 1]$ $\epsilon[1] = \epsilon$ and $\operatorname{sgn} \epsilon = +1$. We call a word w admissible if $\langle w \rangle \neq \emptyset$, and denote the set of all the admissible words with length n by $\mathcal{W}_n$, and $\mathcal{W} = \cup_{n=0}^{\infty} \mathcal{W}_n$.

For a point $x \in [0, 1]$ and a word $w \in \mathcal{W}$, if there exists a point y such that $y \in \langle w \rangle$ and $F^{|w|}(y) = x$, we call wx exists, and denote y by wx.

We will define order on the set of wx. First note that on $\mathcal{A}$, there exists a natural order. Let $w = a_1 \cdots a_n$ and $w' = b_1 \cdots b_m$. Then $wx < w'x$ if one of the followings holds.

(1) $n < m$,

(2) $n = m$ and $a_n \cdots a_1$ is less than $b_m \cdots b_1$ in lexicographical order.

We call the sequence of $\{wx\}$ for which wx exists, a van der Corput sequence generated by F. For convenience, we denote them by

$$w_1x, w_2x, w_3x, \dots .$$

Using above notations, we get

$$P^n f(x) = \beta^{-n} \sum_{|w|=n} f(wx),$$

where we ignore w for which wx does not exist. Thus taking f an indicator function of an interval J,

$$\beta^n P^n 1_J(x) = \#\{wx \in J \colon |w| = n\}. \tag{1}$$

This equals the number of visit to J of wx with $w \in \mathcal{W}_n$.

For a uniformly distributed sequence $\{x_n\}$, we define its discrepancy by

$$D_N = \sup \left| \frac{\#\{w_n x \in J \colon n \leq N\}}{N} - |J| \right|,$$

where supremum is taken over all the intervals and $|J|$ is the Lebesgue measure of J.

Now we get a proposition:

Proposition 1. *Let F be topologically transitive and the absolute value of the slope $\beta > 1$. Then*

(1) *for any $x \in [0,1]$, the van der Corput sequence is uniformly distributed.*
(2) *Let η be the second greatest eigenvalue of P in modulus. Then the order of the discrepancy is less than $(|\eta| + \varepsilon)^{\log N/\log \beta}$.*
(3) *There exists a measurable set E such that $|E| > 0$ and for $x \in E$ the discrepancy is greater than $(|\eta| - \varepsilon)^{\log N/\log \beta}$ for any $\varepsilon > 0$.*

This proposition says that if there exists an eigenvalue in the annulus $\frac{1}{\beta} < |z| < 1$, then our van der Corput sequence is not of low discrepancy. Even if such an eigenvalue does not exist, that is $|\eta| = \frac{1}{\beta}$, to show that our sequence is of low discrepancy ($D_N \sim \frac{\log N}{N}$), we need more detailed discussion, which we will give in the next section.

Proof. From the equation (1) and the fact that 1 is the simple greatest eigenvalue of P and that the dynamical system is mixing, we get

$$\#\{wx \in J \colon |w| = n\} = \beta^n(|J|\rho(x) + o(1)).$$

Taking $J = [0,1]$, we get

$$\#\{\exists wx \colon |w| = n\} = \beta^n(\rho(x) + o(1)).$$

Thus we get the proof of (1). From the assumption of (2), we get for any $\varepsilon > 0$

$$\#\{wx \in J \colon |w| = n\} = \beta^n \left\{|J|\rho(x) + o((|\eta| + \varepsilon|)^n)\right\}.$$

This leads to the proof of (2). Now assume that for almost all $x \in [0,1]$, the discrepancy is less than $|\eta|^{\log N/\log\beta}$. Then, of course, for any word $\langle w \rangle$ the discrepancy is less than $|\eta|^{\log N/\log\beta}$. For any bounded variation function f, there exists a decomposition

$$f(x) = \sum_w C_w 1_{\langle w \rangle}(x),$$

such that

$$\sum_{|w|=n} |C_w| \le V(f).$$

This says there exists $\varepsilon > 0$ and

$$P^n f(x) = \int f(x)\,dx \times \rho(x) + o((|\eta| - \varepsilon)^n).$$

This contradicts that η is an eigenvalue of P on BV. This proves (3).
Q.E.D.

1.1. Symbolic dynamics and generating functions

An endpoint of $\langle a \rangle$, say x, is called Markov endpoint if there exists n such that $\lim_{\substack{y \to x \\ y \in \langle a \rangle}} F^n(y)$ coincides with some endpoint. Note that if an endpoint is eventually periodic, that is, there exists n and m such that

$$\lim_{\substack{y \to x \\ y \in \langle a \rangle}} F^n(y) = \lim_{\substack{y \to x \\ y \in \langle a \rangle}} F^{n+m}(y),$$

then adding all the periodic points as a division points of a partition, eventually periodic endpoint and also the new endpoints become Markov endpoints. Moreover, for a Markov endpoint $x \in \langle a \rangle$ for which $\lim_{y \to x, y \in \langle a \rangle} F^n(y)$ also becomes a Markov endpoint, add $\lim_{y \to x, y \in \langle a \rangle}$

$F^m(y)$ $(m < n)$ as division points of a partition. Then all the new endpoints are Markov endpoints and $\lim_{y\to x, y\in\langle a\rangle} F(y)$ is an endpoint. Hereafter, we assume a partition $\{\langle a\rangle\}_{a\in\mathcal{A}}$ satisfies this condition. We call F Markov if we can choose a partition for which every endpoints are Markov endpoints. Thus, a β-transformation is Markov, if and only if 1 is a Markov endpoint.

We introduce signed symbolic dynamics to express non-Markov transformations. Let for $x \in [0,1]$ define the expansion $a_1^x a_2^x \cdots$ of x by

$$F^n(x) \in \langle a_{n+1}^x \rangle.$$

We identify a point and its expansion. We also introduce

$$x^+ = \lim_{y\uparrow x} a_1^y a_2^y \cdots ,$$
$$x^- = \lim_{y\downarrow x} a_1^y a_2^y \cdots .$$

For an interval J, we denote $(\sup J)^+$ and $(\inf J)^-$ by J^+ and J^-, respectively. Especially, for $a \in \mathcal{A}$, we denote by a^+ and a^- instead of $\langle a\rangle^+$ and $\langle a\rangle^-$. We denote the set of a^σ $(a \in \mathcal{A}, \sigma = \pm)$ by $\tilde{\mathcal{A}}$.

Let us define for an interval J

$$s^J(z,x) = \sum_{n=0}^{\infty} z^n \beta^{-n} \sum_{|w|=n} 1_J(wx).$$

Then, using the Perron-Frobenius operator P associated with F, we get

$$s^J(z,x) = \sum_{n=0}^{\infty} z^n P^n 1_J(x) = (I - zP)^{-1} 1_J(x).$$

We also define

$$s^{y^\sigma}(z,x) = \sum_{n=0}^{\infty} z^n \beta^{-n} \sum_{w\in\mathcal{W}_n} \sigma(y^\sigma, wx)\delta[\langle w[1]\rangle \supset \langle a_1^y\rangle, \exists\theta wx],$$

where θ is the shift to left and

$$\delta[L] = \begin{cases} 1 & \text{if } L \text{ is true,} \\ 0 & \text{if } L \text{ is false,} \end{cases}$$

$$\sigma(y^\sigma, x) = \begin{cases} +\frac{1}{2} & \text{if } y \geq_\sigma x, \\ -\frac{1}{2} & \text{if } y <_\sigma x, \end{cases}$$

$$x <_\sigma y = \begin{cases} x < y & \sigma = +, \\ x > y & \sigma = -. \end{cases}$$

Then it is not difficult to see:

Lemma 1.

$$s^J(z,x) = s^{J^+}(z,x) + s^{J^-}(z,x). \tag{2}$$

Let us define

$$s(z,x) = (s^{a^\sigma}(z,x))_{a\in\mathcal{A},\sigma=\pm},$$
$$\chi(z,x) = (\chi^{a^\sigma}(z,x))_{a\in\mathcal{A},\sigma=\pm},$$
$$\chi^{y^\sigma}(z,x) = \begin{cases} \sigma(y^\sigma,x) & \text{if } \theta y^\sigma = (a_2^y)^{\sigma \operatorname{sgn} a_1^y}, \\ \sum_{n=0}^\infty z^n\beta^{-n}\sigma(\theta^n y^\sigma,x) & \text{otherwise,} \end{cases}$$
$$\phi_{y^\sigma,b^\tau}(z) = \begin{cases} +z\beta^{-1}/2 & \text{if } b^\tau \le_\sigma \theta y^\sigma, \text{ and } \theta y^\sigma = (a_2^y)^{\sigma \operatorname{sgn} a_1^y}, \\ -z\beta^{-1}/2 & \text{if } b^\tau >_\sigma \theta y^\sigma, \text{ and } \theta y^\sigma = (a_2^y)^{\sigma \operatorname{sgn} a_1^y}, \\ \sum_{n=1}^\infty z^n\beta^{-n}\sigma(\theta^n y^\sigma, b^\tau)(\operatorname{sgn} y^\sigma[1,n-1]) & \text{otherwise.} \end{cases}$$

We call $\Phi(z)$ with components ϕ_{a^σ,b^τ} $(a^\sigma, b^\tau \in \tilde{\mathcal{A}})$ the Fredholm matrix associated with F. Moreover, we define $\Phi^J(z)$ a row vector $\sum_{\tau=\pm}\phi_{J^\tau,a^\sigma}(z)$ as a^σ component.

To find eigenvalues of P, we need to find the singularities of $s^{J^\sigma}(z,x)$. For this purpose, we will construct a renewal equation of the form:

Lemma 2. (1)

$$s(z,x) = (I-\Phi(z))^{-1}\chi(z,x). \tag{3}$$

(2) *For an interval $J \subset \langle a\rangle$ for some $a \in \mathcal{A}$,*

$$s^J(z,x) = \Phi^J(z)s(z,x).$$

(3) *For $|z| < \beta$*

$$\det(I-\Phi(z)) = \frac{1}{\zeta(z)}.$$

Using these discussions, in the domain $|z| < \beta$, the singularities of the zeta function, in other word, zero of $\det(I-\Phi(z))$ are the reciprocals of the eigenvalues of the Perron-Frobanius operator. The proof is found in [5].

1.2. Discrepancy

We denote $\tilde{A}_M$ the set of all the Markov endpoints, and define $k = \#\tilde{\mathcal{A}}_M^c$. We divide the Fredholm matrix $\Phi(z)$ into

$$\begin{pmatrix} \Phi_{11}(z) & \Phi_{12}(z) \\ \Phi_{21}(z) & \Phi_{22}(z) \end{pmatrix},$$

Here, $\Phi_{11}(z)$ is $\tilde{A}_M \times \tilde{A}_M$ matirx, $\Phi_{12}(z)$ is $\tilde{A}_M \times \tilde{A}_M^c$ matrix, $\Phi_{21}(z)$ is $\tilde{A}_M^c \times \tilde{A}_M$ matrix and $\Phi_{22}(z)$ is $\tilde{A}_M^c \times \tilde{A}_M^c$ matrix, that is $k \times k$ matrix. Put

$$\Psi(z) = I - \Phi_{21}(I - \Phi_{11}(z))^{-1}\Phi_{12}(z) - \Phi_{22}(z),$$

Then it is not difficult to see

Lemma 3. *If $I - \Phi_{11}(z)$ has inverse, then*

$$\det(I - \Phi(z)) = \det \Psi(z).$$

From the above lemma, we get:

Theorem 2. (1) *Let $(1-z)\zeta(z) = \sum_{n=0}^{\infty} \zeta_n z^n \beta^{-n}$. If ζ_n is bounded, then the discrepancy*

$$D(N) = O\left(\frac{(\log N)^{k+2}}{N}\right).$$

(2) *Assume moreover* $\det(I - \Phi_{11}(z)) \neq 0$ *in $|z| < \beta$. Then the discrepancy*

$$D(N) = O\left(\frac{(\log N)^{k+1}}{N}\right).$$

Proof. First note that $s^J(z,x)$ has simple singularity at $z = 1$. Because for $g \in L^\infty$

$$\int s^J(z,x)g(x)\,dx = \sum_{n=0}^{\infty} z^n \int P^n 1_J(x)g(x)\,dx$$

$$= \sum_{n=0}^{\infty} z^n \int 1_J(x)g(F^n(x))\,dx,$$

and the dynamical system is mixing:

$$\int 1_J(x)g(F^n(x))\,dx \to |J|\rho(x),$$

where $\rho(x)$ is the density of the invariant probability measure. This shows $\det(I - \Phi(z))$ has simple zero at $z = 1$ and

$$\Phi(1)\lim_{z\uparrow 1}(1-z)(I - \Phi(z))^{-1}\chi(1,x) = |J|\rho(x).$$

Hence,

$$\begin{aligned} s^J(z,x) &= \Phi^J(z)(I-\Phi(z))^{-1}\chi(z,x) \\ &= \Phi^J(z)\frac{1}{\det(1-\Phi(z))}\overline{I-\Phi(z)}\chi(z,x) \\ &= \zeta(z)\Phi^J(z)\overline{I-\Phi(z)}\chi(z,x), \end{aligned}$$

where $\overline{I-\Phi(z)}$ is the cofactor of $I-\Phi(z)$. The zeta function $\zeta(z)$ has a simple singularity at $z=1$, so let us denote

$$\eta^J(z,x) = (1-z)\zeta(z)\Phi^J(z)\overline{I-\Phi(z)}\chi(z,x).$$

Then

$$s^J(z,x) = \frac{\eta^J(1,x)}{1-z} + \frac{1}{1-z}(\eta^J(z,x)-\eta^J(1,x)), \tag{4}$$

and $\eta^J(1,x) = |J|\rho(x)$. The latter term of (4) has no singularity in $|z| < \beta$. Now each component of $\Phi^J(z)$, $\chi(z,x)$ has a bounded coefficient of $z^n\beta^{-n}$. At the same time, each component of the cofactor $\overline{I-\Phi(z)}$ is at most k times product of the infinte sequences whose coefficients of $z^n\beta^{-n}$ are bounded. Therefore their coefficients of z^n are at most of order $n^{k+1}\beta^{-n}$. If the assumption of (1) is satisfied, this is the order of the discrepancy for words with length n. The number of visits to J until the words whose length less than or equal to n equals $\sum_{m=0}^n P^m 1_J(x)\beta^m$. On the other hand,

$$\begin{aligned} \sum_{m=0}^{n} P^m 1_J(x)\beta^m &= \sum_{m=0}^{n} \left[|J|\rho(z) + O(m^{k+1}\beta^{-m})\right]\beta^m \\ &= |J|\rho(x)\frac{\beta^{n+1}-1}{\beta-1} + O(n^{k+2}). \end{aligned}$$

Let m be

$$\sum_{k=0}^{m} \#\{wx\colon |w|=k\} \le N < \sum_{k=0}^{m+1} \#\{wx\colon |w|=k\}.$$

If $\sum_{k=0}^m \#\{wx\colon |w|=k\} = N$, we have already proved the assertion. Even for other cases, from the construction there exists $a\in\mathcal{A}$ and from $\sum_{k=0}^m \#\{wx\colon |w|=k\}+1$ our van der Corput sequence has expression wax for $|w|=m$. From this fact, we can apply the same discussion as above, and we get the proof for general cases. See for detail [3].

Take $J = [0,1]$, then we get the number of wx which exists until N is of order β^n, and the discrepancy equals $O(n^{k+2}\beta^{-n})$. This proves (1). If we assume $I - \Phi_{11}(z)$ has inverse for all $|z| \leq \beta$, define

$$\begin{aligned} s_1(z,x) &= (s^{a^\sigma}(z,x))_{a^\sigma \in \tilde{A}_M}, \\ s_2(z,x) &= (s^{a^\sigma}(z,x))_{a^\sigma \in \tilde{A}_M^c}, \\ \chi_1(z,x) &= (\chi^{a^\sigma}(z,x))_{a^\sigma \in \tilde{A}_M}, \\ \chi_2(z,x) &= (\chi^{a^\sigma}(z,x))_{a^\sigma \in \tilde{A}_M^c}. \end{aligned}$$

Then the renewal equation (3) can be divided into

$$\begin{aligned} s_1(z,x) &= \chi_1(z,x) + \Phi_{11}(z)s_1(z,x) + \Phi_{12}(z)s_2(z,x), \\ s_2(z,x) &= \chi_2(z,x) + \Phi_{21}(z)s_1(z,x) + \Phi_{22}(z)s_2(z,x). \end{aligned}$$

Thus, we get

$$\begin{aligned} (5) \qquad & s_1(z,x) \\ & = (I - \Phi_{11}(z))^{-1} \left[I + \Phi_{12}(z)\Psi(z)^{-1}\Phi_{21}(z)(I - \Phi_{11}(z))^{-1}\right] \\ & \quad \times\chi_1(z,x) + (I - \Phi_{11}(z))^{-1}\Phi_{12}(z)\Psi(z)^{-1}\chi_2(z,x), \\ (6) \qquad & s_2(z,x) \\ & = \Psi(z)^{-1}(\chi_2(z,x) + \Phi^{21}(z)(I - \Phi_{11}(z))^{-1}\chi_1(z,x)). \end{aligned}$$

The cofactor of $\Psi(z)$ is the $(k-1)$-times product of the form $\sum_{n=0}^{\infty} b_n z^n$ β^{-n} where b_n is bounded. This proves 2. Q.E.D.

From the above theorem, we get:

Corollary 1. *For a Markov β transformation, our van der Corput sequence is of low discrepancy if there exists no eigenvalues of the Perron-Frobenius operator in the annulus $\frac{1}{\beta} < |z| < 1$.*

§2. Higher Dimensional cases

Now, we proceed to higher dimensional cases. We will consider a piecewise affine transformation F from $[0,1]^d$ into itself with its Jacobian det $|F'|$ equals constant $\beta > 1$. Namely, there exists a finite set $\mathcal{A}$ and

(1) $\langle a \rangle$ is a locally compact set for each $a \in \mathcal{A}$,
(2) $\{\langle a \rangle\}_{a \in \mathcal{A}}$ is a partition of $[0,1]^d$,
(3) on each $\langle a \rangle$, F is an affine transformation, that is, there exists a matrix M_a and a vector v_a and

$$F(x) = M_a x + v_a.$$

Let $\mathcal{B}$ be a set of functions for which there exists an expression

$$f(x) = \sum_{w \in \mathcal{W}} C_w 1_{\langle w \rangle}(x),$$

such that for any $0 < r < 1$

$$\sum_{n=0}^{\infty} r^n \sum_{w \in \mathcal{W}_n} |C_w| < \infty.$$

On $\mathcal{B}$, we define norms

$$||f||_r = \inf \sum_{n=0}^{\infty} r^n \sum_{w \in \mathcal{W}_n} |C_w|,$$

where inf is taken over all expressions

$$f(x) = \sum_{w \in \mathcal{W}} C_w 1_{\langle w \rangle}(x).$$

Then $\mathcal{B}$ becomes a locally convex space with norms $|| \cdot ||_r$ $(0 < r < 1)$. This is the slight extension of BV in one dimensional case.

We consider an order on $\mathcal{A}$, and we can define van der Corput sequences using F as before. Then as in Proposition 1, we get

Proposition 2. *Let F be topologically transitive and its Jacobian $\beta > 1$. Then for any $x \in [0,1]^d$,*

(1) *the van der Corput sequence is uniformly distributed.*
(2) *Let η be the second greatest eigenvalue of P. Then the order of the discrepancy is less than $(|\eta| + \varepsilon)^{\log N/\log \beta}$ for any $\varepsilon > 0$.*
(3) *There exists a measurable set E such that $|E| > 0$ and for $x \in E$ the discrepancy is greater than $(|\eta| - \varepsilon)^{\log N/\log \beta}$ for any $\varepsilon > 0$.*

From this proposition, as in one dimensioanl cases, we at least need to construct a transformation with its essential spectrum radius equals $\frac{1}{\beta}$ to construct low discrepancy sequences. However, the estimate of the essential spectrum radius is crucial in general([9]).

First we consider simple transformation. Let

$$a = \begin{pmatrix} a_1 \\ \vdots \\ a_d \end{pmatrix} \quad (a_i \text{ is either } 0 \text{ or } \tfrac{1}{2})$$

be a symbol in alphabet $\mathcal{A}$. Namely, for $d=2$,

$$\mathcal{A}=\left\{\begin{pmatrix}0\\0\end{pmatrix},\ \begin{pmatrix}\frac{1}{2}\\0\end{pmatrix},\ \begin{pmatrix}0\\\frac{1}{2}\end{pmatrix},\ \begin{pmatrix}\frac{1}{2}\\\frac{1}{2}\end{pmatrix}\right\}.$$

Let for $a\in\mathcal{A}$

$$\langle a\rangle=\prod_{i=1}^{d}\Big[a_i,a_i+\frac{1}{2}\Big).$$

Moreover, we assume for every $a\in\mathcal{A}$

$$F(\langle a\rangle)=[0,1)^d,$$

that is, F is Bernoulli. We assume the Jacobian $\det F'=2^d$ almost everywhere.

We can construct similar renewal equations for general d. Thus we can estimate the discrepancy from them. Because F is Bernoulli, the Fredholm matrix is the structure matrix times $z2^{-d}$. Hence $\det(I-\Phi(z))=1-z$, that is, there exists only one singularity of $\zeta(z)$. Thus the singularity of $s_g^J(z)$ is determined by the singularity of $\Phi^J(z)$ and $\chi_g^J(z)$.

Let $J_0=J$, and define J_n inductively as follows. Let $F(J)$ be the image of J by F. However, we consider overlaps. For example $J=\langle a\rangle\cup\langle b\rangle$ $(a\neq b)$, then $F(J)=2[0,1)^d$. From $F(J_{n-1})$, we delete $\langle a\rangle$ $(a\in\mathcal{A})$ as many as possible. Then we define J_n the remained one, which is a sum of several intervals. Note that the n-th coefficient of

$$\Phi^J(z)(|\langle a\rangle|)_{a\in\mathcal{A}}+\chi_g^J(z)$$

for $g\equiv 1$ equals $2^{-nd}|J_{n-1}|$. Because $|J_n|\le 2^{n(d-1)}$, and the number of words with length n equals 2^{dn}, we get the radii of convergence of the $\Phi^J(z)$ and $\chi_g^J(z)$ are less than or equals to 2.

Theorem 3. *Let F be the transformation mentioned above. If the zeta function has no singularity in $|z|<2$ except 1, then the discrepancy satisfies*

$$D_N=O(N^{-1/d}).$$

When F is the simplest transformation, that is, for $x\in\langle a\rangle$

$$F(x)=\begin{pmatrix}2&\cdots&2\\\vdots&\ddots&\vdots\\2&\cdots&2\end{pmatrix}(x-a).$$

Choose $J = \prod_{i=1}^{d-1}[0,1) \times [0,\alpha)$ for irrational α, then

$$F^n(J) = \begin{cases} \prod_{i=1}^{d-1}[0,1) \times [0,\alpha_n) & \alpha_n \leq \frac{1}{2}, \\ \prod_{i=1}^{d-1}[0,1) \times [\frac{1}{2},\alpha_n) & \alpha_n > \frac{1}{2}, \end{cases}$$

and overlaps 2^n times, where $\alpha_n = 2^n\alpha \pmod 1$. Therefore, we get

$$D_N = O(N^{-1/d}).$$

This says that the van der Corput sequence generated by this F is not of low discrepancy.

2.1. Two and Three dimensional low discrepancy sequences

As we see in the former section, to construct low discrepancy sequences, we need that a transformation is expanding but this is not enough. For two dimensional case, let s_0 is the infinite sequence of 0's, and

$$s_1 = w_1 w_2 \cdots,$$

where w_n is a word with length 2^{n-1} and only the last symbol equals 1 and other symbols are 0, that is,

$$s_1 = 101000100000001 \cdots.$$

We consider the digitwise sum modulo 2 on the set of infinite sequences of 0 and 1. Then the first n digits of

$$s_0, s_1, \theta s_1, \ldots, \theta^{n-1} s_1$$

generate all the words with length n. Define

$$F\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \theta x \\ \theta y \end{pmatrix} + \begin{pmatrix} s_{y_1} \\ s_{x_1} \end{pmatrix},$$

where we identify $x \in [0,1)$ and its binary expansion $x_1 x_2 \cdots$. Then we can prove for $I = [\alpha, \alpha + 2^{-n+m}] \times [\beta, \beta + 2^{-n-m})$ for binary rationals α and β and $m \leq n$,the image of I by F^n does not overlap and $F^n(I) = [0,1)^2$. For example, $I = [0,1) \times [0,1/4)$ $(n = m = 1)$, though θy belongs only to $[0,1/2)$, x_1 takes both 0 and 1. Hence, θy expands all $[0,1)$ by adding s_0 or s_1 depending on where x belongs. Thus $F(I) = [0,1)^2$. From this, we can prove:

Theorem 4 ([10]). *The van der Corput sequence generated by the above F is of low discrepancy.*

When we express

$$F^n\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}x'\\y'\end{pmatrix},$$

and their expansion by $x' = x_1'x_2'\cdots$ and $y' = y_1'y_2'\cdots$, F^n can be expressed

$$\left(\begin{array}{c}x_1'\\x_2'\\\vdots\\\hline y_1'\\y_2'\\\vdots\end{array}\right)=\left(\begin{array}{c}x_{n+1}\\x_{n+2}\\\vdots\\\hline y_{n+1}\\y_{n+2}\\\vdots\end{array}\right)$$

$$+\left(\begin{array}{cccc|cccc}0&0&\cdots&0&i_1&i_2&\cdots&i_n\\0&0&\cdots&0&i_2&i_3&\cdots&i_{n+1}\\\vdots&\vdots&\ddots&\vdots&\vdots&\vdots&\ddots&\vdots\\0&0&\cdots&0&i_n&i_{n+1}&\cdots&i_{2n-1}\\\hline i_1&i_2&\cdots&i_n&0&0&\cdots&0\\i_2&i_3&\cdots&i_{n+1}&0&0&\cdots&0\\\vdots&\vdots&\ddots&\vdots&\vdots&\vdots&\ddots&\vdots\\i_n&i_{n+1}&\cdots&i_{2n-1}&0&0&\cdots&0\end{array}\right)\left(\begin{array}{c}x_n\\x_{n-1}\\\vdots\\x_1\\\hline y_n\\y_{n-1}\\\vdots\\y_1\end{array}\right),$$

where $s_1 = 10100010\cdots = i_1i_2\cdots$ $(i_j = 0, 1)$. In other words,

$$\left(\begin{array}{c}x_1'\\y_1'\\\hline x_2'\\y_2'\\\hline\vdots\end{array}\right)=\left(\begin{array}{c}x_{n+1}\\y_{n+1}\\\hline x_{n+2}\\y_{n+2}\\\hline\vdots\end{array}\right)+\left(\begin{array}{cc|cc|cc|c}0&i_1&0&i_2&0&i_3&\cdots\\i_1&0&i_2&0&i_3&0&\cdots\\\hline 0&i_2&0&i_3&0&i_4&\cdots\\i_2&0&i_3&0&i_4&0&\cdots\\\hline 0&i_3&0&i_4&0&i_5&\cdots\\i_3&0&i_4&0&i_5&0&\cdots\\\hline\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdots\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdots\end{array}\right)\left(\begin{array}{c}x_n\\y_n\\\hline x_{n-1}\\y_{n-1}\\\hline\vdots\\\hline x_1\\y_1\\\hline 0\\\vdots\end{array}\right).$$

For 3 dimensional case, it is not easy to construct by one transformation. We also denote

$$F^n\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}x'\\y'\\z'\end{pmatrix}.$$

Then expressing x', y', z' in binary expansions, we define

$$\begin{pmatrix} x'_1 \\ y'_1 \\ z'_1 \\ \hline x'_2 \\ y'_2 \\ z'_2 \\ \hline \vdots \end{pmatrix} = \begin{pmatrix} x_{n+1} \\ y_{n+1} \\ z_{n+1} \\ \hline x_{n+2} \\ y_{n+2} \\ z_{n+2} \\ \hline \vdots \end{pmatrix} + M \begin{pmatrix} x_n \\ y_n \\ z_n \\ \hline \vdots \\ \hline x_1 \\ y_1 \\ z_1 \\ \hline 0 \\ \vdots \end{pmatrix}.$$

Here, M is an infinite dimensional matrix. One example of M is following.

	x_{-1}	y_{-1}	z_{-1}	x_{-2}	y_{-2}	z_{-2}	x_{-3}	y_{-3}	z_{-3}
x_1	0	1	1	0	1	0	0	1	0
y_1	1	0	1	0	0	1	0	0	1
z_1	1	1	0	1	0	0	1	0	0
x_2	0	1	0	0	0	1	0	1	0
y_2	0	0	1	1	0	0	0	0	1
z_2	1	0	0	0	1	0	1	0	0
x_3	0	1	0	0	1	0	0	0	1
y_3	0	0	1	0	0	1	1	0	0
z_3	1	0	0	1	0	0	0	1	0
x_4	0	0	0	0	0	1	0	1	0
y_4	0	0	0	1	0	0	0	0	1
z_4	0	0	0	0	1	0	1	0	0
x_5	0	0	0	0	1	0	0	1	0
y_5	0	0	0	0	0	1	0	0	1
z_5	0	0	0	1	0	0	1	0	0
x_6	0	0	0	0	0	0	0	1	0
y_6	0	0	0	0	0	0	0	0	1
z_6	0	0	0	0	0	0	1	0	0
x_7	0	0	0	0	0	0	0	1	0
y_7	0	0	0	0	0	0	0	0	1
z_7	0	0	0	0	0	0	1	0	0
x_8	0	0	0	0	0	0	0	0	0
y_8	0	0	0	0	0	0	0	0	0
z_8	0	0	0	0	0	0	0	0	0
x_9	0	0	0	0	0	0	0	0	0
y_9	0	0	0	0	0	0	0	0	0
z_9	0	0	0	0	0	0	0	0	0
x_{10}	0	0	0	0	0	0	0	0	0
y_{10}	0	0	0	0	0	0	0	0	0
z_{10}	0	0	0	0	0	0	0	0	0

	x_{-4}	y_{-4}	z_{-4}	x_{-5}	y_{-5}	z_{-5}
x_1	0	0	1	0	0	0
y_1	1	0	0	0	0	0
z_1	0	1	0	0	0	0
x_2	0	1	0	0	1	0
y_2	0	0	1	0	0	1
z_2	1	0	0	1	0	0
x_3	0	1	0	0	1	0
y_3	0	0	1	0	0	1
z_3	1	0	0	1	0	0
x_4	0	0	1	0	0	1
y_4	1	0	0	1	0	0
z_4	0	1	0	0	1	0
x_5	0	1	1	0	0	0
y_5	1	0	1	0	0	0
z_5	1	1	0	0	0	0
x_6	0	0	1	0	1	1
y_6	1	0	0	1	0	1
z_6	0	1	0	1	1	0
x_7	0	1	0	0	1	1
y_7	0	0	1	1	0	1
z_7	1	0	0	1	1	0
x_8	0	1	0	0	1	1
y_8	0	0	1	1	0	1
z_8	1	0	0	1	1	0
x_9	0	0	1	0	0	0
y_9	1	0	0	0	0	0
z_9	0	1	0	0	0	0
x_{10}	0	0	0	0	1	0
y_{10}	0	0	0	0	0	1
z_{10}	0	0	0	1	0	0

This M has following properties: for any nonnegative integers n and m: the determinant of the minor matrix with coordinates

$$\{x_1, \ldots, x_{n+m}\} \times \{y_{-1}, \ldots, y_{-n}, z_{-1} \ldots, z_{-m}\}$$

or the determinant of the minor matrix with coordinates

$$\{x_1, \ldots, x_n, y_1, \ldots, y_m\} \times \{z_{-1}, \ldots, z_{-n-m}\}$$

do not vanish. M also has symmetry in the permutations x, y and z. This matrix corresponds to s_0 and s_1 in two dimensional case, and this shuffles coordinates, and we get for $k \geq n+m$

$$I = [\alpha, \alpha + 2^{-k-n-m}) \times [\beta, \beta + 2^{-k+n}) \times [\gamma, \gamma + 2^{-k+m})$$

or

$$I = [\alpha, \alpha + 2^{-k-n}) \times [\beta, \beta + 2^{-k-m}) \times [\gamma, \gamma + 2^{-k+n+m}),$$

then

$$F_k(I) = [0, 1)^3,$$

where α, β and γ are binary rationals. Hence, we can prove:

Theorem 5. *The van der Corput sequence generated by $F_1, F_2, \ldots$ is of low discrepancy.*

We have constructed two and three dimensional low discrepancy sequences using dynamical system. However, it seems not so easy to construct higher dimensional cases. Moreover, even for two and three dimensional cases, we have not found good algorithm to generate above sequences.

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Dept. Math. College of Humanities and Sciences
Nihon University
Setagaya-ku, Tokyo, 156–8550
Japan