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Limit theorems for the Mellin transform of $|\zeta(\frac{1}{2}+it)|^2$. II

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Abstract.

A limit theorem in the sense of weak convergence of probability measures in the space of meromorphic functions for the Mellin transform of the square of the Riemann zeta-function is obtained.

§1. Introduction and main results.

Let $s = \sigma + it$ be a complex variable, and let, as usual, $\zeta(s)$ denote the Riemann zeta-function. The modified Mellin transforms of $|\zeta(\frac{1}{2}+it)|$

$$\mathcal{Z}_k(s) = \int_1^\infty |\zeta(\frac{1}{2}+ix)|^{2k} x^{-s} \mathrm{d}x, \quad \sigma > \sigma_0(k),$$

for $k = 2$ was introduced and studied by Y.Motohashi in [12] and [13] in connection with investigation of the moments of the Riemann zeta-function. Later, the function $\mathcal{Z}_2(s)$ was considered in [2]–[4] and [6], while analytic behavior of $\mathcal{Z}_1(s)$ was treated in [4], [5] and [11]. We note that the functions $\mathcal{Z}_2(s)$ and $\mathcal{Z}_1(s)$ have quite different analytic properties.

In [8] we proved limit theorems in the sense of weak convergence of probability measures for the function $\mathcal{Z}_2(s)$, and in [9] a limit theorem on the complex plane $\mathbb{C}$ for the function $\mathcal{Z}_1(s)$ was obtained.

Let $meas\{A\}$ denote the Lebesgue measure of a measurable subset A of the set of real numbers $\mathbb{R}$, and

$$\nu_T^t(....) = \frac{1}{T} meas\{t \in [0,T] : ...\},$$

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where in place of dots a condition satisfied by t is to be written. Here the sign t in ν_T^t only indicates that the measure is taken over $t \in [0, T]$. Denote by $\mathcal{B}(S)$ the class of Borel sets of the space S. Then [8] contains the following statements.

Theorem A. *Let $\frac{7}{8} < \sigma < 1$. Then on $(\mathbb{C}, \mathcal{B}(\mathbb{C}))$ there exists a probability measure $P_{\mathbb{C},\sigma}$ such that the probability measure*

$$\nu_T^t\big(\mathcal{Z}_2(\sigma + it) \in A\big), \quad A \in \mathcal{B}(\mathbb{C}),$$

converges weakly to $P_{\mathbb{C},\sigma}$ as $T \to \infty$.

Let G be a region on $\mathbb{C}$. Denote by $H(G)$ the space of analytic on G functions equipped with the topology of uniform convergence on compacta. Let $\widehat{D} = \{s \in \mathbb{C} : \frac{7}{8} < \sigma < 1\}$.

Theorem B. *On $\big(H(\widehat{D}), \mathcal{B}(H(\widehat{D}))\big)$ there exists a probability measure P_H such that the probability measure*

$$\nu_T^\tau\big(\mathcal{Z}_2(s + i\tau) \in A\big), \quad A \in \mathcal{B}(H(\widehat{D})),$$

converges weakly to P_H as $T \to \infty$.

The main result of [9] is the following theorem.

Theorem C. *Let $\sigma > \frac{1}{2}$. Then on $(\mathbb{C}, \mathcal{B}(\mathbb{C}))$ there exists a probability measure P_σ such that the probability measure*

$$\nu_T^t\big(\mathcal{Z}_1(\sigma + it) \in A\big), \quad A \in \mathcal{B}(\mathbb{C}),$$

converges weakly to P_σ as $T \to \infty$.

This paper is a continuation of [9]. The function $\mathcal{Z}_1(s)$ is meromorphic, therefore its value distribution is better reflected by a limit theorem on the space of meromorphic functions, and we prove a generalization of Theorem C to a functional space.

Let $\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$ be the Riemann sphere with the spheric metric d defined, for $s_1, s_2 \in \mathbb{C}$, by

$$d(s_1, s_2) = \frac{2|s_1 - s_2|}{\sqrt{1 + |s_1|^2}\sqrt{1 + |s_2|^2}},$$

$$d(s_1, \infty) = \frac{2}{\sqrt{1 + |s_1|^2}}, \qquad d(\infty, \infty) = 0.$$

The metric d is compatible with the topology of $\mathbb{C}_\infty$. Denote by $M(G)$ the space of meromorphic on G functions $g\colon\ G \to (\mathbb{C}_\infty, d)$ equipped with the topology of uniform convergence on compacta. In this topology, a sequence $\{g_n(s)\} \in M(G)$ converges to $g(s) \in M(G)$ as $n \to \infty$ if

$$d(g_n(s), g(s)) \underset{n\to\infty}{\longrightarrow} 0,$$

uniformly on compact subsets of G. The set of analytic on G functions $H(G)$ forms a subspace of $M(G)$. Let $D = \{s \in \mathbb{C} : \sigma > \frac{1}{2}\}$.

Theorem 1. *On $(M(D), \mathcal{B}(M(D)))$ there exists a probability measure P such that the probability measure*

$$P_T(A) \overset{def}{=} \nu_T^{\tau}\big(\mathcal{Z}_1(s + i\tau) \in A\big), \quad A \in \mathcal{B}(M(D)),$$

converges weakly to P as $T \to \infty$.

§2. A limit theorem for an integral over a finite interval

The function $\mathcal{Z}_1(s)$ has a double pole at $s = 1$ and simple poles at $s = -(2k-1)$, $k \in \mathbb{N}$. Therefore, the function

$$(1 - 2^{1-s})^2 \mathcal{Z}_1(s) \overset{def}{=} \widehat{\mathcal{Z}}_1(s)$$

is regular in the half-plane $\{s \in \mathbb{C} : \sigma > 0\}$.

Let $a > 1$, and

$$\widehat{\mathcal{Z}}_{1,a,y}(s) = \int_1^a \widehat{\zeta}(x) v(x, y) x^{-s} \mathrm{d}x, \tag{1}$$

where, for $y \geq 1$, $\sigma_1 > \frac{1}{2}$,

$$v(x, y) = \exp\Big\{ - \Big(\frac{x}{y}\Big)^{\sigma_1} \Big\}.$$

The function $\widehat{\zeta}(x)$ will be defined in the next section.

Theorem 2. *Let G be a region on $\mathbb{C}$. Then on $(H(G), \mathcal{B}(H(G)))$ there exists a probability measure $P_{a,y}$ such that the probability measure*

$$P_{T,a,y}(A) \overset{def}{=} \nu_T^{\tau}\big(\widehat{\mathcal{Z}}_{1,a,y}(s + i\tau) \in A\big), \quad A \in \mathcal{B}(H(G)),$$

converges weakly to $P_{a,y}$ as $T \to \infty$.

The proof of Theorem 2 relies on the following statement. Let

$$\Omega_a = \prod_{u \in [1,a]} \gamma_u,$$

where $\gamma_u = \{s \in \mathbb{C} : |s| = 1\} \stackrel{def}{=} \gamma$ for all $u \in [1,a]$. With the product topology and pointwise multiplication the torus Ω_a is a compact topological group. Define the probability measure $Q_{T,a}$ by

$$Q_{T,a}(A) = \nu_T^{\tau}\big((u^{i\tau})_{u\in[1,a]} \in A\big), \quad A \in \mathcal{B}(\Omega_a).$$

Lemma 3. *On $(\Omega_a, \mathcal{B}(\Omega_a))$ there exists a probability measure Q_a such that the probability measure $Q_{T,a}$ converges weakly to Q_a as $T \to \infty$.*

Proof of the lemma is given in [9].

Proof of Theorem 2. For $y_x \in \gamma$, $x \in [1,a]$, define

$$\widehat{y}_x = \begin{cases} y_x \text{ if } y_x \text{ is integrable over } [1,a], \\ \text{an arbitrary integrable function over } [1,a], \text{ otherwise.} \end{cases}$$

Let a function $h : \Omega_a \to H(G)$ be given by the formula

$$h(\{y_x : x \in [1,a]\}) = \int_1^a \widehat{\zeta}(x) v(x,y) x^{-s} \widehat{y_x}^{-1} \mathrm{d}x, \quad y_x \in \gamma.$$

Then in view of (1)

$$\widehat{\mathcal{Z}}_{1,a,y}(s+i\tau) = h\big(\{x^{i\tau} : x \in [1,a]\}\big), \tag{2}$$

and by the Lebesgue theorem of bounded convergence, the function h is continuous. By (2), $P_{T,a,y} = Q_{T,a}h^{-1}$. Therefore, the theorem is a consequence of Lemma 3, continuity of h and Theorem 5.1 of [1].

§3. Approximation of $\widehat{\mathcal{Z}}_1(s)$ in the mean

In this section we approximate the function $\widehat{\mathcal{Z}}_1(s)$ by the absolutely convergent integral in the mean. Let $\sigma > 1$. Then we have that

$$\widehat{\mathcal{Z}}_1(s) = \int_1^\infty \widehat{\zeta}(x) x^{-s} \mathrm{d}x, \tag{3}$$

where

$$\widehat{\zeta}(x) = \begin{cases} |\zeta(\frac{1}{2}+ix)|^2 & \text{if } x \in [1,2), \\ |\zeta(\frac{1}{2}+ix)|^2 - |\zeta(\frac{1}{2}+\frac{ix}{2})|^2 & \text{if } x \in [2,4), \\ |\zeta(\frac{1}{2}+ix)|^2 - |\zeta(\frac{1}{2}+\frac{ix}{2})|^2 + |\zeta(\frac{1}{2}+\frac{ix}{4})|^2 & \text{if } x \in [4,\infty). \end{cases}$$

Let, as usual, $\Gamma(s)$ denote the gamma-function, and $\sigma_1 > \frac{1}{2}$ be the same as above. For $y \geq 1$, define

$$l_y(s) = \frac{s}{\sigma_1}\Gamma\Big(\frac{s}{\sigma_1}\Big)y^s,$$

and let, for $\sigma > \frac{1}{2}$,

$$\widehat{\mathcal{Z}}_{1,y}(s) = \frac{1}{2\pi i}\int\limits_{\sigma_1 - i\infty}^{\sigma_1 + i\infty} \widehat{\mathcal{Z}}_1(s+z)l_y(z)\frac{\mathrm{d}z}{z}.$$

By the choice of σ_1 we have that $\sigma_1 + \sigma > 1$, therefore, in view of (3), for $\Re z = \sigma_1$,

$$\widehat{\mathcal{Z}}_1(s+z) = \int\limits_1^\infty \widehat{\zeta}(x)x^{-(s+z)}\mathrm{d}x.$$

Now define

$$b_y(x) = \frac{1}{2\pi i}\int\limits_{\sigma_1 - i\infty}^{\sigma_1 + i\infty} \widehat{\zeta}(x)\frac{l_y(z)\mathrm{d}z}{zx^z}.$$

Using the well-known estimates for the gamma-function, hence we find that

$$\text{(4)} \qquad b_y(x) \ll |\widehat{\zeta}(x)|x^{-\sigma_1}\int\limits_{-\infty}^{\infty} |l_y(\sigma_1 + it)|\mathrm{d}t \ll |\widehat{\zeta}(x)|x^{-\sigma_1}.$$

Since

$$\int\limits_1^T |\zeta(\frac{1}{2} + it)|^2\mathrm{d}t \ll T\log T,$$

the definition of $\widehat{\zeta}(x)$ shows that

$$\int\limits_1^T |\widehat{\zeta}(x)|\mathrm{d}x \ll T\log T.$$

Therefore, (4) implies the absolute convergence of the integral

$$\int\limits_1^\infty b_y(x)x^{-s}\mathrm{d}x$$

for $\sigma > \frac{1}{2}$. Hence

$$(5)\int_1^\infty b_y(x)x^{-s}\mathrm{d}x = \frac{1}{2\pi i}\int_{\sigma_1-i\infty}^{\sigma_1+i\infty}\left(\frac{l_y(z)}{z}\int_1^\infty \widehat{\zeta}(x)\frac{\mathrm{d}x}{x^{s+z}}\right)\mathrm{d}z = \widehat{\mathcal{Z}}_{1,y}(s).$$

An application of the Mellin formula

$$\frac{1}{2\pi i}\int_{b-i\infty}^{b+i\infty}\Gamma(s)c^{-s}\mathrm{d}s = \mathrm{e}^{-c}, \quad b, c > 0,$$

and the definition of $b_y(x)$ and $v(x,y)$ yield

$$b_y(x) = \widehat{\zeta}(x)v(x,y).$$

Consequently, by (5)

$$\widehat{\mathcal{Z}}_{1,y}(s) = \int_1^\infty \widehat{\zeta}(x)v(x,y)x^{-s}\mathrm{d}x,$$

the integral being absolutely convergent for $\sigma > \frac{1}{2}$.

Theorem 4. *Let K be a compact subset of the half-plane D. Then*

$$\lim_{y\to\infty}\limsup_{T\to\infty}\frac{1}{T}\int_0^T \sup_{s\in K}\left|\widehat{\mathcal{Z}}_1(s+i\tau) - \widehat{\mathcal{Z}}_{1,y}(s+i\tau)\right|\mathrm{d}\tau = 0.$$

Proof. First we change the contour of integration in the definition of $\widehat{\mathcal{Z}}_{1,y}(s)$. The integrand has a simple pole at $z = 0$. Let σ belong to $[\frac{1}{2}+\epsilon, A]$, $\epsilon > 0$, $A > \frac{1}{2}+\epsilon$, when $s \in K$. We take $\sigma_2 = \frac{1}{2}+\frac{\epsilon}{2}$. Then in view of the estimate [4]

$$\mathcal{Z}_1(\sigma+it) \ll_\epsilon t^{1-\sigma+\epsilon},$$

which is valid for $0 \le \sigma \le 1$, $t \ge t_0 > 0$ (the paper [5] contains a more precise bound for $\mathcal{Z}_1(s)$), using the residue theorem we find that

$$(6)\qquad \widehat{\mathcal{Z}}_{1,y}(s) = \frac{1}{2\pi i}\int_{\sigma_2-\sigma-i\infty}^{\sigma_2-\sigma+i\infty}\widehat{\mathcal{Z}}_1(s+z)l_y(z)\frac{\mathrm{d}z}{z} + \widehat{\mathcal{Z}}_1(s).$$

We take a simple closed contour L lying in D and enclosing the set K, and let $|L|$ be the length of L and δ denote the distance of L from the set K. Then the Cauchy integral formula yields

$$\sup_{s\in K} \left|\widehat{\mathcal{Z}}_1(s+i\tau) - \widehat{\mathcal{Z}}_{1,y}(s+i\tau)\right| \le \frac{1}{2\pi\delta} \int\limits_L \left|\widehat{\mathcal{Z}}_1(z+i\tau) - \widehat{\mathcal{Z}}_{1,y}(z+i\tau)\right| |\mathrm{d}z|.$$

Hence, for sufficiently large T,

$$\frac{1}{T}\int\limits_0^T \sup_{s\in K} \left|\widehat{\mathcal{Z}}_1(s+i\tau) - \widehat{\mathcal{Z}}_{1,y}(s+i\tau)\right| \mathrm{d}\tau \ll$$

$$\frac{1}{T\delta}\int\limits_L |\mathrm{d}z| \int\limits_0^{2T} \left|\widehat{\mathcal{Z}}_1(\Re z+i\tau) - \widehat{\mathcal{Z}}_{1,y}(\Re z+i\tau)\right| \mathrm{d}\tau \ll$$

$$\frac{|L|}{T\delta} \sup_{\substack{\sigma\\ s\in L}} \int\limits_0^{2T} \left|\widehat{\mathcal{Z}}_1(\sigma+i\tau) - \widehat{\mathcal{Z}}_{1,y}(\sigma+i\tau)\right| \mathrm{d}\tau. \tag{7}$$

Now we choose the contour L so that, for $s \in L$, the inequalities

$$\sigma \ge \frac{1}{2} + \frac{3\epsilon}{4} \quad \text{and} \quad \delta \ge \frac{\epsilon}{4} \tag{8}$$

should be satisfied. By (6) we have that

$$\widehat{\mathcal{Z}}_1(\sigma+it) - \widehat{\mathcal{Z}}_{1,y}(\sigma+it) \ll \int\limits_{-\infty}^{\infty} \left|\widehat{\mathcal{Z}}_1(\sigma_2+it+i\tau)\right| |l_y(\sigma_2-\sigma+i\tau)| \mathrm{d}\tau.$$

Therefore, for σ defined by (8),

$$\frac{1}{T}\int\limits_0^{2T} \left|\widehat{\mathcal{Z}}_1(\sigma+i\tau) - \widehat{\mathcal{Z}}_{1,y}(\sigma+i\tau)\right| \mathrm{d}\tau \ll$$

$$\int\limits_{-\infty}^{\infty} |l_y(\sigma_2-\sigma+i\tau)| \Bigg(\frac{1}{T}\int\limits_{-|\tau|}^{|\tau|+2T} \left|\widehat{\mathcal{Z}}_1(\sigma_2+it)\right| \mathrm{d}t\Bigg) \mathrm{d}\tau. \tag{9}$$

By the estimate in [4]

$$\int\limits_1^T \left|\mathcal{Z}_1(\sigma+it)\right|^2 \mathrm{d}t \ll_\epsilon T^{2-2\sigma+\epsilon}$$

with $\frac{1}{2} \leq \sigma \leq 1$ we obtain that

$$\int_0^T \left|\widehat{\mathcal{Z}}_1(\sigma_2 + it)\right| \mathrm{d}t \ll T.$$

This together with (9) shows that

$$\frac{1}{T} \sup_{\substack{\sigma \\ s \in L}} \int_0^{2T} \left|\widehat{\mathcal{Z}}_1(\sigma + i\tau) - \widehat{\mathcal{Z}}_{1,y}(\sigma + i\tau)\right| \mathrm{d}\tau \ll$$

$$\sup_{\substack{\sigma \\ s \in L}} \int_{-\infty}^{\infty} |l_y(\sigma_2 - \sigma + i\tau)|(1 + |\tau|)\mathrm{d}\tau \ll$$

$$\sup_{\sigma \in [-A, -\frac{\varepsilon}{4}]} \int_{-\infty}^{\infty} |l_y(\sigma + it)|(1 + |t|)\mathrm{d}t = o(1)$$

as $y \to \infty$. This and (7) prove the theorem.

§4. A limit theorem for the function $\widehat{\mathcal{Z}}_{1,y}(s)$

In this section we will prove a limit theorem in the space $H(D)$ for the function $\widehat{\mathcal{Z}}_{1,y}(s)$.

Theorem 5. *On* $\big(H(D), \mathcal{B}(H(D))\big)$ *there exists a probability measure* P_y *such that the probability measure*

$$\nu_T^{\tau}\big(\widehat{\mathcal{Z}}_{1,y}(s + i\tau) \in A\big), \quad A \in \mathcal{B}(H(D)),$$

converges weakly to P_y *as* $T \to \infty$.

Proof. By Theorem 2 with $G = D$ the probability measure $P_{T,a,y}$ converges weakly to a measure $P_{a,y}$ as $T \to \infty$. The first step of the proof consists of the observation that the family of probability measures $\{P_{a,y}\}$ is tight, for the definitions, see [1], for fixed y.

On a certain probability space $(\Omega, \mathcal{B}(\Omega), \mathbb{P})$ define a random variable θ_T by the formula

$$\mathbb{P}(\theta_T \in A) = \frac{1}{T} \int_0^T I_A \mathrm{d}t, \quad A \in \mathcal{B}(\mathbb{R}).$$

Here I_A denotes the indicator function of the set A. Define

$$X_{T,a,y}(s) = \widehat{\mathcal{Z}}_{1,a,y}(s + i\theta_T).$$

Then Theorem 2 implies the relation

$$X_{T,a,y}(s) \xrightarrow[T\to\infty]{\mathcal{D}} X_{a,y}(s), \tag{10}$$

where $X_{a,y}(s)$ is an $H(D)$-valued random element having the distribution $P_{a,y}$, and $\xrightarrow[T\to\infty]{\mathcal{D}}$ means the convergence in distribution.

Let $\{K_l\}$ be a sequence of compact subsets of D such that

$$D = \bigcup_{l=1}^{\infty} K_l,$$

$K_l \subset K_{l+1}$, $l \in \mathbb{N}$, and if $K \subset D$ is a compact, then $K \subseteq K_l$ for some l. Define, for $f, g \in H(D)$,

$$\rho(f,g) = \sum_{l=1}^{\infty} 2^{-l} \frac{\rho_l(f,g)}{1+\rho_l(f,g)},$$

where $\rho_l(f,g) = \sup\limits_{s\in K_l} |f(s) - g(s)|$. Then ρ is a metric on $H(D)$ which induces its topology of uniform convergence on compacta.

Let $M_l > 0$, $l \in \mathbb{N}$. Then by the Chebyshev inequality

$$\begin{aligned} &P_{T,a,y}\big(\{g \in H(D) : \sup_{s\in K_l} |g(s)| > M_l\}\big) = \\ &\nu_T^{\tau}\big(\sup_{s\in K_l} \big|\widehat{\mathcal{Z}}_{1,a,y}(s+i\tau)\big| > M_l\big) \le \\ &\frac{1}{M_l T} \int_0^T \sup_{s\in K_l} \big|\widehat{\mathcal{Z}}_{1,a,y}(s+i\tau)\big| \mathrm{d}\tau. \end{aligned} \tag{11}$$

The integral defining $\widehat{\mathcal{Z}}_{1,y}(s)$ converges absolutely on D, hence the convergence is uniform on compact subsets of D. Consequently,

$$\sup_{a\ge 1} \limsup_{T\to\infty} \frac{1}{T} \int_0^T \sup_{s\in K_l} \big|\widehat{\mathcal{Z}}_{1,a,y}(s+i\tau)\big| \mathrm{d}\tau \le R_l < \infty. \tag{12}$$

Now we put $M_l = R_l 2^l \epsilon^{-1}$. Then (11) and (12) yield

$$\limsup_{T\to\infty} P_{T,a,y}\big(\{g \in H(D) : \sup_{s\in K_l} |g(s)| > M_l\}\big) \le \frac{\epsilon}{2^l}, \quad l \in \mathbb{N}. \tag{13}$$

The function $h : H(D) \to \mathbb{R}$ given by the formula $h(g) \stackrel{def}{=} \sup\limits_{s\in K_l} |g(s)|$, $g \in H(D)$, is clearly continuous, therefore, Theorem 2 shows that the probability measure

$$\nu_T^{\tau}\Big(\sup_{s\in K_l}\big|\widehat{\mathcal{Z}}_{1,a,y}(s+i\tau)\big|\in A\Big),\quad A\in\mathcal{B}(\mathbb{R}),$$

converges weakly to $P_{a,y}h^{-1}$ as $T\to\infty$. This, the properties of the weak convergence and (13) show that

$$\begin{aligned} &P_{a,y}\big(\{g\in H(D):\sup_{s\in K_l}|g(s)|>M_l\}\big)\le \\ (14)\qquad &\liminf_{T\to\infty}P_{T,a,y}\big(\{g\in H(D):\sup_{s\in K_l}|g(s)|>M_l\}\big)\le\frac{\epsilon}{2^l},\quad l\in\mathbb{N}.\end{aligned}$$

By the compactness principle, the set $H_\epsilon=\{g\in H(D):\sup_{s\in K_l}|g(s)|\le M_l,\ l\in\mathbb{N}\}$ is compact, and in virtue of (14)

$$P_{a,y}(H_\epsilon)\ge 1-\epsilon$$

for all $a>1$, i.e. the family of probability measures $\{P_{a,y}\}$ is tight. Hence, by the Prokhorov theorem [1], $\{P_{a,y}\}$ is relatively compact.

The absolute convergence on D of the integral for $\widehat{\mathcal{Z}}_{1,y}(s)$ implies

$$\lim_{a\to\infty}\widehat{\mathcal{Z}}_{1,a,y}(s)=\widehat{\mathcal{Z}}_{1,y}(s)$$

uniformly on compact subsets of D. Hence

$$\begin{aligned} &\lim_{a\to\infty}\limsup_{T\to\infty}\nu_T^{\tau}\big(\rho\big(\widehat{\mathcal{Z}}_{1,a,y}(s+i\tau),\widehat{\mathcal{Z}}_{1,y}(s+i\tau)\big)\ge\epsilon\big)\le \\ (15)\qquad &\lim_{a\to\infty}\limsup_{T\to\infty}\frac{1}{\epsilon T}\int_0^T\rho\big(\widehat{\mathcal{Z}}_{1,a,y}(s+i\tau),\widehat{\mathcal{Z}}_{1,y}(s+i\tau)\big)\mathrm{d}\tau=0.\end{aligned}$$

Define

$$X_{T,y}(s)=\widehat{\mathcal{Z}}_{1,y}(s+i\theta_T).$$

In view of (15)

$$(16)\qquad \lim_{a\to\infty}\limsup_{T\to\infty}\mathbb{P}\big(\rho\big(X_{T,a,y}(s),X_{T,y}(s)\big)\ge\epsilon\big)=0.$$

Since the family $\{P_{a,y}\}$ is relatively compact, there exists a sequence $\{P_{a_1,y}\}\subset\{P_{a,y}\}$ such that $P_{a_1,y}$ converges weakly to some probability measure P_y on $(H(D),\mathcal{B}(H(D)))$ as $a_1\to\infty$. Therefore,

$$X_{a_1,y}\ \underset{a_1\to\infty}{\overset{\mathcal{D}}{\longrightarrow}}\ P_y.$$

This and relations (10) and (16) show that all hypotheses of Theorem 4.2 of [1] are satisfied. Therefore,

$$X_{T,y} \xrightarrow[T\to\infty]{\mathcal{D}} P_y,$$

and the proof is completed.

§5. A limit theorem for the function $\widehat{\mathcal{Z}}_1(s)$

Theorem 5 implies the following statement.

Theorem 6. *On $(H(D), \mathcal{B}(H(D)))$ there exists a probability measure $\widehat{P}$ such that the probability measure*

$$\widehat{P}_T(A) = \nu_T^\tau\big(\widehat{\mathcal{Z}}_1(s+i\tau) \in A\big), \quad A \in \mathcal{B}(H(D)),$$

converges weakly to $\widehat{P}$ as $T \to \infty$.

Proof. By Theorem 5 we have that

$$X_{T,y}(s) \xrightarrow[T\to\infty]{\mathcal{D}} X_y(s), \tag{17}$$

where $X_y(s)$ is an $H(D)$-valued random element having the distribution P_y. Let $M_l > 0$, $l \in \mathbb{N}$. Then by the Chebyshev inequality

$$P_{T,y}\big(\{g \in H(D) : \sup_{s\in K_l} |g(s)| > M_l\}\big) \leq$$

$$\frac{1}{M_l T}\int_0^T \sup_{s\in K_l} \big|\widehat{\mathcal{Z}}_{1,y}(s+i\tau)\big| \mathrm{d}\tau. \tag{18}$$

Since

$$\int_0^T \big|\zeta\big(\frac{1}{2}+it\big)\big|^4 \mathrm{d}t \ll T\log^4 T,$$

the integral

$$\int_1^\infty \big|\zeta\big(\frac{1}{2}+ix\big)\big|^4 x^{-2\sigma}\mathrm{d}x,$$

converges for $\sigma > \frac{1}{2}$. Therefore, the integral

$$\int_1^\infty \big|\widehat{\zeta}(x)\big|^2 x^{-2\sigma}\mathrm{d}x$$

also converges for $\sigma > \frac{1}{2}$. Hence by the Cauchy integral formula, for some $\sigma > \frac{1}{2}$,

$$\sup_{y\geq 1}\limsup_{T\to\infty}\frac{1}{T}\int_0^T \sup_{s\in K_l}\big|\widehat{\mathcal{Z}}_{1,y}(s+i\tau)\big|\mathrm{d}\tau \ll_l$$
$$\sup_{y\geq 1}\limsup_{T\to\infty}\frac{1}{T}\int_0^T \big|\widehat{\mathcal{Z}}_{1,y}(\sigma+it)\big|\mathrm{d}t \ll_l$$
$$\sup_{y\geq 1}\limsup_{T\to\infty}\left(\frac{1}{T}\int_0^T \big|\widehat{\mathcal{Z}}_{1,y}(\sigma+it)\big|^2\mathrm{d}t\right)^{\frac{1}{2}} \ll_l$$
$$\sup_{y\geq 1}\left(\int_1^\infty \big|\widehat{\zeta}(x)\big|^2 v(x,y)x^{-2\sigma}\mathrm{d}x\right)^{\frac{1}{2}} \leq R_l < \infty.$$

This and (18) with the same M_l and H_ϵ as above show that

$$P_y(H_\epsilon) \geq 1-\epsilon$$

for all $y \geq 1$. This means that the family $\{P_y\}$ is tight, and therefore, it is relatively compact.

By Theorem 4 we find that

$$\lim_{y\to\infty}\limsup_{T\to\infty}\nu_T^\tau\big(\rho\big(\widehat{\mathcal{Z}}_{1,y}(s+i\tau),\widehat{\mathcal{Z}}_1(s+i\tau)\big)\geq\epsilon\big) \leq$$
$$\lim_{y\to\infty}\limsup_{T\to\infty}\frac{1}{\epsilon T}\int_0^T \rho\big(\widehat{\mathcal{Z}}_{1,y}(s+i\tau),\widehat{\mathcal{Z}}_1(s+i\tau)\big)\mathrm{d}\tau = 0. \tag{19}$$

Let

$$X_T(s) = \widehat{\mathcal{Z}}_1(s+i\theta_T).$$

Then by (19)

$$\lim_{y\to\infty}\limsup_{T\to\infty}\mathbb{P}\big(\rho\big(X_T(s),X_{T,y}(s)\big)\geq\epsilon\big) = 0. \tag{20}$$

Since the family $\{P_y\}$ is relatively compact, we can find a sequence $\{P_{y_1}\} \subset \{P_y\}$ such that P_{y_1} converges weakly to some probability measure $\widehat{P}$ on $(H(D),\mathcal{B}(H(D)))$ as $y_1 \to \infty$. Hence,

$$X_{y_1} \xrightarrow[y_1\to\infty]{\mathcal{D}} \widehat{P}.$$

This, (17) and (20) and Theorem 4.2 of [1] prove the theorem.

§6. Proof of Theorem 1

The function

$$g_1(s) \overset{def}{=} \left(1 - 2^{1-s}\right)^2$$

is a Dirichlet polynomial, therefore the probability measure

$$\nu_T^{\tau}\big(g_1(s+i\tau) \in A\big), \quad A \in \mathcal{B}(H(D)),$$

converges weakly to some probability measure P_{g_1} on $(H(D), \mathcal{B}(H(D)))$ as $T \to \infty$, for the proof, see, for example [7], Chapter 5. Using this and Theorem 6 we obtain by a modified Cramér-Wald criterion, see, for example, [10], that the probability measure

$$P_{T,g_1,\widehat{\mathcal{Z}}_1}(A) \overset{def}{=} \nu_T^{\tau}\Big(\big(g_1(s+i\tau), \widehat{\mathcal{Z}}_1(s+i\tau)\big) \in A\Big), \quad A \in \mathcal{B}(H^2(D)),$$

also converges weakly to some probability measure $P_{g_1,\widehat{\mathcal{Z}}_1}$ on $(H^2(D), \mathcal{B}(H^2(D)))$ as $T \to \infty$.

Now define a function $h : H^2(D) \to M(D)$ by the formula

$$h(g, f) = \frac{f}{g}, \quad g, f \in H(D).$$

Since the metric d satisfies the equality

$$d\left(\frac{1}{f}, \frac{1}{g}\right) = d(f, g),$$

the function h is continuous. Moreover, by the definition of $\widehat{\mathcal{Z}}_1(s)$, we have that $P_T = P_{T,g_1,\widehat{\mathcal{Z}}_1} h^{-1}$. Therefore, Theorem 5.1 of [1] shows that P_T converges weakly to $P_{g_1,\widehat{\mathcal{Z}}_1} h^{-1}$ as $T \to \infty$. The theorem is proved.

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