Advanced Studies in Pure Mathematics 49, 2007
Probability and Number Theory — Kanazawa 2005
pp. 171–184

On simultaneous Diophantine approximation to periodic points related to modified Jacobi-Perron algorithm

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Abstract.

For each (α, β) which is a periodic point related to modified Jacobi-Perron algorithm and $\mathbb{Q}(\alpha)$ has a complex embedding, we claim the following facts: the limit set of $\{(\sqrt{q_n}(q_n\alpha - p_n), \sqrt{q_n}(q_n\beta - r_n)|n = 1, 2, \ldots\}$ is a finite union of similar ellipses, where (p_n, q_n, r_n) is the n-th convergent $(p_n/q_n, r_n/q_n)$ of (α, β) by the modified Jacobi-Perron algorithm but for some (α, β) the ellipse given above is not the nearest ellipse in the limit set of $\{(\sqrt{q}(q\alpha - p), \sqrt{q}(q\beta - r)|q \in \mathbb{Z}, q > 0\}$ which is a union of similar ellipses.

§1. Introduction

We denote by $\mathbb{C}$, $\mathbb{R}$, $\mathbb{Q}$ and $\mathbb{Z}$ the set of all complex numbers, the set of all real numbers, the set of all rational numbers and the set of all integers respectively. Let $\beta_1, \beta_2, \ldots, \beta_n \in \mathbb{R}$ be linearly independent over $\mathbb{Q}$. Then, it is well known that there exist infinitely many $q \in \mathbb{Z}(q > 0)$ such that $q^{\frac{1}{n}}\|q\beta_i\| < 1$ for any integer i with $1 \le i \le n$. We consider the limit set of points:

$$\{(q^{\frac{1}{n}}\|q\beta_1\|, \ldots, q^{\frac{1}{n}}\|q\beta_n\|)|q \in \mathbb{Z}, q > 0\},$$

which is denoted by $\lim(\beta_1, \beta_2, \ldots, \beta_n)$, where $\|x\| = x - m$ and m is the nearest integer to $x \in \mathbb{R}$. For $n = 1$, using the continued fraction expansion of β_1, we know the nearest point in $\lim(\beta_1)$ to the origin, that is, let $\gamma = \limsup_{m\to\infty} q_{2m+1}\|q_{2m+1}\beta_1\|$ and $\gamma' = \liminf_{m\to\infty} q_{2m}\|q_{2m}\beta_1\|$, then $\lim(\beta_1) \cap [\gamma, \gamma'] = \{\gamma, \gamma'\}$, where (p_m, q_m) is the m-convergent of β_1.

Received March 16, 2006.
Revised June 3, 2006.
2000 *Mathematics Subject Classification.* Primary 11J70; Secondary 11J13.

W. Adams [1] determined the $\lim(\beta_1, \beta_2)$ for specific β_1, β_2 by using algebraic number theory.

Theorem(W. Adams[1]). *Let $1, \beta_1, \beta_2$ be a basis for a real cubic number field. Let us define matrix A by*

$$A = \begin{pmatrix} 1 & \beta_1 & \beta_2 \\ 1 & \beta_1^{\tau_1} & \beta_2^{\tau_1} \\ 1 & \beta_1^{\tau_2} & \beta_2^{\tau_2} \end{pmatrix}, \tag{1}$$

where τ_1, τ_2 are non trivial embeddings of $\mathbb{Q}(\beta_1)$ into $\mathbb{C}$. Let us define a quadratic form $F(x, y)$ by

$$F(x, y) = (\alpha_1^{\tau_1} x + \alpha_2^{\tau_1} y)(\alpha_1^{\tau_2} x + \alpha_2^{\tau_2} y),$$

where

$$({}^t A)^{-1} = \begin{pmatrix} \alpha_0 & \alpha_1 & \alpha_2 \\ \alpha_0^{\tau_1} & \alpha_1^{\tau_1} & \alpha_2^{\tau_1} \\ \alpha_0^{\tau_2} & \alpha_1^{\tau_2} & \alpha_2^{\tau_2} \end{pmatrix}.$$

Let

$$M = \mathbb{Z}\alpha_0 \oplus \mathbb{Z}\alpha_1 \oplus \mathbb{Z}\alpha_2,$$

$$N^+(M) = \{N(\alpha) \mid \alpha \in M, \alpha > 0\},$$

where $N(\alpha) = \alpha\alpha^{\tau_1}\alpha^{\tau_2}$.

Then, we have

$$\lim(\beta_1, \beta_2) = \bigcup_{c \in N^+(M)} \{(x, y) \mid F(x, y) = c\}.$$

By Theorem of W.Adams, if $1, \beta_1, \beta_2$ is a basis for a real cubic number field and $\mathbb{Q}(\beta_1)$ has a complex embedding, $\lim(\beta_1, \beta_2)$ is a union of similar ellipses whose center are at the origin. If the modified Jacobi-Perron algorithm ([7],[10]) admits (β_1, β_2) as a fixed point, it computes the nearest ellipse in $\lim(\beta_1, \beta_2)$ to the origin ([5]).

Furthermore, in [5] it is conjectured that the modified Jacobi-Perron algorithm gives the nearest ellipse for each (β_1, β_2) which is purely periodic point by the modified Jacobi-Perron algorithm and has a complex embedding. In this paper, we show that for such (β_1, β_2) the limit set of $\{(\sqrt{q_n}(q_n\beta_1 - p_n), \sqrt{q_n}(q_n\beta_2 - r_n) | n = 1, 2, \ldots\}$ is a finite union of similar ellipses, where (p_n, q_n, r_n) is the n-th convergent $(p_n/q_n, r_n/q_n)$ of (β_1, β_2) by the modified Jacobi-Perron algorithm. We also prove that, for some case, the nearest ellipse to the origin among them is not equal to the nearest ellipse to the origin in $\lim(\beta_1, \beta_2)$. Therefore, the conjecture ([5]) is not true.

Some closely related results appear in [2, 3].

Acknowledgement. The authors are grateful to the referee for helpful comments.

§2. Modified Jacobi-Perron algorithm

Let us define an algorithm called modified Jacobi-Perron algorithm, which is introduced by Podsypanin [10] as follows: Let X be the domain given by $\{(x,y) \in [0,1] \times [0,1) | 1, x, y \text{ are linearly independent over } \mathbb{Q}\}$ and let us define the transformation T on X by

$$(2) \qquad T(x,y) = \begin{cases} (\frac{y}{x}, \frac{1}{x} - \left[\frac{1}{x}\right]) & \text{if } (x,y) \in X_0 \\ (\frac{1}{y} - \left[\frac{1}{y}\right], \frac{x}{y}) & \text{if } (x,y) \in X_1 \end{cases}$$

where $X_0 = \{(x,y) \in X | x > y\}$ and $X_1 = \{(x,y) \in X | x < y\}$. We define the integer valued functions $a(\ ,\ )$ and $\epsilon(\ ,\ )$ on X^2 by

$$a(x,y) = \begin{cases} \left[\frac{1}{x}\right] & \text{if } (x,y) \in X_0 \\ \left[\frac{1}{y}\right] & \text{if } (x,y) \in X_1, \end{cases}$$

$$\epsilon(x,y) = \begin{cases} 0 & \text{if } (x,y) \in X_0 \\ 1 & \text{if } (x,y) \in X_1. \end{cases}$$

We have for each $(\alpha, \beta) \in X$ a sequence of digits $(a_n(\alpha,\beta), \epsilon_n(\alpha,\beta)) := (a(T^{n-1}(\alpha,\beta), \epsilon(T^{n-1}(\alpha,\beta))$ for $n \in \mathbb{Z}$ with $n > 0$. For simplicity, $a_n(\alpha,\beta)$ and $\epsilon_n(\alpha,\beta)$ are denoted by a_n and ϵ_n respectively.

The triple $(X, T, a(\alpha,\beta), \epsilon(\alpha,\beta))$ is called modified Jacobi-Perron algorithm. We denote (α_n, β_n) by $T^n(\alpha,\beta)$. For the modified Jacobi-Perron algorithm, we introduce a transformation $(\bar{X}, \bar{T})$ which is called a natural extension of the modified Jacobi-Perron algorithm as follows: let $\bar{X} = X \times X$ and let us define the transformation $\bar{T}$ on $\bar{X}$ by

$$(3) \qquad \bar{T}(x,y,s,t) = (T(x,y), T'_{a_1,\epsilon_1}(s,t)),$$

where for $a > 0$, $\epsilon \in \{0,1\}$ and $(u,v) \in \mathbb{R}^2$ with $u, v \geq 0$,

$$T'_{a_1,\epsilon_1}(s,t) = \begin{cases} \left(\frac{t}{a_1+s}, \frac{1}{a_1+s}\right) & \text{if } \epsilon = 0, \\ \left(\frac{1}{a_1+t}, \frac{s}{a_1+t}\right) & \text{if } \epsilon = 1. \end{cases} \tag{4}$$

$(\bar{X}, \bar{T})$ was first introduced in [7].

For $(\alpha, \beta, \gamma, \delta) \in \bar{X}$ we denote $(\alpha_n, \beta_n, \gamma_n, \delta_n)$ by $T^n(\alpha, \beta, \gamma, \delta)$. We define θ_n and η_n for $n = 1, 2, \dots$ by

$$\theta_n(\alpha,\beta) = \max\{\alpha_{n-1}, \beta_{n-1}\},$$

$$\eta_n(\alpha,\beta,\gamma,\delta) = \begin{cases} \gamma_{n-1} + a_n & \text{if } \epsilon_n = 0, \\ \delta_{n-1} + a_n & \text{if } \epsilon_n = 1. \end{cases}$$

Let us define the family of matrices as follows: for each (a, ϵ) with $a \in \mathbb{N}, \epsilon \in \{0,1\}$

$$A_{(a,\epsilon)} = \begin{cases} \begin{pmatrix} a & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} & \text{if } \epsilon = 0, \\ \\ \begin{pmatrix} a & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} & \text{if } \epsilon = 1. \end{cases} \tag{5}$$

We define $M_n(\alpha, \beta)$ by

$$M_n(\alpha,\beta) = \begin{pmatrix} q_n(\alpha,\beta) & q'_n(\alpha,\beta) & q''_n(\alpha,\beta) \\ p_n(\alpha,\beta) & p'_n(\alpha,\beta) & p''_n(\alpha,\beta) \\ r_n(\alpha,\beta) & r'_n(\alpha,\beta) & r''_n(\alpha,\beta) \end{pmatrix} = \prod_{1\leq i\leq n} A_{(a_i,\epsilon_i)}. \tag{6}$$

Then, we have the following formulae.

Lemma 1 ([7]). *For $n \in \mathbb{N}$,*

$$\begin{pmatrix} 1 \\ \alpha_{n-1} \\ \beta_{n-1} \end{pmatrix} = \theta_n(\alpha,\beta) A_{(a_n,\epsilon_n)} \begin{pmatrix} 1 \\ \alpha_n \\ \beta_n \end{pmatrix},$$

$$\begin{pmatrix} 1 \\ \gamma_n \\ \delta_n \end{pmatrix} = \eta_n^{-1}(\alpha,\beta,\gamma,\delta) A^t_{(a_n,\epsilon_n)} \begin{pmatrix} 1 \\ \gamma_{n-1} \\ \delta_{n-1} \end{pmatrix}.$$

From Lemma 1 we see easily following formulae.

Lemma 2 ([7]). *For $n \in \mathbb{Z}$ with $n > 0$,*

$$\begin{pmatrix} 1 \\ \alpha \\ \beta \end{pmatrix} = \prod_{1 \le i \le n} \theta_i(\alpha, \beta) M_n(\alpha, \beta) \begin{pmatrix} 1 \\ \alpha_n \\ \beta_n \end{pmatrix},$$

$$\begin{pmatrix} 1 \\ \gamma \\ \delta \end{pmatrix} = \prod_{1 \le i \le n} \eta_i(\alpha, \beta, \gamma, \delta) M_n^t(\alpha, \beta)^{-1} \begin{pmatrix} 1 \\ \gamma_n \\ \delta_n \end{pmatrix}$$

Then, roughly speaking, $(\frac{p_n(\alpha,\beta)}{q_n(\alpha,\beta)}, \frac{r_n(\alpha,\beta)}{q_n(\alpha,\beta)})$ gives a simultaneous approximation of (α, β) (for example, see [4]).

§3. Periodic points

In this section, we assume that $(\alpha, \beta) \in X$ satisfies $T^m(\alpha, \beta) = (\alpha, \beta)$ for some integer $m > 0$. On the assumption, α is a cubic number and $\mathbb{Q}(\alpha)$ has a complex embedding. Let τ_0 be the trivial embedding of $\mathbb{Q}(\alpha)$ into $\mathbb{R}$. Let τ_1, τ_2 be non trivial embeddings of $\mathbb{Q}(\alpha)$ into $\mathbb{C}$ and $\tau_1 \neq \tau_2$. We note that $\overline{u^{\tau_1}} = u^{\tau_2}$ for any $u \in \mathbb{Q}(\alpha)$, where $\overline{x}$ is the complex conjugate of x. We set $\gamma = \prod_{1 \le i \le m} \theta_i(\alpha, \beta)$. From Lemma 2 we have $M_m(\alpha, \beta)(1, \alpha, \beta)^t = \gamma^{-1}(1, \alpha, \beta)^t$. We denote $M_m(\alpha, \beta), \theta_n(\alpha, \beta)$ and $\eta_n(\alpha, \beta, \gamma, \delta)$ by M, θ_n and η_n respectively. We have following Lemma.

Lemma 3.
(1) $\gamma^{\tau_0}\gamma^{\tau_1}\gamma^{\tau_2} = \gamma|\gamma^{\tau_1}|^2 = 1$,
(2) $\gamma^{-1} > 1$,
(3) $\gamma^{-1} = \prod_{1 \le i \le m} \eta_i$,
(4) $\mathbb{Q}(\alpha) = \mathbb{Q}(\beta) = \mathbb{Q}(\gamma)$.

Proof. The term $\gamma^{\tau_0}\gamma^{\tau_1}\gamma^{\tau_2}$ is the coefficient term in the characteristic polynomial of M ; since M is the product of matrices of determinant 1 according to (5) and (6), we have assertion (1). We can prove the rest of the assertions easily. □

Lemma 4. *Let $(u, v, w)^t$ be a non trivial eigenvector related to $M_m^t(\alpha, \beta)$ and the eigenvalue γ^{-1}. Then, $u \neq 0$ and $(\frac{v}{u}, \frac{w}{u}) \in X$ and $\mathbb{Q}(\alpha) = \mathbb{Q}(\frac{v}{u}) = \mathbb{Q}(\frac{w}{u})$.*

Proof. Since γ^{-1} is the dominant eigenvalue of M^t, it is not difficult to see that $u, v, w \geq 0$ or $u, v, w \leq 0$ by using Perron-Frobenius Theorem. We assume that $u, v, w \geq 0$ without loss of generality. We set $(c_{ij}) = M^t$. Then, we see easily $c_{12} > 0$ or $c_{13} > 0$. Therefore, if $u = 0$, then $v = 0$ or $w = 0$, which contradicts that γ is the cubic number. Hence, we have $u \neq 0$. Since $(1, \frac{v}{u}, \frac{w}{u})$ is an eigenvector related to M^t and the eigenvalue γ^{-1}, we see that $\frac{v}{u}, \frac{w}{u} \in \mathbb{Q}(\alpha)$. On the setting ${}^*\alpha = \frac{v}{u}$ and ${}^*\beta = \frac{w}{u}$, since $M^t(1, {}^*\alpha^{\tau_i}, {}^*\beta^{\tau_i})^t = (\gamma^{-1})^{\tau_i}(1, {}^*\alpha^{\tau_i}, {}^*\beta^{\tau_i})^t$ for $i = 1, 2, 3$, we see that $(1, {}^*\alpha^{\tau_i}, {}^*\beta^{\tau_i})^t$ for $i = 1, 2, 3$ are linearly independent on $\mathbb{C}$. Therefore, $1, {}^*\alpha, {}^*\beta$ are linearly independent on $\mathbb{Q}$. On the other hand, from the fact that $(1, {}^*\alpha, {}^*\beta)$ is the eigenvector of M^t with the eigenvalue γ^{-1} we see that ${}^*\alpha, {}^*\beta \in \mathbb{Q}(\gamma)$. By using Lemma 3 we have $\mathbb{Q}(\alpha) = \mathbb{Q}({}^*\alpha) = \mathbb{Q}({}^*\beta)$.

We set $(1, {}^*\alpha(0), {}^*\beta(0))^t = (1, {}^*\alpha, {}^*\beta)^t$. For each positive integer k, we set $c(k)(1, {}^*\alpha(k), {}^*\beta(k))^t = \prod_{1 \leq i \leq k} A^t_{(a_{k+1-i}, \epsilon_{k+1-i})}(1, {}^*\alpha, {}^*\beta)^t$. Then, it is not difficult to see that ${}^*\alpha(k)$ and ${}^*\beta(k)$ are positive and $1, {}^*\alpha(k)$ and ${}^*\beta(k)$ are linearly independent on $\mathbb{Q}$ for each integer k, which implies that $\mathbb{Q}(\alpha) = \mathbb{Q}({}^*\alpha(k)) = \mathbb{Q}({}^*\beta(k))$. By using (4) and (5) we have $T'_{a_{k+1}, \epsilon_{k+1}}({}^*\alpha(k), {}^*\beta(k)) = ({}^*\alpha(k+1), {}^*\beta(k+1))$ for each k. From the fact that $(1, {}^*\alpha, {}^*\beta)$ is the eigenvector related to M^t, we see that $({}^*\alpha(k), {}^*\beta(k)) = ({}^*\alpha(k+m), {}^*\beta(k+m))$ for each $k \geq 0$. By (5) and ${}^*\alpha(k), {}^*\beta(k) > 0$ we see that if $\max\{{}^*\alpha(k), {}^*\beta(k)\} < 1$, then $\max\{{}^*\alpha(k+1), {}^*\beta(k+1)\} < 1$ and if $\max\{{}^*\alpha(k), {}^*\beta(k)\} > 1$, then $\max\{{}^*\alpha(k), {}^*\beta(k)\} > \max\{{}^*\alpha(k+1), {}^*\beta(k+1)\}$. Therefore, we see that $\max\{{}^*\alpha(m), {}^*\beta(m)\} < \max\{1, {}^*\alpha(0), {}^*\beta(0)\}$ holds. Finally, since $({}^*\alpha(0), {}^*\beta(0)) = ({}^*\alpha(m), {}^*\beta(m))$, we have $\max\{{}^*\alpha(m), {}^*\beta(m)\} < 1$, which implies that $\max\{{}^*\alpha(k), {}^*\beta(k)\} < 1$ holds for any $k \geq 0$. □

$(1, {}^*\alpha, {}^*\beta)^t$ is denoted the non trivial eigenvector related to $M^t_m(\alpha, \beta)$ and the eigenvalue γ^{-1}. Then, we have the following lemma.

Lemma 5. *For any positive integer n,*

$$\begin{pmatrix} 1 & {}^*\alpha_n^{\tau_0} & {}^*\beta_n^{\tau_0} \\ 1 & {}^*\alpha_n^{\tau_1} & {}^*\beta_n^{\tau_1} \\ 1 & {}^*\alpha_n^{\tau_2} & {}^*\beta_n^{\tau_2} \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ \alpha_n^{\tau_0} & \alpha_n^{\tau_1} & \alpha_n^{\tau_2} \\ \beta_n^{\tau_0} & \beta_n^{\tau_1} & \beta_n^{\tau_2} \end{pmatrix} = \begin{pmatrix} \delta^{\tau_0} & 0 & 0 \\ 0 & \delta^{\tau_1} & 0 \\ 0 & 0 & \delta^{\tau_2} \end{pmatrix},$$

where $(\alpha_n, \beta_n, {}^*\alpha_n, {}^*\beta_n) = \bar{T}^n(\alpha, \beta, {}^*\alpha, {}^*\beta)$ *and* $\delta = 1 + {}^*\alpha_n\alpha_n + {}^*\beta_n\beta_n$.

Proof. We set $M(n) = \prod_{1 \leq i \leq m} A_{(a_{n+i}, \epsilon_{n+i})}$. Then, it is easily seen that $M(n)(1, \alpha_n, \beta_n)^t = \gamma^{-1}(1, \alpha_n, \beta_n)^t$ and $M(n)(1, {}^*\alpha_n, {}^*\beta_n)^t =$

$\gamma^{-1}(1, {}^*\alpha_n, {}^*\beta_n)^t$. Therefore, we have

$$\begin{pmatrix} 1 & {}^*\alpha_n^{\tau_0} & {}^*\beta_n^{\tau_0} \\ 1 & {}^*\alpha_n^{\tau_1} & {}^*\beta_n^{\tau_1} \\ 1 & {}^*\alpha_n^{\tau_2} & {}^*\beta_n^{\tau_2} \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ \alpha_n^{\tau_0} & \alpha_n^{\tau_1} & \alpha_n^{\tau_2} \\ \beta_n^{\tau_0} & \beta_n^{\tau_1} & \beta_n^{\tau_2} \end{pmatrix}$$
$$= \begin{pmatrix} 1 & {}^*\alpha_n^{\tau_0} & {}^*\beta_n^{\tau_0} \\ 1 & {}^*\alpha_n^{\tau_1} & {}^*\beta_n^{\tau_1} \\ 1 & {}^*\alpha_n^{\tau_2} & {}^*\beta_n^{\tau_2} \end{pmatrix} M(n)^{-1}M(n) \begin{pmatrix} 1 & 1 & 1 \\ \alpha_n^{\tau_0} & \alpha_n^{\tau_1} & \alpha_n^{\tau_2} \\ \beta_n^{\tau_0} & \beta_n^{\tau_1} & \beta_n^{\tau_2} \end{pmatrix}$$
$$= \begin{pmatrix} \gamma^{\tau_0} & (\gamma\,{}^*\alpha_n)^{\tau_0} & (\gamma\,{}^*\beta_n)^{\tau_0} \\ \gamma^{\tau_1} & (\gamma\,{}^*\alpha_n)^{\tau_1} & (\gamma\,{}^*\beta_n)^{\tau_1} \\ \gamma^{\tau_2} & (\gamma\,{}^*\alpha_n)^{\tau_2} & (\gamma\,{}^*\beta_n)^{\tau_2} \end{pmatrix} \begin{pmatrix} (\gamma^{-1})^{\tau_0} & (\gamma^{-1})^{\tau_1} & (\gamma^{-1})^{\tau_2} \\ (\gamma^{-1}\alpha_n)^{\tau_0} & (\gamma^{-1}\alpha_n)_n^{\tau_1} & (\gamma^{-1}\alpha_n)^{\tau_2} \\ (\gamma^{-1}\beta_n)^{\tau_0} & (\gamma^{-1}\beta_n)^{\tau_1} & (\gamma^{-1}\beta_n)^{\tau_2} \end{pmatrix}.$$

Form above formula and the fact that $\gamma^{\tau_i} \neq \gamma^{\tau_j}$ with $i \neq j$ we have Lemma. □

For each $n \in \mathbb{Z}$ with $n \geq 0$, P_n is defined by $\{(x, y, z) \in \mathbb{R}^3 | x + {}^*\alpha_n y + {}^*\beta_n z = 0\}$ and L_n is defined by $\{t(1, \alpha_n, \beta_n) \in \mathbb{R}^3 | t \in \mathbb{R}\}$.

We define $\rho_n(x, y, z)$ for each $n \in \mathbb{Z}$ with $n \geq 0$ and $(x, y, z) \in \mathbb{R}^3$ by $\rho_n(x, y, z) = |x + ({}^*\alpha_n)^{\tau_1} y + ({}^*\beta_n)^{\tau_1} z|$. Then, we have following Lemma.

Lemma 6.

(1) *For any* $\mathbf{u} \in P_n$ *with* $\mathbf{u} \neq 0$, $\rho_n(\mathbf{u}) > 0$.
(2) *For any* $\mathbf{w} \in \mathbb{R}^3$ *and any* $\mathbf{v} \in L_n$ $\rho_n(\mathbf{w} + \mathbf{v}) = \rho_n(\mathbf{w})$.
(3) *For any* $\mathbf{w} \in \mathbb{R}^3$ $\rho_n(\mathbf{w}) = |\eta_{n+1}^{\tau_1}| \rho_{n+1}(A_{(a_{n+1}, \epsilon_{n+1})}^{-1} \mathbf{w})$.

Proof. First, we assume that $\rho_n(\mathbf{u}') = 0$ for some $\mathbf{u}' = (u'_x, u'_y, u'_z) \in P_n$. Then, we see that $|u'_x + ({}^*\alpha_n)^{\tau_i} u'_y + ({}^*\beta_n)^{\tau_i} u'_z| = 0$ for $i = 0, 1, 2$. Therefore, we have $u'_x = u'_y = u'_z = 0$ and we have (1). Secondly, let $\mathbf{w} = (w_x, w_y, w_z)$ and $\mathbf{v} = t(1, \alpha_n, \beta_n) \in L_n$. Then, using Lemma 5 we have

$$\begin{aligned} &\rho_n(\mathbf{w} + \mathbf{v}) \\ &= |(w_x + t) + ({}^*\alpha_n)^{\tau_1}(w_y + t\alpha_n) + ({}^*\beta_n)^{\tau_1}(w_z + t\beta_n)| \\ &= |w_x + ({}^*\alpha_n)^{\tau_1} w_y + ({}^*\beta_n)^{\tau_1} w_z + t(1 + ({}^*\alpha_n)^{\tau_1}\alpha_n + ({}^*\beta_n)^{\tau_1}\beta_n)| \\ &= \rho_n(\mathbf{w}). \end{aligned}$$

Therefore, we have (2). For the proof of (3), let $\mathbf{w} = (w_x, w_y, w_z)$, then, we have

$$\begin{aligned}
\rho_n(\mathbf{w}) &= |w_x + ({}^*\alpha_n)^{\tau_i} w_y + ({}^*\beta_n)^{\tau_i} w_z| \\
&= \left| (1, ({}^*\alpha_n)^{\tau_i}, ({}^*\beta_n)^{\tau_i}) \begin{pmatrix} w_x \\ w_y \\ w_z \end{pmatrix} \right| \\
&= \left| (1, ({}^*\alpha_n)^{\tau_i}, ({}^*\beta_n)^{\tau_i}) A_{(a_{n+1},\epsilon_{n+1})} A^{-1}_{(a_{n+1},\epsilon_{n+1})} \begin{pmatrix} w_x \\ w_y \\ w_z \end{pmatrix} \right| \\
&= |\eta_{n+1}^{\tau_1}| \rho_{n+1}(A^{-1}_{(a_{n+1},\epsilon_{n+1})} \mathbf{w}).
\end{aligned}$$

□

By Lemma 6 we remark that $|x\,{}^*\alpha^{\tau_1} + y\,{}^*\beta^{\tau_1}|^2$ is a positive definite quadratic form.

Lemma 7. *For each $n \in \mathbb{Z}$ with $n > 0$, we have $\rho_0(q_n, p_n, r_n) = |\prod_{1 \le i \le n} \eta_i^{\tau_1}|$.*

Proof. By Lemma 6 and an easy recurrence, we have

$$\begin{aligned}
\rho_0(q_n, p_n, r_n) &= \rho_0(\prod_{1 \le i \le n} A_{(a_i,\epsilon_i)} \mathbf{e}_1) \\
&= |\eta_1^{\tau_1} \rho_1(\prod_{2 \le i \le n} A_{(a_i,\epsilon_i)}) \mathbf{e}_1)| \\
&= |\prod_{1 \le i \le n} \eta_i^{\tau_1}|,
\end{aligned}$$

where $\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$. □

Let π_n be the projection map to P_n along L_n and π be the projection map to $\{(x, y, z) \in \mathbb{R}^3 | x = 0\}$ along L_0.

Lemma 8. *For each $n \in \mathbb{Z}$ with $n > 0$, we have $|(p_n - q_n\alpha)\,{}^*\alpha^{\tau_1} + (r_n - q_n\beta)\,{}^*\beta^{\tau_1}| = |\prod_{1 \le i \le n} \eta_i^{\tau_1}|$.*

Proof. Since $\pi(q_n, p_n, r_n) = (0, p_n - q_n\alpha, r_n - q_n\beta)$, by using Lemma 6 we have

$$\begin{aligned}\rho_0(q_n, p_n, r_n) &= \rho_0(0, p_n - q_n\alpha, r_n - q_n\beta) \\ &= |(p_n - q_n\alpha)\,{}^*\alpha^{\tau_1} + (r_n - q_n\beta)\,{}^*\beta^{\tau_1}|.\end{aligned}$$

Therefore, using Lemma 7 we obtain Lemma 8. □

Lemma 9. *There exists a positive constant $C_1(\alpha, \beta)$ such that for any $n \in \mathbb{Z}$ with $n \geq 0$*

$$|(p_n - q_n\alpha)\,{}^*\alpha^{\tau_1} + (r_n - q_n\beta)\,{}^*\beta^{\tau_1}| \leq C_1(\alpha, \beta)\gamma^{\frac{n}{2m}}.$$

Proof. We set $C_1(\alpha, \beta) = \prod_{1\leq i\leq m} \max\{1, |\eta_j^{\tau_i}|\}$. Using the fact that $\sqrt{\gamma} = |\prod_{1\leq i\leq m} \eta_j^{\tau_1}|$ and $\eta_{j+m} = \eta_j$ for each $j > 0$, we have

$$\begin{aligned}|(p_n - q_n\alpha)\,{}^*\alpha^{\tau_1} + (r_n - q_n\beta)\,{}^*\beta^{\tau_1}| &= |\prod_{1\leq i\leq n} \eta_i^{\tau_1}| \\ &\leq C_1(\alpha, \beta)\gamma^{\frac{n}{2m}}.\end{aligned}$$

□

From the fact that $|x\,{}^*\alpha^{\tau_1} + y\,{}^*\beta^{\tau_1}|^2$ is a positive definite quadratic form and by Lemma 9 we have the following lemma.

Lemma 10. *There exists a positive constant $C_2(\alpha, \beta)$ such that for any $n \in \mathbb{Z}$ with $n \geq 0$*

$$|p_n - q_n\alpha| \leq \frac{C_2(\alpha, \beta)}{\sqrt{q_n}},$$

$$|r_n - q_n\beta| \leq \frac{C_2(\alpha, \beta)}{\sqrt{q_n}}.$$

We remark that the above formulae hold for each periodic point (α, β) related to Jacobi-Perron algorithm (see [9]).

Lemma 11. *For each $n \geq 1$, $q_n + {}^*\alpha p_n + {}^*\beta r_n = \prod_{1\leq i\leq n} \eta_i$ holds.*

Proof. It is easy to see that

$$
\begin{aligned}
q_n + {}^*\alpha p_n + {}^*\beta r_n &= (1\ \ {}^*\alpha\ \ {}^*\beta)\begin{pmatrix} q_n \\ p_n \\ r_n \end{pmatrix} \\
&= (1\ \ {}^*\alpha\ \ {}^*\beta)\prod_{1\le i\le n} A_{(a_i,\epsilon_i)}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \\
&= \prod_{1\le i\le n}\eta_i (1\ \ {}^*\alpha\ \ {}^*\beta)\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \\
&= \prod_{1\le i\le n}\eta_i .
\end{aligned}
$$

□

We define $\mathbf{u_n}, \mathbf{v_n}$ and $\mathbf{r_n}$by

$$
\begin{aligned}
\mathbf{u_n} &= \frac{1}{2}\left(\begin{pmatrix} 1 \\ \alpha_n^{\tau_1} \\ \beta_n^{\tau_1} \end{pmatrix} + \begin{pmatrix} 1 \\ \alpha_n^{\tau_2} \\ \beta_n^{\tau_2} \end{pmatrix}\right), \\
\mathbf{v_n} &= \frac{1}{2i}\left(\begin{pmatrix} 1 \\ \alpha_n^{\tau_1} \\ \beta_n^{\tau_1} \end{pmatrix} - \begin{pmatrix} 1 \\ \alpha_n^{\tau_2} \\ \beta_n^{\tau_2} \end{pmatrix}\right), \\
\mathbf{r_n} &= \begin{pmatrix} 1 \\ \alpha_n \\ \beta_n \end{pmatrix}.
\end{aligned}
$$

By Lemma 5 we see that $\mathbf{u_n}, \mathbf{v_n} \in P_n$. We set $(\gamma^{\tau_1})^{-1} = \sqrt{\gamma}e^{2\pi i\theta}$ with $0 \le \theta < 1$. From the fact that $M(1, \alpha^{\tau_1}, \beta^{\tau_1})^t = (\gamma^{\tau_1})^{-1}(1, \alpha^{\tau_1}, \beta^{\tau_1})^t$ and $M(1, \alpha^{\tau_2}, \beta^{\tau_2})^t = (\gamma^{\tau_2})^{-1}(1, \alpha^{\tau_2}, \beta^{\tau_2})^t$ we have

$$
M(\mathbf{u_n}, \mathbf{v_n}) = (\mathbf{u_n}, \mathbf{v_n})\begin{pmatrix} \sqrt{\gamma}\cos 2\pi\theta & \sqrt{\gamma}\sin 2\pi\theta \\ -\sqrt{\gamma}\sin 2\pi\theta & \sqrt{\gamma}\cos 2\pi\theta \end{pmatrix}. \tag{7}
$$

Lemma 12. *θ is irrational.*

Proof. We suppose that θ is rational. We set $\theta = \frac{k}{l}$, where $k, l \in \mathbb{Z}$ and $l > 0$. From (7) we see that $(\gamma^{\tau_1})^l, (\gamma^{\tau_2})^l \in \mathbb{R}$. Since $(\gamma^{\tau_1})^l \in$

$\mathbb{Q}(\gamma^{\tau_1}) \cap \mathbb{R}$, we see $(\gamma^{\tau_1})^l \in \mathbb{Q}$. Therefore, $\gamma^l \in \mathbb{Q}$. By using Lemma 3 we see that γ^l is the unit in $\mathbb{Q}(\gamma)$. Therefore, we have $\gamma^l = \pm 1$. But it contradicts that $0 < \gamma < 1$. □

Theorem 13. *For each* $0 \leq k \leq m-1$, *the limit set of* $\{\sqrt{q_n}(p_n - q_n\alpha, r_n - q_n\beta) | n \equiv k \mod m\}$ *as* $n \to \infty$ *is the following ellipse*

$$\{(x,y) \in \mathbb{R}^2 || x\, {}^*\alpha^{\tau_1} + y\, {}^*\beta^{\tau_1}|^2 = \frac{\prod_{1 \leq i \leq k} n(\eta_i)}{1 + \alpha\, {}^*\alpha + \beta\, {}^*\beta}\}, \tag{8}$$

which is denoted by $E(k)$.

Proof. We see that $|x\, {}^*\alpha^{\tau_1} + y\, {}^*\beta^{\tau_1}|^2$ is a positive definite quadratic form, which is noticed as the remark following Lemma 6. Therefore, the set (8) is an ellipse. From Lemma 8 and 11 we have

$$\begin{aligned}
&|\sqrt{q_n}(p_n - q_n\alpha)\, {}^*\alpha^{\tau_1} + \sqrt{q_n}(r_n - q_n\beta)\, {}^*\beta^{\tau_1}|^2 \\
&= q_n |(p_n - q_n\alpha)\, {}^*\alpha^{\tau_1} + (r_n - q_n\beta)\, {}^*\beta^{\tau_1}|^2 \\
&= \frac{q_n}{q_n + {}^*\alpha p_n + {}^*\beta r_n}(q_n + {}^*\alpha p_n + {}^*\beta r_n) | \prod_{1 \leq i \leq n} \eta_i^{\tau_1}|^2 \\
&= \frac{q_n}{q_n + {}^*\alpha p_n + {}^*\beta r_n} \prod_{1 \leq i \leq n} n(\eta_i).
\end{aligned}$$

Therefore, by using Lemma 10 we have

$$\begin{aligned}
&\lim_{\substack{n \equiv k \\ n \to \infty}} |\sqrt{q_n}(p_n - q_n\alpha)\, {}^*\alpha^{\tau_1} + \sqrt{q_n}(r_n - q_n\beta)\, {}^*\beta^{\tau_1}|^2 \\
&= \frac{1}{1 + \alpha\, {}^*\alpha + \beta\, {}^*\beta} \prod_{1 \leq i \leq k} n(\eta_i).
\end{aligned}$$

Thus, the limit set of $\{\sqrt{q_n}(p_n - q_n\alpha, r_n - q_n\beta) | n \equiv k \mod m\}$ as $n \to \infty$ is included in $E(k)$. We define c_k, d_k, e_k by

$$\begin{pmatrix} q_k \\ p_k \\ r_k \end{pmatrix} = c_k \mathbf{u}_0 + d_k \mathbf{v}_0 + e_k \mathbf{r}_0.$$

We see easily that $c_k \neq 0$ or $d_k \neq 0$. Then, for $n = ml + k$ we have

$$\begin{aligned}\begin{pmatrix} q_n \\ p_n \\ r_n \end{pmatrix} &= \prod_{1 \leq i \leq n} A_{(a_i,\epsilon_i)} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \\ &= M^l \begin{pmatrix} q_k \\ p_k \\ r_k \end{pmatrix} \\ &= \begin{pmatrix} \mathbf{u}_0 & \mathbf{v}_0 & \mathbf{r}_0 \end{pmatrix} \begin{pmatrix} \sqrt{\gamma^l}\cos 2\pi\theta & \sqrt{\gamma^l}\sin 2\pi\theta & 0 \\ -\sqrt{\gamma^l}\sin 2\pi\theta & \sqrt{\gamma^l}\cos 2\pi\theta & 0 \\ 0 & 0 & \frac{1}{\gamma^l} \end{pmatrix} \begin{pmatrix} c_k \\ d_k \\ e_k \end{pmatrix}.\end{aligned}$$

Therefore, we have

$$\begin{aligned}\begin{pmatrix} 0 \\ p_n - q_n\alpha \\ r_n - q_n\beta \end{pmatrix} &= \pi \begin{pmatrix} q_n \\ p_n \\ r_n \end{pmatrix} \\ &= \begin{pmatrix} \pi(\mathbf{u}_0) & \pi(\mathbf{v}_0) \end{pmatrix} \begin{pmatrix} \sqrt{\gamma^l}\cos 2\pi\theta & \sqrt{\gamma^l}\sin 2\pi\theta \\ -\sqrt{\gamma^l}\sin 2\pi\theta & \sqrt{\gamma^l}\cos 2\pi\theta \end{pmatrix} \begin{pmatrix} c_k \\ d_k \end{pmatrix},\end{aligned}$$

which yields Theorem 13 by using Lemma 12. □

Similarly, we have the following corollary.

Corollary 14. *Let j_i $(1 \leq i \leq 3)$ be non negative integers and $j_i > 0$ for some i. For j_i $(1 \leq i \leq 3)$ and any positive integer n, we define p_n^*, q_n^* and r_n^* by*

$$\begin{pmatrix} q_n^* \\ p_n^* \\ r_n^* \end{pmatrix} = \prod_{1 \leq i \leq n} A_{(a_i,\epsilon_i)} \begin{pmatrix} j_1 \\ j_2 \\ j_3 \end{pmatrix}.$$

Then, for each $0 \leq k \leq m-1$ the limit set of $\{\sqrt{q_n^*}(p_n^* - q_n^*\alpha, r_n^* - q_n^*\beta) | n \equiv k \mod m\}$ as $n \to \infty$ is the following ellipse

(9)

$$\{(x,y) \in \mathbb{R}^2 | |x\,{}^*\alpha^{\tau_1} + y\,{}^*\beta^{\tau_1}|^2 = \frac{n(j_1 + {}^*\alpha_k j_2 + {}^*\beta_k j_3) \prod_{1 \leq i \leq k} n(\eta_i)}{1 + \alpha\,{}^*\alpha + \beta\,{}^*\beta}\}.$$

In [5] we conjecture that the modified Jacobi-Perron algorithm gives the best simultaneous approximation to the points (β_1, β_2) such that $1, \beta_1, \beta_2$ is a basis for a real cubic number field, $\mathbb{Q}(\beta_1)$ has a complex embedding and (β_1, β_2) is a purely periodic point by the modified Jacobi-Perron algorithm. But we have a following counter example.

Counter Example. Let γ be the real root of $x^3 + 8x^2 + 16x - 1$. Then, $\mathbb{Q}(\gamma)$ has a complex embedding. Let $\alpha = \frac{3}{\gamma+3}$ and $\beta = \gamma$. (α, β) is the purely periodic point by the modified Jacobi-Perron algorithm and the digits are given as follows: $(a_1, \epsilon_1) = (1, 0), (a_2, \epsilon_2) = (1, 1), (a_3, \epsilon_3) = (2, 0)$, $(a_4, \epsilon_4) = (3, 0)$ and $(a_{n+4}, \epsilon_{n+4}) = (a_n, \epsilon_n)$ for each $n \in \mathbb{Z}$ with $n > 0$.

Then, we have the following table.

n	0	1	2	3
${}^*\alpha_n$	$\frac{2\gamma}{1-\gamma}$	$\frac{2+\gamma}{2\gamma+7}$	$\frac{1+\gamma}{2}$	γ
${}^*\beta_n$	$\frac{1}{\gamma+3}$	$\frac{1-\gamma}{\gamma+1}$	$\frac{1-\gamma}{2\gamma+6}$	$\frac{2}{\gamma+5}$
η_n		$\frac{\gamma+1}{1-\gamma}$	$\frac{2}{\gamma+1}$	$\frac{5+\gamma}{2}$
$n(\eta_n)$		$\frac{5}{12}$	$\frac{4}{5}$	$\frac{3}{4}$

Let $\mu = 3 + 2\,{}^*\alpha_0$. For any positive integer n we define p_n^*, q_n^* and r_n^* by

$$\begin{pmatrix} q_n^* \\ p_n^* \\ r_n^* \end{pmatrix} = \prod_{1 \le i \le n} A_{(a_i, \epsilon_i)} \begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix}.$$

Then, we have $n(\mu) = \frac{1}{6}$. Since $n(\mu) < \min\{\prod_{1 \le i \le k} n(\eta_i) | i = 1, 2, 3\}$, by using Theorem 13 and Corollary 14 we see that the ellipse defined from $\{p_n^*, q_n^*, r_n^*\}_{n \equiv 1 \bmod 4}$ as in Corollary 14 is nearer to the origin than the ellipses defined from $\{p_n, q_n, r_n\}$ as in Theorem 13. We remark that $p_{4j}^* = p_{4j+1}^* + p_{4j+2}^*, q_{4j}^* = q_{4j+1}^* + q_{4j+2}^*$ and $r_{4j}^* = r_{4j+1}^* + r_{4j+2}^*$ for each $j \in \mathbb{Z}$ with $j \ge 0$. In our paper [6] in preparation we will show that under some conditions for (α, β) the nearest ellipses to the origin in $\lim(\alpha, \beta)$ are given as intermediate convergents of modified Jacobi-Perron algorithm.

References

[1] W. W. Adams, Simultaneous asymptotic diophantine approximations to a basis of a real cubic number field, J. Number Theory., **1** (1969), 179–194.

[2] N. Chevallier, Best simultaneous Diophantine approximations of some cubic algebraic numbers, J. Théor. Nombres Bordx., **14** (2002), 403–414.

[3] N. Chekhova, P. Hubert and A. Messaoudi, Propriétés combinatoires, ergodiques et arithmétiques de la substitution de Tribonacci, J. Théor. Nombres Bordx., **13** (2001), 371–394.

[4] T. Fujita, Sh. Ito, M. Keane and M. Ohtsuki, On almost everywhere exponential convergence of modified Jacobi-Perron algorithm: a corrected proof, Ergodic theory an dynamical systems, **16** (1996), 1345–1352.

[5] S. Ito, J. Fujii, H, Higashino and S. Yasutomi, On simultaneous approximation to (α, α^2) with $\alpha^3 + k\alpha - 1 = 0$, J. Number Theory., **1** (2003), 255–283.

[6] S. Ito, M. Furukado and S. Yasutomi, Best simultaneous diophantine approximation of cubic numbers related to modified Jacobi-Perron algorithm, in preparation.

[7] S. Ito and M. Ohtsuki, Modified Jacobi-Perron algorithm and generating Markov partitions for special hyperbolic toral automorphisms, Tokyo J. Math., **16** (1993), 441–472.

[8] Sh. Ito and M. Ohtsuki, Paralleogram tilings and Jacobi-Perron algorithm, Tokyo J. Math., **17** (1994), 33–58.

[9] O. Perron, Grundlagen für eine Theorie des Jacobischen Kettenbruchalgorithmus, Math. Ann., **64** (1907), 1–76.

[10] E. V. Podsypanin, A generalization of continued fraction algorithm that is related to the ViggoBrun algorithm. (Russian), Studies in Number Theory (LOMI), **4**., Zap. Naucn. Sem. Leningrad. Otdel. Mat. Inst. Steklov, **67** (1977), 184–194.

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