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On lacunary trigonometric product

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Abstract.

We prove the law of the iterated logarithm for gap series under weak regularity condition and apply to the lacunary trigonometric product.

§1. Introduction

Let f be an $\mathbf{R}$-valued function on $\mathbf{R}$ satisfying

$$(1)\qquad f(t+1) = f(t), \quad \int_0^1 f(t)\,dt = 0, \quad \|f\|_2^2 = \int_0^1 |f(t)|^2\,dt < \infty.$$

Denote by $S(f;N)$ the N-th partial sum of Fourier series of f, and put $R(f;N) = f - S(f;N)$ and $\|f\|_* = \|f\|_2 + \sum_{k=1}^{\infty} \|R(f;k)\|_2/k$.

We prove the theorem below:

Theorem 1. *Let* $\{n_k\}$ *be a sequence of positive integers satisfying* $n_{k+1}/n_k \geq q > 1$. *If*

$$(2)\qquad \|R(f;N)\|_2 = O((\log N)^{-2}(\log\log N)^{-\alpha})$$

for some $\alpha > 0$, *then*

$$(3)\qquad \varlimsup_{N\to\infty} \frac{1}{\sqrt{2N\log\log N}} \sum_{k=1}^{N} f(n_k t) \leq C_q \|f\|_2^{1/2}\|f\|_*^{1/2} \quad a.e.$$

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where C_q is a constant depending only on q. If $n_{k+1}/n_k \to \infty$, then

$$\varlimsup_{N\to\infty} \frac{1}{\sqrt{2N\log\log N}} \sum_{k=1}^{N} f(n_k t) = \|f\|_2 \quad a.e. \tag{4}$$

If

$$f \in L^{16+\alpha} \qquad and \tag{5}$$

$$\|R(f;N)\|_2 = O((\log N)^{-1}(\log\log N)^{-1-\alpha}) \tag{6}$$

for some $\alpha > 0$, then the same conclusions hold.

We apply the above theorem to functions

$$\log|2\sin\pi x| = -\sum_{\nu=1}^{\infty} \frac{\cos 2\pi\nu}{\nu}, \quad \log|2\cos\pi x| = \sum_{\nu=1}^{\infty}(-1)^{\nu-1}\frac{\cos 2\pi\nu}{\nu}$$

which are unbounded and L^2-1/2-Hölder continuous. (Cf. Lemma 5 of [11]) In case when $n_{k+1}/n_k > q > 1$, we have

$$\varlimsup_{k\to\infty} \left|\prod_{j=1}^{k} 2\cos(\pi n_j x)\right|^{1/\sqrt{k\log\log k}} < C < \infty, \tag{7}$$

and when $n_{k+1}/n_k \to \infty$, we have

$$\varlimsup_{k\to\infty} \left|\prod_{j=1}^{k} 2\cos(\pi n_j x)\right|^{1/\sqrt{k\log\log k}} = e^{\pi/\sqrt{6}}.$$

These results remain valid if we replace cosine functions by sines.

The above gap conditions are best possible in the following sense. It is proved in [10] that for any $\rho_k \downarrow 0$, there exists $\{n_k\}$ such that $n_{k+1}/n_k > \rho_k$ and the law of the iterated logarithm (7) does not hold.

Now we make a little survey on the studies of this fields. The central limit theorem corresponding to (4) was proved by Kac [13] by assuming uniform Hölder continuity on function f, and later the condition was weakened to (6) by Takahashi [15]. (The condition in [15] seems to stronger than (6), but it is clear that (6) is enough to convey the proof given by Takahashi.)

As to the law of the iterated logarithm (3) and (4), Takahashi [16] and [17] proved by assuming Hölder continuity. Philipp [14] proved for functions of bounded variations and Berkes [2] proved for bounded L^2-Hölder continuous functions. Our conditions are much weaker than these, which is clear from the inequality (10). Although extra integrability condition (5) is assumed, it should be emphasised that the law of the iterated logarithm was proved under the same regularity condition (6) as the central limit theorem. And it is also noted that application to product of lacunary trigonometric is made possible by our results.

At the end we remark that functions satisfying conditions (5) and (6) exist. Actually, by taking a sequence $\{a_k\}$ satisfying $\sum_{k=N}^{\infty} a_k^2 \sim (\log N)^{-2}(\log\log N)^{-2-2\alpha}$ and consider function defined by the random series $\sum \pm a_n \cos 2\pi n x$ then it belongs to L^r for all $r > 0$ almost surely. (Ch. 5 Theorem 8.16 of Zygmund [19]).

Our proof goes along with the direction given by Takahashi [17]. The uses of various truncation techniques make it possible to give a new result.

§2. The proof of the theorem

We put δ, δ', δ'', and $\mu(k)$ as follows: In the case the condition (2) is assumed, we put $\delta = 1$, $0 < \delta'' = \delta' < 1/30$, and $\mu(k) = k^{2\delta'}$. In the case the conditions (5) and (6) are assumed, take δ'' between $1/(30+2\alpha)$ and $1/30$, and take $0 < \delta' < 1/15$ small enough to satisfy $1/(30+2\alpha) < \delta'' - \delta'$, put $\mu(k) = 2^{k^{\delta'}}$ and $\delta = \delta'' - \delta'$.

In both cases, we have

$$3^{M^{1/7}}/\mu(2M^{15/7}) \to \infty, \qquad (M \to \infty). \tag{8}$$

We may assume $n_{k+1}/n_k \geq 3$. For $M > 0$, put

$$\xi_M(t) = (t \wedge M) \vee (-M) \quad \text{and} \quad \eta_M(t) = t - \xi_M(t).$$

Lemma 2. *If* $\|R(f;N)\|_2 = O((\log N)^{-1}(\log\log N)^{-\gamma})$ *for some* $\gamma > 0$, *then* $\sup_{M>0} \|R(\xi_M(f);N)\|_2 = O((\log N)^{-1}(\log\log N)^{-\gamma})$. *For any* M_k, *we have*

$$\sum_{k=1}^{\infty} \|R(\xi_{M_k}(f);3^k)\|_2 \leq C_0 \|f\|_*, \tag{9}$$

where C_0 *is an absolute constant.*

Proof. Let us recall the notion of L^2-modulus of continuity $\omega^{(2)}(\varepsilon, g)$ of function g. It is given by $\omega^{(2)}(\varepsilon, g) = \sup_{|h|<\varepsilon} \|g(\,\cdot\, + h) - g(\,\cdot\,)\|_2$, and have close relations with the decay order of $\|R(g;N)\|_2$: by (3.3) of pp. 241 of Zygmund [19] and by (2.6) of pp. 160 of Bari [1], we have

$$\|R(f;N)\|_2 \leq C_1 \omega^{(2)}(1/N, f) \leq C_2 \frac{1}{N} \sum_{k=0}^{N-1} \|R(f;k)\|_2, \tag{10}$$

where C_1 and C_2 are absolute constants. Thus $\|R(\xi_M(f);N)\|_2 \le C_1\omega^{(2)}(1/N,\xi_M(f))$. Because of $|\xi_M(t)-\xi_M(s)| \le |t-s|$, we have

$$\|\xi_M(f(\cdot+h)) - \xi_M(f(\cdot))\|_2 \le \|f(\cdot+h) - f(\cdot)\|_2$$

and see $\omega^{(2)}(\varepsilon,\xi_M(f)) \le \omega^{(2)}(\varepsilon,f)$. We also have

$$\begin{aligned} C_1\omega^{(2)}(1/N,f) &\le C_2\frac{1}{N}\sum_{k=0}^{N-1}\|R(f;k)\|_2 \\ &= \frac{1}{N}\Bigg(\sum_{0\le k<\sqrt{N}} + \sum_{\sqrt{N}\le k<N}\Bigg)O((\log k)^{-1}(\log\log k)^{-\gamma}) \\ &= O((\log N)^{-1}(\log\log N)^{-\gamma}). \end{aligned}$$

Thus we have $\sup_M \omega^{(2)}(1/N,\xi_M(f)) = O((\log N)^{-1}(\log\log N)^{-\gamma})$ and eventually have the conclusion. The proof of (9) is given as follows:

$$\sum_{k=1}^{\infty}\|R(\xi_{M_k}(f);3^k)\|_2 \le C_2\sum_{k=1}^{\infty}\frac{1}{3^k}\sum_{l=0}^{3^k-1}\|R(f;l)\|_2 \le C_2\|f\|_*.$$

Q.E.D.

Lemma 3. *We have*

$$\sum_{k=1}^{N}\eta_{k^\delta}(f(n_kt)) = o(\sqrt{N\log\log N}) \quad a.e. \tag{11}$$

Proof. Let $\beta = 1$ or $15+\alpha$, we have $f \in L^{1+\beta}$ and $\delta > 1/(2\beta)$. Since $|\eta_M(f)| \le |f|\mathbf{1}_{\{|f|\ge M\}} \le |f|^{1+\beta}/M^\beta$, we have

$$\int_0^1\sum_{k=1}^{\infty}\frac{|\eta_{k^\delta}(f(n_kt))|}{\sqrt{k\log\log k}}\,dt \le \sum_{k=1}^{\infty}\frac{\int_0^1|f|^{\beta+1}\,dt}{k^{\beta\delta}\sqrt{k\log\log k}} < \infty.$$

By Kronecker's lemma, we have the conclusion. Q.E.D.

Lemma 4. *We have*

$$\sum_{k=1}^{N}R(\xi_{k^\delta}(f);\mu(k))(n_kt) = o\big(\sqrt{N\log\log N}\,\big) \quad a.e. \tag{12}$$

Proof. For $g \in L^2$, put $\widehat{g}(x) = 0$ for $x \notin \mathbf{Z}$ and denote $\operatorname{Spec} g = \{\nu \mid \widehat{g}(\nu) \neq 0\}$ and $|\operatorname{Spec} g| = \{|\nu| \mid \widehat{g}(\nu) \neq 0\}$. We first prove by assuming (2). If $j \geq k$, then we have

$$\begin{aligned}\left|\int h(kt)g(jt)\,dt\right| &= \left|\sum_{kn+jm=0} \widehat{h}(n)\widehat{g}(m)\right| \leq \sum_{m\in\operatorname{Spec} g} |\widehat{h}(-mj/k)\widehat{g}(m)| \\ &\leq \Bigl(\sum_{|m|\geq \min|\operatorname{Spec} g|} |\widehat{h}(mj/k)|^2\Bigr)^{1/2}\Bigl(\sum_{m\in\operatorname{Spec} g} |\widehat{g}(m)|^2\Bigr)^{1/2} \\ &\leq \|R(h;\min|\operatorname{Spec} g|j/k)\|_2\|g\|_2.\end{aligned}$$

Therefore if $j \geq k$, by Lemma 2 and $n_j/n_k \geq 3^{j-k}$ we have

$$\begin{aligned}&\left|\int R(\xi_{j^\delta}(f);\mu(j))(n_j t)R(\xi_{k^\delta}(f);\mu(k))(n_k t)\,dt\right| \\ &\leq \|R(\xi_{j^\delta}(f);\mu(j))\|_2\|R(\xi_{k^\delta}(f);\mu(j)n_j/n_k)\|_2 \\ &= O((\log j)^{-2}(\log\log j)^{-\alpha}(\log j + (j-k))^{-2}).\end{aligned}$$

Hence we have

$$\begin{aligned}&\int\Bigl(\sum_{k=A+1}^{B} \frac{R(\xi_{k^\delta}(f);\mu(k))(n_k t)}{(k\log\log k)^{1/2}}\Bigr)^2 dt \\ &\leq 2\sum_{k=A+1}^{B}\sum_{l=0}^{\infty}\left|\int \frac{R(\xi_{k^\delta}(f);\mu(k))(n_k t)R(\xi_{(k+l)^\delta}(f);\mu(k+l))(n_{k+l}t)}{(k\log\log k)^{1/2}((k+l)\log\log(k+l))^{1/2}}\,dt\right| \\ &= O\Bigl(\sum_{k=A+1}^{B}\sum_{l=0}^{\infty} \frac{1}{k(\log k)^2(\log\log k)^{1+\alpha}(\log k + l)^2}\Bigr) \\ &= O\Bigl(\sum_{k=A+1}^{B} \frac{1}{k(\log k)^3(\log\log k)^{1+\alpha}}\Bigr) \to 0 \qquad (A, B \to \infty).\end{aligned}$$

Thus the series $\sum \frac{R(\xi_{k^\delta}(f);\mu(k))(n_k t)}{(k\log\log k)^{1/2}}$ converges in L^2-sense. Thanks to

$$\begin{aligned}&\int\sum_{A=0}^{\infty}\Bigl(\sum_{k=2^A+1}^{\infty} \frac{R(\xi_{k^\delta}(f);\mu(k))(n_k t)}{(k\log\log k)^{1/2}}\Bigr)^2 dt \\ &= O\Bigl(\sum_{A=0}^{\infty}\sum_{k=2^A}^{\infty} \frac{1}{k(\log k)^3(\log\log k)^{1+\alpha}}\Bigr) \\ &= O\Bigl(\sum_{k=1}^{\infty} \frac{1}{k(\log k)^2(\log\log k)^{1+\alpha}}\Bigr) = O(1),\end{aligned}$$

we see that $\sum_{k=1}^{2^A} \frac{R(\xi_{k^\delta}(f);\mu(k))(n_k t)}{(k\log\log k)^{1/2}}$ converges a.e. as $A \to \infty$.

For $g_n \in L^2$ and $A_n \in \mathbf{R}$ satisfying $\|g_m - g_n\|_2^2 \le A_m - A_n$ for $m \ge n$, Menchoff's inequality claims $\|\max_{n\le N} g_n\|_2^2 \le C_3(\log N)^2(A_N - A_0)$, where C_3 is an absolute constant. Applying this, we have

$$\int \Big(\max_{2^A < n \le 2^{A+1}} \sum_{k=2^A+1}^{n} \frac{R(\xi_{k^\delta}(f);\mu(k))(n_k t)}{(k\log\log k)^{1/2}} \Big)^2 dt$$
$$= O\Big(\sum_{k=2^A+1}^{2^{A+1}} \frac{1}{k\log k(\log\log k)^{1+\alpha}} \Big).$$

Since it is summable in A, by Beppo-Levi's theorem, we see that the integrand tends to 0 a.e. as $A \to \infty$. Thus the series $\sum_{k=1}^{\infty} \frac{R(\xi_{k^\delta}(f);\mu(k))(n_k t)}{(k\log\log k)^{1/2}}$ converges a.e. By Kronecker's lemma, we have the conclusion.

Next, we prove by assuming (5) and (6). By noting Lemma 2 again, we have the estimate

$$\Big| \int R(\xi_{j^\delta}(f);\mu(j))(n_j t) R(\xi_{k^\delta}(f);\mu(k))(n_k t)\, dt \Big|$$
$$= O(j^{-\delta'}(j-k)^{-1}(\log(j-k))^{-1-\alpha}).$$

In the same way as before we can complete the proof. Q.E.D.

Lemma 5. *We have*

$$\|S(\xi_{k^\delta}(f);\mu(k))\|_\infty = O(\|f\|_2 k^{\delta''}). \tag{13}$$

Proof. When (2) is assumed, it is clear from the Schwartz inequality:
$\|S(g;M)\|_\infty \le \|g\|_2 M^{1/2}$. When (5) and (6) are assumed, it is derived from the inequality $\|S(f,N)\|_\infty \le C_4\|f\|_\infty \log N$, where C_4 is an absolute constant. This inequality is proved in the same way as the proof of Th. 11.9 of Ch. II in Zygmund [19]. Q.E.D.

Lemma 6. $e^x \le (1 + x + x^2/2)e^{|x|^3}$ *for all* $x \in \mathbf{R}$.

Proof. By expressing both sides by power series, it is clear for $x \ge 0$. Elementary calculus shows that $e^x \le 1 + x + x^2/2$ holds for all $x \le 0$. Q.E.D.

Lemma 7. *Put* $\Xi_f = (\sqrt{C_0}\,\|f\|_2\|f\|_*)^{1/2}$. *There exists* M_0 *such that, for all* $M \ge M_0$, *for all* $0 < \lambda < M^{-1/2}\log M$, *and for all* $N \le$

$M^{15/7}$, *we have*

$$\int_0^1 \exp\left(\lambda \sum_{k=N+1}^{N+M} S(\xi_{k^\delta}(f);\mu(k))(n_k t)\right) dt < 2\exp\Big(2(\lambda \Xi_f)^2 M\Big).$$

Proof. Let $m^7 \le M < (m+1)^7$, $m(2L+2) \le M < m(2L+4)$ and

$$U_l(t) = \sum_{k=lm+N+1}^{(l+1)m+N} S(\xi_{k^\delta}(f);\mu(k))(n_k t)$$

Because of

$$\max_{k \le N+M} \|S(\xi_{k^\delta}(f);\mu(k))\|_\infty = O((N+M)^{\delta''}) = O(M^{(15/7)\delta''}),$$

we have

$$\begin{aligned}
&\lambda\left|\sum_{k=N+1}^{N+M} S(\xi_{k^\delta}(f);\mu(k))(n_k t) - \sum_{l=0}^{2L+1} U_l(t)\right| \\
&\le \lambda \sum_{k=m(2L+2)+N+1}^{N+M} \|S(\xi_{k^\delta}(f);\mu(k))\|_\infty = O(\lambda m M^{(15/7)\delta''}) \\
&= O(M^{-1/2+1/7+(15/7)\delta''}\log M) = o(1).
\end{aligned}$$

Similarly, $\lambda \max_{l \le L} |U_{2l}(t)| = O(M^{-1/2+1/7+(15/7)\delta''}\log M)$ and

$$(14)\quad \lambda^3 \sum_{l=0}^{L} |U_{2l}(t)|^3 = O(M^{3(-1/2+1/7+(15/7)\delta'')+6/7}(\log M)^3) = o(1).$$

Thus for $M \ge M_0$, we have

$$\begin{aligned}
&\int_0^1 \exp\left(\lambda \sum_{k=N+1}^{N+M} S(\xi_{k^\delta}(f);\mu(k))(n_k t)\right) dt \\
&< \sqrt{2}\int_0^1 \exp\left(\lambda \sum_{l=0}^{2L+1} U_l(t)\right) dt \\
&\le \sqrt{2}\left(\int_0^1 \exp\left(2\lambda \sum_{l=0}^{L} U_{2l}(t)\right) dt \int_0^1 \exp\left(2\lambda \sum_{l=0}^{L} U_{2l+1}(t)\right) dt\right)^{1/2}.
\end{aligned}$$

By Lemma 6 and (14), for $M \ge M_0$, we have

$$\exp\left(2\lambda \sum_{l=0}^{L} U_{2l}(t)\right) \le \sqrt{2}\prod_{l=0}^{L}(1 + 2\lambda U_{2l} + 2\lambda^2 U_{2l}^2)$$

Denote the Fourier series of $S(\xi_{k^\delta}(f);\mu(k))$ by

$$S(\xi_{k^\delta}(f);\mu(k)) = \sum_{\nu\le\mu(k)} \rho_{k,\nu}\cos(2\pi\nu t+\gamma_{k,\nu}),$$

and denote

$$\Phi(k,j) = \left\{ (r,s) \;\middle|\; \begin{array}{c} |n_{k+N}s - n_{j+N}r| \le n_{lm+N}, \\ 0 < s \le \mu(N+k), \\ 0 < r \le \mu(N+j) \end{array} \right\}$$

$$W_l(t) = \sum_{k=lm+2}^{(l+1)m} \sum_{j=lm+1}^{k-1} \sum_{(r,s)\in\Phi(k,j)} \rho_{N+j,r}\rho_{N+k,s}\cos(2\pi(n_{N+k}s - n_{N+j}r)t + \gamma_{N+k,s} - \gamma_{N+j,r})$$

$$V_l = 2\lambda U_l + \lambda^2\left(2U_l^2 - \sum_{k=lm+1}^{(l+1)m}\sum_{s=1}^{\mu(N+k)} \rho_{N+k,s}^2 - 2W_l(t)\right).$$

Let $l \le 2L+1$ and $lm+2 \le k \le (l+1)m$. Because of

$$S(\xi_{(N+k)^\delta}(f);\mu(N+k))^2(n_{N+k}t) - \frac{1}{2}\sum_{s=1}^{\mu(N+k)} \rho_{N+k,s}^2$$
$$= 2\sum_{1\le s<r\le\mu(N+k)} \rho_{N+k,s}\rho_{N+j,r}\cos(2\pi n_{N+k}st+\gamma_{N+k,s})\cos(2\pi n_{N+k}rt+\gamma_{N+j,r}),$$

if we expand into trigonometric polynomial, frequencies all belong to

$$[\,n_{N+k}, 2n_{N+k}\mu(N+k)\,] \subset [\,n_{lm+N}, 2n_{(l+1)m+N}\mu(2M^{15/7})\,]$$

We also see that frequencies of V_l all belong to the last interval. By (8), $\frac{n_{lm+N}}{2\mu(2M^{15/7})n_{(l-1)m+N}} > \frac{3^m}{2\mu(2M^{15/7})} \ge \frac{3^{M^{1/7}}}{2\mu(2M^{15/7})} \to \infty$ as $M\to\infty$, and hence $\{V_{2l}\}$ satisfies

$$\int_0^1 V_{2l_1}(t)\dots V_{2l_\kappa}(t)\,dt = 0 \quad (\kappa\in\mathbf{N}, l_1<\dots<l_\kappa) \tag{15}$$

when M is large enough. If $lm+1 \le j < k \le (l+2)m$, we have $n_{lm+N}/n_{j+N} \le 1/3$ and $\Phi(k,j) \subset \{(r,s) \mid |sn_{k+N}/n_{j+N} - r| < 1/3\}$, and hence

$$\sum_{(r,s)\in\Phi(k,j)} |\rho_{N+j,r}\rho_{N+k,s}| \le \left(\sum_{s=1}^{\infty}\rho_{N+k,s}^2 \sum_{s=1}^{\infty}\rho_{N+j,sn_{N+k}/n_{N+j}}^2\right)^{1/2}$$
$$\le \|\xi_{(N+k)^\delta}(f)\|_2\|R(\xi_{(N+j)^\delta}(f);n_{N+k}/n_{N+j})\|_2$$

Therefore, by (9) we have

$$\sum_{k=lm+2}^{(l+1)m}\sum_{s=1}^{\mu(N+k)} \rho_{N+k,s}^2 + \|W_l\|_\infty$$
$$\le \|f\|_2 \sum_{k=lm+2}^{(l+1)m}\left(\|f\|_2 + \sum_{j=lm+1}^{k-1} \|R(\xi_{(N+j)^\delta}(f);3^{k-j})\|_2\right)$$

and hence $1+2\lambda U_l + 2\lambda^2 U_l^2 \le 1 + V_l + 2(\lambda\Xi_f)^2 m$. Thus by (15),

$$\int_0^1 \exp\left(2\lambda\sum_{l=0}^{L}U_{2l}(t)\right)dt \le \sqrt{2}\int_0^1\prod_{l=0}^{L}(1+V_l(t)+2(\lambda\Xi_f)^2)\,dt$$
$$= \sqrt{2}\prod_{l=0}^{L}(1+2(\lambda\Xi_f)^2 m) \le \sqrt{2}\exp\left(\sum_{l=0}^{L}2(\lambda\Xi_f)^2 m\right).$$

If we replace $2l$ by $2l+1$, it is still valid. Thus for $M\ge M_0$, we have

$$\int_0^1 \exp\left(\lambda\sum_{k=N+1}^{N+M} S(\xi_{k^\delta}(f);\mu(k))(n_k t)\right)dt < 2\exp\left(\sum_{l=0}^{2L+1}2(\lambda\Xi_f)^2 m\right),$$

which is less than $2\exp\bigl(2(\lambda\Xi_f)^2 M\bigr)$. Q.E.D.

Lemma 8. *Let $\psi(M) < (2\Xi_f\log M)^2$. For all $M\ge M_0$ and $N\le M^{15/7}$, we have*

$$\left|\left\{t;\sum_{k=N+1}^{N+M} S(\xi_{k^\delta}(f);\mu(k))(n_k t)\ge 2\Xi_f\sqrt{M\psi(M)}\right\}\right| \le 2e^{-\psi(M)/2}.$$

Proof. By Putting $\lambda = (2\Xi_f)^{-1/2}\psi^{1/2}(M)M^{-1/2}$, applying Lemma 7 and Markov's inequality, we have the above estimate. Q.E.D.

Lemma 9. *We have*

$$\overline{\lim_{m\to\infty}}\frac{1}{\sqrt{2^{m+1}(1+\varepsilon)\log m}}\sum_{k=1}^{2^m} S(\xi_{k^\delta}(f);\mu(k))(n_k t) \le 2\Xi_f \quad a.e.$$

Proof. By putting $M = 2^m$, $N = 0$ and $\psi(2^m) = 2(1+\epsilon)\log m$ and by applying previous lemma, we have

$$\left|\left\{t; \sum_{k=1}^{2^m} S(\xi_{k^\delta}(f);\mu(k))(n_k t) \geq 2\Xi_f\sqrt{2^{m+1}\log m}\right\}\right| \leq 2m^{-1-\epsilon},$$

and Borel-Cantelli Lemma proves the conclusion. Q.E.D.

Lemma 10. *We have*

$$\overline{\lim_{m\to\infty}} \max_{N<2^m} \frac{1}{\sqrt{2^{m+1}\log m}} \sum_{k=2^m+1}^{2^m+N} S(\xi_{k^\delta}(f);\mu(k))(n_k t) \leq 6\Xi_f \quad a.e.$$

Proof. Let $2^{[(7/15)m]} > M_0$ and $(1+1/\log m)/2 < 9/16$. Let $\epsilon > 0$ and take m large as $2(m-l) + 2(1+\epsilon)\log m < (2\Xi_f \log 2^l)^2 \quad (m > l \geq [(7/15)m])$. Put

$$X_l(t) = 0 \vee \max_{r=0}^{2^{m-l}-1} \sum_{k=2^m+r2^l+1}^{2^m+(r+1)2^l} S(\xi_{k^\delta}(f);\mu(k))(n_k t).$$

Then $\sum_{l=0}^{[(7/15)m]-1} X_l$ equals to a sum of at most $2^{[(7/15)m]-1}$ many terms among $S(\xi_{k^\delta}(f);\mu(k))(n_k t)$ $(2^m < k \leq 2^{m+1})$. Because of (13), we have

$$\begin{aligned}&\max_{N<2^m} \sum_{k=2^m+1}^{2^m+N} S(\xi_{k^\delta}(f);\mu(k))(n_k t)\\ &\leq \sum_{l=0}^{m-1} X_l(t) = O(2^{[(7/15)m]-1}\|f\|_2 2^{\delta'' m}) + \sum_{l=[(7/15)m]}^{m-1} X_l(t)\end{aligned}$$

Let us put $\psi(2^l) = 2(m-l) + 2(1+\epsilon)\log m$ $(m > l > [(7/15)m])$. Then the condition of Lemma 8 is satisfied and we have $N \leq 2^{(15/7)l}$

$$\left|\left\{t; \sum_{k=N+1}^{N+2^l} S(\xi_{k^\delta}(f);\mu(k))(n_k t) \geq 2\Xi_f\sqrt{2^l\psi(2^l)}\right\}\right| \leq 2e^{-(m-l)}m^{-1-\epsilon}.$$

By noting $N < 2^m \leq 2^{(15/7)l}$, we see that $|E_l| \leq 2^{m-l+1}e^{-(m-l)}m^{-1-\epsilon}$ for $m > l \geq [(7/15)m]$, where $E_l = \{t; X_l(t) \geq 2\Xi_f\sqrt{2^l\psi(2^l)}\}$. Thus we have $\sum_{m=1}^{\infty}\sum_{l=[(7/15)m]}^{m-1} |E_l| < \infty$ and $t \notin \bigcup_{l=[(7/15)m]}^{m-1} E_l$ for large m a.e. If $t \notin \bigcup_{l=[(7/15)m]}^{m-1} E_l$, we have

$$\sum_{l=[(7/15)m]}^{m-1} X_l(t) \leq 2\Xi_f \sum_{l=[(7/15)m]}^{m-1} \sqrt{2^l\psi(2^l)}.$$

Because of

$$\sqrt{\frac{2^l\psi(2^l)}{2^{l+1}\psi(2^{l+1})}} \le \sqrt{\frac{1}{2}\Big(1+\frac{2}{\psi(2^{l+1})}\Big)} \le \sqrt{\frac{1}{2}\Big(1+\frac{1}{\log m}\Big)} < \frac{3}{4} \quad (l < m),$$

we have

$$\begin{aligned}\sum_{l=[(7/15)m]}^{m-1} \sqrt{2^l\psi(2^l)} &\le \sqrt{2^{m-1}\psi(2^{m-1})}\Big(1+\frac{3}{4}+\big(\frac{3}{4}\big)^2+\cdots\Big)\\ &\le 4\sqrt{2^m(1+(1+\epsilon)\log m)} < 3\sqrt{2^{m+1}(1+(1+\epsilon)\log m)}.\end{aligned}$$

Thus we have the conclusion. Q.E.D.

By Lemmas 9 and 10 we have

$$\overline{\lim_{N\to\infty}} \sum_{k=1}^{N} \frac{1}{\sqrt{2N\log\log N}} S(\xi_{k^\delta}(f);\mu(k))(n_k t) \le 8\Xi_f.$$

Combining this with (11) and (12), we have

$$\overline{\lim_{N\to\infty}} \frac{1}{\sqrt{2N\log\log N}} \sum_{k=1}^{N} f(n_k t) \le 8\Xi_f,$$

which is the first assertion of our theorem. By applying this to $\pm R(f;A)$,

$$\overline{\lim_{N\to\infty}} \frac{1}{\sqrt{2N\log\log N}} \sum_{k=1}^{N} \pm R(f;A)(n_k t) \le 8\Xi_{R(f;A)}.$$

If $n_{k+1}/n_k \to \infty$, we have

$$\overline{\lim_{N\to\infty}} \frac{1}{\sqrt{2N\log\log N}} \sum_{k=1}^{N} S(f;A)(n_k t) = \|S(f;A)\|_2.$$

$\sum f(n_k x) = \sum S(f;A)(n_k x) + \sum R(f;A)(n_k x)$ implies

$$\|S(f;A)\|_2 - 8\Xi_{R(f;A)} \le \overline{\lim_{N\to\infty}} \frac{\sum_{k=1}^{N} f(n_k t)}{\sqrt{2N\log\log N}} \le \|S(f;A)\|_2 + 8\Xi_{R(f;A)}.$$

By letting $A \to \infty$, we have the second assertion.

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