# The Chern-Finsler connection and Finsler-Kähler manifolds

Tadashi Aikou

*To the memory of Professor Makoto Matsumoto*

**Abstract.**

In this paper, we shall discuss the theory of connection in complex Finsler geometry, i.e., the Chern-Finsler connection $\nabla$ and its applications. In particular, we shall investigate (1) the ampleness of holomorphic vector bundles over a compact complex manifold which is based on the study due to [Ko1], (2) some special class of complex Finsler metrics and its characterization in terms of torsion and curvature of $\nabla$, and in the last section, (3) the characterization of Finsler-Kähler manifolds in terms of the Cartan connection $D$ which is naturally induced on the real tangent bundle from $\nabla$.

## §1. Introduction

Let $M$ be a smooth manifold, and $\pi : E \rightarrow M$ a vector bundle over $M$. A *Finsler metric* on $E$ is a smooth assignment of a norm $\| \cdot \|_x$ to each fibre $E_x = \pi^{-1}(x)$. If we set $\|X\| = L(X)$, then the function $L : E \rightarrow M$ satisfies the following conditions:

(1) $L(X) \geq 0$, and $L(X) = 0$ if and only if $X = 0$,
(2) $L(\lambda X) = \lambda L(X)$ for $^\forall \lambda \in \mathbf{R}^+ = \{\lambda \in \mathbf{R} : \lambda > 0\}$,
(3) $L(X)$ is smooth on $E^\times$,
(4) $L(X + Y) \leq L(X) + L(Y)$.

We suppose that $E$ admits a complex $J$. A *complex Finsler metric* on $(E, J)$ is a Finsler metric satisfying

(1.1) $$\| (aI_E + bJ) X \|_x = \sqrt{a^2 + b^2}\|X\|_x$$

for all $X \in E_x$ and for all $a, b \in \mathbf{R}$, where $I_E$ is the identity morphism of $E$. Let $E_x \otimes \mathbf{C} = E_x^{1,0} \oplus \overline{E_x^{1,0}}$ be the canonical decomposition. Then the condition (1.1) is equivalent to

$$(1.2) \qquad \|(a + \sqrt{-1}b)X\|_x = \sqrt{a^2 + b^2}\|X\|_x$$

for all $X \in E_x^{1,0}$ and for all $a, b \in \mathbf{R}$.

Let $M$ be a complex manifold of $\dim_{\mathbf{C}} M = n$, and $\pi : E \to M$ a holomorphic vector bundle over $M$. For every $X \in E_z \subset E_z \otimes \mathbf{C}$, there exists a unique $v \in E_z^{1,0} \subset E_z \otimes \mathbf{C}$ satisfying $X = v + \bar{v} = 2\mathrm{Re}(v)$. Then we set $F(v) = L(X)/2$. The function $F$ satisfies

   **(F1)** $F$ is smooth on $E^{\times}$, the outside of the zero-section of $E$,
   **(F2)** $F(v) \geq 0$, and $F(v) = 0$ if and only if $v = 0$,
   **(F3)** $F(\lambda v) = |\lambda|F(v)$ for all $\lambda \in \mathbf{C}$ and $v \in E$.

The pair $(E, F)$ is called a *complex Finsler bundle*. In this paper, we furthermore suppose the following assumption:

   **(F4)** the pull-back of the real $(1,1)$-form $\sqrt{-1}\partial\bar{\partial}G$ to each fibre is positive-definite, where we set $G = F^2$.

Such a Finsler metric $F$ is said to be *strongly pseudoconvex*.

The interest in complex Finsler geometry has been motivated by the *Kobayashi metric* intrinsically defined in a bounded strictly convex domain in $\mathbf{C}^n$. This metric is holomorphically invariant, and it plays an important role in the theory of several complex variables. Recently Nishikawa[Ni] has started to study the harmonic maps in complex Finsler geometry to give a differential geometric proof of Frankel conjecture.

The interest in complex Finsler geometry also arises from the study of holomorphic vector bundles. The multiplier group $\mathbf{C}^{\times} = \mathbf{C}\backslash\{0\}$ acts on $E^{\times}$ by multiplication, and the projective bundle $P(E) \to M$ associated with $E$ is defined by $P(E) = E^{\times}/\mathbf{C}^{\times}$. Then the *tautological line bundle* $L(E) \to P(E)$ is defined by $L(E) = \{(V, v) \in P(E) \times E \mid v \in V\}$, and the *hyperplane bundle* $H(E) \to P(E)$ by $P(E) = L(E)^*$. Since $L(E)$ is obtained from $E$ by blowing up the zero section of $E$, the manifold $L(E)^{\times}$ is biholomorphic to $E^{\times}$. Thus, any complex Finsler metric on $E$ is identified as a Hermitian metric on the tautological line bundle $L(E)$. The bundle $E$ is said to be *ample* in the sense of Hartshone if $H(E)$ is an ample line bundle over $P(E)$. The negativity of $E$ is equivalent to the ampleness of its dual $E^*$, and is characterized by the existence of a complex Finsler metric on $E$ of negative curvature. Since the Chern class $c_1(L(E))$ is represented by the first Chern form $c_1(L(E), F)$ for a metric $F$ on $L(E)$, $E$ is negative if and only if $E$ admits a Finsler metric $F$ satisfying $c_1(L(E), F) < 0$. This condition is written in terms of curvatures of Chern-Finsler connection of $(E, F)$ (cf. [Ko1]).

We denote by $\mathbf{R}^+$ the set of positive real numbers which defines a multiplier group. Since $\mathbf{R}^+$ is naturally identified with the group $\{cI \in GL(r, \mathbf{R}); \ c \in \mathbf{R}^+\} \subset GL(r, \mathbf{R})$, $\mathbf{R}^+$ acts on the total space as a subalgebra of $\mathrm{End}(E)$. We denote this action by $L : \mathbf{R}^+ \times E \to E$, and we write $L_\lambda v = \lambda v$ for $^\forall \lambda \in \mathbf{R}^+$. A *connection* is a selection of a horizontal subspace $\mathcal{H}_v$ at each point $v \in E$ in $\mathbf{R}^+$-invariant way:

$$(1.3) \qquad \mathcal{H}_{\lambda v} = (L_\lambda)_* \mathcal{H}_v.$$

**Definition 1.1.** A *connection* (in the sense of Ehresmann) on $E$ is a selection of a horizontal subspace $\mathcal{H}_v$ at each point $v \in E$. A connection is said to be *linear* if the selection is $GL(r, \mathbf{R})$-invariant.

If a connection is selected, we have the splitting

$$(1.4) \qquad TE = \mathcal{V}_E \oplus \mathcal{H}_E,$$

where $\mathcal{H}_E = \coprod_{v \in E} \mathcal{H}_v$ is called the *horizontal subbundle* of $TE$. We denote by $\theta : TE \to \mathcal{V}_E$ the projection, i.e.,

$$(1.5) \qquad \theta(Z) = Z$$

for every $Z \in \mathcal{V}_E$. The horizontal bundle $\mathcal{H}_E$ is defined by $\mathcal{H}_E = \ker \theta$. For a vector field $X$ on $M$, a section $X^{\mathcal{H}}$ of $\mathcal{H}_E$ such tha $\pi_* X^{\mathcal{H}} = X$ is called the *horizontal lift* of $X$.

**Definition 1.2.** The *integrability tensor* of a connection $\theta$ on $E$ is a section $\Theta$ of $\wedge^2 \pi^* T^* M \otimes \mathcal{V}_E$ defined by

$$(1.6) \qquad \Theta(X, Y) = [X, Y]^{\mathcal{H}} - \left[ X^{\mathcal{H}}, Y^{\mathcal{H}} \right]$$

for all vector fields $X, Y$ on $M$.

We can easily check the identity

$$\theta \left( \left[ X^{\mathcal{H}}, Y^{\mathcal{H}} \right] \right) = \left[ X^{\mathcal{H}}, Y^{\mathcal{H}} \right] - [X, Y]^{\mathcal{H}},$$

which implies

$$(1.7) \qquad \Theta(X, Y) = d\theta \left( X^{\mathcal{H}}, Y^{\mathcal{H}} \right) = -\theta \left( \left[ X^{\mathcal{H}}, Y^{\mathcal{H}} \right] \right)$$

for all vector fields $X, Y$ on $M$. The horizontal subbundle $\mathcal{H}_E$ is said to be *integrable* if $\left[ X^{\mathcal{H}}, Y^{\mathcal{H}} \right] \in \mathcal{H}_E$ for all vector fields $X, Y$ on $M$. Since $\mathcal{H}_E$ is defined by $\mathcal{H}_E = \ker \theta$, (1.7) implies

**Proposition 1.1.** *The horizontal subbundle $\mathcal{H}_E$ is integrable if and only if $\Theta$ vanishes identically.*

**Definition 1.3.** For a smooth curve $c : I = [0,1] \to M$ in the base manifold $M$, a curve $\tilde{c}_\zeta : I \to E$ starting at a point $\zeta \in E_{c(0)}$ is called the *horizontal lift* of $c$ with respect to $\theta$ if it satisfies $\tilde{c}_\zeta(0) = \zeta$, $\pi \circ \tilde{c}_\zeta(t) = c(t)$ and

$$(1.8) \qquad\qquad\qquad \tilde{c}_\zeta^* \theta = 0.$$

Since $\dim I = 1$, this differential equation is integrable and thus, the horizontal lift $\tilde{c}_\zeta$ exists for every curve $c = c(t)$ in $M$ and a point $\zeta \in E_{c(0)}$.

**Definition 1.4.** The diffeomorphism $P_c : E_{c(0)} \ni \zeta \to \tilde{c}_\zeta(1) \in E_{c(1)}$ is called the *parallel displacement* of $\zeta$ along $c$ with respect to $\theta$.

We denote by $\mathcal{A}^k(M)$ the sheaf of germs of smooth $k$-forms on $M$, and by $\mathcal{A}^k(M, E)$ the sheaf of germs of smooth $k$-forms on $M$ with values in $E$. We merely denote $\mathcal{A}^0(M, E)$ by $\mathcal{A}(M, E)$. A *covariant derivative* $\nabla$ of $E$ is a morphism $\nabla : \mathcal{A}(M, E) \to \mathcal{A}^1(M, E)$ such that

$$(1.9) \qquad\qquad\qquad \nabla(fs) = df \otimes s + f\nabla s$$

for every smooth function $f$ on $M$ and every $s \in \mathcal{A}(M, E)$. Any covariant derivative $\nabla$ is naturally extended to a morphism $\nabla : \mathcal{A}^k(M, E) \to \mathcal{A}^{k+1}(M, E)$ by setting

$$(1.10) \qquad\qquad \nabla(\varphi \otimes s) = d\varphi \otimes s + (-1)^k \varphi \wedge \nabla s$$

for $\varphi \in \mathcal{A}^k(M)$ and $s \in \mathcal{A}(M, E)$.

We shall show that any covariant derivative $\nabla$ on $E$ induces a linear connetion. To this end, we introduce the notion of *tautological section* of $\mathcal{V}_E$. The action $L_\bullet$ of $\mathbf{R}^+$ induces a vertical vector filed $\mathcal{E}$ defined by

$$(1.11) \qquad\qquad\qquad \mathcal{E}(v) = (v, v)$$

for all $v \in E$. This vector field $\mathcal{E}$ on $E$ is called the *tautological section* of $\mathcal{V}_E$.

We denote by $\tilde{E}$ the pull-back bundle $\pi^* E$ and by $\tilde{\nabla} : \mathcal{A}(E, \tilde{E}) \to \mathcal{A}^1(E, \tilde{E})$ the pull-back $\pi^* \nabla$ of any covariant derivative $\nabla$ on $E$. Because of $\mathcal{V}_E \cong \tilde{E}$, $\mathcal{E}$ is also considered as a section of $\tilde{E}$. Then we can check that $\theta \in \mathcal{A}^1(E, \tilde{E})$ defined by

$$(1.12) \qquad\qquad\qquad \theta = \tilde{\nabla} \mathcal{E}$$

induces a linear connection in the sense of Definition 1.1. Conversely, any linear connection $\theta$ in the sense of Definition 1.1 defines a covariant

derivative $\nabla : \mathcal{A}(M, E) \to \mathcal{A}^1(M, E)$. In fact, for any curve $c = c(t)$ tangents to $X \in T_{c(0)}M$,

$$(1.13) \qquad \nabla_X v = \frac{d}{dt} \left[ P_{c(t)}^{-1} \left( v(c(t)) \right) \right]_{t=0}$$

defines a covariant derivative $\nabla$ on $E$. Consequently any linear connection $\theta$ on $E$ is equivalent to a covariant derivative $\nabla$ on $E$ (cf. [Be]). Thus, in the sequel, we shall use the terminology "connection" for a covariant derivative.

## 1.1. Torsions of connections

We shall recall the definition of torsion of affine connection $\nabla$ on the tangent bundle $TM$. A differential form $T_{\nabla} \in \mathcal{A}^2(M, TM)$ defined by

$$(1.14) \qquad T_{\nabla}(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y]$$

is called the *torsion* of $(TM, \nabla)$. Let $\eta \in \mathcal{A}^1(M, TM)$ be the *canonical form* of $TM$, i.e., $\eta$ is defined by $\eta(X) = X$ for all vector field $X$ on $M$. Then $T_{\nabla}$ is given by

$$(1.15) \qquad T_{\nabla} = \nabla \eta.$$

Hence the torsion $T_{\nabla}$ of $(TM, \nabla)$ is given by *the covariant derivative of the canonical form $\eta$*.

In the case of a general vector bundle $E$ with a connection $\nabla$, the pull-back $\tilde{\nabla}$ defines a connection on the pull-back bundle $\tilde{E}$, and furthermore, a connection $\theta$ is introduced by (1.12). Then, since (1.5) is satisfied, $\theta$ is considered as the canonical form of $\tilde{E}$. Similarly to (1.15), we set

$$(1.16) \qquad \tilde{T} = \tilde{\nabla} \theta,$$

and we call $\tilde{T}$ the *torsion form* of $(\tilde{E}, \tilde{\nabla})$.

**Proposition 1.2.** *The torsion form* $\tilde{T} \in \mathcal{A}^2(E, \tilde{E})$ *of* $(\tilde{E}, \tilde{\nabla})$ *is given by the integrability tensor* $\Theta$ *of* $\mathcal{H}_E$.

*Proof.* It is enough to show that $\tilde{T}\left(X^{\mathcal{H}}, Y^{\mathcal{H}}\right) = \Theta(X, Y)$. By definition, we have

$$\begin{aligned} \tilde{T}\left(X^{\mathcal{H}}, Y^{\mathcal{H}}\right) &= \tilde{\nabla}_{X^{\mathcal{H}}} \theta \left(Y^{\mathcal{H}}\right) - \tilde{\nabla}_{Y^{\mathcal{H}}} \theta \left(X^{\mathcal{H}}\right) - \theta \left([X^{\mathcal{H}}, Y^{\mathcal{H}}]\right) \\ &= -\theta \left([X^{\mathcal{H}}, Y^{\mathcal{H}}]\right) \\ &= \Theta(X, Y). \end{aligned}$$

The horizontal subbundle $\mathcal{H}_E$ is naturally identified with the pull-back bundle $\widetilde{TM} = \pi^*TM$. For the differential $\pi_* : TE \to \widetilde{TM}$, we have $\ker \pi_* = \mathcal{V}_E$, and so the differential $\pi_*$ is considered as the canonical form of $\widetilde{TM}$. However, there are no natural ways to introduce a connection on $\widetilde{TM}$ from $\nabla$ except the case of $E = TM$ with an affine connection $\nabla$. In the case of $TM$, we have

$$(1.17) \qquad \tilde{\nabla}\pi_* \left( X^{\mathcal{H}}, Y^{\mathcal{H}} \right) = \pi^* T_\nabla(X, Y),$$

and thus $\tilde{\nabla}\pi_* = 0$ if and only if $T_\nabla = 0$.

Furthermore, since $TE \cong \mathcal{V}_E \oplus \widetilde{TM}$, the $TE$-valued 1-form $\tilde{\eta} = \theta + \pi_*$ on $E$ is considered as the canonical form of $TE$. In the case of $E = TM$, the connection $\nabla$ on $E$ also induces a connection $\tilde{\nabla}$ on $\widetilde{TM}$, and we can define the covariant derivatives $\tilde{\nabla}\tilde{\eta}$ by

$$(1.18) \qquad \tilde{\nabla}\tilde{\eta} = \tilde{\nabla}\theta + \tilde{\nabla}\pi_* = \Theta + \pi^* T_\nabla.$$

Thus $\tilde{\nabla}\tilde{\eta} = 0$ if and only if the affine connection $\nabla$ on $TM$ is symmetric and the horizontal subbundle $\mathcal{H}_{TM}$ defined by $\nabla$ is integrable.

## 1.2.   Curvatures of connections

Let $\nabla$ be a connection of a vector bundle $\pi : E \to M$.

**Definition 1.5.** The curvature $R \in \mathcal{A}^2(M, \mathrm{End}(E))$ of $(E, \nabla)$ is defined by $R = \nabla \circ \nabla$, i.e.,

$$(1.19) \qquad R(X, Y)s = \nabla_X \nabla_Y s - \nabla_Y \nabla_X s - \nabla_{[X,Y]}s$$

for every $s \in \mathcal{A}(M, E)$ and $X, Y \in \mathcal{A}(M, TM)$. If $R$ vanishes identically, then $(E, \nabla)$ is said to be *flat*.

Let $\tilde{\nabla}$ be the connection on $\tilde{E}$ induced from $\nabla$. Then, identifying $s \in \mathcal{A}(M, E)$ as a section of $\tilde{E}$, the curvature $\tilde{R}$ of $\tilde{\nabla}$ is defined by $\tilde{\nabla}^2 s = \tilde{R}s$, i.e.,

$$\tilde{R}\left( X^{\mathcal{H}}, Y^{\mathcal{H}} \right)s = \tilde{\nabla}_{X^{\mathcal{H}}}\tilde{\nabla}_{Y^{\mathcal{H}}}s - \tilde{\nabla}_{Y^{\mathcal{H}}}\tilde{\nabla}_{X^{\mathcal{H}}}s - \tilde{\nabla}_{[X^{\mathcal{H}}, Y^{\mathcal{H}}]}s.$$

Then, because of (1.12) and (1.16), we get

$$\tilde{R}\left( X^{\mathcal{H}}, Y^{\mathcal{H}} \right)\mathcal{E} = \tilde{T}\left( X^{\mathcal{H}}, Y^{\mathcal{H}} \right) = \Theta(X, Y).$$

Since $\tilde{\nabla}$ is flat if and only if $\nabla$ is flat, we have

**Proposition 1.3.** *A connection $\nabla$ on a vector bundle $E$ is flat if and only if its horizontal bundle $\mathcal{H}_E$ is integrable.*

## §2. Hermitian connections and Kähler manifolds

### 2.1. Hermitian metrics and Hermitian connections

Let $(E, J)$ be a complex vector bundle over a smooth manifold $M$, namely $J$ is an endmorphism of $E$ satisfying

$$(2.1) \qquad J \circ J = -I_E,$$

where $I_E$ is the identity morphism of $E$.

On a complex vector bundle $(E, J)$, the complex scalar multiplication on $E$ is defined by $(a + \sqrt{-1}b) \cdot v := (aI_E + bJ)v$ for all $a + \sqrt{-1}b \in \mathbf{C}$ and $v \in \mathcal{A}(M, E)$. By this definition, each fibre $E_x$ is considered as a complex vector space of complex dimension $r$.

We denote by $E \otimes \mathbf{C}$ the complexification of $E$. Then $J$ is canonically extended on $E \otimes \mathbf{C}$ by setting $J(u + \sqrt{-1}v) = Ju + \sqrt{-1}Jv$ for all $u + \sqrt{-1}v \in \mathcal{A}(M, E \otimes \mathbf{C})$. Since $J$ satisfies (2.1), the eigenvalues are $\pm\sqrt{-1}$. We denote by $E^{1,0}$ and $E^{0,1}$ the eigen-vector space corresponding to $\sqrt{-1}$ and $-\sqrt{-1}$ respectively. Then $E \otimes \mathbf{C}$ is decomposed as

$$(2.2) \qquad E \otimes \mathbf{C} = E^{1,0} \oplus E^{0,1},$$

and $E$ is identified with $E^{1,0}$ by the mapping $\varrho : E \to E^{1,0}$ defined by

$$(2.3) \qquad \varrho(v) = \frac{1}{2}\left(v - \sqrt{-1}Jv\right).$$

By definition, the following is trivial:

$$(2.4) \qquad \varrho(Jv) = \sqrt{-1}\varrho(v)$$

A smooth inner product $g$ on a complex vector bundle $(E, J)$ is said to be a *Hermitian metric* if

$$(2.5) \qquad g(Ju, Jv) = g(u, v)$$

is satifsied for all sections $u, v$ of $E$. Then, if we set

$$(2.6) \qquad h\left(\varrho(u), \varrho(v)\right) = g(u, v) + \sqrt{-1}g(u, Jv),$$

we can easily check the following conditions:

(1) $h(\xi, \eta)$ is $\mathbf{C}$-linear in $\xi$,
(2) $h(\xi, \eta) = \overline{h(\eta, \xi)}$,
(3) $h(\xi, \xi) \geq 0$, and the equality holds if and only if $\xi = 0$.

Conversely, if a map $h : E^{1,0} \times E^{1,0} \to \mathbf{C}$ satisfying the conditions above, we can easily check that the real part $g$ of $h$ is an inner product of $(E, J)$ satisfying (2.5). Consequently, a Hermitian metric on a complex vector bundle $E$ is characterized by the three conditions above. The pair $(E, h)$ is called a *Hermitian bundle* over $M$.

A connection $\nabla$ on $E$ said to be *complex* if it satisfies

$$(2.7) \qquad\qquad\qquad \nabla J = 0.$$

**Definition 2.1.** Let $(E, h)$ be a Hermitian bundle. Then a complex connection $\nabla$ is said to be a *Hermitian connection* of $(E, h)$ if it satisfies the metrical condition

$$(2.8) \qquad\qquad dh(\xi, \eta) = h(\nabla\xi, \eta) + h(\xi, \nabla\eta)$$

for all $\xi, \eta \in \mathcal{A}(M, E)$.

If the base manifold $M$ is a complex manifold, then there exists a special class of complex vector bundles.

**Definition 2.2.** A complex vector bundle $\pi : E \to M$ over a complex manifold $M$ is called a *holomorphic vector bundle* if it admits local trivializations $\{(U, \varphi_U)\}$, $\varphi_U : \pi^{-1}(U) \to U \times \mathbf{C}^r$, whose transition co-cycles $g_{UV} = \varphi_U \circ \varphi_V^{-1} : U \cap V \to GL(r, \mathbf{C})$ are holomorphic. In the case of $r = 1$, the bundle $E$ is called a *holomorphic line bundle*.

In the sequel, we suppose that $M$ is a complex manifold of copmplex dimension $n$, and $\pi : E \to M$ is a holomorphic vector bundle of rank$(E) = r$. In the sequel, we denote by $T^{1,0}M$ and $T^{1,0}E$ the holomorphic tangent bundles of the base manifold $M$ and the total space $E$ respectively:

$$TM \otimes \mathbf{C} = T^{1,0}M \oplus \overline{T^{1,0}M}, \quad TE \otimes \mathbf{C} = T^{1,0}E \oplus \overline{T^{1,0}E}.$$

According to this decomposition, we get the decomposition $\mathcal{A}^1(M, E) = \mathcal{A}^{1,0}(M, E) \oplus \mathcal{A}^{0,1}(M, E)$. Hence the connection $\nabla$ is also decomposed as $\nabla = \nabla^{1,0} + \nabla^{0,1}$, where $\nabla^{1,0} : \mathcal{A}(M, E) \to \mathcal{A}^{1,0}(M, E)$ and $\nabla^{0,1} : \mathcal{A}(M, E) \to \mathcal{A}^{0,1}(M, E)$. A connection $\nabla$ is said to be *of $(1, 0)$-type* if $\nabla^{0,1} = \bar{\partial}$. It is known that a holomorphic vector bundle $E$ admits a $(1, 0)$-type connection (cf. [Ko2]).

We shall work on a local trivialization $\varphi_U : \pi^{-1}(U) \to \mathbf{C}^r$ of $E$, where $U \subset M$ is endowed with complex coordinate $z = (z^1, \cdots, z^n)$. We take a local holomorphic frame field $s_U = (s_1, \cdots, s_r)$ on $U$. With respect to $s_U$, the connection form $\omega = (\omega_j^i)$ of a connection is defined

by $\nabla s_j = \sum s_i \otimes \omega_j^i$. If we set $h_{i\bar{j}} = h(s_i, s_j)$ on $U$, we have $h_{i\bar{j}} = \overline{h_{j\bar{i}}}$ since $h$ is Hermitian. Then we have

**Proposition 2.1.** *Let $E$ be a holomorphic vector bundle with a Hermitian metric $h$ over a complex manifold $M$. There exists a unique Hermitian connection $\nabla$ of $(1,0)$-type on $(E, h)$, i.e., the connection form $\omega = (\omega_j^i)$ is given by*

$$(2.9) \qquad \omega_j^i = \sum h^{i\bar{m}} \partial h_{j\bar{m}},$$

*where $(h^{i\bar{m}})$ is the inverse matrix of $(h_{i\bar{m}})$ so that $\sum h^{i\bar{m}} h_{j\bar{m}} = \delta_j^i$.*

Let $R = \nabla \circ \nabla$ be the curvature of the Hermitian connection $\nabla$ of $(E, h)$. Then we have

$$\nabla \circ \nabla s_j = \nabla \left( \sum s_i \otimes \omega_j^i \right) = \sum s_i \otimes \left( d\omega_j^i + \sum \omega_m^i \wedge \omega_j^m \right).$$

The 2-form $\Omega_j^i$ defined by

$$(2.10) \qquad \Omega_j^i = d\omega_j^i + \sum \omega_m^i \wedge \omega_j^m$$

is called the *curvature form* of $\nabla$. Since the Hermitian connection $\nabla$ is of $(1,0)$-type, we have

$$\nabla \circ \nabla = \left( \nabla^{1,0} + \bar{\partial} \right) \circ \left( \nabla^{1,0} + \bar{\partial} \right) = \nabla^{1,0} \circ \nabla^{1,0} + \left( \bar{\partial} \circ \nabla^{1,0} + \nabla^{1,0} \circ \bar{\partial} \right).$$

Then we have

**Proposition 2.2.** *The curvature $R$ of the Hermitian connection $\nabla$ of $(E, h)$ is a section of $\mathcal{A}^{1,1}(M, \mathrm{End}(E))$, i.e., the curvature form $\Omega_j^i$ is given by*

$$(2.11) \qquad \Omega_j^i = \bar{\partial} \omega_j^i.$$

By this proposition, the curvature form $\Omega_j^i$ is of the form

$$(2.12) \qquad \Omega_j^i = \sum R_{j\alpha\bar{\beta}}^i(z) dz^\alpha \wedge d\bar{z}^\beta,$$

where the tensor field $R_{j\alpha\bar{\beta}}^i$ is called the *curvature tensor* of $\nabla$.

By setting $v = \sum \zeta^i s_i$ for every $v \in \pi^{-1}(U)$, we introduce a local complex coordinate $(z, \zeta) = (z^1, \cdots, z^n, \zeta^1, \cdots, \zeta^r)$ on $\pi^{-1}(U)$. Then the tautological section $\mathcal{E}$ of $\mathcal{V}_E$ is expressed on $\pi^{-1}(U)$ as

$$(2.13) \qquad \mathcal{E}(z, \zeta) = \sum \zeta^i s_i,$$

where we consider $s_U = (s_1, \cdots, s_r)$ as local holomorphic frame field of $\tilde{E}$. As shown in §1, any connection $\nabla$ in $E$ defines a connection $\theta$ in the sense of Definition 1.1.

Let $\nabla$ be the Hermitian connection of a Hermitian bundle $(E, h)$. Since the pull-back $\tilde{\nabla}$ is also of $(1,0)$-type and $\mathcal{E}$ is holomorphic, the connection $\theta$ defined by (1.12) is of $(1,0)$-type, i.e., $\theta \in \mathcal{A}^{1,0}(E, \tilde{E})$. If we set $\theta = \sum s_i \otimes \theta^i$, the $(1,0)$-forms $\theta^i$ are given by

$$(2.14) \qquad\qquad \theta^i = d\zeta^i + \sum \omega^i_j \zeta^j.$$

The torsion $\tilde{T}$ of $(\tilde{E}, \tilde{\nabla})$ is defined by (1.16). Since $\tilde{T} = \tilde{\nabla}^2 \mathcal{E} = \tilde{R}\mathcal{E}$, the form $\tilde{T} \in \mathcal{A}^{1,1}(E, \tilde{E})$ is given by

$$\tilde{T} = \sum s_i \otimes \left( \sum \tilde{\Omega}^i_j \zeta^j \right) = \sum s_i \otimes \sum \left( R^i_{j\alpha\bar{\beta}}(z) \zeta^j dz^\alpha \wedge d\bar{z}^\beta \right).$$

### 2.2. Kähler manifolds

Let $E$ be the holomorphic tangent bundle $T^{1,0}M$ of a complex manifold $M$, and $h$ a Hermitian metric on $T^{1,0}M$. If we denote by $\nabla$ the Hermitian connection of $(M, h)$, $\nabla$ of $(M, h)$ also induces a connection $\tilde{\nabla}$ on the pull-back $\widetilde{T^{1,0}M} = \pi^* T^{1,0}M$.

Since the vertical subbundle $\mathcal{V}_{T^{1,0}M}$ is naturally identified with the pull-back bundle $\widetilde{T^{1,0}M}$, the covariant derivative $\tilde{\nabla}\pi_*$ of $\pi_*$ is given by $\tilde{\nabla}\pi_* = \pi^* T_\nabla$ for the torsion $T_\nabla$ of $(T^{1,0}M, \nabla)$. A Hermitian manifold $(M, h)$ is said to be *Kähler* if its Hermitian connection $\nabla$ is torsion free, i.e., $T_\nabla = 0$. Hence we have

**Proposition 2.3.** *A Hermitian manifold $(M, h)$ is a Kähler manifold if and only if $\tilde{\nabla}\pi_*$ vanishes identically.*

For the isomorphism $\varrho : TM \longrightarrow T^{1,0}M$ defined by (2.3), the real part $g$ of $h$ is defined by

$$g(X, Y) = \frac{1}{2} \left\{ h(\varrho(X), \varrho(Y)) + \overline{h(\varrho(Y), \varrho(X))} \right\}$$

for all sections $X, Y \in \mathcal{A}(M, TM)$, and it defines a Riemannian metric on $TM$. We shall compare the Hermitian connection $\nabla$ of $(M, h)$ and the Levi-Civita connection of $(M, g)$. The Hermitian metric $\nabla$ of $(M, h)$ induces a connection $D$ on $(M, g)$ by setting $\nabla\varrho(X) = \varrho(DX)$. Then $D$ is a metrical connection of $(M, g)$. In fact, because of (2.8), we have

$$dh\left(\varrho(X), \varrho(Y)\right) = h\left(\nabla(\varrho(X), (\varrho(Y)) + h\left((\varrho(X), \nabla(\varrho(Y))\right).$$

Taking the real part of both sides implies $dg(X,Y) = g(DX,Y) + g(X,DY)$, therefore $D$ is metrical with respect to $g$. The torsion $T_D$ of $(TM, D)$ is given by

$$T_D(X,Y) = \varrho^{-1}\left(\nabla_{\varrho(X)}\varrho(Y) - \nabla_{\varrho(Y)}\varrho(X) - [\varrho(X),\varrho(Y)]\right)$$
$$= \varrho^{-1}\left(T_\nabla\left(\varrho(X),\varrho(Y)\right)\right).$$

Hence, Proposition 2.3. implies

**Proposition 2.4.** *A Hermitian manifold* $(M, h)$ *is a Kähler manifold if and only if the Hermitian connection* $\nabla$ *coincides with the Levi-Civita connection* $D$ *of* $(M, g)$.

## §3. Ample vector bundles

Let $E$ be a holomorphic vector bundle of $\mathrm{rank}(E) = r+1 \ (\geq 2)$ over a compact complex manifold $M$, and $P(E)$ the associated with $E$. We denote by $[v]$ the point of $P(E)$ corresponding to $v = (z, \zeta) \in E$.

**Definition 3.1.** A holomorphic vector bundle $E$ over a compact complex manifold $M$ is said to be *negative* if its tautological line bundle $L(E)$ is negative, and $E$ is said to be *ample* if its dual $E^*$ is negative.

Let $z = (z^\alpha) \ (1 \leq \alpha \leq n)$ be a local complex coordinate system on $M$, and $\zeta = (\zeta^i) \ (0 \leq i \leq r)$ the complex coordinate system on the fibre $E_z = \pi^{-1}(z)$ with respect to a local frame field $s = \{s_0, \cdots, s_r\}$. With respect to such a local coordinate system, we use the notations $\partial_\alpha = \partial/\partial z^\alpha$ and $\partial_j = \partial/\partial\zeta^j$. We denote by $\partial_{\bar\alpha}$ and $\partial_{\bar\jmath}$ their conjugate.

We suppose that $E$ admits a Hermitian metric $h = (h_{i\bar\jmath})$, where we set $h_{i\bar\jmath}(z) = h(s_i, s_j)$. The Hermitian connection $\nabla$ of $(E, h)$ is given by (2.9), and its curvature $R$ is given (2.11) and (2.12). For all non-zero $v = (z, \zeta) \in E_z$ and $X \in T_z M$, we set

$$R_z(v, X) = h\left(R(X, X)v, v\right) = \sum R_{i\bar\jmath\alpha\bar\beta}(z)\zeta^i\bar\zeta^j X^\alpha\bar{X}^\beta,$$

where we put $R_{i\bar\jmath\alpha\bar\beta} = \sum h_{m\bar\jmath} R^m_{i\alpha\bar\beta}$.

**Definition 3.2.** A Hermitian vector bundle $(E, h)$ over a compact complex manifold $M$ is said to be *negative* (resp. *positive*) if $R_z(v, X) < 0$ (resp. $R_z(v, X) > 0$) for all non-zero $v \in E_z$ and $X \in T_z M$ at every point $z \in M$.

We show a sufficient condition for $(E, h)$ to be negative.

**Theorem 3.1.** *If a Hermitian vector bundle $(E, h)$ over a compact complex manifold $M$ is negative, then $E$ is negative.*

*Proof.* We define a function $G$ on $E^\times$ by $G(z, \zeta) = h(\mathcal{E}, \mathcal{E}) = \sum h_{i\bar{j}}(z)\zeta^i \bar{\zeta}^j$. We set $G_j = G/|\zeta^j|^2$ on each $U_j = \{[\zeta] \in P(E) \mid \zeta^j \neq 0\}$. Then $\{G_j\}$ satisfy the relation $|\zeta^j|^2 G_j = |\zeta^i|^2 G_i$ on $U_i \cap U_j \neq \phi$. Hence the family $\{G_j\}$ defines a Hermitian metric on $L(E)$. By direct calculations, we see that the curvature form $\bar{\partial}\partial \log G_j = \bar{\partial}\partial \log G$ of this metric is given by

$$\bar{\partial}\partial \log G_j = \begin{pmatrix} \dfrac{1}{G}\sum R_{i\bar{j}\alpha\bar{\beta}}\zeta^i \bar{\zeta}^j & O \\ O & -\partial_i \partial_{\bar{j}}(\log G) \end{pmatrix}.$$

Hence $R_z(v, X) < 0$ implies that $L(E)$ and thus $E$ is negative.

$$\text{Q.E.D.}$$

The converse of this fact is not true except the case of $\dim M = 1$([Um]). Kobayashi[Ko1] characterized negativity of holomorphic vector bundles in terms of complex Finsler metrics (see Theorem 4.1 below).

**Remark 3.1.** In [Sc], some examples of complex surface with negative tangent bundle are constructed. Let $M$ be a compact connected complex surface and $C$ a compact Riemann surface. Let $p : M \longrightarrow C$ be a surjective holomorphic map of maximal rank. The exact sequence $0 \longrightarrow \mathcal{V}_M \overset{i}{\longrightarrow} T^{1,0}M \overset{p_*}{\longrightarrow} \pi^* T^{1,0}C \longrightarrow 0$, where $\mathcal{V}_M = \ker p_*$ yields an exact sequence

$$0 \longrightarrow T^{1,0}X \overset{i}{\longrightarrow} (T^{1,0}M)|_X \overset{p_*}{\longrightarrow} \pi^* T_x C \longrightarrow 0$$

for any point $x \in C$ and $X = p^{-1}(x)$. This sequence leads to the following exact sequence of cohomology groups

$$\cdots \longrightarrow H^0(T^{1,0}M|_X) \longrightarrow H^0(p^*(T_x C)) \overset{\delta_x}{\longrightarrow} H^1(T^{1,0}X) \longrightarrow \cdots,$$

where $\delta_x : H^0(p^*(T_x C)) \longrightarrow H^1(T^{1,0}X)$ is the *Kodaira-Spencer map*. Such a complex surface $M$ is called a *Kodaira surface* if $\delta_x \neq 0$ for any point $x \in C$. Schneider[Sc] proved that any Kodaira surface $M$ has negative tangent bundle $T^{1,0}M$.

## §4.   Chern-Finsler connection and Kobayashi's theorem

Let $E$ be a holomorphic vector bundle of $\text{rank}(E) = r + 1$ $(r \geq 1)$ over a complex manifold $M$. Let $F$ be a strongly pseudoconvex Finsler

metric on $E$. Then, the Hermitian matrix $(H_{i\bar{j}})$ defined by

$$(4.1) \qquad H_{i\bar{j}}(z,\zeta) = \partial_i \partial_{\bar{j}} F^2$$

is smooth on $E^\times$ and positive-definite. We also identify the local holomorphic frame field $s = \{s_0, \cdots, s_r\}$ with the one of the pull-back bundle $\tilde{E}$. Then $\tilde{E}$ admits a Hermitian metric $H$ defined by

$$(4.2) \qquad H(Z,W) = \sum H_{i\bar{j}} Z^i \overline{W^j}$$

for all $Z = \sum Z^i s_i$ and $W = \sum W^j s_j$.

Let $\nabla : \mathcal{A}(E^\times, \tilde{E}) \to A^1(E^\times, \tilde{E})$ be the Hermitian connection of $(\tilde{E}, H)$, i.e., $\nabla$ is the unique connection satisfying $\nabla^{0,1} = \bar{\partial}$ and

$$(4.3) \qquad dH(Z,W) = H(\nabla Z, W) + H(Z, \nabla W)$$

for all $Z, W \in \mathcal{A}(E^\times, \tilde{E})$.

**Definition 4.1.** ([Ab-Pa]) The Hermitian connection $\nabla$ on $(\tilde{E}, H)$ is called the *Chern-Finsler connection* of $(E, F)$.

The connection form $\omega = (\omega_j^i)$ of $\nabla$ with respect to $s$ is defined by $\nabla s_j = \sum s_i \otimes \omega_j^i$, and the connection forms are given by

$$(4.4) \qquad \omega_j^i = \sum H^{i\bar{m}} \partial H_{j\bar{m}} = \sum \gamma_{j\alpha}^i dz^\alpha + \sum C_{jk}^i d\zeta^k,$$

where we put $\gamma_{j\alpha}^i(z,\zeta) = \sum H^{i\bar{m}} \partial_\alpha H_{j\bar{m}}$ and $C_{jk}^i(z,\zeta) = \sum H^{i\bar{m}} \partial_k H_{j\bar{m}}$ for the inverse matrix $(H^{i\bar{m}})$ of $(H_{i\bar{m}})$. We note that the coefficients $C_{jk}^i = \omega_j^i(s_k)$ satisfy the symmetric properties

$$(4.5) \qquad C_{jk}^i = C_{kj}^i.$$

By definitions, it is easily shown that a complex Finsler metric $F$ is Hermitian, i.e., $F^2 = \sum h_{i\bar{j}}(z)\zeta^i \bar{\zeta}^j$ for some Hermitian metric $h = (h_{i\bar{j}})$ on $E$ if and only if $C_{jk}^i \equiv 0$.

In the sequel of this section, we shall discuss the negativity (or ampleness) of holomorphic vector bundles, and we show an outline of the proof of Kobayashi's theorem.

Since the Chern class $c_1(L(E))$ of $L(E)$ is expressed in terms of the curvature $\bar{\partial}\partial \log F^2$ of a Hermitian metric $F$ on $L(E)$, $L(E)$ is negative if and only if there exists a complex Finsler metric $F$ on $E$ satisfying

$$(4.6) \qquad \frac{\sqrt{-1}}{2\pi} \bar{\partial}\partial \log F^2 < 0$$

or equivalently this real $(1,1)$-form $\omega_{P(E)} = \sqrt{-1}\partial\bar{\partial}\log F^2$ defines a Kähler metric on the base manifold $P(E)$.

Conversely, in general, the projective bundle $P(E)$ associated with an arbitrary vector bundle $E$ is not a Kähler manifold. Since each fibre $P(E_z) \cong \mathbf{P}^{r-1}$ is a Kähler manifold, it has only the structure of pseudo-Kähler manifold. For an arbitrary convex Finsler metric $F$ on $E$, the $(1,1)$-form $\omega_{P(E)}$ is a pseudo-Kähler metric on $P(E)$. Kobayashi's characterization is obtained by analyzing the positivity of $\omega_{P(E)}$ (cf. [Ai2]).

For the curvature $R$ of $\nabla$, we define a $(1,1)$-form $\Psi$ on $E^\times$ by

$$(4.7) \qquad \Psi = \frac{H(R\mathcal{E}, \mathcal{E})}{\|\mathcal{E}\|^2}.$$

To investigate the negativity of $\sqrt{-1}\partial\bar{\partial}\log F^2$, the following is useful.

**Lemma 4.1.** *The curvature $\bar{\partial}\partial\log F^2$ of $(L(E), F)$ is given by*

$$(4.8) \qquad \bar{\partial}\partial\log F^2 = \Psi - \sum \partial_i\partial_{\bar{j}}(\log F^2)\theta^i \wedge \bar{\theta}^j,$$

*where the $(1,0)$-forms $\theta^i$ are defined by (2.14) for the Chern-Finsler connection $\nabla$.*

Since the second term of (4.8) is negative definite in the vertical direction in $P(E)$, the curvature of $(L(E), F)$ is negative if and only if $\Psi$ is negative.

**Theorem 4.1.** ([Ko1]) *A holomorphic vector bundle $E$ is negative if and only if $E$ admits a strongly pseudoconvex Finsler metric $F$ with negative curvature $\Psi$.*

We shall state a characterization of negative vector bundles due to [Ca-Wo]. We denote by $\odot^m E$ the symmetric product of $E$. Then, we need the following Grothendieck's identification:

$$(4.9) \qquad H^p(P(E), L) \cong H^p(M, \odot^m E^*)$$

for all $p \geq 0$ and $m \geq 0$. Let $\gamma : H^p(P(E), L) \longrightarrow H^p(M, \odot^m E^*)$ be the isomorphism. The bases $\{\sigma^0, \cdots, \sigma^N\}$ of $H^0(P(E), L)$ is identified with a bases $\{\omega^0, \cdots, \omega^N\}$ of $H^0(M, \odot^m E^*)$ by setting $\gamma^*\omega^b = \sigma^b$. Then a Hermitian metric $h^{\otimes m}$ on $\odot^m E$ is defined by

$$(4.10) \qquad h^{\otimes m}(A, B) = \sum_{b=0}^{N} \omega^b(A)\bar{\omega}^b(\bar{B})$$

for all section $A, B$ of $\odot^m E$. Then it induces a Finsler metric $F$ on $E$ by setting

$$(4.11) \qquad F(v) = \left[h^{\otimes m}(\otimes^m v, \otimes^m v)\right]^{1/2m} = \sqrt[2m]{h^{\otimes m}(\otimes^m v, \otimes^m v)}$$

In [Ca-Wo], it is proved that the metric $h^{\otimes m}$ on $\odot^m E$ has negative curvature, and thus the Finsler metric $F$ defined by (4.11) has negative curvature $\Psi$. Summarizing the discussion above, we have

**Theorem 4.2.** ([Ca-Wo]) *Let $\pi : E \to M$ be a holomorphic vector bundle over a compact complex manifold $M$ such that* $\mathrm{rank}(E) \geq 2$. *The following statements are equivalent.*

(1)    $E^*$ *is ample,*

(2)    $E$ *admits a strongly pseudoconvex Finsler metric with negative curvature $\Psi$,*

(3)    *there exists a sufficiently large $m \in \mathbf{Z}$ and a Hermitian metric $h^{\otimes m}$ on the symmetric product $\odot^m E$ with negative curvature.*

In the rest of this section, we shall investigate some vanishing theorems in complex Finsler geometry. To this end, we show a Weitzenböck formula. The covariant derivative $\nabla v = \sum \nabla_\alpha v \otimes dz^\alpha$ of a holomorphic section $v$ of $E$ is defined by $\nabla v = v^* \nabla \mathcal{E}$:

$$\nabla v = \sum s_i \otimes \left(dv^i + \sum \omega^i_j(v) v^j\right)$$

If we set $f(z) = v^* H(\mathcal{E}, \mathcal{E}) = v^* F^2$, then we have

**Proposition 4.1.** ([Ai3]) *Let $v$ be a non-vanishing holomorphic section of $E$. Then we have*

$$(4.12) \qquad\qquad \partial\bar{\partial} f = H(\nabla v, \nabla v) - H(Rv, v)$$

We suppose that $M$ admits a Hermitian metric $h$.

**Definition 4.2.** Let $(E, F)$ be a strongly pseudoconvex Finsler bundle over a compact Hermitian manifold $(M, h)$. The *mean curvature $K$* of $(E, F)$ is defined by the $h$-trace of $H(Rv, v)$:

$$(4.13) \qquad\qquad K(v, v) = \mathrm{tr}_h H(Rv, v).$$

We set $h = \sum h_{\alpha\bar{\beta}} dz^\alpha \otimes d\bar{z}^\beta$. By taking the $h$-trace of (4.12) we get the Weitzenböck formula:

$$(4.14) \qquad\qquad \sum h^{\alpha\bar{\beta}} \frac{\partial^2 f}{\partial z^\alpha \partial \bar{z}^\beta} = \|\nabla v\|^2 - K(v, v),$$

where we put $\|\nabla v\|^2 = \sum H_{i\bar{j}}(\nabla_\alpha v^i \overline{\nabla_\beta v^j})h^{\alpha\bar{\beta}}$. Then Hopf's maximal principle implies the following theorem.

**Theorem 4.3.** ([Ai3]) *Let* $(E, F)$ *be a strongly pseudoconvex Finsler bundle over a compact Hermitian manifold* $(M, h)$, *and* $K$ *the mean curvature of* $(E, F)$. *Then*

> (1)   *if* $K$ *is semi-negative everywhere on* $E$, *then every holomorphic section* $v$ *of* $E$ *is parallel with respect to* $\nabla$ *and satisfies* $K(v, v) = 0$,
>
> (2)   *if* $K$ *is semi-negative everywhere on* $E$ *and negative definite on some points in* $E$, *then* $E$ *admits no non-trivial holomorphic sections, i.e.,* $H^0(M, \mathcal{O}(E)) = 0$.

As an application of Theorem 4.1 and 4.3, we shall show the following Kobayashi's vanishing theorem:

**Theorem 4.4.** ([Ko1]) *Let* $\pi : E \to M$ *be a negative vector bundle over a compact complex manifold* $M$. *Then* $E$ *has no non-trivial holomorphic sections:* $H^0(M, \mathcal{O}(E)) = 0$.

*Proof.* We suppose that $E$ is negative. Let $h = \sum h_{\alpha\bar{\beta}}(z)dz^\alpha \otimes d\bar{z}^\beta$ be a Hermitian metric on $M$. Then, by Theorem 4.1, $E$ admits a strongly pseudoconvex Finsler metric $F$ with negative curvature $\Psi$. For a non-trivial local holomorphic section $v$ of $E$, we have $K(v, v) = \|v\|^{-2} \sum h^{\alpha\bar{\beta}}\Psi_{\alpha\bar{\beta}}$ from (4.14), which show that the negativity of $\Psi$ implies the one of $K(v, v)$, hence, from second assertion in Theorem 4.3, we have $H^0(M, \mathcal{O}(E)) = 0$.

$$\text{Q.E.D.}$$

## §5.   Differential geometry of complex Finsler bundles

Let $E$ be a holomorphic vector bundle over $M$ with a strongly pseudoconvex Finsler metric $F$ and the Chern-Finsler connection $\nabla$. Let $\mathcal{V}_{E^\times}$ be the vertical bundle of $T^{1,0}E^\times$, i.e., $\mathcal{V}_{E^\times} = \ker \pi_*\{T^{1,0}E^\times \to \widetilde{T^{1,0}M}\}$. The bundle $\mathcal{V}_{E^\times}$ is naturally identified with $\tilde{E}$ over $E^\times$. Then, similarly to (1.12), we define a connection $\theta$ by

$$(5.1) \qquad\qquad\qquad \theta = \nabla\mathcal{E}.$$

By the identity $\sum C^i_{jk}\zeta^j \equiv 0$, we can easily obtain $\theta(Z) = \nabla_Z\mathcal{E} = Z$ for every $Z \in \mathcal{A}(E^\times, \mathcal{V}_{E^\times})$, i.e., it is a morphism $\theta : T^{1,0}E^\times \to \mathcal{V}_{E^\times}$ satisfying $\theta \circ i = \text{identity}$. Furthermore, by the homogeneity (**F3**), this form $\theta$ is invariant by the action of $\mathbf{C}^\times$ on $E$. Thus $\theta$ defined by (5.1) is a connection of $\pi$:

$$0 \longrightarrow \mathcal{V}_{E^\times} \underset{\theta}{\overset{i}{\rightleftarrows}} T^{1,0}E^\times \overset{\pi_*}{\longrightarrow} \widetilde{T^{1,0}M} \longrightarrow 0.$$

If we define the *horizontal subbundle* $\mathcal{H}_{E^\times} \subset T^{1,0}E^\times$ by $\mathcal{H}_{E^\times} = \ker\theta$, the tangent bundle $T^{1,0}_{E^\times}$ is decomposed as $T^{1,0}E^\times = \mathcal{V}_{E^\times} \oplus \mathcal{H}_{E^\times}$.

For every vector field $X \in \mathcal{A}(E^\times, T^{1,0}E^\times)$, we denote by $X^{\mathcal{V}} = \theta(X)$ the vertical part, and by $X^{\mathcal{H}} = X - \theta(X)$ the horizontal part of $X$. Then the differential operator $d$ is also decomposed as $d = d^{\mathcal{V}} + d^{\mathcal{H}}$, where $d^{\mathcal{H}}f(X) = df(X^{\mathcal{H}})$, $d^{\mathcal{V}}f(X) = df(X^{\mathcal{V}})$ for every smooth function $f \in C^\infty(E^\times)$. Furthermore the partial derivation are also decomposed as $\partial = \partial^{\mathcal{H}} + \partial^{\mathcal{V}}$ and $\bar{\partial} = \bar{\partial}^{\mathcal{H}} + \bar{\partial}^{\mathcal{V}}$. Then, the integrability tensor $\Theta$ of $\theta$ is defined by (1.6) for all $X, Y \in \mathcal{A}(E^\times, T^{1,0}E^\times)$. If $\Theta \equiv 0$, then $\mathcal{H}_{E^\times}$ is integrable.

Let $P_c$ be the parallel displacement with respect to $\theta$ along a smooth curve $c$ in $M$.

**Definition 5.1.** If $P_c : (E_{c(0)}, H_{c(0)}) \to (E_{c(1)}, H_{c(1)})$ is an isometry for every curve $c$ in $M$, then we say $(E, F)$ has *isometric fibres*.

It is known that $(E, F)$ has isometric fibres if and only if each fibre is a totally geodesic submanifold, and the necessary and sufficient condition for this is given by

(5.2) $$[\mathcal{L}_{X^{\mathcal{H}}}H]^{\mathcal{V}} \equiv 0$$

for every vector field $X \in \mathcal{A}(E^\times, T^{1,0}E^\times)$(cf. [Is-Ko]).

Since the tautological section $\mathcal{E}$ is given by (2.13), the connection $\theta = \sum s_i \otimes \theta^i$ is given by $\theta^i = \nabla\zeta^i$:

(5.3) $$\theta^i = d\zeta^i + \sum \omega_j^i \zeta^j = d\zeta^i + \sum N_\alpha^i dz^\alpha.$$

Here, since the homogeneity (**F3**) of $F$ implies $\sum C_{jk}^i \zeta^j = 0$, we put $N_\alpha^i = \sum \gamma_{j\alpha}^i \zeta^j$. These forms $\theta^i$ satisfy the relations $\theta^i(s_j) = \delta_j^i$, and thus $\theta = \{\theta^0, \cdots, \theta^r\}$ is the dual frame field of $s = (s_0, \cdots, s_r)$.

On the other hand, the bundle $\tilde{T}^{1,0}M$ is naturally isomorphic to the horizontal bundle $\mathcal{H}_{E^\times}$ as a complex vector bundle by sending

$$\widetilde{T^{1,0}M} \ni \frac{\partial}{\partial z^\alpha} \longrightarrow \left(\frac{\partial}{\partial z^\alpha}\right)^{\mathcal{H}} = \frac{\partial}{\partial z^\alpha} - \theta\left(\frac{\partial}{\partial z^\alpha}\right) \in \mathcal{H}_{E^\times}.$$

In the sequel we set $X_\alpha = (\partial/\partial z^\alpha)^{\mathcal{H}}$ $(\alpha = 1, \cdots, n)$.

According to the splitting $X = X^{\mathcal{H}} + X^{\mathcal{V}}$, we have the decomposition $\nabla = \nabla^{\mathcal{H}} + \nabla^{\mathcal{V}}$ of $\nabla$, where we define $\nabla_X^{\mathcal{H}} = \nabla_{X^{\mathcal{H}}}$ and $\nabla_X^{\mathcal{V}} = \nabla_{X^{\mathcal{V}}}$ for every vector field $X$ on $E$. If we put $\Gamma_{j\alpha}^i = \omega_j^i(X_\alpha)$, then we have

$$(5.4) \quad \Gamma_{j\alpha}^i = \sum H^{i\bar{m}} X_\alpha H_{j\bar{m}} = \sum H^{i\bar{m}} \left( \partial_\alpha H_{j\bar{m}} - \sum \partial_l H_{j\bar{m}} N_\alpha^l \right).$$

Using these notations, we have $\omega_j^i = \sum \Gamma_{j\alpha}^i dz^\alpha + \sum C_{jk}^i \theta^k$, and

$$(5.5) \quad \nabla^{\mathcal{H}} s_j = \sum s_i \otimes \left( \sum \Gamma_{j\alpha}^i dz^\alpha \right), \quad \nabla^{\mathcal{V}} s_j = \sum s_i \otimes \left( \sum C_{jk}^i \theta^k \right).$$

## §6. Torsion of Chern-Finsler connection

Let $\nabla$ be the Chern-Finsler connection of the Finsler bundle $(E, F)$, and $\theta$ the connection of $E$ defined by (5.1). The canonical form $\tilde{\eta}$ of the holomorphic tangent bundle of the total space $E$ is expressed by

$$(6.1) \quad \tilde{\eta} = \sum s_j \otimes \theta^j + \sum X_\alpha \otimes dz^\alpha = \theta + \pi_*$$

with respect to the connection $\theta$. Similarly to (1.16), we shall define

**Definition 6.1.** The *torsion* of $(\tilde{E}, \nabla)$ is a $\tilde{E}$-valued 1-form on $E^\times$ defined by

$$(6.2) \quad T = \nabla\theta = d\theta + \omega \wedge \theta.$$

By this definition, the torsion $T$ is given by

$$T(X, Y) = \nabla_X \theta(Y) - \nabla_Y \theta(X) - \theta([X, Y])$$

for all vector fields $X, Y$ on $E^\times$. The torsion form $\Omega^i$ with respect to $s = (s_0. \cdots, s_r)$ is given by

$$(6.3) \quad \Omega^i = d\theta^i + \sum \omega_j^i \wedge \theta^j.$$

According to the decomposition of $\nabla$, the torsion $T$ of $(\tilde{E}, \nabla)$ is decomposed as $T = T^{\mathcal{HH}} + T^{\mathcal{HV}}$:

$$T^{\mathcal{HH}}(X, Y) = T(X^{\mathcal{H}}, Y^{\mathcal{H}}), \quad T^{\mathcal{HV}}(X, Y) = T(X^{\mathcal{H}}, Y^{\mathcal{V}}) + T(X^{\mathcal{V}}, Y^{\mathcal{H}}).$$

We note that (4.5) implies $T^{\mathcal{VV}} \equiv 0$. Since

$$T^{\mathcal{HH}}(X, Y) = -\theta([X^{\mathcal{H}}, Y^{\mathcal{H}}]) = \Theta(X, Y),$$

$T^{\mathcal{HH}} \equiv 0$ if and only if the horizontal subbundle $\mathcal{H}_E$ is integrable. For the mixed part $T^{\mathcal{HV}}$, we know that $T^{\mathcal{HV}} \equiv 0$ if and only if $E$ has isometric fibres.

**Proposition 6.1.** ([Ai8]) *Let* $T = T^{\mathcal{H}\mathcal{H}} + T^{\mathcal{H}\mathcal{V}}$ *be the torsion of the Chern-Finsler connection* $\nabla$. *Then*

(1)   $T^{\mathcal{H}\mathcal{H}}$ *vanishes if and only if* $\mathcal{H}_E$ *is integrable,*
(2)   $T^{\mathcal{H}\mathcal{V}}$ *vanishes if and only if* $\pi : E \to M$ *has isometric fibres.*

In a previous paper [Ai2], we have investigated a complex Finsler bundle $(E, F)$ which is *modeled on a complex Minkowski space*. Such a bundle $(E, F)$ is characterized by the vanishing of $T^{\mathcal{H}\mathcal{V}}$. Furthermore, we proved the following

**Theorem 6.1.** ([Ai2]) *If* $(E, F)$ *is modeled on a complex Minkowski space, then there exists a Hermitian metric* $h_F$ *on* $E$ *such that* $\nabla^{\mathcal{H}} = \pi^* D$ *for the Hermitian connection* $D$ *of* $(E, h_F)$.

We shall write down the torsion tensor field of $\nabla$. The torsion $T = T^{\mathcal{H}\mathcal{H}} + T^{\mathcal{H}\mathcal{V}}$ is given by $T^{\mathcal{H}\mathcal{V}} = \bar{\partial}^{\mathcal{H}}\theta$ and $T^{\mathcal{H}\mathcal{V}} = \bar{\partial}^{\mathcal{V}}\theta$. With respect to a local holomorphic frame field $s = \{s_0, \cdots, s_r\}$, the horizontal part $T^{\mathcal{H}\mathcal{H}}$ and mixed part $T^{\mathcal{H}\mathcal{V}}$ are given as as follows:

$$T^{\mathcal{H}\mathcal{H}} = \sum s_i \otimes \left( \sum R^i_{\alpha\bar{\beta}} dz^\alpha \wedge d\bar{z}^\beta \right)$$

and

$$T^{\mathcal{H}\mathcal{V}} = \sum s_i \otimes \left( \sum R^i_{\alpha\bar{j}} dz^\alpha \wedge \bar{\theta}^j \right),$$

where the *torsion tensors* $R^i_{\alpha\bar{\beta}}$ and $R^i_{\alpha\bar{j}}$ are defined by

(6.4)            $R^i_{\alpha\bar{\beta}} = -X_{\bar{\beta}} N^i_\alpha, \quad R^i_{\alpha\bar{j}} = -\partial_{\bar{j}} N^i_\alpha$

respectively. The homogeneity **(F3)** implies the following.

**Proposition 6.2.** *The torsion tensor* $R^i_{\alpha\bar{j}}$ *satisfies*

(6.5)                          $\sum R^i_{\alpha\bar{j}} \bar{\zeta}^j \equiv 0,$

(6.6)                          $R_{\alpha\bar{i}\bar{j}} = R_{\alpha\bar{j}\bar{i}},$

*where we set* $R_{\alpha\bar{i}\bar{j}} = \sum H_{m\bar{i}} R^m_{\alpha\bar{j}}$.

## §7.    Curvature of Chern-Finsler connection

We denote by $R = \nabla \circ \nabla$ the curvature of the Chern-Finsler connection $\nabla$. By definition, $R$ is computed by the formula (1.19) for all vector fields $X, Y$ on $E^\times$ and section $s$ of $\tilde{E}$. Since $\nabla$ is the Hermitian connection of $(\tilde{E}, H)$, $R$ is a section of $\mathcal{A}^{1,1}(E^\times, \mathrm{End}(\tilde{E}))$, and thus the curvature form $\Omega^i_j$ is given by (2.11).

The curvature $R$ is also decomposed as $R = R^{\mathcal{HH}} + R^{\mathcal{HV}} + R^{\mathcal{VV}}$ into the sum of horizontal part $R^{\mathcal{HH}} = \nabla^{\mathcal{H}} \circ \nabla^{\mathcal{H}}$, mixed part $R^{\mathcal{HV}} = \nabla^{\mathcal{H}} \circ \nabla^{\mathcal{V}} + \nabla^{\mathcal{V}} \circ \nabla^{\mathcal{H}}$ and vertical part $R^{\mathcal{VV}} = \nabla^{\mathcal{V}} \circ \nabla^{\mathcal{V}}$. Since $\theta$ is defined by (5.1), the Ricci identity $\nabla \circ \nabla \mathcal{E} = R\mathcal{E}$ implies

$$(7.1) \qquad R^{\mathcal{HH}}\mathcal{E} = T^{\mathcal{HH}}, \;\; R^{\mathcal{HV}}\mathcal{E} = T^{\mathcal{HV}}, \;\; R^{\mathcal{VV}}\mathcal{E} \equiv 0.$$

Since the torsion form $\Omega^i$ and curvature form $\Omega^i_j$ satisfy the relation $\Omega^i = \sum \Omega^i_j \zeta^j$, the identity (7.1) implies that $T = \bar{\partial}\theta$.

We shall write down the curvature tensor field of $\nabla$. With respect to a local holomorphic frame field $s = \{s_0, \cdots, s_r\}$, the horizontal part $R^{\mathcal{HH}}$, mixed part $R^{\mathcal{HV}}$ and vertical part $R^{\mathcal{VV}}$ are given as follows:

$$R^{\mathcal{HH}}s_j = \sum s_i \otimes \left( \sum R^i_{j\alpha\bar{\beta}} dz^\alpha \wedge d\bar{z}^\beta \right),$$

$$R^{\mathcal{HV}}s_j = \sum s_i \otimes \left( \sum R^i_{j\alpha\bar{k}} dz^\alpha \wedge \bar{\theta}^k + R^i_{jk\bar{\beta}} \theta^k \wedge d\bar{z}^\beta \right),$$

$$R^{\mathcal{VV}}s_j = \sum s_i \otimes \left( \sum R^i_{jk\bar{l}} \theta^k \wedge \bar{\theta}^l \right),$$

where we put

$$(7.2) \qquad R^i_{j\alpha\bar{\beta}} = -X_{\bar{\beta}} \Gamma^i_{j\alpha} - \sum C^i_{jl} R^l_{\alpha\bar{\beta}},$$

$$(7.3) \qquad R^i_{j\alpha\bar{k}} = -\partial_{\bar{k}} \Gamma^i_{j\alpha} - \sum C^i_{jl} R^l_{\alpha\bar{k}}, \;\; R^i_{jk\bar{\beta}} = X_{\bar{\beta}} C^i_{jk},$$

$$(7.4) \qquad R^i_{jk\bar{l}} = -\partial_{\bar{l}} C^i_{jk}.$$

Then the homogeneity condition (**F3**) implies

$$(7.5) \qquad \sum \zeta^j R^i_{j\alpha\bar{\beta}} = R^i_{\alpha\bar{\beta}}, \;\; \sum \zeta^j R^i_{j\alpha\bar{k}} = R^i_{\alpha\bar{k}},$$

$$(7.6) \qquad \sum \zeta^j R^i_{jk\bar{\beta}} = \sum \zeta^j R^i_{jk\bar{l}} = 0.$$

Furthermore, by direct calculations, we have

$$(7.7) \qquad R^i_{j\alpha\bar{\beta}} = \partial_j R^i_{\alpha\bar{\beta}} - \sum R^i_{\alpha l}\overline{R^l_{\beta\bar{j}}} - \sum C^i_{jl}R^l_{\alpha\bar{\beta}}.$$

We shall show an application of this expression.

**Proposition 7.1.** *If $R^{\mathcal{H}\mathcal{H}} \equiv 0$, then $(E, F)$ is modeled on a complex Minkowski space, and its associated Hermitian metric $h_F$ is flat.*

*Proof.* First we shall prove that $(E, F)$ is modeled on a complex Minkowski space $(R^i_{\alpha\bar{j}} \equiv 0)$. From the assumption and (7.5), the horizontal subbundle $\mathcal{H}_{E^\times}$ is integrable. Then, (7.7) implies $\sum R^i_{\alpha\bar{m}}\overline{R^m_{\beta\bar{j}}} = 0$. For any section $Z$ of $\tilde{E}$, we define $\varphi(Z) \in \mathcal{A}^{1,0}(E^\times, \tilde{E})$ by

$$\varphi(Z) = \sum s_i \otimes \left( \sum R^i_{\alpha\bar{j}}\overline{Z^j}dz^\alpha \right).$$

If we fix an arbitrary Hermitian metric $h = \sum h_{\alpha\bar{\beta}}(z)dz^\alpha \otimes d\bar{z}^\beta$ on $M$, the norm $\|\varphi(Z)\|$ is computed as follows:

$$\begin{aligned}
\|\varphi(Z)\|^2 &= \sum H_{i\bar{j}}\left( \sum R^i_{\alpha\bar{l}}\bar{Z}^l\overline{R^j_{\beta\bar{m}}\bar{Z}^m} \right) h^{\alpha\bar{\beta}} \\
&= \sum \left( \sum R_{\alpha\bar{j}\bar{l}}\bar{Z}^l\overline{R^j_{\beta\bar{m}}\bar{Z}^m} \right) h^{\alpha\bar{\beta}} \\
&= \sum \left( \sum R_{\alpha\bar{l}\bar{j}}\bar{Z}^l\overline{R^j_{\beta\bar{m}}\bar{Z}^m} \right) h^{\alpha\bar{\beta}} \\
&= \sum H_{i\bar{l}}\left( \sum R^i_{\alpha\bar{m}}\overline{R^m_{\beta\bar{k}}}Z^k\bar{Z}^l \right) h^{\alpha\bar{\beta}} \\
&= 0
\end{aligned}$$

for every $Z \in \mathcal{A}(E, \tilde{E})$. Consequently we have $R^i_{\alpha\bar{j}} \equiv 0$, i.e., $T^{\mathcal{H}\mathcal{V}} \equiv 0$.

Denoting by $D$ the Hermitian connection of the associated Hermitian metric $h_F$, the flatness of $h_F$ is obtained from $0 = R^{\mathcal{H}\mathcal{H}} = \pi^*D \circ D$.

Q.E.D.

By this proposition, we know that if $R^{\mathcal{H}\mathcal{H}} \equiv 0$, then $(E, F)$ is modeled on a complex Minkowski space. Then, from Theorem 6.1, the horizontal part $\nabla^{\mathcal{H}}$ is given by $\nabla^{\mathcal{H}} = \pi^*D$ for the Hermitian connection $D$ of a Hermitian metric $h_F$ on $E$. Thus we have $\Gamma^i_{j\alpha} = \Gamma^i_{j\alpha}(z)$, and consequently we get $R^i_{j\alpha\bar{k}} \equiv 0$.

We shall state some properties of curvature of $\nabla$. First we state

**Definition 7.1.** A strongly pseudoconvex Finsler metric $F$ is said to be *flat* if $F$ has the form $F = F(\zeta)$ at around of every point of $M$ with respect to a suitable local holomorphic frame field $s = (s_0, \cdots, s_r)$.

Then we have

**Theorem 7.1.** *A strongly pseudoconvex Finsler metric $F$ is flat if and only if the horizontal curvature $R^{\mathcal{H}\mathcal{H}}$ vanishes identically.*

*Proof.* Setting $\Gamma^i_j = \sum \Gamma^i_{j\alpha} dz^\alpha$, from (5.5) we have $\nabla^{\mathcal{H}} s_j = \sum s_i \otimes \Gamma^i_j$. The flatness of $F$ is equivalent to $\gamma^i_{j\alpha} = 0$, and from (5.4), this is also equivalent to $\Gamma^i_{j\alpha} = 0$ which implies $R^{\mathcal{H}\mathcal{H}} \equiv 0$.

Conversely, we suppose that $R^{\mathcal{H}\mathcal{H}} = 0$. Then, from Proposition 7.1, $(E, F)$ is modeled on a complex Minkowski space, and its associated Hermitian metric $h_F$ is flat. Then Theorem 6.1 implies that $\Gamma^i_j$ is given by the pull-back $\pi^* \gamma^i_j$ for the connection form $\gamma^i_j$ of the associated Hermitian metric $h_F$, and this implies that $\bar{\partial}^{\mathcal{H}} \Gamma^i_j = \pi^* \bar{\partial} \gamma^i_j = 0$. Now, if we take another local holomorphic frame field $\tilde{s}_j = \sum s_i A^i_j(z)$, we have

$$\nabla^{\mathcal{H}} \tilde{s}_j = \sum s_i \otimes \left( d^{\mathcal{H}} A^i_j + \Gamma^i_l A^l_j \right) = \sum s_i \otimes \left( dA^i_j + \pi^* \gamma^i_l A^l_j \right).$$

Then, because of

$$\begin{aligned}
d(dA^i_j) &= -\sum \left( d(\pi^* \gamma^i_l) A^l_j + (\pi^* \gamma^i_l) \wedge dA^l_j \right) \\
&= -\sum \pi^* \left( \partial \gamma^i_l + \gamma^i_m \wedge \gamma^i_l + \bar{\partial} \gamma^i_l \right) A^l_j \\
&= -\sum (\pi^* \bar{\partial} \gamma^i_l) A^l_j,
\end{aligned}$$

the integrability condition for $dA^i_j + \pi^* \gamma^i_l A^l_j = 0$ is satisfied, and thus, if $R^{\mathcal{H}\mathcal{H}} \equiv 0$, we have $\nabla^{\mathcal{H}} \tilde{s}_j = 0$ with respect to $\tilde{s}_j = \sum s_i A^i_j$. Consequently we have $\tilde{\gamma}^i_{j\alpha} = 0$.

$$\text{Q.E.D.}$$

Similarly to Theorem 7.1, we have

**Theorem 7.2.** *A strongly pseudoconvex Finsler metric $F$ is Hermitian, i.e., $F^2 = \sum h_{i\bar{j}}(z) \zeta^i \bar{\zeta}^j$ for a Hermitian metric $h$ on $E$ if and only if the vertical part $R^{\mathcal{V}\mathcal{V}}$ of the curvature of $R$ vanishes identically.*

## §8.  Complex Finsler manifolds

Let $M$ be a complex manifold of $\dim_{\mathbf{C}} M = n$. In this section, we shall investigate the case where a strongly pseudoconvex Finsler metric $F$ is given on the holomorphic tangent bundle $T^{1,0}M$. We call the pair $(M, F)$ a *complex Finsler manifold*. In the case of $E = T^{1,0}M$,

we identify $\{\partial/\partial z^1, \cdots, \partial/\partial z^n\}$ a local holomorphic frame field of the bundle $\widetilde{T^{1,0}M}$, and so the Chern-Finsler connection $\nabla$ is denoted by

$$\nabla \frac{\partial}{\partial z^j} = \sum \frac{\partial}{\partial z^i} \otimes \omega_j^i$$

for the connection form $\omega_j^i$ of $\nabla$.

### 8.1. Holomorphic sectional curvature

Let $\Delta(r) = \{\eta \in \mathbf{C} : |\eta| < r\}$ be the disk of radius $r$ in $\mathbf{C}$ with the *Poincaré metric*

$$g_r = \frac{4r^2}{(r^2 - |\eta|^2)^2} d\eta \otimes d\bar{\eta}.$$

For every point $(z, \zeta) \in T^{1,0}M^\times$, there exists a holomorphic map $\varphi : \Delta(r) \to M$ satisfying $\varphi(0) = z$ and

(8.1) $$\varphi_*(0) := \varphi_*\left(\left(\frac{\partial}{\partial \eta}\right)_{\eta=0}\right) = \zeta.$$

Then, the pull-back $\varphi^*F$ defines a Hermitian metric in a neighborhood of the origin by $\varphi^*F^2 = E(\eta)d\eta \otimes d\bar{\eta}$, where we put $E(\eta) = F^2(\varphi(\eta), \varphi_*(\eta))$. The Gauss curvature $K_{\varphi^*F}(z, \zeta)$ is defined by

$$K_{\varphi^*F}(z, \zeta) = -\left(\frac{1}{E}\frac{\partial^2 \log E}{\partial \eta \partial \bar{\eta}}\right)_{\eta=0}.$$

**Definition 8.1.** ([Ro]) The *holomorphic sectional curvature* $K_F$ of $(M, F)$ at $(z, \zeta) \in T^{1,0}M^\times$ is defined by

$$K_F(z, \zeta) = \sup_{\varphi} \left\{K_{\varphi^*F}(z, \zeta) \mid \varphi(0) = z, \; \varphi_*(0) = \zeta\right\},$$

where $\varphi$ ranges over all holomorphic maps from a small disk into $M$ satisfying $\varphi(0) = z$ and (8.1).

Then $K_F$ has a computable expression in terms of the curvature tensor of the Chern-Finsler connection $\nabla$.

**Proposition 8.1.** ([Ai1]) *The holomorphic sectional curvature $K_F$ of $(M, F)$ at $(z, \zeta) \in T^{1,0}M^\times$ is given by*

(8.2) $$K_F(z, \zeta) = \frac{\Psi(\mathcal{E}, \mathcal{E})}{\|\mathcal{E}\|^2} = \frac{1}{\|\mathcal{E}\|^4}\sum R_{i\bar{j}k\bar{l}}(z, \zeta)\zeta^i \bar{\zeta}^j \zeta^k \bar{\zeta}^l,$$

*where $R_{i\bar{j}k\bar{l}} = \sum H_{m\bar{j}}R_{ik\bar{l}}^m$ is the curvature tensor of the Finsler connection $\nabla$ on $(T^{1,0}M, F)$.*

Then we have the Schwarz-type lemma:

**Proposition 8.2.** ([Ai1]) *Let $F$ be a strongly pseudoconvex Finsler metric on the holomorphic tangent bundle of a complex manifold $M$. Suppose that its holomorphic sectional curvature $K_F(z, \zeta)$ at every point $(z, \zeta) \in T^{1,0}M^\times$ is bounded above by a negative constant $-k$. Then, for every holomorphic map $\varphi : \Delta(r) \to M$ satisfying $\varphi(0) = z$ and (8.1), we have*

$$(8.3) \qquad\qquad g_r \geq k\varphi^* F^2.$$

The *Kobayashi metric* $F_M$ on a complex manifold $M$ is a positive semi-definite pseudo metric defined by

$$(8.4) \qquad F_M(z, \zeta) = \inf_{\varphi} \left\{ \tfrac{1}{r} \mid \varphi(0) = z, \ \varphi_*(0) = \zeta \right\}.$$

In general, $F_M$ is not smooth. $F_M$ is only upper semi-continuous, i.e., for every $X \in T^{1,0}M$ and every $\epsilon > 0$ there exists a neighborhood $U$ of $X$ such that $F_M(Y) < F_M(X) + \epsilon$ for all $Y \in U$. Even though $F_M$ is not a Finsler metric in our sense, the *decreasing principle* shows the importance of the Kobayashi metric, i.e., for every holomorphic map $\varphi : N \to M$, we have the inequality

$$(8.5) \qquad\qquad F_N(X) \geq F_M(\varphi_*(X)).$$

This principle shows that $F_M$ is holomorphically invariant, i.e., if $\varphi : N \to M$ is biholomorphic, then we have $F_N = \varphi^* F_M$. In this sense, $F_M$ is an intrinsic metric on complex manifolds. It is well-known that, if $M$ is a strongly convex domain with smooth boundary in $\mathbf{C}^n$, then $F_M$ is a pseudoconvex Finsler metric in our sense (cf. [Le])

A complex manifold $M$ is said to be *Kobayashi hyperbolic* if its Kobayashi metric $F_M$ is a metric in the usual sense. If $M$ admits a pseudoconvex Finsler metric $F$ whose holomorphic sectional curvature $K_F$ is bounded above by a negative constant $-k$, then (8.3) implies the inequality

$$(8.6) \qquad\qquad F_M \geq kF,$$

and thus $M$ is Kobayashi hyperbolic.

**Theorem 8.1.** ([Ko1]) *Let $M$ be a compact complex manifold. If its holomorphic tangent bundle $T^{1,0}M$ is negative, then $M$ is Kobayashi hyperbolic.*

*Proof.* We suppose that $T^{1,0}M$ is negative. Then, Theorem 4.1 implies that there exists a pseudoconvex Finsler metric $F$ on $T^{1,0}M$ with negative-definite $\Psi$. By the definition (4.7), the negativity of $\Psi$ and (8.2) imply

$$K_F(z,\zeta) = \frac{\Psi(\mathcal{E},\mathcal{E})}{\|\mathcal{E}\|^2} < 0.$$

Since $M$ is compact, $P(E)$ is also compact. Moreover, since $K_F$ is a function on $P(E)$, the negativity of $K_F$ shows that $K_F$ is bounded by a negative constant $-k$. Hence we obtain (8.6), and $M$ is Kobayashi hyperbolic.

<div align="right">Q.E.D.</div>

### 8.2. Finsler-Kähler manifolds

In this subsection, we shall generalize the Kählerity of Hermitian metrics to complex Finsler geometry. We define the *Kähler form* $\varpi$ of $(M, F)$ by

$$(8.7) \qquad \varpi = \sqrt{-1} \sum H_{i\bar{j}}(z,\zeta) dz^i \wedge d\bar{z}^j.$$

We can easily show that $d\varpi = 0$ if and only if $F^2 = \sum h_{i\bar{j}}(z)\zeta^i\bar{\zeta}^j$ for a Kähler metric $h = \sum h_{i\bar{j}}(z) dz^i \otimes d\bar{z}^j$ on $M$.

Let $\mathcal{E}$ be the tautological vector field over $T^{1,0}M$ and $\theta$ the connection of $\pi : T^{1,0}M \to M$ defined by (5.1).

**Definition 8.2.** ([Ai1]) A strongly pseudoconvex Finsler metric $F$ is said to be *Finsler-Kähler* if the following is satisfied:

$$(8.8) \qquad d^{\mathcal{H}}\varpi = 0.$$

**Remark 8.1.** In [Ab-Pa], a strongly pseudoconvex Finsler metric $F$ satisfying (8.8) is called a *strongly* Finsler-Kähler metric.

If we denote by $X_j$ $(j = 1, \cdots, n)$ the vector field $(\partial/\partial z^j)^{\mathcal{H}}$, then $F$ is a Finsler-Kähler metric if and only if

$$(8.9) \qquad X_j H_{i\bar{k}} = X_i H_{j\bar{k}}.$$

The connection form $\omega_j^i$ of the Chern-Finsler connection $\nabla$ is given by $\omega_j^i = \sum \Gamma_{jk}^i dz^k + \sum C_{jk}^i \theta^k$ with $\Gamma_{jk}^i = \sum H^{i\bar{m}} X_j H_{k\bar{m}}$ and $C_{jk}^i = \sum H^{i\bar{m}} \partial_j H_{k\bar{m}}$, and the condition (8.8) is equivalent to

$$(8.10) \qquad \Gamma_{jk}^i = \Gamma_{kj}^i.$$

We remark that the coefficients $C^i_{jk}$ always satisfy the symmetric properties (4.5).

**Remark 8.2.** The differential $\pi_*$ of the projection $\pi : T^{1,0}M \to M$ is considered as the canonical form of $\tilde{T}^{1,0}M$, and it has the form

$$\pi_* = \sum \frac{\partial}{\partial z^j} \otimes dz^j,$$

where $\{dz^1, \cdots, dz^n\}$ is, of course, considered as the dual frame field of the frame field $\{X_1, \cdots, X_n\}$ for the horizontal bundle $\mathcal{H}_{T^{1,0}M^\times}$. Since

$$\nabla \pi_* = \sum \frac{\partial}{\partial z^i} \otimes \left( \sum \Gamma^i_{jk} dz^k + \sum C^i_{jk} \theta^k \right) \wedge dz^j,$$

$\nabla \pi_*$ vanishes if and only if $F^2 = \sum g_{i\bar{j}}(z) \zeta^i \bar{\zeta}^j$ for a Kähler metric $h = \sum h_{i\bar{j}}(z) dz^i \otimes d\bar{z}^j$ on $M$. The condition (8.8) is equivalent to

$$(8.11) \qquad\qquad\qquad \nabla^{\mathcal{H}} \pi_* = 0.$$

For the complex structure $J$ on $TM$, we also denote by the same notation $J$ the lifted complex structure on $\widetilde{TM} = \pi^*TM$. Then $\widetilde{TM}$ is naturally identified with $\widetilde{T^{1,0}M}$ via the isomorphism

$$\varrho : \widetilde{TM} \ni Y \longrightarrow \varrho(Y) = \frac{1}{2} \left( Y - \sqrt{-1} JY \right) \in \widetilde{T^{1,0}M}.$$

We denote by $G$ the real part of $H$:

$$(8.12) \qquad G = \frac{1}{2} \left[ H\left( \varrho(Y), \varrho(Z) \right) + \overline{H\left( \varrho(Y), \varrho(Z) \right)} \right].$$

Then $G$ is an inner product on $\widetilde{TM}$, and it satisfies

$$(8.13) \qquad H\left( \varrho(Y), \varrho(Z) \right) = G(Y, Z) - \sqrt{-1} G\left( JY, Z \right)$$

for all $Y, Z \in \mathcal{A}(TM^\times, \widetilde{TM})$, i.e., $G$ is a Hermitian metric on $(\widetilde{TM}, J)$. The imaginary part of $H$ us given by

$$\begin{aligned}
G(JY, Z) &= \frac{\sqrt{-1}}{2} \left[ H\left( \varrho(Y), \varrho(Z) \right) - \overline{H\left( \varrho(Y), \varrho(Z) \right)} \right] \\
&= \frac{\sqrt{-1}}{2} \left[ H\left( \varrho(Y), \varrho(Z) \right) - H\left( \varrho(Z), \varrho(Y) \right) \right] \\
&= \varpi(\rho(Y), \rho(Z)),
\end{aligned}$$

where $\varpi$ in the last line is the Kähler form defined by (8.7).

Under the isomorphism $\varrho$, the Chern-Finsler connection $\nabla$ on $\widetilde{T^{1,0}M}$ is considered as c connection on the bundle $\widetilde{TM}$:

(8.14)
$$\nabla\varrho(Y) = \varrho(\nabla Y).$$

This connection $\nabla$ on $\widetilde{TM}$ is metrical with respect to $G$. In fact, we have

$$dH(\varrho(Z), \varrho(W)) = H(\nabla\varrho(Z), \varrho(W)) + H(\varrho(Z), \nabla\varrho(W))$$
$$= H(\varrho(\nabla Z), \varrho(W)) + H(\varrho(Z), \varrho(\nabla W)).$$

Taking the real part of both sides, we have

$$dG(Z, W) = G(\nabla Z, W) + G(Z, \nabla W),$$

and thus $\nabla$ is metrical with respect to $G$.

Since $\nabla$ satisfies $\nabla\sqrt{-1}\varrho(Y) = \sqrt{-1}\nabla\varrho(Y)$ and the multiplication by $\sqrt{-1}$ is identified with the operator $J$, we have

$$\sqrt{-1}\nabla\varrho(Y) = \nabla\sqrt{-1}\varrho(Y) = \nabla\varrho(JY) = \varrho(\nabla(JY)),$$

and

$$\sqrt{-1}\nabla\varrho(Y) = \varrho(J(\nabla Y)).$$

Thus the lifted complex structure $J$ is $\nabla$-parallel:

(8.15)
$$\nabla J \equiv 0.$$

Consequently, the connection $\nabla$ on $\widetilde{TM}$ defined by (8.14) is the Hermitian connection of $(\widetilde{TM}, G)$, and so we call $\nabla$ on $\widetilde{TM}$ is the *Chern-Finsler connection* on $(\widetilde{TM}, G)$.

For the connection $\theta$ defined by (5.1), the corresponding connection of $\widetilde{TM}$ is denoted by the same symbol $\theta$, i.e.,

$$\theta(\varrho(Z)) = \varrho(\theta(Z))$$

for every $Z \in \mathcal{A}(TM^\times, \widetilde{TM})$. Then $\theta$ defines a splitting $T(TM^\times) = \mathcal{V}_{TM^\times} \oplus \mathcal{H}_{TM^\times}$, and the differential operator $d$ also splits as $d = d^{\mathcal{H}} + d^{\mathcal{V}}$. Then, for every horizontal real $k$-form $\Theta$, we have

$$(d^{\mathcal{H}}\Theta)(Z_1, \cdots, Z_{k+1})$$
$$= \sum_{j=1}^{k+1}(-1)^{j-1}Z_j\Theta(Z_1, \cdots, \hat{Z}_j, \cdots, Z_{k+1})$$
$$+ \sum_{1 \le i < j \le k+1}(-1)^{i+j}\Theta([Z_i, Z_j]^H, \cdots, \hat{Z}_i, \cdots, \hat{Z}_j, \cdots, Z_{k+1})$$

for all sections $Z_1, \cdots, Z_{k+1} \in \mathcal{A}(TM^\times, \mathcal{H}_{TM^\times})$, where $\hat{Z}_i$ means that $Z_i$ is to be omitted.

In the sequel of this subsection, to avoid the confusion of notations, we shall identify the bundle $\widetilde{TM}$ with the horizontal bundle $\mathcal{H}_{TM^\times}$.

$$
\begin{array}{ccc}
\mathcal{H}_{TM^\times} & \xrightarrow[\;(\bullet)^{\mathcal{H}}\;]{\pi_*} & \widetilde{TM} \\[2mm]
\varrho \downarrow & & \downarrow \varrho \\[2mm]
\mathcal{H}_{T^{1.0}M^\times} & \xrightarrow[\;(\bullet)^{\mathcal{H}}\;]{\pi_*} & \widetilde{T^{1,0}M}
\end{array}
$$

We suppose that a connection $D = D^{\mathcal{H}} + D^{\mathcal{V}}$ is given on $\widetilde{TM}$. Since the torsion $D^{\mathcal{H}}\pi_*$ is given by

$$
(D^{\mathcal{H}}\pi_*)(Z, W) = D_Z^{\mathcal{H}}\pi_*(W) - D_W^{\mathcal{H}}\pi_*(Z) - \pi_*([Z, W])
$$

under the identification $\widetilde{TM} \cong \mathcal{H}_{TM^\times}$, the condition $D^{\mathcal{H}}\pi_* = 0$ is given by

(8.16) $$ D_Z^{\mathcal{H}}W - D_W^{\mathcal{H}}Z - [Z, W]^{\mathcal{H}} \equiv 0 $$

for all $Z$, $W \in \mathcal{A}(TM^\times, \widetilde{TM})$. Then we have

**Proposition 8.3.** *We suppose that a connection $D$ satisfies the symmetric property* (8.16). *Then*

(1)    *$D$ satisfies*

$$
(d^{\mathcal{H}}\Theta)(Z_1, \cdots, Z_{k+1}) = \sum_{j=1}^{k+1}(-1)^{j-1}(D_{Z_j}^{\mathcal{H}}\Theta)(Z_1, \cdots, \hat{Z}_j, \cdots, Z_{k+1})
$$

     *for any horizontal real $k$-form $\Theta$ and all horizontal vector fields $Z_1, \cdots, Z_{k+1}$.*

(2)    *Any $D^{\mathcal{H}}$-parallel horizontal form is $d^{\mathcal{H}}$-closed.*

**Definition 8.3.** A connection $D$ on the bundle $\widetilde{TM}$ is called the *Cartan connection* of $(\widetilde{TM}, G)$ if it satisfies the following conditions.

(1)    $D$ is metrical, i.e., $DG \equiv 0$,
(2)    $D$ satisfies (8.16), i.e., $D^{\mathcal{H}}\pi_* \equiv 0$,
(3)    $D$ satisfies $T^{\mathcal{V}\mathcal{V}} \equiv 0$,

where $T^{\mathcal{V}\mathcal{V}}$ is the vertical part of the torsion $D\theta$ of $D$.

**Remark 8.3.** The Cartan connection $D$ of $(\widetilde{TM}, G)$ is uniquely determined. However we remark that the Cartan connection $D$ in our sense is *not* the Cartan connection in the usual sense in [Ma], since $G$ is a *generalized Finsler metric, not* a real Finsler metric in the usual sense (cf. [Ic]).

**Remark 8.4.** The Chern-Finsler connection $\nabla$ of $(\widetilde{TM}, G)$ is metrical and it satisfies the symmetric property $T^{\mathcal{V}\mathcal{V}} \equiv 0$, but *not* necessarily the condition (8.16).

Identifying $\widetilde{T^{1,0}M}$ with $\widetilde{TM}$ via the morphism $\varrho$, we have constructed two metrical connections $\nabla$ and $D$ on $(\widetilde{TM}, G)$. Then we have a characterization of Finsler-Kähler metrics.

**Theorem 8.2.** *Let $(M, F)$ be a complex Finsler manifold. Then the following conditions are equivalent.*

(1)  *The Cartan connection $D$ coincides with the Chern-Finsler connection $\nabla$.*

(2)  *The lifted complex structure $J$ is parallel with respect to $D$:*

$$(8.17) \qquad\qquad D^{\mathcal{H}} J = 0.$$

(3)  *The Kähler form $\varpi$ is parallel with respect to $D$:*

$$(8.18) \qquad\qquad D^{\mathcal{H}} \varpi = 0.$$

(4)  *$(M, F)$ is a Finsler-Kähler manifold.*

*Proof.* (1) $\longrightarrow$ (2) is obvious, since the assumption $D = \nabla$ and (8.15) imply (8.17).

(2) $\longrightarrow$ (3) is proved as follows. For all $X, Y, Z \in \mathcal{A}(TM^\times, \widetilde{TM})$, we have

$$
\begin{aligned}
&(D^{\mathcal{H}}_X \varpi)(Y, Z) \\
&= X \varpi(Y, Z) - \varpi(D^{\mathcal{H}}_X Y, Z) - \varpi(Y, D^{\mathcal{H}}_X Z) \\
&= X G(JY, Z) - G(J D^{\mathcal{H}}_X Y, Z) - G(JY, D^{\mathcal{H}}_X Z) \\
&= G(D^{\mathcal{H}}_X(JY), Z) + G(JY, D^{\mathcal{H}}_X Z) - G(J D^{\mathcal{H}}_X Y, Z) - G(JY, D^{\mathcal{H}}_X Z) \\
&= 0,
\end{aligned}
$$

since $DG = 0$, and thus the condition (8.17) implies (8.18).

(3) $\longrightarrow$ (4) is proved as follows. Since $D$ satisfies the symmetric property (8.16), the second assertion in Proposition 8.3 implies (8.8).

(4) $\longrightarrow$ (1) is proved as follows. Since the Chern-Finsler connection $\nabla$ of $(\widetilde{TM}, G)$ is metrical and satisfies $T^{\mathcal{V}\mathcal{V}} \equiv 0$, it is enough to prove that $\nabla$ satisfies $\nabla^{\mathcal{H}}\pi_* = 0$. Because of

$$
\begin{aligned}
&(\nabla^{\mathcal{H}}\pi_*)(\varrho(Y), \varrho(Z)) \\
&= \nabla^{\mathcal{H}}_{\varrho(Y)}\pi_*(\varrho(Z)) - \nabla^{\mathcal{H}}_{\varrho(Z)}\pi_*(\varrho(Y)) - \pi_*([\varrho(Y), \varrho(Z)]) \\
&= \varrho\left(\nabla^{\mathcal{H}}_Y \pi_*(Z)\right) - \varrho\left(\nabla^{\mathcal{H}}_Z \pi_*(Y)\right) - \varrho\left(\pi_*([Y, Z])\right) \\
&= \varrho\left(\nabla^{\mathcal{H}}_Y \pi_*(Z) - \nabla^{\mathcal{H}}_Z \pi_*(Y) - \pi_*([Y, Z])\right) \\
&= \varrho\left((\nabla^{\mathcal{H}}\pi_*)(Y, Z)\right)
\end{aligned}
$$

for all $Y$, $Z \in \mathcal{A}(TM^\times, \widetilde{TM})$, the assumption (8.18) on $\tilde{T}^{1,0}M$ implies $\nabla^{\mathcal{H}}\pi_* = 0$ on $\widetilde{TM}$, and thus the Chern-Finsler connection $\nabla$ on $(\widetilde{TM}, G)$ coincides with the Cartan connection $D$.

$$\text{Q.E.D.}$$

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*Department of Mathematics and Computer Science*
*Kaogoshima University*
*Kagoshima*
*Japan*