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Finsler geometry in the tangent bundle

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Abstract.

Linear and metrical connections of a Riemannian space, whose indicatrices are ellipsoids, are established in the tangent bundle. Indicatrices of Finsler spaces are smooth, starshaped and convex hypersurfaces. They do not transform, in general, into each other by linear transformations, and thus they do not admit linear metrical connections in the tangent bundle. This necessitates the introduction of line-elements yielding the dependence of the geometric objects not only of points x but also of the direction y. Therefore, the apparatus (connections, covariant derivatives, curvatures, etc.) of Finsler geometry becomes inevitably a little more complicated.

Nevertheless there are a number of problems which need no line-elements. Such are those, which concern the metric only (arc length, area, angle, geodesics, etc.) and also the investigation of those important special Finsler spaces, which allow linear metrical connections in the tangent bundle.

In this paper we want to present results which use the tangent bundle TM only, and do not need TTM or VTM or line-elements. These investigations often admit direct geometrical considerations. Longer proofs are only sketched or omitted.

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I. Relation between distance spaces and Finsler spaces

§ 1. Finsler spaces, distance spaces

1. A Finsler space $F^n = (M, \mathcal{F})$ is a couple of an n-dimensional manifold M and a Finsler structure (fundamental function, metric function)

$$\mathcal{F} : TM \to R^+ = [0, \infty), \ (p, y) \mapsto \mathcal{F}(p, y) \geq 0, \quad p \in M, \ y \in T_pM,$$

which satisfies the following requirements:

(F i) $\mathcal{F} \in C^0$ on TM, and $\mathcal{F} \in C^\infty$ on the slit tangent bundle $TM \setminus 0 = \{(p, y) \mid y \neq 0\}$ (regularity)

(F ii) $\mathcal{F}(p, \lambda y) = \lambda \mathcal{F}(p, y)$, $\lambda \in R^+$ ((first order) positive homogeneity)

(F iii) $\frac{\partial^2 \mathcal{F}^2}{\partial y^i \partial y^j}(p, y) v^i v^j > 0$, $\forall v \neq 0 \in T_pM$, $i, j = 1, \ldots, n$ (strong convexity).

In place of (F ii) a more restrictive requirement is

(F iv) $\mathcal{F}(p, \lambda y) = |\lambda| \mathcal{F}(p, y)$, $\lambda \in R$ ((first order) absolute homogeneity).

At the early stage of Finsler geometry (F iv) was usually supposed.

The Finsler norm of $y \in T_pM$ is defined by $\|y\|_F := \mathcal{F}(p, y)$, and the Finsler arc length of a piecewise differentiable (this will always be supposed) curve $c : [a, b] \to M$, $t \mapsto c(t) \approx x(t)$ is given by the integral

$$\text{(a)} \qquad s = \int_a^b \mathcal{F}(c, \dot{c}) dt = \int_a^b (g_{ij}(x, \dot{x}) \dot{x}^i \dot{x}^j)^{1/2} dt,$$

where

$$g_{ij}(x, y) := \frac{1}{2} \frac{\partial^2 \mathcal{F}^2}{\partial y^i \partial y^j}(x, y)$$

(x^i are coordinates of p in a local chart $U \subset M$). If $\mathcal{F}^2(x, \dot{x}) = g_{ij}(x) \dot{x}^i \dot{x}^j$, then F^n reduces to a Riemannian space $V^n = (M, g)$, and we have

$$\text{(b)} \qquad s = \int_a^b (g_{ij}(x) \dot{x}^i \dot{x}^j)^{1/2} dt.$$

Conditions (a) and (b) are very similar, except that in the Riemannian case the integrand of (b) is the square root of a quadratic form in $\dot{x}$, while in the Finsler case the same expression need not to be quadratic in $\dot{x}$. Since both Finsler and Riemann geometries are built on the arc length of curves, on the base of the strong similarity of (a) and (b) we can say that Finsler geometry is just Riemannian geometry without the quadratic restriction. This witty statement is due to S. S. Chern [C]. Indeed, the two geometries have many basic similarities. Chern deems Finsler geometry to be the geometry of the XXI th century.

(F ii–iv) have simple and important geometrical meanings. (F ii) is equivalent to the invariance of the arc length against orientation-preserving parameter transformations of the curves. Also, (F ii) implies that the graph $z = \mathcal{F}(p_0, y) \subset R^{n+1} = (T_{p_0}M)(y) \times R(z)$ is a cone centered at p_0. The orthogonal projection in R^{n+1} of $(z = \mathcal{F}(p_0, y)) \cap (z = 1)$ on $T_{p_0}M$ is the indicatrix $\mathcal{I}(p_0) := \{y \in T_{p_0}M \mid \mathcal{F}(p_0, y) = 1\}$ of F^n at p_0. $\mathcal{I}(p)$ plays a role similar to that of the unit sphere of the Euclidean space E^n.

Condition (F iii) is equivalent to the triangle inequality in $T_{p_0}M$ with respect to the Finsler norm:
(F iii')

$$\mathcal{F}(p_0, y_1) + \mathcal{F}(p_0, y_2) > \mathcal{F}(p_0, y_1 + y_2), \quad \forall\, y_1, y_2 \in T_{p_0}M, \quad y_2 \neq \lambda y_1.$$

The tangent plane $T_{p_0}M$ endowed with the Finsler norm $\|y\|_F = \mathcal{F}(p_0, y)$ is a Minkowski space $\mathcal{M}^n = (T_{p_0}M, \mathcal{F}(p_0, y))$. So a Finsler space makes any of its tangent spaces $T_{p_0}M$ into a Minkowski space. In the case of a Riemannian space V^n the indicatrices are ellipsoids, and the induced Minkowski spaces are Euclidean spaces.

Remark that $\mathcal{F} \in C^0$ at $y = 0$. However, in view of (F ii, iii) more cannot be achieved. Indeed, if we had $\mathcal{F} \in C^1$ at $y = 0$, then the cone $z = \mathcal{F}(p_0, y)$ would be a hyperplane through the origin of R^{n+1}, and because of $\mathcal{F}(p_0, y) \geq 0$ this should be $T_{p_0}M$. Then $z = \mathcal{F}(p_0, y) \equiv 0$, which is not compatible with (F iii).

Condition (F iv) is equivalent to the invariance of the Finsler arc length s against any reparametrizations of the curves, including the change of the orientation. It is also equivalent to $\mathcal{F}(p, y) = \mathcal{F}(p, -y)$, $\forall\, p, y$. In this case $\mathcal{M}^n = (T_{p_0}M, \mathcal{F}(p_0, y))$ is a Banach space.

2. A distance space (M, ϱ) is a set M and a distance function $\varrho : M \times M \to D$ associating with any ordered pair p, q an element $\varrho(p, q)$ of the "distance set" D. In most cases, as in our case too, D consists of the non-negative reals R^+ or a subset of them. If ϱ has still the properties: a) $\varrho(p, q) = 0 \iff p = q$ (positive definiteness), b) $\varrho(p, q) = \varrho(q, p)$ (symmetry), and c) $\varrho(p, q) + \varrho(q, r) \geq \varrho(p, r)$ (triangle inequality), then (M, ϱ) is called a metric space ([B] sec. 8). If c) may fail, then ϱ and also (M, ϱ) are semi-metric. They are genuine semi-metric if c) really fails. If a) and c) are satisfied, but b) may fail, then ϱ and (M, ϱ) are called quasi-metric [RS] (or genuine quasi-metric if b) really fails).

Distance spaces were introduced by K. Menger, and developped by L. Blumenthal, H. Busemann, M. Fréchet and others. Distance spaces were used in investigations of geometric problems without differentiability conditions [e.g. Be]. They often appear also in recent topological studies, e.g. in investigations on the metrizability of topological spaces, etc. ([K], [RR], [RS], [St]).

§ 2. Distance functions induced by Finsler spaces

Let M be a connected manifold, and $\Gamma(p,q)$, $p,q \in M$ the collection of all equally oriented curves $c(t)$, $a \leq t \leq b$ emanating from p and terminating at q. Then a Finsler space $F^n = (M, \mathcal{F})$ determines by

$$(1) \qquad \varrho^F(p,q) := \inf \int_{\Gamma(p,q)} F(c, \dot{c})dt, \quad c(a) = p, \ c(b) = q$$

a distance function ϱ^F. It induces a distance function to a Finsler metric and a distance space to a Finsler space

$$\mathcal{F}(x,y) \mapsto \varrho^F(p,q) \quad \text{and} \quad F^n = (M, \mathcal{F}) \mapsto (M, \varrho^F).$$

We want to answer the naturally arising questions: Does conversely any distance space (M, ϱ) determine a Finsler space $(M, \mathcal{F}) : \varrho \overset{?}{\mapsto} \mathcal{F}$? Which of the ϱ do this? We also want to find those distance spaces (M, ϱ) and those relations $\varrho \mapsto \mathcal{F}$ for which

$$\varrho \mapsto \mathcal{F} \overset{(1)}{\mapsto} \varrho^F = \varrho.$$

If this is satisfied, then the initial ϱ must possess the properties of ϱ^F.

So we first recall some properties of ϱ^F (cf. [BCS] Chap. 6, esp sec. 6.4). Clearly

$$(\text{R i}) \qquad \varrho^F(p,q) \geq 0 \quad \text{and} \quad \varrho^F(p,q) = 0 \iff p = q$$

(the positive definiteness of ϱ^F). If $\mathcal{F}$ is absolute homogeneous, then

$$(\text{R ii}) \qquad \varrho^F(p,q) = \varrho^F(q,p)$$

(the symmetry of ϱ^F). This is true, since in the case of (F iv) the arc lengths of curves $c \in \Gamma_{(p,q)}$ and $c \in \Gamma_{(q,p)}$ are independent of the orientation. Nevertheless without the absolute homogeneity (F iv) $\varrho^F(p,q)$

may differ from $\varrho^F(q,p)$. This can easily be seen on a Minkowski space $\mathcal{M}^n = (R^n, \mathcal{F})$, whose indicatrix is non-symmetric.

Also, every ϱ^F satisfies

(R iii) $$\varrho^F(p,q) + \varrho^F(q,r) \geq \varrho^F(p,r), \quad p,q,r \in M$$

(the triangle inequality), since by (1) for apropriate $c_1 \in \Gamma_{(p,q)}$ and $c_2 \in \Gamma_{(q,r)}$ with arc length I_1 and I_2 we obtain

$$\varrho^F(p,r) \leq \dot{I}_1 + \dot{I}_2 \leq \varrho^F(p,q) + \varepsilon + \varrho^F(q,r) + \varepsilon, \quad 0 < \varepsilon \to 0.$$

Moreover, (R iii) holds also for spaces $(M, \mathcal{F})$, where $\mathcal{F}$ is not strongly convex (i.e. (F iii) or (F iii') is not satisfied; these spaces are not Finsler spaces in our sense). This happens if the indicatrices $\mathcal{F}(p_0, y) = 1$, $p_0 \in M$ are star-shaped, smooth, but non-convex. Arc lengths s of curves and distance functions ϱ^F can be formed even in this case, and our considerations described in the previous paragraph also remain alive. Thus (R iii) is valid as well. It means also that neither of these distance functions can be genuine semi-metric. – Nevertheless we can present differential geometric examples for genuine semi-metric spaces, if ϱ is given in another way. Let us consider a Minkowski space $\mathcal{M}^n = (R^n, \mathcal{F})$ in an adapted coordinate system (x) (see p. 182 of this article or [M] p. 158) with a symmetric, star-shaped, smooth and non-convex indicatrix $\mathcal{I}$, and define a distance function $\varrho(x_1, x_2)$ by the Minkowski norm of the vector $\overrightarrow{x_1, x_2}$:

$$\varrho(x_1, x_2) := \|\overrightarrow{x_1 x_2}\|_M.$$

Then (R ii) is satisfied because of the symmetry of $\mathcal{I}$, but the triangle inequality (R iii) is not, since $\mathcal{I}$ is non-convex.

So (M, ϱ^F) is a metric space provided $\mathcal{F}$ is absolutely homogeneous, and it is a genuine quasi-metric space if $\mathcal{F}$ is only positively homogeneous. Further on (M, ϱ) is supposed to be quasi-metric. Metric (M, ϱ) are included as special case.

What differentiability properties has $\varrho^F(p,q)$?

Using in F^n a geodesic polar coordinate system (r, φ) in a neighbourhood $U \subset M$ around p_0, we find that $\varrho^F(p_0, q) = r$. This shows that $\varrho^F(p_0, q) \in C^0$ at $q = p_0$, $\varrho^F(p_0, q) \notin C^1$ at $q = p_0$, and $\varrho^F(p_0, q) \in C^\infty$ on the punctured domain $U \setminus 0$ $(r \neq 0)$.

Let $q(t)$, $0 \leq t \leq a$ be a geodesic of F^n with $q(0) = p_0$ and $\lim_{t \to 0} \dot{q}(t) = y_0 \neq 0$. Then

(2')
$$\lim_{t\to 0}\left[\frac{d}{dt}\varrho^F(p_0, q(t))\right] = \lim_{t\to 0}\left[\frac{d}{dt}\int_0^t F(q(\tau), \dot{q}(\tau))d\tau\right] = \mathcal{F}(p_0, y_0) > 0.$$

Hence $\frac{d}{dt}\varrho^F(p_0, q(t)) = \frac{d}{dt}\big|_{q(t),\dot q(t)}\varrho^F(p_0, q)$ is the directional derivative of ϱ^F at $q(t)$ in the direction $\dot q(t)$. Since directional derivatives depend on the point and the direction only, $q(t)$ in (2') can be replaced by any other $c(t)$, $0 \le t$ emanating from $p_0 = c(0)$, and having at p_0 the (one sided) tangent y_0. Then

$$\text{(2)} \qquad \lim_{t\to 0}\left[\frac{d}{dt}\varrho^F(p_0, c(t))\right] = \mathcal{F}(p_0, y_0), \quad y_0 = \lim_{t\to 0}\frac{dc}{dt}.$$

The relation (2) is basically the content of the Busemann-Mayer theorem ([BM] p. 186, in a more comfortable form in [BCS] p. 153, or [S] p. 72).

Thus we obtain

(R iv) (a) $\varrho^F(p_0, q) \in C^0$ at $q = p_0$

(b) $\varrho^F(p_0, q) \in C^\infty$ in an open domain around, but without p_0.

(c) There exists $\lim_{t\to 0}\frac{d}{dt}\big|_{c(t),\dot c(t)}\varrho^F(p_0, q)$ for any $(c(t), 0 \le t$ emanating from $p_0 = c(0)$. The value of this limit is $\mathcal{F}(p_0, y_0)$, $y_0 = \lim_{t\to 0}\frac{dc}{dt}$, which is positive if $y_0 \ne 0$, of class C^0 if $y_0 = 0$.

It follows from the properties of the directional derivatives that

$$\text{(R v)} \quad \lim_{t\to 0}\frac{d}{dt}\Big|_{\bar c(t),\dot{\bar c}(t)}\varrho^F(p_0, q) = \lambda \lim_{t\to 0}\frac{d}{dt}\Big|_{c(t),\dot c(t)}\varrho^F(p_0, q), \quad \lambda \in R^+,$$

where $\bar c(0) = c(0)$ and $\dot{\bar c}(t) = \lambda \dot c(0)$. Hence (R v) is a consequence of (R iv).

Let $c_1(t)$, $c_2(t)$, $c_3(t)$, $0 \le t$ be curves emanating from p_0 with non-null and non-parallel tangents $\dot c_1(0) = y_1$, $\dot c_2(0) = y_2$, $\dot c_3(0) = y_1 + y_2$. From (2), (F iii) and (R iv,c) we obtain

$$\text{(R vi)} \quad \begin{aligned} &\lim_{t\to 0}\frac{d}{dt}\Big|_{c_1(t),\dot c_1(t)}\varrho^F(p_0, q) + \lim_{t\to 0}\frac{d}{dt}\Big|_{c_2(t),\dot c_2(t)}\varrho^F(p_0, q) > \\ &\qquad > \lim_{t\to 0}\frac{d}{dt}\Big|_{c_3(t),\dot c_3(t)}\varrho^F(p_0, q). \end{aligned}$$

This is somewhat stronger than the local triangle axiom (see [W] p. 56).

Conditions (R iv–vi) hold also for $\varrho^F(q, p_0)$.

We can summarize these statements in

Proposition 1. *The distance function ϱ^F derived from an F^n by* (1) *possesses the properties* (R i, iii–vi). *The condition* (R ii) *is added iff $\mathcal{F}$ is absolutely homogeneous.*

The often appearing properties (Ri, iii–vi) will be denoted by (R*).

§ 3. Finsler spaces induced by distance functions

Further on we suppose that ϱ in place of ϱ^F staisfies (R*).

Given a quasi-metric distance space (M, ϱ), we want to define a correspondence

$$(3) \qquad \varrho(p_0, q) \mapsto \overline{\mathcal{F}}(p_0, y), \quad \forall p_0 \in M; \quad (M, \varrho) \mapsto (M, \overline{\mathcal{F}})$$

with the natural requirement that in case of $\varrho = \varrho^F$ the Finsler metric $\overline{\mathcal{F}}$ corresponding to $\varrho = \varrho^F$ by (3) is just that $\mathcal{F}$ from which ϱ^F originates by (1):

$$\left(\mathcal{F} \overset{(1)}{\longmapsto}\right) \varrho^F \overset{(3)}{\longmapsto} \overline{\mathcal{F}} = \mathcal{F}.$$

We know that between ϱ^F and $\mathcal{F}$ the relation (2) subsists. Hence (3) must have the form

$$(4) \qquad \overline{\mathcal{F}}(p, y) := \lim_{t \to 0} \left[\frac{d}{dt} \varrho(p, c(t))\right], \quad y = \lim_{t \to 0} \frac{dc}{dt},$$

where $c(t)$, $0 \leq t$, $c(0) = p$ is a curve emanating from p.

It follows that $\overline{\mathcal{F}}$ defined by (4) is a Finsler metric. By (R iv,c) the function $\overline{\mathcal{F}}(p, y)$ is non-negative, it is of class C^∞ if $\dot{c}(0) = y \neq 0$, and of class C^0 if $\dot{c}(0) = y = 0$. Thus $\overline{\mathcal{F}}(p, y)$ of (4) satisfies (F i). By (R v) it satisfies (F ii). Finally because (2) and (R vi) it satisfies also (F iii). Thus we obtain

Proposition 2. *If $\varrho(p, q)$ satisfies* (R*), *then $\overline{\mathcal{F}}(p, y)$ defined by* (4) *is a Finsler metric. If* (R ii) *is also satisfied, then $\overline{\mathcal{F}}$ is absolutely homogeneous.*

Without any of the conditions (R*) on ϱ, the function $\overline{\mathcal{F}}(p, y)$ defined by (4) may not be a Finsler metric.

By (1), (2) and (4) we have $\mathcal{F} \overset{(1)}{\longmapsto} \varrho^F \overset{(4)}{\longmapsto} \mathcal{F} \overset{(1)}{\longmapsto} \varrho^F$. This means that (1) and (4) are map and inverse map. Thus they induce between $\{\mathcal{F}\}$ and $\{\varrho^F\}$ (over a given M) a $1:1$ relation. Nevertheless (4) assigns, for every ϱ (which satisfies (R*)), an $\mathcal{F}$ (the bar is omitted) and thus $\varrho \overset{(4)}{\longmapsto} \mathcal{F} \overset{(1)}{\longmapsto} \varrho^F$. We show that in this sequence $\varrho^F \neq \varrho$ may occur. This fact is expressed by the

Theorem 1. *$\{\varrho^F\}$ is a proper part of $\{\varrho\}$, where ϱ satisfy* (R*).

This can be proved by giving an example, where ϱ induces by (4) a Finsler metric $\mathcal{F}(p, y)$, yet the ϱ^F obtained from this $\mathcal{F}$ by (1) differs from the initial ϱ, that is $\varrho^F \neq \varrho$.

First we give a 1-dimensional example. Let $M = R^1 = R$ be the Euclidean line E^1, and (x) the canonical coordinates on it. Let $\varrho(0,x)$, $x \in [0,\infty)$ be a strictly increasing C^∞ function with strictly decreasing first derivative, $\varrho(0,0) = 0$, and satisfying

$$\lim_{x\to 0^+} \frac{d}{dx}\varrho(0,x) = 1 \tag{5}$$

(e.g. $\varrho(0,x) = \ln(x+1)$). We define ϱ for $\overline{x} < 0$ by

$$\varrho(0,\overline{x}) = \varrho(0,|\overline{x}|), \tag{6}$$

and for $x_0 \neq 0$ by

$$\varrho(x_0,x) = \varrho(0,x-x_0). \tag{7}$$

The functions $\varrho(x_0,x)$ for different x_0 are parallel translates of each other. One can prove that they satisfy (R i–vi). In consequence from (R i–iii) it follows that (M,ϱ) is a metric space. According to Proposition 2 this ϱ generates by (4) a Finsler space $F^1 = (R^1,\mathcal{F})$, ((F iii) with the sign of equality). By (4), (6) and (7) we have $\mathcal{F}(x_0,a) = \mathcal{F}(x_0,-a)$. Thus $\mathcal{F}$ is absolutely homogeneous. By (4) and (7) one can see that $\mathcal{F}(x,a)$ is independent of x. Therefore F^1 is a Minkowski space with symmetric indicatrix, and because of $n = 1$ it is a Euclidean space E^1. Hence $\varrho^F(x_1,x_2) = |x_1 - x_2|$. Nevertheless, by the integral mean theorem

$$\varrho(x_1,x_2) = \int_{x_1}^{x_2} \varrho'(x_1,z)dz = |x_1 - x_2|\varrho'(x_1,z^*), \quad z^* \in (x_1,x_2).$$

By (5), (7) and the strict decrease of $\varrho'(x_1,z)$ on $z > x_1$ we obtain

$$\lim_{z\to x_1^+} \varrho'(x_1,z) = 1 > \varrho'(x_1,z^*).$$

Thus

$$\varrho(x_1,x_2) < |x_1 - x_2| = \varrho^F(x_1,x_2), \quad \text{i.e. } \varrho \overset{(4)}{\longmapsto} \mathcal{F} \overset{(1)}{\longmapsto} \varrho^F \neq \varrho.$$

The discussed 1-dimensional example can be extended to $M = R^n$. Let now the graph $z = \varrho(0,x) \subset R^{n+1}(x,z)$ of the new distance function $\varrho(0,x)$, $x \in R^n$ be the rotation around z-axis of the graph $z = \varrho(0,x^1) \subset R^1(x) \times R^1(z) \equiv R^2(x^1,z)$ of the previous 1-dimensional example, and let $z = \varrho(a,x)$, $x \in R^n$ be the parallel translate of it with $\overrightarrow{0a}$ of $z = \varrho(0,x) \subset R^{n+1}$. Then, again, we have $\varrho \overset{(4)}{\mapsto} \mathcal{F} \overset{(1)}{\mapsto} \varrho^F \neq \varrho$.

Similar examples can be constructed on manifolds M different from R^n, provided that M admits a locally Minkowski structure. This is possible iff M admits an open cover $M = \bigcup_{\alpha} U_\alpha$ by local charts, and on each U_α there exists a coordinate system (x_α), such that the transitions $(x_\alpha) \longleftrightarrow (x_\beta)$ on $U_\alpha \cap U_\beta$ are linear ([T1] sec. 2). The torus has this property, but the sphere does not ([BC] p. 250; [BCS] p. 14).

§ 4. Conditions for $\varrho = \varrho^F$

Further on we suppose that in $F^n = (M, \mathcal{F})$ any pair of points $p, q \in M$ can be connected by a (short) geodesic $g(t)$, $a \leq t \leq b$, $g(a) = p$, $g(b) = q$ whose arc length is $\varrho^F(p,q)$. This is certainly true if F^n is geodesically complete. (In this case the infimum in (1) is a minimum.)

Starting with an arbitrary ϱ (which satisfies (R*), it may happen that

$$\varrho \overset{(4)}{\longmapsto} \mathcal{F} \overset{(1)}{\longmapsto} \varrho^F \neq \varrho$$

(i.e. ϱ^F may differ from ϱ), as it was shown by the examples of the previous section. We look for conditions assuring

$$\varrho \overset{(4)}{\longmapsto} \mathcal{F} \overset{(1)}{\longmapsto} \varrho^F = \varrho.$$

First we show the parallelity of certain vector fields. Let $g(t)$, $t \in [0,T]$ be a short geodesic of $F^n = (M, \mathcal{F})$ from p_0 to q. Then for any t_1, $0 < t_1 < t < T$

$$\varrho^F(p_0, g(t)) = \varrho^F(p_0, g_1) + \varrho^F(g_1, g(t)), \quad g_1 = g(t_1).$$

From this

$$\left[\frac{d}{dt}\varrho^F(p_0, g(t)\right]_{|t_1} = \lim_{t \to t_1^+} \left[\frac{d}{dt}\varrho^F(g_1, g(t))\right]. \tag{8}$$

Consider the *distance surface* of F^n attached to p_0 given by

$$\Theta^F_{p_0} : z = \varrho^F(p_0, q) \subset U(q) \times R^1(z),$$

where $U(q) \subset M$ is a coordinate neighbourhood of p_0. The point p_0 is the vertex (cape) of $\Theta^F_{p_0}$. The curve $\xi_0(t) := (g(t), \varrho^F(p_0, g(t)) \subset U \times R^+$ lies on $\Theta^F_{p_0}$, and $\xi_1(t) := (g(t), \varrho^F(g_1, g(t)) \subset \Theta^F_{g_1}$. By (8) their tangents, $\dot{\xi}_0(t_1)$ and $\lim_{t \to t_1^+} \dot{\xi}_1(t) =: \dot{\xi}_1^+(t_1)$, are parallel, i.e.

Proposition 3. $\dot{\xi}_0(t_1) \| \dot{\xi}_1^+(t_1), \ \forall \, t_1 \in (0,T)$.

Consider the projection $\pi : U \times R^+ \to U$, $(p, z) \mapsto p$. Then

$$d\pi\dot{\xi}_0(t_1) = d\pi\dot{\xi}_1^+(t_1) = \dot{g}(t_1),$$

where $\dot{\xi}_0(t_1)$ and $\dot{\xi}_1^+(t_1)$ are the lifts of $\dot{g}(t_1)$ to $T_{\xi_0(t_1)}\Theta^F_{p_0}$ resp. $\lim_{t\to t_1^+} T_{\xi_1(t)}$ $\Theta^F_{g_1}$.

In a distance space with (R*) the notion of geodesic can be replaced to a certain extent by that of "parallelity curve". Let us consider a curve $p(t)$, $t \in [0, T]$. Along this there exists a family of distance surfaces $\theta^{\varrho}_{p(t_0)}$: with $z = \varrho(p(t_0), q)$ (t_0 is the parameter of the family) and curves $\zeta_0(t)$, $\zeta_1(t)$ on $\theta^{\varrho}_{p(t_0)}$ over $p(t)$ similarly to $\xi_0(t)$ and $\xi_1(t)$. If $\zeta_0(t) \in C^1$, and

$$\dot{\zeta}_0(t_1) \| \dot{\zeta}_1^+(t_1), \quad \forall t_1 \in (0, T),$$

then $p(t)$ is called a *parallelity curve.*

One can prove the following

Theorem 2. *For any curve $c(t)$, $t \in [0, T]$ of a distance space (M, ϱ) satisfying* (R*)*, and for the Finsler metric $\mathcal{F}$ determined by ϱ according to* (4) *we obtain*

(a) $\varrho(c(0), c(T)) \leq \int_0^T \mathcal{F}(c, \dot{c})dt$

(b) *if $c(t)$ is a parallelity curve, then*

$$\varrho(c(\tau), c(t)) = \int_\tau^t \mathcal{F}(c, \dot{c})du, \quad 0 \leq \tau < t < T \tag{9}$$

(c) *if along $c(t)$ (9) holds for $\forall\tau$, t, $0 \leq \tau < t < T$, then $c(t)$ is a parallelity curve.*

Corollary. *In a Finsler space parallelity curves and short geodesics coincide.*

As we have shown, a distance space (M, ϱ) with (R*) determines an $F^n = (M, \mathcal{F})$, and this F^n determines a ϱ^F:

$$\varrho \overset{(4)}{\longmapsto} \mathcal{F} \overset{(1)}{\longmapsto} \varrho^F. \tag{10}$$

Theorem 3. *In (10) $\varrho^F = \varrho$ iff any short geodesic of F^n (determined by ϱ) is a parallelity curve of (M, ϱ).*

In other words: the distances in (M, ϱ) coincide with the distances of a Finsler space iff the short geodesics of the Finsler space are parallelity curves of the distance space (Proof in [T4]).

In Theorem 3 we required the parallelity property on curves determined by F^n, and not on curves determined directly by the distance space (M, ϱ). Now we replace the parallelity property (the condition of Theorem 3) by another one, which is expressed directly in terms of the (M, ϱ).

Let us consider two points a, b of a distance space (M, ϱ) (with (R^*)), and a sphere $S_a^\varrho(t) := \{q \in M \mid \varrho(a, q) = t\}$ around a with radius $t \leq r = \varrho(a, b)$. Then there exists another sphere $S_b^\varrho(\tau) := \{q \in M \mid \varrho(q, b) = \tau\}$ such that the two spheres osculate each other from outside at a common point $\sigma(t)$ of $S_a^\varrho(t)$ and $S_b^\varrho(\tau)$. (These spheres are actually "forward" and "backward" metric spheres (see [BCS] p. 149, 155).) If $\sigma(t)$, $t \in [0, r]$ is a C^1 curve, then it will be called *osculation curve*, and we obtain the

Theorem 4. *The distances $\varrho(p, q)$ in a distance space (M, ϱ) (whose ϱ satisfies (R^*)) coincide with the distances $\varrho^F(p, q)$ of a Finsler space $(M, \mathcal{F})$ iff any osculation curve is a parallelity curve in (M, ϱ).*

Because of the triangle inequality (R iii) we obtain $r \leq t + \tau$ along any osculation curve $\sigma(t; a, b)$. If $r = t + \tau$, $\forall\, 0 < t < r$, then $\sigma(t; a, b)$ is called straight ([BC]) or a Hilbert curve ([BM] p. 170). In a Finsler space osculation curves are short geodesics.

II. Angle in Minkowski and Finsler spaces

Area in Minkowski spaces was given by Busemann [Bu] and studied and often used by others. Infinitesimally a Finsler space is a Minkowski space. So if we can measure area in a Minkowski space, then by integration we obtain the area (of a domain) of a Finsler space. The same holds also for submanifolds. We consider the angle of two vectors in a tangent space of the base manifold of a Finsler space. This angle in Minkowski (or Finsler) spaces attracted less interest. Since the Finsler space makes its tangent space into a Minkowski space, measuring of angles in a Finsler space reduces to that in a Minkowski space. We show that they are applicable in measuring the deviation of a Finsler space from being Riemanian. Also it can be proved that a diffeomorphism between two Finsler spaces is an isometry iff it keeps angle (in the above sense) and area, similary to the well known result of Riemannian geometry.

§ 1. Angle

Given a Finsler space $F^n = (M, \mathcal{F})$ we consider an angle $\alpha = \measuredangle(a,b)$ between two rays $a, b \in T_{p_0}M$ emanating from the origin $0 = p_0$ of $T_{p_0}M$. Here $T_{p_0}M$ is an n-dimensional vector space $\mathcal{V}^n$ and, a and b span a two-dimensional linear subspace Σ of $T_{p_0}M$, provided a is not parallel to $b : a \nparallel b$. If $a \parallel b$, then we assign to the pair a, b a 2-dimensional linear subspace Σ of T_pM trough the straight line $g \supset a, b$. The convex domain of Σ bounded by a and b will be denoted by A. This is unambigous if $a \nparallel b$. If $a = b$, then $A = \emptyset$. In the case when $a, b \subset g$, $a \neq b$, then g cuts Σ into Σ^+ and Σ^-. Therefore $A = \Sigma^+$ or $A = \Sigma^-$.

Let $B^n_{x_0}(1) := \{y \mid \mathcal{F}(x_0, y) \le 1\} \subset T_{x_0}M$ be the indicatrix body of F^n at $x_0 \in M$. The Finsler space F^n makes each $T_{x_0}M$ into a Minkowski space $\mathcal{M}^n_{x_0}$ with indicatrix body $B^n_{x_0}(1)$ and with the Minkowski functional $\mathcal{F}(y) = \mathcal{F}(x_0, y) : T_{x_0}M \to R^+$. Then $B^n_x(1)$ is a Minkowski ball of radius 1, and $\partial B^n_x(1) = \mathcal{I}$ is the indicatrix (hyper) surface. By $B^2_x = B^n_x(1) \cap \Sigma$, it follows that $\mathcal{M}^n_x$ (or F^n) induces on $\Sigma \subset T_xM$ a two-dimensional Minkowski metric and thus an $\mathcal{M}^2_x$. Remark that $B^2_x \cap A = D$ is a segment of the indicatrix body of $\mathcal{M}^2_x$ belonging to $\measuredangle\alpha(a,b)$.

Let $\{e_1, e_2\}$ be an arbitrary basis in the real vector space $\mathcal{V}^2 \approx \Sigma \subset T_xM$. Then $y = \sum_{i=1}^2 y^i e_i$. Let $\Psi : \Sigma \to R^2$ be a mapping given by $\Psi(y) = (y^1, y^2) \in R^2$. Considering $\Psi(e_i)$ as an orthonormal system, R^2 becomes a Euclidean space E^2. We denote the Minkowski area in $\mathcal{M}^2_x$ by $\|\cdot\|_M$, and the Euclidean area in E^2 by $\|\cdot\|_E$. Then the 2-dimensional Minkowski area of D in $\mathcal{M}^n_x$ is the Minkowski area of D in $\mathcal{M}^2$:

(11)

$$\|D\|_{\mathcal{M}} = \int_{\mathbb{D}} \sigma dy^1\, dy^2, \quad \sigma = \frac{\pi}{\|\mathbb{B}^2\|_E}, \quad \mathbb{B}^2 = \Psi(B^2_x), \quad \mathbb{D} = \Psi(D).$$

(Z. Shen [S1] §1.3, or H. Busemann [Bu], H. Rund [Ru] Chap. I, §8, D. Bao – S. S. Chern – Z. Shen [BCS], §1.4, and many other places.) Since $\int_{\mathbb{D}} dy^1 dy^2$ is the Euclidean area of $\mathbb{D}$, the relation (11) is equivalent to

$$(11') \qquad \|D\|_{\mathcal{M}} = \frac{\pi\|\mathbb{D}\|_E}{\|\mathbb{B}^2\|_E}.$$

Formulas (11) and (11') are true for any domain $\mathcal{G} \subset \Sigma$ in place of D.

The Minkowski measure of the angle $\measuredangle\alpha(a,b)$ can be defined as follows:

Definition.

$$(12) \qquad \measuredangle_{\mathcal{M}}\alpha(a,b) := \epsilon 2\|D\|_{\mathcal{M}}. \quad \epsilon = 1 \text{ or } -1.$$

The sign ϵ depends on the orientation of the angle.

The angle $\measuredangle_{\mathcal{M}}\alpha$ can be expressed by the Minkowski functional $\mathcal{F}$ and the data of the two legs a and b. Let (r,φ) be a polar coordinate system in E^2, $e(\varphi)$ a unit vector in E^2 with polar coordinates $(1,\varphi)$, and $\mathcal{I}^2 := \partial B_x^2(1)$ the indicatrix curve of $\mathcal{M}_x^2$. Let $y \in \mathcal{I}^2$, $\Psi(\mathcal{I}^2) = (r(\varphi),\varphi)$, and $\Psi^{-1}(e(\varphi)) = \overline{e}(\varphi)$. Then $1 = \mathcal{F}(y(\varphi)) = \mathcal{F}(r(\varphi)\overline{e}(\varphi)) = r(\varphi)\mathcal{F}(\overline{e}(\varphi))$. Thus $r(\varphi) = \mathcal{F}^{-1}(\overline{e}(\varphi))$. The Euclidean area of $\mathbb{B}^2$ is

$$\|\mathbb{B}^2\|_E = \int_{\varphi=0}^{2\pi} \frac{1}{2} r^2(\varphi)d\varphi = \frac{1}{2}\int_0^{2\pi} \frac{1}{\mathcal{F}^2(\overline{e}(\varphi))}d\varphi$$

$$= \frac{1}{2}\int_0^{2\pi} \left(y(\varphi) \wedge \frac{dy}{d\varphi}\right) d\varphi = \frac{1}{2}\int_0^{2\pi} (y^1 dy^2 - y^2 dy^1).$$

Hence

$$\epsilon\|D\|_{\mathcal{M}} = 2\pi \left[\int_0^{2\pi} \mathcal{F}^{-2}(\overline{e}(\varphi))d\varphi\right]^{-1} \cdot \int_{\mathbb{D}} r\, dr\, d\varphi. \tag{13.a}$$

Or, in another from

$$\epsilon\|D\|_{\mathcal{M}} = 2\pi \left[\int_0^{2\pi} \mathcal{F}^{-2}(\overline{e}(\varphi))d\varphi\right]^{-1} \cdot \int_{\varphi=\varphi_1}^{\varphi_2} \left[\int_{r=0}^{r(\varphi)} r\, dr\right] d\varphi, \tag{13.b}$$

$$\epsilon = \begin{cases} +1 & \text{if} \quad \varphi_1 < \varphi_2 \\ -1 & \text{if} \quad \varphi_2 < \varphi_1, \end{cases}$$

where $0 \le \varphi_1,\ \varphi_2 \le 2\pi$ denote the directions of the two legs a, b of α.

If $\mathcal{M}_x^n$ is a Euclidean space E^n, then (12) reduces to the Euclidean measure $\measuredangle_E\alpha$ of the angle α. Indeed, if $\mathcal{M}_x^n = E^n$, then $\mathbb{B}^2$ is the Euclidean unit ball. Now $\Psi^{-1}(e(\varphi)) = y(\varphi) \in \mathcal{I}^2 \Rightarrow \mathcal{F}^{-1}(y(\varphi)) = 1 \Rightarrow \|D\|_{\mathcal{M}} \overset{(13.b)}{=} 2\pi(2\pi)^{-1}(\varphi_2-\varphi_1)\frac{1}{2}$ and by (12) we have $\measuredangle_{\mathcal{M}}\alpha = \varphi_2 - \varphi_1$, which is the Euclidean measure of α. Thus $\measuredangle_{\mathcal{M}}\alpha = 2\epsilon\|D\|_{\mathcal{M}}$ is a generalization of the Euclidean measure of α. – If $\partial\mathbb{B}^2$ is an ellipse $\mathcal{E}$, then there is a linear (Minkowski) isomorphism i of $\Sigma = \mathcal{V}^2$, which takes $\mathcal{E}$ into the unit circle of E^2, and $\mathcal{M}_x^2$ into a $\overline{\mathcal{M}}_x^2$. Since $\|\cdot\|_{\mathcal{M}}$ is a Haar measure which is preserved by linear isomorphisms, we obtain that $\measuredangle_{\mathcal{M}}\alpha = \measuredangle_{\overline{\mathcal{M}}}\overline{\alpha} = \measuredangle_E i\alpha$.

$\|D\|_{\mathcal{M}}$ of (13.b) is positive if $\varphi_2 > \varphi_1$, and negative if $\varphi_2 < \varphi_1$. Therefore $\measuredangle_{\mathcal{M}}\alpha$ has a sign, and because of the additivity of the second integral in (13.b), the angle $\measuredangle_{\mathcal{M}}\alpha$ is also additive: $\measuredangle_{\mathcal{M}}\alpha(a,b) + \measuredangle_M\alpha(b,c) = \measuredangle_{\mathcal{M}}\alpha(a,c)$. However $\measuredangle_{\mathcal{M}}\alpha$ is symmetric in the sense that $|\measuredangle_{\mathcal{M}}(a,b)| = |\measuredangle_{\mathcal{M}}(b,a)|$.

Let us consider the case of the straight angle. In this case $a \cup b = g$ is a line through $0 \in T_xM$. Let $\measuredangle(a, b) = \alpha^+$ be the straight angle with the domain $\Sigma_g^+ = A^+$, and $\measuredangle(b, a) = \alpha^-$ the straight angle with the domain $\Sigma_g^- = A^-$. Because of the additivity we have

$$\measuredangle_{\mathcal{M}}\alpha^+ + \measuredangle_{\mathcal{M}}\alpha^- = 2\pi, \qquad \forall\, g.$$

Therefore the equality $\measuredangle_{\mathcal{M}}\alpha^+ = \measuredangle_{\mathcal{M}}\alpha^-$ of the Minkowski measure of the two straight angles implies $\|B^2 \cap A^+\|_E = \|^+\mathbb{D}^2\|_E = \|^-\mathbb{D}^2\|_E = \|B^2 \cap A^-\|_E$, and conversely. In other words: $\measuredangle_{\mathcal{M}}\alpha^+ = \measuredangle_{\mathcal{M}}\alpha^-$ iff g bisects B^2.

If $\mathbb{B}^2$ is symmetric, then every line g through O bisects B^2. We show that also conversely, if every g through O bisects B^2, then $\mathbb{B}^2$ is symmetric. Suppose that $\mathbb{B}^2$ is non-symmetric. Then there exists a φ_0, such that in the applied polar coordinate system $r(\varphi_0) > r(\varphi_0 + \pi)$, where $(r(\varphi), \varphi) \in \partial\mathbb{B}^2$, $\forall\, \varphi$. A g is fixed by its direction φ_0. Then for every $g(\varphi_0)$ we have

$$\frac{1}{2}\int_{\varphi_0}^{\varphi_0+\pi} r^2(\varphi)d\varphi = \frac{1}{2}\|\mathbb{B}^2\|_E, \quad \forall 0 \leq \varphi < \pi.$$

Especially

$$\int_{\varphi_0-\epsilon}^{\varphi_0-\epsilon+\pi} r^2(\varphi)d\varphi = \int_{\varphi_0+\epsilon}^{\varphi_0+\epsilon+\pi} r^2(\varphi)d\varphi,$$

and hence it follows

$$\int_{\varphi_0-\epsilon}^{\varphi_0+\epsilon} r^2(\varphi)d\varphi = \int_{\varphi_0-\epsilon+\pi}^{\varphi_0+\epsilon+\pi} r^2(\varphi)d\varphi.$$

By the integral mean value theorem we obtain

$$2\epsilon r^2(\varphi_1) = 2\epsilon r^2(\varphi_2), \qquad \begin{array}{l} \varphi_0 - \epsilon \leq \varphi_1 \leq \varphi_0 + \epsilon \\ \varphi_0 - \epsilon + \pi \leq \varphi_2 \leq \varphi_0 + \epsilon + \pi, \end{array}$$

and because of the continuity of $r(\varphi)$, the limit $\epsilon \to 0$ yields $r(\varphi_0) = r(\varphi_0 + \pi)$ in contradiction to our assumption. Therefore $\mathbb{B}^2$, and thus also B^2 is symmetric. This is equivalent to the absolute homogeneity of $\mathcal{F}$.

These statements are summed up in

Theorem 5. $\measuredangle_{\mathcal{M}}\alpha = \epsilon 2\|D\|_{\mathcal{M}}$ *is an additive, symmetric measure of the angles in Finsler spaces. In a Euclidean space this reduces to the Euclidean measure of* α. *Moreover* $\measuredangle_{\mathcal{M}}\widetilde{\alpha} = \pm\pi$ *for every straight angle* $\widetilde{\alpha}$ *if and only if the Finsler metric is absolute homogeneous.*

§ 2. Isometry between F^n and $\overline{F}^n$

Let $F^n = (M, \mathcal{F})$ and $\overline{F}^n = (\overline{M}, \overline{\mathcal{F}})$ be two Finsler spaces, $\varphi : M \to \overline{M}$ a diffeomorphism, $\mathcal{I}(p_0) := \{y \in T_{p_0}M \mid \mathcal{F}(p_0, y) = 1\}$ and $\overline{\mathcal{I}}(\overline{p}_0) := \{\overline{y} \in T_{\overline{p}_0}\overline{M} \mid \overline{\mathcal{F}}(\overline{p}_0, \overline{y}) = 1\}$, $\overline{p}_0 = \varphi(p_0)$ are indicatrix hypersurfaces (indicatrices) of F^n and $\overline{F}^n$ resp. $\mathcal{I}(p) \cap \Sigma = \mathcal{I}^2(p)$ is the indicatrix of $\mathcal{M}_p^2$, and $\overline{\mathcal{I}}(\overline{p}) \cap \overline{\Sigma} = \overline{\mathcal{I}}^2(\overline{p})$, $\overline{p} = \varphi(p)$, $\overline{\Sigma} = \varphi(\Sigma)$ is the indicatrix of $\overline{\mathcal{M}}_{\overline{p}}^2$. The mapping φ is an isometry iff

$$(d\varphi)\mathcal{I}(p) = \overline{\mathcal{I}}(\overline{p}), \quad \forall\, p \in M. \tag{14}$$

Theorem 6. *The diffeomorphism $\varphi : M \to \overline{M}$ is an isometry between the Finsler spaces F^n and $\overline{F}^n$ iff φ keeps angle and (2-dimensional) area.*

Proof. **A**) Suppose that φ is an isometry. By (14) we obtain

$$(d\varphi)\mathcal{I}^2(p) = (d\varphi)\mathcal{I}(p) \cap (d\varphi)\Sigma = \overline{\mathcal{I}}(\overline{p}) \cap \overline{\Sigma} = \overline{\mathcal{I}}^2(\overline{p}). \tag{15}$$

The linear spaces Σ and $\overline{\Sigma}$ equipped with Euclidean metrics are in fact $E^2$and $\overline{E}^2$, respectively. Then by (11')

$$\|D\|_{\mathcal{M}} = \|D\|_{\mathcal{M}^2} = \pi \frac{\|\mathbb{D}\|_E}{\|\mathbb{B}^2\|_E}, \tag{15'}$$

and since $d\varphi$ is a linear mapping which keeps the ratio of areas we obtain

$$\|D\|_{\mathcal{M}} = \pi \frac{\|(d\varphi)\mathbb{D}\|_{\overline{E}}}{\|(d\varphi)\mathbb{B}^2\|_{\overline{E}}}.$$

(Strictly speaking, $d\varphi$ should be replaced here by $(d\varphi)^* := \overline{\Psi} \circ d\varphi \circ \Psi^{-1}$; $\overline{\Psi} : \overline{\Sigma} \to \overline{E}^2$, $\overline{y} \mapsto (\overline{y}^1, \overline{y}^2)$.)

Finally, in consequence of (15'), we obtain

$$\|D\|_{\mathcal{M}} = \pi \frac{\|\overline{\mathbb{D}}\|_{\overline{E}}}{\|\overline{\mathbb{B}}^2\|_{\overline{E}}} = \|\overline{D}\|_{\overline{\mathcal{M}}}, \quad \overline{D} = (d\varphi)D.$$

This means that φ keeps (2-dimensional) area. (It is easy to see that an isometry keeps also the k-dimensional $(1 \le k \le n)$ area.)

According to (12) the measure $\measuredangle_{\mathcal{M}}\alpha$ is defined by area. Thus, if φ keeps area, then φ keeps angle too. Indeed, we know that $\measuredangle_{\mathcal{M}}\alpha \overset{(12)}{=} 2\epsilon\|D\|_{\mathcal{M}}$ and $\measuredangle_{\overline{\mathcal{M}}}\overline{\alpha} = 2\epsilon\|\overline{D}\|_{\overline{\mathcal{M}}}$, where $\overline{\alpha} = (d\varphi)\alpha$. Then, from $\|D\|_{\mathcal{M}} = \|(d\varphi)D\|_{\overline{\mathcal{M}}}$ (φ keeps area) we obtain $\measuredangle_{\mathcal{M}}\alpha = \measuredangle_{\overline{\mathcal{M}}}\overline{\alpha}$, that is φ keeps angle too.

B) Suppose that φ keeps area (a) and angle (n). In the following, the notations (a), (n) mean these assumptions. Let us denote $(d\varphi)\mathcal{I}^2(p) =: \widetilde{\mathcal{I}}(p)$. Then $\overline{F}^n$ determines the indicatrix $\overline{\mathcal{I}}^2(\overline{p}) = \overline{\mathcal{I}}(\overline{p}) \cap \overline{\Sigma}$. We denote $(d\varphi)^{-1}\overline{\mathcal{I}}^2(\overline{p}) =: \widehat{\mathcal{I}}(p) \subset \Sigma$ and $\widehat{\mathcal{I}}(p) \cap A =: \widehat{D}$ (A is the domain of the angle $\measuredangle\alpha(a,b)$).

The application $d\varphi$ maps a, b into $\widetilde{a}$, $\widetilde{b}$, and the domain A into $\widetilde{A}$. Furthermore $\widetilde{D} := \widetilde{A} \cap \widetilde{\mathcal{I}}(\overline{p})$, and $\overline{D} := \widetilde{A} \cap \overline{\mathcal{I}}^2(\overline{p})$. Moreover $(d\varphi)^{-1}$ takes $\widetilde{a}$, $\widetilde{b}$ into a, b respectively, as well as $\widetilde{\mathcal{I}}(\overline{p})$ into $\mathcal{I}^2(p)$.

By (a) and (n) we obtain

$$\|D\|_{\mathcal{M}} \overset{(a)}{=} \|\widetilde{D}\|_{\overline{\mathcal{M}}}$$

$$\|D\|_{\mathcal{M}} \overset{(n)}{=} \|\overline{D}\|_{\overline{\mathcal{M}}} \overset{(a)}{=} \|\widehat{D}\|_{\mathcal{M}}.$$

Thus we obtain

$$(16) \qquad \frac{\|\mathbb{D}\|_E}{\|\mathbb{B}^2(p)\|_E} = \|D\|_{\mathcal{M}} = \|\widehat{D}\|_{\mathcal{M}} = \frac{\|\widehat{\mathbb{D}}\|_E}{\|\mathbb{B}^2(p)\|_E} \Longrightarrow \|\mathbb{D}_E\| = \|\widehat{\mathbb{D}}\|_E.$$

Let c be a ray in Σ, with $c \cap \mathcal{I}^2(p) = C$, and $c \cap \widehat{\mathcal{I}} = \widehat{C}$. Suppose that at a point p there exists a c such that $C \neq \widehat{C}$, and let us say that $\widehat{C}$ is outside B_p^2. Then, because of the continuity, there exists a ray $h(\neq c)$, such that the whole arc $\widehat{C}\widehat{H}$ ($\widehat{H} := h \cap \widehat{\mathcal{I}}$) is outside B_p^2. Then the segment $D(c,h)$ of B_p^2 is a proper part of the segment $\widehat{D}(c,h)$ bounded by c, h and $\widehat{\mathcal{I}}$. Then $\|D(c,h)\|_E < \|\widehat{D}(c,h)\|_E$, which contradicts (16). Therefore we must have $C = \widehat{C}$, for $\forall\, c, p$. Then $\partial B_p^2 = \mathcal{I}^2(p) = \widehat{\mathcal{I}}$. Consequently we obtain $(d\varphi)\mathcal{I}^2(p) = \widetilde{\mathcal{I}} = (d\varphi)\widehat{\mathcal{I}} = \overline{\mathcal{I}}^2(\overline{p})$, $\forall p \in M$. This yields (14), and thus φ is an isometry.

§ 3. Deviation of Finsler spaces from Riemannian spaces

There are known several conditions which imply the reduction of an F^n to a Riemannian space V^n. Such a condition is the vanishing of the Cartan tensor C_{ijk} or the constantness of the distortion $\tau(x,y)$ [S3]. Many other quantities, such as the S-curvature [S2], Landsberg curvature, Cartan torsion, etc. can be coupled with this problem. Also, recall that a Finsler space is a Riemann space iff the indicatrices are ellipsoids. We want to present conditions expressed by the Minkowskian angle which imply the reduction of the indicatrices to ellipsoids.

We consider a Finsler space $F^n = (M, \mathcal{F})$ and its tangent space, as a Minkowski space $\mathcal{M}^n = (T_pM, \mathcal{F}(p,y))$, and a 2-dimensional linear subspace Σ of $T_{p_0}M$. Here $T_{p_0}M$ can be identified with a vector space $\mathcal{V}^n$ or the coordinate space $R^n(x)$ which can be equipped with a Euclidean metric, yielding $E^n(x)$. Let B^n be the indicatrix body of $\mathcal{M}^n$, $\partial B^n = \mathcal{I}$ the indicatrix surface, and $\mathcal{I} \cap \Sigma = \mathcal{I}^2$ is the indicatrix of the $\mathcal{M}^2$ induced by $\mathcal{M}^n$ on Σ. If F^n is a Riemannian space V^n, then $\mathcal{M}^2$ is a Euclidean space, and $\mathcal{I}$ reduces to an ellipse. In this case Minkowskian and Euclidean angle are the same, $\measuredangle_{\mathcal{M}}\alpha(a,b) = \measuredangle_E\alpha(a,b)$, and it equals π iff α is a straight angle, i.e. its two legs a, b are two half lines of a straight line $g : a \cup b = g$. As we have seen

$$\measuredangle_{\mathcal{M}}\alpha(a,b) = \pi \quad \text{if} \quad a \cup b = g, \qquad \forall\, g \subset \Sigma \subset T_{p_0}M, \quad \forall\, p \in M$$

is necessary for an F^n to be a V^n.

Given an arbitrary ray $a \subset \Sigma$. Let $\overline{a}$ be the other ray, such that $a \cup \overline{a}$ is a line g, and let $b \subset \Sigma$ be such that $\forall_{\mathcal{M}}\alpha(a,b) = \pi$. Then b depends on a, and $|\measuredangle_{\mathcal{M}}(b,\overline{a})| =: f(a) \geq 0$ is a function of $a \subset \Sigma$. It follows that $f(a) = 0$, $\forall\, a \subset \Sigma$ is necessary for $F^n = V^n$. Let (r,ν) be a Minkowskian polar coordinate system in Σ, where $r = \mathcal{F}(p_0, y)$ for a $y \in \Sigma$, and $\nu = \measuredangle_{\mathcal{M}}(0y, d_0)$ the Minkowskian angle between the ray $0y$ and an initial direction (initial ray) d_0. Then

(17)

$$\mathcal{G}(p,\Sigma) := \int_{\nu=0}^{2\pi} \overline{f}(\nu)d\nu = 0, \;\; f(a(\nu)) \equiv \overline{f}(\nu), \quad \forall \Sigma \subset T_{p_0}M, \quad \forall\, p_0 \in M$$

is necessary for $F^n = V^n$. This and sec. 1 of this Chaper yield

Proposition 4. *The condition* (17) *is equivalent to the following:* 1) $b = \overline{a}, \forall\, a$, 2) $\measuredangle_{\mathcal{M}}(a,\overline{a}) = \pi, \forall\, a$, 3) *any* g *bisects* $\mathcal{I}^2$, 4) $\mathcal{I}(p)$ *is symmetric,* 5) F^n *is absolutely homogeneous.*

All these are necessary for a Finsler space to be Riemannian. Hence, $\mathcal{G}(p,\Sigma) \geq 0$ measures the deviation of an F^n from being absolutely homogeneous in $\Sigma \subset T_pM$.

We want to obtain sufficient conditions for $F^n = V^n$. Our tool for this will be the difference between Minkowski orthogonality and transversality. Since the properties listed in Proposition 4 are necessary, we suppose that the indicatrices are symmetric. Let $g = a \cup \overline{a}$, $h = b \cup \overline{b}$ be lines and rays in $\Sigma \subset T_pM$, where $\mathcal{M}_p^2 = (T_pM, \mathcal{F}(p,y))$, and F^n as above. Our considerations will be restricted to Σ. Because of the symmetry of $\mathcal{I}^2(p)$ the Minkowskian perpendicularities $a \bot_{\mathcal{M}} b$, i.e. $\measuredangle_{\mathcal{M}}\alpha(a,b) = \frac{\pi}{2}$, $a \bot_{\mathcal{M}} \overline{b}$, $\overline{a} \bot_{\mathcal{M}} b$, $\overline{a} \bot_{\mathcal{M}} \overline{b}$ are equivalent. They mean $g \bot_{\mathcal{M}} h$. So, in the case of the symmetry of $\mathcal{I}^2(p)$ we can speak of the perpendicularity of lines in place of rays. Denoting by $g^{\bot}$ a line perpendicular to g, we obtain $(g^{\bot})^{\bot} \parallel g$. – Another notion is transversality. Let

$g \cap \mathcal{I}^2(p) = \mathcal{G}, \mathcal{G}'$. Then the tangent $T_{\mathcal{G}}\mathcal{I}^2(p) =: g^*$ is called transversal to g. Because of the symmetry of $\mathcal{I}^2(p)$, $T_{\mathcal{G}'}\mathcal{I}^2(p) =: (g')^*$ is parallel to g^*. Also, any line parallel to g^* is said to be transversal to g. So we can speak of transversality of a direction to another direction. Nevertheless, this relation is not symmetric, that is the direction transversal to g^* is in general not g, i.e. $(g^*)^* \nparallel g$. The relation

$$(g^*)^* \parallel g, \quad \forall\, g \subset \Sigma \tag{18}$$

means that in $\mathcal{M}_p^2$ the transversality operation * is involutive.

A strictly convex, closed, differentiable curve with O in its interior, and with the property (18) is called a Radon curve. Every ellipse is a Radon curve, but not conversely. This shows that if the indicatrices of an $F^2 = (M, \mathcal{F})$ satisfy (18) at every point $p \in M$, then these indicatrices need not be ellipses, and thus F^2 needs not be a Riemannian space $V^2 = (M, g)$.

We claim that if $n > 2$ and (18) is satisfied in every Σ with respect to $\mathcal{I}^2(p)$, then F^n is a V^n. Indeed, under these conditions every $\mathcal{I}^2(p) = \mathcal{I}(p) \cap \Sigma$ is a Radon curve. Then in T_pM every cylinder osculating to B_p^n osculates along a planar curve [T2]. In this case, according to W. Blaschke ([B1] pp. 157–159), every $\mathcal{I}(p)$ is an ellipsoid, and thus $F^n = V^n$.

If $F^n = V^n$, then $\forall\, \mathcal{I}^2(p)$ is an ellipse, and (18) is satisfied. But $(g^\perp)^\perp \parallel g$ is always true if $\mathcal{I}^2(p)$ is symmetric. Hence in case of $F^n = V^n$ $g^\perp \parallel g^*$ for any g.

If $\measuredangle_{\mathcal{M}}\alpha(g^*, g^\perp) = 0$, i.e. if $g^* = g^\perp$, then (18) hold good for $(g^\perp)^\perp \parallel g$ is true. Hence

$$\begin{gathered} K(p,\Sigma) := \int_0^\pi |\measuredangle_{\mathcal{M}}\alpha(g^*(\nu), g^\perp(\nu))|\,d\nu = 0, \\ g \subset \Sigma, \quad \forall\, \Sigma \subset T_pM,\ \forall\, p \in M \end{gathered} \tag{19}$$

is sufficient for $F^n = V^n$. Conversely, (19) is always satisfied in a Riemannian space V^n. Thus we obtain

Theorem 7. *An absolutely homogeneous Finsler space F^n, $n > 2$ reduces to a Riemann space V^n if and only if $K(p, \Sigma) = 0$, $\forall \Sigma \subset T_{p_0}M$, $\forall\, p \in M$.*

The deviation of an absolutely homogeneous F^n from being a Riemannian space on $\Sigma \subset T_{p_0}M$ can be measured by $K(\Sigma)$. Thus $K(\Sigma)$ can be considered as a kind of sectional curvature. The deviation at a point $p_0 \in M$ can be measured by the integral

$$\mathcal{G}(p_0) = \frac{1}{\int_{\mathcal{G}_{n,2}} d\sigma} \int_{\mathcal{G}_{n,2}} K(\Sigma) d\sigma \geq 0,$$

where $\mathcal{G}_{n,2}$ is the Grassmann manifold of the 2-dimensional linear subspaces of $T_{p_0}M$, and $d\sigma$ is a positive measure on $\mathcal{G}_{n,2}$, such that $\int_{\mathcal{G}_{n,2}} d\sigma$ is finite and invariant with respect to linear transformations in $T_{p_0}M$. The deviation of F^n from being Riemannian on M (the global case) can be measured by the integral

$$H(M) = \frac{1}{\int_M d\mu} \int_M \mathcal{G}(x) d\mu \geq 0,$$

where $d\mu$ is the Finsler volume element, and $\int_M d\mu$ is supposed to be finite.

III. Metrical connections in TM for an F^n

Besides the metric another very important notion of a metrical differential geometry, especially of Finsler geometry, is parallelism. In order to develope Finsler geometry in a way more or less similar to Riemannian geometry (covariant derivation, curvature theory, etc.) a metrical and linear (or at least homogeneous (see [KB])) connection is indispensable. However for Finsler spaces there do not exist, in general, linear mappings of the tangent spaces taking indicatricies into indicatrices, consequently there do not exist linear metrical connections $\Gamma(p)$ in the tangent bundle TM, in contrast to Riemann spaces. To solve this problem line-elements $(p, y) \in TM$ and Finsler vectors $\xi(p, y) \in VTM \subset TTM$ were introduced. This allowed the introduction of a metrical linear connection $\Gamma(p, y)$ in the Finsler vector bundle VTM. Nevertheless the dimension of the base space TM of this vector bundle is $2n$, while the rank of the bundle is n. This is sometimes unconvenient, and makes the apparatus of Finsler geometry more complicated than that of Riemannian geometry.

Euclidean, Riemannian and Minkowskian spaces, as special Finsler spaces, allow metrical linear connections. We want investigate which other Finsler spaces allow still linear metrical connections in TM, and what are the special features of their geometry. Also we touch upon several related questions, such as affine deformation, and locally Minkowski spaces.

§ 1. Affine deformation, locally Minkowski spaces

Let us consider a Finsler space $F^n = (M, \mathcal{F})$. Because of the positive homogeneity of $\mathcal{F}$, the knowledge of the indicatrix bundle $\pi : I \to M$, $\pi^{-1}(p) = \mathcal{I}(p)$ is equivalent to the knowledge of the structure function $\mathcal{F}(p, y)$. Thus we can write (M, I) in place of $(M, \mathcal{F})$.

Let $\mathcal{A} = \{a(p)\}$ be a field of linear automorphisms (i.e. centroaffine transformations) of the tangent spaces:

$$a(p) : T_pM \to T_p(M).$$

It is easy to see that $\mathcal{A}$ of class C^∞ exist over any paracompact manifold. Moreover, if $\mathcal{A}$ is given over a chart of M, then it can be extended to the whole M in a C^∞ manner. Then

$$a(p)\mathcal{I}(p) = \overline{\mathcal{I}}(p), \tag{20}$$

and we obtain the *affinely deformed* Finsler space $\overline{F}^n = (M, \overline{I}) = (M, \overline{\mathcal{F}}) = \mathcal{A}F^n$. The relation (20) is a kind of gauge transformation.

Given two Riemannian spaces $V_0^n = (M, Q_0)$ and $V^n = (M, Q)$ (where Q_0 and Q denote the indicatrices) over the same base manifold, clearly there exist $a(p)$, such that $a(p)Q_0 = Q$. So every V^n is the affine deformation of a single V_0^n (over the same manifold). Nevertheless this is not true anymore for Minkowski-, locally Minkowski- and Finsler-spaces.

Locally Minkowski spaces $\ell\mathcal{M}^n$ play important role in the search for Finsler spaces admitting metrical linear connections $\Gamma(p)$ in TM. An $F^n = (M, \mathcal{F})$ is locally Minkowskian if $\forall p_0 \in M$ has a chart $U_{p_0}(x)$ in which $\mathcal{F}(p, y)$ restricted to $U_{p_0} : \mathcal{F}(p, y) \restriction U_{p_0} = \mathcal{F}(y)$ is independent of p. This coordinate system (x) is called *adapted.* If (x) is adapted on U, then any other coordinate system (z) on U is adapted iff the transformation $(x) \longleftrightarrow (z)$ is linear ([M] p. 158). – If M has an open covering by local charts $M = \cup_\alpha U_\alpha(x_\alpha)$ with coordinate system (x_α), such that $(x_\alpha) \longleftrightarrow (x_\beta)$ is linear on $U_\alpha \cap U_\beta$, then M will be called *affine differentiable manifold* (affine manifold, for short).

Theorem 8. *A manifold M admits a locally Minkowski structure iff M is an affine manifold.*

Proof. **A)** Suppose that M is an affine manifold. We show that there exists a Finsler metric $\mathcal{F}(p, y)$, such that $(M, \mathcal{F})$ is a locally Minkowski space. Let $U_\alpha(x_\alpha)$, $\alpha \in A$ be an atlas of M with coordinate systems (x_α) on U_α, such that the transitions $x_\alpha \longleftrightarrow x_\beta$ are linear. Consider two points $r, q \in M$, and a curve $c(t)$, $0 \le t \le 1$, connecting $c(0) = r$ and $c(1) = q$. The curve $c(t)$ is a closed set covered by U_α, of which already finitely many cover $c(t)$. We can omit part of them, then truncate and renumber them in such a way that for the remaining $U_i,\ i = 1, 2, \dots, N$ we have $r \in U_1,\ q \in U_N,\ U_i \cap U_{i+1} \neq \emptyset$, and $U_i \cap U_j = \emptyset,\ j = 1, \dots, N$, but $j \neq i$ and $j \neq i+1$. Each of these U_i carries a coordinate system (x_i), such that the transition between them is linear.

On $U(x_1) \cap U(x_2) = U_{12}$ the transition between x_1^j and x_2^k is $x_1^j = a_\ell^j x_2^\ell + a^j = f^j(x_2)$. It means also a coordinate transformation $(x_2) \to (x_1)$ on $U_2(x_2)$. Thus we can extend the coordinate system (x_1) from U_1 to $U_1 \cup U_2$. This process can be continued, and the coordinate system (x_1) can be extended over $\cup_{i=1}^N U_i = U$. From now on (x_1) on U will be denoted by (x).

Let $\mathcal{I}(r) \subset T_x M$ be an indicatrix surface in $T_r U$. Then $\mathcal{I}(r)$ corresponds to a regular, positively homogeneous, strongly convex Minkowski functional $\mathcal{F}_1(y)$ i.e. $\mathcal{I}(r) \Longleftrightarrow \mathcal{F}_1(y),\ y \in T_r U$.

We extend this $\mathcal{F}_1(y)$ over the whole $U(x)$ by

$$\mathcal{F}(x, y) := \mathcal{F}_1(y) \quad x \in U,\ y \in T_x U. \tag{21}$$

This $\mathcal{F}(x, y)$ is independent of x. Thus we obtain a Minkowski space on U.

Choosing two other points $\overline{r} \in U$ and $\overline{q} \notin U$, and a curve $\overline{c}(t)$ joining $\overline{r}$ with $\overline{q}$, we can repeat our previous construction. Using the previous notation, but with a dash $^-$, we obtain a domain $\overline{U} \supset \overline{c}(t)$ (corresponding to $U \supset c(t)$), equipped with a coordinate system $(\overline{x})$, and we construct an $\overline{\mathcal{F}}(\overline{x}, \overline{y})$ on $\overline{U}$ such that $\overline{\mathcal{F}} = \mathcal{F}$ on $U \cap \overline{U}$. The function $\overline{\mathcal{F}}$ in the coordinate system $(\overline{x})$ is independent of $\overline{x}$. If $U \cap \overline{U}$ is homeomorphic to R^n, then we can introduce on $U \cup \overline{U}$ a common coordinate system (x). On $U \cap \overline{U} = U_{12}(\overline{x})$ we have $\overline{x}^k = a_\ell^k x^\ell + a^k = f^k(x)$. Similarly $\overline{y}^k = a_\ell^k y^\ell$ for $\overline{y} \in T_{\overline{x}} U_{12}$, or in a matrix form $\overline{y} = J(\overline{x}, x) y$, where $J(\overline{x}, x)$ is the Jacobi matrix $\frac{\partial(\overline{x})}{\partial(x)} = a_\ell^k$, which is constant. Then we define $\overline{\mathcal{F}}$ on U_{12} as

$$\overline{\mathcal{F}}(p, \overline{y}) \restriction U_{12} \equiv \overline{\mathcal{F}}_{12}(\overline{x}, \overline{y}) = \overline{\mathcal{F}}(f(x), J(\overline{x}, x) y) := \mathcal{F}(x, y) = \mathcal{F}_1(y). \tag{22}$$

Nevertheless, $\mathcal{F}$ is independent of x on $U_{12} \subset U$. Hence also $\overline{\mathcal{F}}_{12}$ is independent of $\overline{x}$. We denote it by $\overline{\mathcal{F}}_2(\overline{y})$, and we define

$$\overline{\mathcal{F}}(\overline{x}, \overline{y}) := \overline{\mathcal{F}}_2(\overline{y}) \quad \text{on } \overline{U}(\overline{x}).$$

Thus $\overline{\mathcal{F}}_2(\overline{y}) = \mathcal{F}_1(y)$ on U_{12}. Then by $\mathcal{F}(p, y) \restriction \overline{U} \backslash U := \overline{\mathcal{F}}_2(J(\overline{x}, x) y)$ we can extend $\mathcal{F}$ to $U \cup \overline{U}$. By (22) also $\overline{\mathcal{F}} = \mathcal{F}$ on U_{12}. So the constructed $\mathcal{F}$ is a Minkowski functional on $U \cup \overline{U}$.

If $U \cap \overline{U}$ consists of several disjunct domains $U_1^*, U_2^*, \dots$ homeomorphic to R^n, then the introduction of a common coordinate system (x) on $U \cup \overline{U}$ may be impossible. The coordinates of $p \in (U \cap \overline{U}) \setminus U_1^* = U^*$ on $U(x)$ may differ from the coordinates obtained for them on $\overline{U}(\overline{x}(x))$. However $\overline{\mathcal{F}}(\overline{x}, \overline{y}) = \overline{\mathcal{F}}_2(\overline{y})$, on $\overline{U}$, $\overline{\mathcal{F}}_2(\overline{y}) = \mathcal{F}_1(y)$ on U_{12}, and $\mathcal{F}_1(y) \overset{(22)}{=} \mathcal{F}(p, y)$ on U. Thus $\overline{\mathcal{F}}(\overline{p}, \overline{y}) = \mathcal{F}(p, y)$ also on U^*. Moreover $\mathcal{F} \restriction U = \mathcal{F}_1(y)$ and $\mathcal{F} \restriction \overline{U} = \overline{F}_2(\overline{y})$ are independent of p, and the transition functions on $U \cap \overline{U}$ are linear, since the Jacobi matrix $J(\overline{x}, x)$ is constant because of the supposed affine character of M. So $\mathcal{F}$ is again a Minkowski functional on $U \cup \overline{U}$.

Continuing this construction with further points $r^* \in U \cup \overline{U}$, $q^* \notin U \cup \overline{U}$ and connecting curves $c^*(t) \subset \overline{\overline{U}}$ until $U \cup \overline{U} \cup \overline{\overline{U}} \cdots = M$, we obtain a regular, positively homogeneous, strongly convex $\mathcal{F}(p, y)$, $p \in M$, such that $\mathcal{F}(x, y) \restriction U_\alpha(x_\alpha)$ is independent of x_α $\forall \alpha \in A$. Thus $(M, \mathcal{F})$ is a locally Mikowski space over the affine differentiable manifold M.

B) If M admits a locally Minkowski structure $(M, \mathcal{F}) = \ell\mathcal{M}^n$ then the charts $U_\alpha(x_\alpha)$ with adapted coordinate systems (x_α) form an open cover of M, and $(x_\alpha) \restriction U_{\alpha\beta}$ and $(x_\beta) \restriction U_{\alpha\beta}$ are adapted coordinate systems on $U_{\alpha\beta} = U_\alpha \cap U_\beta$. Thus $(x_\alpha) \restriction U_{\alpha\beta} \longleftrightarrow (x_\beta) \restriction U_{\alpha\beta}$ are linear transformations. Then M is an affine manifold. Q.E.D.

Not every manifold is an affine manifold. Consider the Euclidean sphere $S^2 \subset E^3$ covered by two charts $H(x)$ and $\overline{H}(\overline{x})$ (e.g. two hemispheres, one of them extended a little beyond the equator). On $H \cap \overline{H}$ $\frac{\partial}{\partial \overline{x}^i} = a_i^k \frac{\partial}{\partial x^k}$, $\det |a_i^k| \neq 0$. If S^2 with $H(x)$, $\overline{H}(\overline{x})$ is an affine manifold, then $a_i^k = \text{const.}$, and thus $y := \frac{\partial}{\partial \overline{x}^1}$ on $\overline{H}$ and $y := a_1^k \frac{\partial}{\partial x^k}$ on H is a continuous, never vanishing vector field on S^2, what is impossible. Hence $a_i^k \neq \text{const.}$, and S^2 with $H(x)$ and $H(\overline{x})$ is no affine manifold. This is a concrete simple example. A result of Bao and Chern ([BC], p. 250) gives still more. According to this a compact, boundaryless manifold with an Euler characteristic $\lambda \neq 0$ admits no locally Minkowski structure, and then, by Theorem 8 it is not an affine manifold.

§ 2. Linear metrical connections in TM

Every Riemannian space over a paracompact manifold admits linear metrical connections. Also Minkowski spaces do so. It is not difficult to see that locally Minkowski spaces also belong to this family. All these are non-Riemannian special Finsler spaces. What other Finsler spaces do still allow linear metrical connections in TM?

Theorem 9. *Affinely deformated locally Minkowski spaces $\mathcal{A}\ell\mathcal{M}^n$ admit linear metrical connections in TM. Moreover, if a Finsler space $F^n = (M, \mathcal{F})$ admits a linear metrical connection in TM, then it is the affine deformation of a locally Minkowski space $\ell\mathcal{M}^n$, provided its base space M is an affine manifold.*

Proof. **A)** Let the Finsler space $F^n = (M, \overline{I})$ be an affine deformation of a locally Minkowski space $\ell\mathcal{M}^n = (M, I) : F^n = \mathcal{A}\ell\mathcal{M}^n$. Then $\overline{\mathcal{I}}(p) \overset{(20)}{=} a(p)\mathcal{I}(p)$, $a(p) \in \mathcal{A}$, $\forall p \in M$, where $\overline{\mathcal{I}}(p)$ and $\mathcal{I}(p)$ are indicatrices of F^n and $\ell\mathcal{M}^n$ respectively. Let p_0, q_0 be two points of M, and $c(t)$, $0 \leq t \leq 1$, $c(0) = p_0$, $c(1) = q_0$ a curve connecting them. Since $\ell\mathcal{M}^n$ is a locally Mikowski space, then M is an affine manifold (Theorem 8), and thus there exists a domain $U \ni p_0, q_0$, such that $c(t) \subset U$ with an adapted coordinate system (x) on it (see the previous section). We consider (x) as an affine coordinate system on U, and $t(x_0, x)$ as the parallel translation in TU:

$$t(x_0, x) : T_{x_0}U \to T_xU, \quad x_0 \approx p_0, x \approx p \in U.$$

Then $t(x_0, x)\mathcal{I}(x_0) = \mathcal{I}(x)$, is an adapted coordinate system for (x).

Furthermore we obtain

$$(a(x) \circ t(x_0, x) \circ a^{-1}(x_0))\overline{\mathcal{I}}(x_0) \equiv g(x_0, x)\overline{\mathcal{I}}(x_0) = \overline{\mathcal{I}}(x),$$

$$g(x_0, x) := a(x) \circ t(x_0, x) \circ a^{-1}(x_0).$$

Let $e_i(x_0) = e_i^j(x_0)\frac{\partial}{\partial x^j} \in \overline{\mathcal{I}}(x_0)$, $i, j = 1, 2, \ldots, n$ be a frame (x_0, e_0) of $T_{x_0}U = T_{x_0}M$. Then $g(x_0, x)e_i(x_0) =: e_i(x) \in \overline{\mathcal{I}}(x)$, and (x, e) is a frame of T_xU. The frame bundle P over U is

$$\pi : P \to U, \quad \pi^{-1}(x) \cong \mathcal{G}\ell(n) = \{e, f, g, \ldots\}, \quad e = (e_j^i), \quad P = \{(x, e)\}.$$

Let $\sigma : U \to P$, $x \mapsto (x, e(x)), e(x) = g(x_0, x)e(x_0)$ be a section of P. Then $\varrho_g(x, e) := (x, ge)$ (ge is the matrix multiplication) is a fiber preserving transitive transformation in P, and

$$d\varrho_g\big(T_{(x,e)}\sigma\big) =: H(x, ge) \in T_{(x,ge)}P$$

yields a horizontal distribution H in TP, and thus induces a linear connection Γ_P on P. This Γ_P determines a linear connection $\Gamma(x)$ in TM (in the vector bundle associated to P). In this connection the parallel translate of $e_i(x_0)$ along $c(t)$, i.e. $\mathcal{P}_c^\Gamma e_i(x_0)$, is $g(x_0, x)e_i(x_0) = e_i(x)$, and $g(x_0, x)\overline{\mathcal{I}}(x_0) = a(x) \circ t(x_0, x) \circ a^{-1}(x_0)\overline{\mathcal{I}}(x_0) = a(x) \circ t(x_0, x)\mathcal{I}(x_0) =$

$a(x)\mathcal{I}(x) = \overline{\mathcal{I}}(x)$. Thus $\Gamma(x)$ is also metrical with respect to $F^n = \mathcal{A}\ell\mathcal{M}^n$.

The distribution H is integrable, therefore σ is an integral manifold of H. Then the obtained $\Gamma(x)$ is integrable too, and each $e_i(x) = g(x_0, x)e_i(x_0)$ is an absolute parallel vector field, i.e. an integral manifold of the distribution determined by $\Gamma(x)$ in TM. Now the curvature $R(x)$ of $\Gamma(x)$ vanishes. This can be seen also from the fact that at the construction of the absolute parallel vector field $e_i(x)$, and thus also at parallel translation of $e_i(x_0)$, no route connecting x_0 and x was used.

There are also other curves $\widehat{c}(t)$ connecting p_0 and q_0, and to each of them belongs a $\widehat{U}$ with an adapted coordinate system $(\widehat{x})$, and a connection $\widehat{\Gamma}(\widehat{x})$ as constructed above. We show that $\Gamma = \widehat{\Gamma}$ on $U \cap \widehat{U}$. This is true if $e_i(q_0)$ constructed on U equal $\widehat{e_i}(q_0)$ constructed on $\widehat{U}$, and thus $\sigma(q_0) = \widehat{\sigma}(q_0)$, $q_0 \in U \cap \widehat{U}$. Nevertheless $U \cap \widehat{U} = U^*$ may have disjunct components U_j^*, i.e. $U^* = U_1^* \cup U_2^* \cup \dots$, $p_0 \in U_1^*$, $q_0 \in U_j^*$. Since both (x) and $(\widehat{x})$ are adapted coordinate systems on U_1^*, after an appropriate linear transformation $(ltr) : (\widehat{x}) \to (\overline{x})$ on $\widehat{U}$ we obtain $x = \overline{x}$ on U_1^*. Also we obtain a new coordinate system on $\widehat{U}$. We denote $\widehat{U}$ with the new coordinates by $\overline{U}(\overline{x})$. This does not mean that $\overline{x} = x$ on the other U_j^* $(j \neq 1)$. Nevertheless M is an affine manifold, and thus the Jacobi matrix $J(x, \overline{x}) = \left(\frac{\partial x}{\partial \overline{x}}\right)$ is the same on the whole U^*. Moreover, it is the unit matrix on U_1^*, and therefore $J(x, \overline{x})$ too is the unit matrix on U^*. We know that $e_i(q)$ constructed from $e_i(p_0)$ on $U(x)$ is

$$e_i(q) = a(q) \circ t(p_0, q) \circ a^{-1}(p_0)e_i(p_0), \tag{23}$$

and $\overline{e}_i(q)$ constructed from $e_i(p_0)$ on $\overline{U}(\overline{x})$ is

$$\overline{e}_i(q) = a(q) \circ \overline{t}(p_0, q) \circ a^{-1}(p_0)e_i(p_0), \tag{24}$$

where $\overline{t}$ means parallel translation on $\overline{U}(\overline{x})$. Both parallel translations t and $\overline{t}$ keep the components of the parallel translated vectors. Thus (23) an (24) have the same components in T_qU and $T_q\overline{U}$. Since $J(x, \overline{x})$ is the unit matrix on U^*, we obtain $e_i(q) = \overline{e}_i(q)$. Consequently $\Gamma = \overline{\Gamma} \equiv \widehat{\Gamma}$.

The above said is true for any $q \in M$. Thus we obtain an unambiguous linear metrical connection $\Gamma(x)$ in TM for $F^n = \mathcal{A}\ell\mathcal{M}^n$.

B) Suppose that a Finsler space $F^n = (M, \overline{I})$ over an affine manifold M admits a linear metrical connection $\Gamma(x)$ in TM. Then there exists for M an atlas $\{U_\alpha(x_\alpha)\}$, such that the transitions $(x_\alpha) \longleftrightarrow (x_\beta)$ are linear. We show that this F^n is the affine deformation of a locally Minkowski space $\ell\mathcal{M}^n = (M, I)$.

First we give the $\ell\mathcal{M}^n$. Choose a point $r \in M$, and take the indicatrix $\overline{\mathcal{I}}(r)$ of F^n at r. Starting with this as $\mathcal{I}(r) \in I$, the construction applied in the proof of Theorem 8 part A yields a locally Minkowski space $\ell\mathcal{M}^n = (M, I)$. As we have seen, for any $p \in M$ there is a curve $c(t)$ from r to p, and a domain $U \supset c(t)$ with a coordinate system (x) adapted with respect to $\ell\mathcal{M}^n$. Let $t(p,r)$ denote the parallel translation in (x) from p to r. Then $t(p,r)\mathcal{I}(p) = \mathcal{I}(r) = \overline{\mathcal{I}}(r)$, since (x) is adapted. Let $\mathcal{P}_c^\Gamma$ be the parallel transport along c according to the linear metrical connection Γ of F^n. Then $\mathcal{P}_c^\Gamma \circ t(p,r) : T_pM \to T_pM$ is an affine transformation $a(p) : \overline{\mathcal{I}}(p) \to \mathcal{I}(p)$. Thus we obtain an $\mathcal{A} = \{a(p)\}$ on M, such that $a(p)\mathcal{I}(p) = \overline{\mathcal{I}}(p)$. In the adapted coordinate system (x) $\overline{\mathcal{I}}$ is independent of x, and thus of class C^∞. Also $\mathcal{I}(x) \in C^\infty$. Hence also $\mathcal{A} = \{a(p)\}$ are C^∞. This means that $F^n = \mathcal{A}\ell\mathcal{M}^n$. Q.E.D.

The content of this section is closely related to the results of Y. Ichijyo ([I1–3]). He investigated Finsler spaces modeled on Minkowski spaces. Nevertheless he used an approach completely different from the ours.

We have seen that any affinely deformated locally Minkowski space $\mathcal{A}\ell\mathcal{M}^n$ admits a linear metrical connection with vanishing curvature $R(x)$. This means that we have

Proposition 5. *Any affinely deformated locally Minkowski space $\mathcal{A}\ell\mathcal{M}^n$ is parallelizable.*

Given an affinely deformated locally Minkowski space $\mathcal{A}\ell\mathcal{M}^n$, with $\mathcal{A} = \{a(p)\}$, one can determine on an adapted coordinate sytem (x) the local components of the just constructed linear metrical connection $\Gamma(x)$. The parallel translate of a vector $\xi_0 = \xi(x(t_0))$ along a curve $x(t)$ according to the metrical linear connection $\Gamma(x)$ is

$$\overline{\xi}(t) = a(x(t)) \circ t(x_0, x(t)) \circ b(x_0)\xi_0,$$

where $b(x)$ is the inverse matrix of $a(x)$, and $x_0 = x(t_0)$. In components it is $\overline{\xi}^i(t) = a_k^i(x(t))b_j^k(x_0)\xi_0^j$, since the parallel translation t does not alter the vector components. Then

$$\frac{d\overline{\xi}^t}{dt}(t_0) = \frac{\partial a_k^i}{\partial x^r}(x(t_0))\frac{dx^r}{dt}(t_0)b_j^k(x_0)\xi_0^j$$

must have the form

$$= \Gamma_j{}^i{}_r(x_0)\xi_0^j\frac{dx^r}{dt}(t_0), \quad \forall\, x_0 = x(t_0),$$

where ${\Gamma_j}^i{}_r$ are the components of the linear metrical connection $\Gamma(x)$. Hence, in an adapted coordinate system (x) we obtain

$$ {\Gamma_j}^i{}_r(x) = \frac{\partial a^i_k}{\partial x^r}(x) b^k_j(x). \tag{25} $$

We remark that there are Finsler spaces other than affine deformations of locally Minkowski spaces, which admit linear metrical connections in TM. We know that every Riemannian space V^n over a paracompact manifold admits such connections. Nevertheless not every V^n is the affine deformation of a locally Euclidean space (the indicatrices of a V^n are affine images of a sphere), for not every M admits a locally Euclidean structure. It is easy to see that for this it is necessary and sufficient that M is an affine manifold (see Theorem 8). The following question arises: are there Finsler spaces not V^n and not $\mathcal{A}\ell\mathcal{M}^n$ which admit linear metrical connections in TM?

A Finsler space $\overset{*}{F}{}^n = (M, \overset{*}{\mathcal{F}})$, where M is the coordinate space $R^n(x)$ (or a domain of it), is said to be of *1-form metric* (the local case), if there exits a function $\mathcal{F}$ of n variable, such that

$$ \overset{*}{\mathcal{F}}(x,y) = \mathcal{F}(a(x)y), $$

where $(a(x)y)^i = a^i_k(x)y^k$, $\det|a^i_k(x)| \neq 0$. The name comes from the fact that for fix i any $a^i_k(x)y^k$ is a 1-form. Such spaces, as line-element spaces, were introduced and investigated by M. Matsumoto and H. Shimada [M], [MS1–2]. These spaces are also affine deformations of Minkowski spaces [T3], [KT]. This is true also in the global case.

Consider a manifold M (in the sequel no more restricted to R^n), an atlas $\{U_\alpha(x_\alpha)\}$ of it and a Finsler metric $\mathcal{F} : TM \to R^+$, such that $\mathcal{F}(x_\alpha, y_\alpha) = \mathcal{F}_\alpha(y_\alpha)$, that is $\mathcal{F} \restriction U_\alpha$ is independent of x_α. (Then $F^n = (M, \mathcal{F})$ is a locally Minkowski space $\ell\mathcal{M}^n$.) On $U_{\alpha\beta} = U_\alpha \cap U_\beta$ we have $\mathcal{F}_\alpha(y_\alpha) = \mathcal{F}_\beta(y_\beta)$, and $y^i_\beta = \frac{\partial x^i_\beta}{\partial x^k_\alpha} y^k_\alpha$, or $y_\beta = J(x_\beta, x_\alpha)y_\alpha$ for short. Thus

$$ \mathcal{F}_\alpha(y_\alpha) = \mathcal{F}_\beta(J(x_\beta, x_\alpha)y_\alpha). \tag{26} $$

The left side of (26) is independent of x_α, x_β. Then so is the right side too. Hence $J(x_\beta, x_\alpha)$ must be constant. This means that $(x_\alpha) \longleftrightarrow (x_\beta)$ is linear, and thus M is an affine manifold.

Now consider a Finsler space $\overset{*}{F}{}^n = (M, \overset{*}{\mathcal{F}}) = (M, \overset{*}{I})$, an atlas $\{U_\alpha(x_\alpha)\}$ for M, and suppose that there exists on M a function $\mathcal{F}$

of n variable, such that on each $U_\alpha(x_\alpha)$ we have

$$\overset{*}{\mathcal{F}} \restriction U_\alpha = \overset{*}{\mathcal{F}}(x_\alpha, y_\alpha) = \mathcal{F}(b(x_\alpha)y_\alpha) \equiv \mathcal{F}_\alpha(\overline{y}_\alpha),$$
$$(b(x_\alpha)y_\alpha)^i = b^i_k(x_\alpha)y^k_\alpha = \overline{y}^i_\alpha.$$

On $U_{\alpha\beta} = U_\alpha \cap U_\beta$ we obtain

$$\overset{*}{\mathcal{F}} \restriction U_{\alpha\beta} = \mathcal{F}_\alpha(\overline{y}_\alpha) = \mathcal{F}_\beta(\overline{y}_\beta) = \mathcal{F}_\beta(J(x_\beta, x_\alpha)\overline{y}_\alpha)$$
$$\overline{y}_\beta = J(x_\beta, x_\alpha)\overline{y}_\alpha, \quad \overline{y}^i_\beta = \frac{\partial x^i_\beta}{\partial x^k_\alpha}\,\overline{y}^k_\alpha.$$

Then, according to the previous paragraph, M is an affine manifold, and $\overset{*}{F}{}^n$ is called a Finsler space of *1-form metric* (the global case). $(M, \overset{*}{\mathcal{F}})$ is a Finsler space, and since $\overset{*}{\mathcal{F}} \restriction U_\alpha = \mathcal{F}_\alpha(y_\alpha)$, it is a locally Minkowski space $\ell\mathcal{M}^n = (M, I)$. Let us choose a chart $U_\alpha(x_\alpha)$, and denote it by $U(x)$. If $\overset{*}{\mathcal{F}}(x, y) = 1$, then y is an element of $\overset{*}{\mathcal{I}}(x)$ the indicatrix of $\overset{*}{F}{}^n$ at x. Moreover $b(x)y$ is an element of $\mathcal{I}(x)$ the indicatrix of $\ell\mathcal{M}^n$, which is independent of x on U. Now let $\mathcal{A} = \{a(x)\}$ be such that $a^i_k(x) := (b^{-1})^i_k$, and thus $b(x)y =: \overline{y}$, or in other form $y = a(x)\overline{y}$. Then we have $\overline{y} \in \mathcal{I}$ and $a(x)\overline{y} \in \overset{*}{\mathcal{I}}(x)$. This means that $\overset{*}{\mathcal{I}}(x) = a(x)\mathcal{I}$ on U, and similarly on every U_α. Thus, every Finsler space $\overset{*}{F}{}^n$ of 1-form metric is the affine deformation of a locally Minkowski space: $\overset{*}{F}{}^n = \mathcal{A}\ell\mathcal{M}^n$.

In our investigations $a(p) : T_pM \to T_pM$ was a centroaffine transformation, which keeps the origo O (the null vector of $\mathcal{V}^n \approx T_pM$). Such deformations were considered also by M. Anastasiei ([A1–2]). However $a(p)$ can indicate also a genuine affine transformation: a centroaffine transformation followed by a translation, which takes O into another point $C \in R^n \approx T_pM$. Understanding $\mathcal{A}$ and $a(p)$ in this sense, we obtain the indicatrices of a Randers space $\mathcal{R}^n = (R^n, \mathcal{F})$ over $R^n(x)$ by the affine deformation of the Euclidean space $E^n(x)$, i.e. $\mathcal{R}^n = \mathcal{A}E^n$. For an $\mathcal{R}^n = (M, \mathcal{F})$ this is true locally only. Globally $\mathcal{R}^n = \mathcal{A}\ell E^n$ is not true in general. Randers spaces are the most simple, but the most important examples of not absolutely homogeneous Finsler spaces. They are often investigated also recently (see [BRS], [Ba], [Mi] and also [H] with relation to the affine deformation). Randers spaces also have a close relation to physics ([In], [IT]) etc.

§ 3. Affine automorphisms of the indicatrices

Any Riemannian space $V^n = (R^n, Q)$ is the affine deformation of the Euclidean space $E^n(R^n, S)$, i.e. $V^n = \mathcal{A}E^n$, $Q = aS$, where Q means ellipsoids, S is a sphere, and $a \in \mathcal{A}$. Nevertheless this relation can be realized by a set of $\mathcal{A}$. Let namely $r(p)$, $p \in R^n$ be a rotation in $T_pM \approx E^n$, then $a(p)r(p)S \equiv \overline{a}(p)S = Q(p)$, and $V^n = \overline{\mathcal{A}}E^n$, $\overline{\mathcal{A}} = \{\overline{a}(p)\}$. Given $F^n = \mathcal{A}\ell\mathcal{M}^n = (M, I)$, an orientation preserving (centro) affine transformation $k(p) : T_pM \to T_pM$ is an *affine authomorphism* of $\mathcal{I}(p)$ if

$$k(p)\mathcal{I}(p) = \mathcal{I}(p),$$

i.e. if $k(p)$ takes the indicatrix $\mathcal{I}(p)$ as a whole into itself.

The parallel translation $\mathcal{P}_c^\Gamma$ in an $\mathcal{A}\ell\mathcal{M}^n$ according to $\Gamma(x)$ given by (25), along a curve c from p to q is an affine transformation $a(p,q) : T_pM \to T_qM$, and takes $\mathcal{I}(p)$ into $\mathcal{I}(q)$. This means that all indicatrices of an $\ell\mathcal{M}^n$, and of any $\mathcal{A}\ell\mathcal{M}^n$ are in affine relation. Thus, if $k(p)$ is an affine automorphism of $\mathcal{I}(p)$, then $a(p,q) \circ k(p) \circ a^{-1}(p,q)$ is an affine automorphism for $\mathcal{I}(q)$.

If an indicatrix $\mathcal{I}_0$ of an $\ell\mathcal{M}^n$ has only the identity as affine automorphism, then $\mathcal{I}$ is called *rigid* (with respect to the affine automorphisms). If there exists $k \neq$ id., such that $k\mathcal{I}_0 = \mathcal{I}_0$, then $\mathcal{I}_0$ is called *mobile*. In these cases every indicatrix of $\ell\mathcal{M}^n$ and $\mathcal{A}\ell\mathcal{M}^n$is rigid (resp. mobile). If in an $\mathcal{A}\ell\mathcal{M}^n$ there exists a closed curve $c \subset M$, $p \in c$, and a $\Gamma(x)$ given by (25), such that $\mathcal{P}_c^\Gamma \neq$ id, then $\mathcal{A}\ell\mathcal{M}^n$ and its indicatrices are called Γ-*mobile*, otherwise $\mathcal{A}\ell\mathcal{M}^n$ is Γ-*rigid.* If $\mathcal{A}\ell\mathcal{M}^n$ is Γ-mobile, then there exists an $y_0 \in \mathcal{I}(p)$, such that $\mathcal{P}_c^\Gamma y_0 = y_1 \neq y_0$. Moreover, if c is contractible to p, and $c(\nu)$, $0 \le \nu \le 1$, $c(0) = c$, $c(1) = p$ is a continuous family of curves, and if $\nu \to 1$, then $\mathcal{P}_{c(\nu)}^\Gamma y_0 = y_1(\nu) \to y_0$, and $\mathcal{P}_{c(\nu)}^\Gamma = k(\nu)$ yields infinitely many different affine automorphisms of $\mathcal{I}(p)$. These automorphisms belong to the holonomy group of Γ.

Proposition 6. *If in an $\mathcal{A}\ell\mathcal{M}^n$ there exits a $\mathcal{P}_c^\Gamma \neq$ id with a contractible closed curve c, then $\mathcal{A}\ell\mathcal{M}^n$ has infinitely many different affine automorphisms k.*

Clearly any Γ-mobile $\mathcal{A}\ell\mathcal{M}^n$ is mobile, and any rigid $\mathcal{A}\ell\mathcal{M}^n$ is Γ-rigid, but not conversely. Consider an $\mathcal{M}^2 = (R^2(x), I)$, where $\mathcal{I}(p_0) \subset T_{p_0}R^2 \approx E^2$ is a regular planar polygon with N-vertices (i.e. regular N-gon). (This $\mathcal{I}(p_0)$ is actually no indicatrix, but the vertices can be rounded, and the edges slightly curved, and thus we obtain a smooth indicatrix, which admits only N affine automorphisms.) So this $\mathcal{M}^2$ is mobile, but it is not Γ-mobile with any linear metrical connection Γ. This is also true for any affine deformation $\mathcal{A}\mathcal{M}^2$ of the considered $\mathcal{M}^2$.

If $\Gamma(x)$ is a connection of $F^n = \mathcal{A}\ell\mathcal{M}^n$ constructed by (25) and $k(p)$ is an affine automorphism, then $\overline{\mathcal{A}} = \{\overline{a} = a \circ k\}$ yields by (25) another linear metrical connection $\overline{\Gamma}(x)$ on F^n. Both of them have vanishing curvature: $R(x) = \overline{R}(x) = 0$. However, if $\mathcal{A}\ell\mathcal{M}^n$ admits a field $k(p, \mu)$ of affine automorphisms C^∞ in (p, μ), where μ is a parameter, then there exist linear metrical connections $\Gamma(x)$ with not vanishing curvature $R(x)$.

We show the following

Theorem 10. *A Γ-mobile $\mathcal{A}\ell\mathcal{M}^2 = (M, I)$ on a simply connected M is a Riemanian space V^2.*

Proof. $\mathcal{A}\ell\mathcal{M}^2$ is an affine deformation of an $\ell\mathcal{M}^2 = (M, \overset{*}{I})$ and it makes the tangent space T_pM (p is fixed) into a Mikowski space $\mathcal{M}_p^2 = (T_pM, \mathcal{I}(p))$. Since $\mathcal{A}\ell\mathcal{M}^2$ is Γ-mobile, and M is simply connected, there are infinitely many affine automorphisms $k(p, \nu) = k(\nu)$. Each $k(\nu)$ is a linear automorphism of T_pM and at the same time an isometry with respect to the metric of $\mathcal{M}_p^2$, for $k(\nu)\mathcal{I}(p) = \mathcal{I}(p)$. Then by a result of P. Gruber ([G], or [Th] p. 83) $\mathcal{I}(p)$ is an ellipse. This holds at every point of M. Hence $\mathcal{A}\ell\mathcal{M}^2$ is a V^2.

Since $\ell\mathcal{M}^2$ is a locally Minkowski space, M is an affine manifold. Thus there exists on M a locally Euclidean space $\ell E^2 = (M, S)$, where S are congruent spheres in each adapted coordinate system. Then to each $S(p)$ there exists an affine transformation $a(p)$ taking $S(p)$ into $Q(p)$. Hence $\mathcal{A}\ell\mathcal{M}^2$ is the affine deformation of an ℓE^2. Q.E.D.

This theorem can be extended to higher dimensions. Consider an $\mathcal{A}\ell\mathcal{M}^n = (M, I)$, and the set $Y(y_*)$ of those points $y \in \mathcal{I}(p)$ (p is fixed), to which we can parallel translate y_* by a $\mathcal{P}_c^\Gamma$, where Γ is any possible linear metrical connection of $\mathcal{A}\ell\mathcal{M}^n$, and c is any closed curve through p. Denote by $L^r(p)$ the smallest linear subspace of T_pM containing $Y(y_*)$. r means the dimension of this subspace. r may depend on y_*. Let m be the maximum of $r(y_*)$. Then

$$L^m(p) \cap \mathcal{I}(p) = Q^{m-1}(p) \tag{27}$$

is an $(m-1)$-dimensional ellipsoid. (The proof is omitted.) After an appropriate linear transformation Q^{m-1} is a sphere S^{m-1}, and $\mathcal{I}(p)$ is a rotation surface containing an $(m-1)$-dimensional sphere. This yields

Theorem 11. *If $\mathcal{A}\ell\mathcal{M}^n$ is Γ-mobile, then its indicatrices are affine images of rotation surfaces in $T_pM \approx E^n$ containing a sphere S^{m-1}. The dimension of this sphere depends on the size of the Γ-mobility of $\mathcal{A}\ell\mathcal{M}^n$.*

If $\mathcal{A}\ell\mathcal{M}^n$ is Γ-rigid, then it has a single $\Gamma(x)$ determined by (25). The curvature of this Γ vanishes. If also the torsion of this $\Gamma(x)$ vanishes, then parallel vector fields with respect to this Γ have constant components in an appropriate coordinate system. Consequently the indicatrices $\mathcal{I}(p)$ of $\mathcal{A}\ell\mathcal{M}^n$, which are parallel translate of each other with respect to Γ, are independent of p in this coordinate system, that is $\mathcal{A}\ell\mathcal{M}^n$ is a locally Minkowski space. On the other hand, any $V^n = (M, Q)$ over an affine manifold is the affine deformation of a locally Euclidean space: $V^n = \mathcal{A}\ell E^n$, and these V^n are maximally Γ-mobile. Conversely, if an $\mathcal{A}\ell\mathcal{M}^n$ is maximally mobile, then $m = n$ in (27). Since $L^n(p) \cap \mathcal{I}(p) = \mathcal{I}(p)$, also these $\mathcal{A}\ell\mathcal{M}^n$ are Riemannian spaces. Thus Minkowski spaces, and Riemannian spaces are extreme cases of certain $\mathcal{A}\ell\mathcal{M}^n$ spaces belonging to Γ-rigidity, resp. to maximal Γ-mobility.

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