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Riemann-Finsler surfaces

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Dedicated to the memory of Professor Makoto Matsumoto

Abstract.

This paper study the Gauss-Bonnet theorem for Finsler surfaces with smooth boundary. This is a natural generalization of the Gauss-Bonnet theorem for Riemannian surfaces with smooth boundary as well as an extension of the Gauss-Bonnet theorem for boundaryless Finsler surfaces. The paper starts with an introduction in the Finsler geometry of surfaces with emphasis on the Berwald and Landsberg surfaces.

§1. Introduction

Riemann-Finsler geometry is a domain of modern differential geometry that cannot be ignored by anyone who wants to have a complete picture of the geometrical properties of a differentiable manifold.

Regarding Riemannian geometry as a particular case of a more general geometry, namely Riemann-Finsler geometry, we might expect to generalize many results from Riemannian geometry to the more general case of a Finsler metric.

One of the most important topics in Riemannian geometry is the study of the relation between the curvature of the Riemannian metric and the topology of the manifold. This is mainly achieved through the well-known Gauss-Bonnet-Chern theorem. The theorem and its consequences are especially interesting in the case of Riemannian surfaces (see [SST2003] for a comprehensive exposition).

The Gauss-Bonnet theorem was extended by D. Bao and S. S. Chern to the case of boundaryless Finslerian manifolds ([BC1996]). For the case of Landsberg surfaces the Gauss-Bonnet theorem is stated in a particular

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form that can be regarded as a direct generalization to the Finslerian case of the Riemannian classical result.

However, as far as we know, there are no attempts to prove the Gauss-Bonnet theorem for Finsler manifolds (in particular Finsler surfaces) with boundary.

The purpose of this paper is two-fold. First, we give a self-contained introduction of the geometry of Riemann-Finsler surfaces, and second, we prove the Gauss-Bonnet theorem for Landsberg surfaces with smooth boundary. Unfortunately we are not able yet to provide examples and applications of this theory because of the lack of examples of Landsberg surfaces that are not Berwald ones (see D. Bao's and Z. Shen's papers in this volume for a detailed discussion on this matter).

The paper is organized as follows. We begin by recalling the basic properties of Minkowski planes in §2. We continue by discussing the Riemannian length of the indicatrix of a Minkowski norm in §3. Some essential differences between the Minkowskian and Euclidean cases are pointed out. The basics of Finsler surfaces are exposed in §4 and the Chern connection is described in §5. We present the special status of Landsberg and Berwald surfaces among other Finsler surfaces in §6.

The following sections lead to the final aim of this paper: the Gauss-Bonnet theorem for Landsberg surfaces with smooth boundary. In §7 we study the geodesic curvature tensor and the signed curvature of a curve on a Finsler surface, and in §8 we prove the just announced theorem.

§2. Minkowski planes

Minkowski planes are one of the simplest Finslerian surfaces. They are at the same time generalizations of Riemannian planes.

Definition 2.1. A **Minkowski plane** is the vector space $\mathbf{R}^2$ endowed with a Minkowski norm. A **Minkowski norm** on $\mathbf{R}^2$ is a non-negative real valued function

$$F : \mathbf{R}^2 \to [0, \infty)$$

with the properties

(1) F is C^∞ on $\widetilde{\mathbf{R}^2} = \mathbf{R}^2 \setminus \{0\}$,
(2) **1-positive homogeneity** : $F(\lambda y) = \lambda F(y),\ \forall \lambda > 0,\ y \in \mathbf{R}^2$,
(3) **strong convexity:** the Hessian matrix

$$g_{ij}(y) = \frac{1}{2}\frac{\partial^2 F^2(y)}{\partial y^i \partial y^j} \tag{2.1}$$

is positive definite on $\widetilde{\mathbf{R}^2}$.

If $F(-y) = F(y)$, or equivalently $F(\lambda y) = |\lambda| \cdot F(y)$, $\lambda \in \mathbf{R}$, F is called **reversible** or **absolute homogeneous**. In this case the Minkowski norm is a norm in the sense of functional analysis.

Remark 2.1. From the above definition it follows:

(1) $F(y) > 0$ for all $y \neq 0$,
(2) $F(y_1 + y_2) \leq F(y_1) + F(y_2)$, for any $y_1, y_2 \in \mathbf{R}^2$,
(3) the **indicatrix** $S := \{y \in \mathbf{R}^2 : F(y) = 1\}$ is a closed, strictly convex, smooth curve around the origin $y = 0$,
(4) $w^i \dfrac{\partial F}{\partial y^i}(y) \leq F(w), \quad y \neq 0, \quad w \in \mathbf{R}^2$,

(see [BCS2000] for details.)

Define now the **Cartan tensor** of a Minkowski norm by

$$A_{ijk}(y) := \frac{F(y)}{4} \frac{\partial^3 F^2(y)}{\partial y^i \partial y^j \partial y^k}, \quad i, j, k \in \{1, 2\}. \tag{2.2}$$

It is obvious that F is Euclidean if and only if $A \equiv 0$.

The Minkowski norm F on $\mathbf{R}^2$ induces a Riemannian metric $\hat{g}$ on the punctured plane $\widetilde{R}^2$ by

$$\hat{g} := g_{ij}(y) dy^i \otimes dy^j. \tag{2.3}$$

Remark that the Riemannian manifold $(\widetilde{R}^2, \hat{g})$ is flat, i.e. the Gaussian curvature of $\hat{g}$ vanishes on $\widetilde{R}^2$. This is a peculiarity of the two dimensional case (see [BCS2000]).

The outward pointing normal to the indicatrix is

$$\hat{n}_{out} = \frac{y}{F(y)} = \frac{y^i}{F(y)} \cdot \frac{\partial}{\partial y^i}. \tag{2.4}$$

Indeed, let us consider $y^i = y^i(t)$ to be a unit speed parametrization of the indicatrix S. By derivation with respect to t of the formula $g_{ij}(y) y^i y^j = 1$ one obtains

$$g_{ij}(y) y^i \dot{y}^j = 0,$$

where the dot notations means derivative with respect to t.

In the following let us consider the indicatrix S as a Riemannian submanifold of the punctured Riemannian manifold $(\widetilde{R}^2, \hat{g})$, with the induced Riemannian metric h, and let $y(t) = (y^1(t), y^2(t))$ be a unit speed (with respect to h) parametrization of S.

Suppose $\widetilde{R}^2$ is identified with $(0, \infty) \times S$ by

$$y \mapsto \big(F(y), \frac{y}{F(y)}\big).$$

Then $\hat{g}$ admits the block decomposition

$$\hat{g} = dr \otimes dr + r^2 h,$$

where $r = F(y)$ and h is the induced Riemannian metric on S.

The **Cartan scalar** $I : \widetilde{R}^2 \to \mathbf{R}$ is defined by

$$I(y) = A_{ijk}(y)\frac{dy^i}{dt}\frac{dy^j}{dt}\frac{dy^k}{dt}. \tag{2.5}$$

The scalar I is also called the **main scalar** by some authors (see for example [M1986]).

This definition extends to all $\widetilde{R}^2$ by requiring that I be constant along each ray that emanates from the origin of $\mathbf{R}^2$.

Obviously, F is Euclidean if and only if $I \equiv 0$. In other words, the Cartan scalar I "measures" the deviation of F from an Euclidean inner product.

Indeed, every unit speed parametrization $y(t)$ of the indicatrix (S, h) must satisfy the following ODE:

$$\ddot{y} + I\dot{y} + y = 0, \tag{2.6}$$

where $\dot{y} = \dfrac{dy}{dt}$, $\ddot{y} = \dfrac{d^2y}{dt^2}$ ([R1959]).

The volume form of the Riemannian metric $\hat{g}$ is

$$dV = \sqrt{g}dy^1 \wedge dy^2, \tag{2.7}$$

where $\sqrt{g} = \sqrt{det(g_{ij})}$, and the induced Riemannian volume form on the submanifold S is

$$ds = \sqrt{g}(y^1\dot{y}^2 - y^2\dot{y}^1)dt. \tag{2.8}$$

Along S the 1-form ds coincides with

$$d\theta = \frac{\sqrt{g}}{F^2}(y^1dy^2 - y^2dy^1). \tag{2.9}$$

The parameter θ is called the **Landsberg angle**.

Remark 2.2. (1) The formula $ds = \sqrt{g}(y^1\dot{y}^2 - y^2\dot{y}^1)dt$ is valid as long as the underlying parametrization traces S out in a positive manner.

(2) The **Riemannian length** of the indicatrix is therefore defined by

$$L := \int_S ds$$

and it is typically NOT equal to 2π as in the case of Riemannian surfaces. This fact was remarked for the first time by M. Matsumoto [M1986].

§3. The Riemannian Length of the Indicatrix

Let us consider again the Minkowski plane (M, F) and the indicatrix $S = \{y \in \widetilde{\mathbf{R}^2} : F(y) = 1\}$, a closed convex curve in the plane.

The Riemannian length of the indicatrix S is an integral where the integration domain also depends on F. One would like however to work with integrals over the *standard unit circle*

$$\mathbf{S}^1 = \{y \in \widetilde{\mathbf{R}^2} : (y^1)^2 + (y^2)^2 = 1\}, \tag{3.1}$$

even with the price of a more complicated integrand.

One has ([BCS2000])

Lemma 3.1. **(Computational lemma)**
The indicatrix length in a Minkowski plane can be computed by

$$L = \int_{\mathbf{S}^1} \frac{\sqrt{g}}{F^2}(y^1 dy^2 - y^2 dy^1). \tag{3.2}$$

Indeed, the 1-form

$$d\theta = \frac{\sqrt{g}}{F^2}(y^1 dy^2 - y^2 dy^1) \tag{3.3}$$

is a closed 1-form on $\widetilde{\mathbf{R}^2}$. By the use of Stokes' theorem one can easily see that integrating this over S and $\mathbf{S}^1$ one obtains the same answer.

Remark 3.1. A local straightforward computation shows that

$$\frac{\sqrt{g}}{F^2}(y^1 \dot{y}^2 - y^2 \dot{y}^1) = \sqrt{g_{ij}(y)\dot{y}^i \dot{y}^j}, \tag{3.4}$$

in other words, the formula (3.2) means to measure the Riemannian arc length of the indicatrix, regarded as a curve in $\widetilde{\mathbf{R}^2}$, by the Riemannian metric $\hat{g}$ (see [BCS2000] for details).

Example 1. ([BCS2000]) Consider a Randers- Minkowski norm of Numata's type

$$F(y^1, y^2) = \sqrt{(y^1)^2 + (y^2)^2} + By^1$$

on $\mathbf{R}^2$, where $B \in [0, 1)$ is a constant parameter.

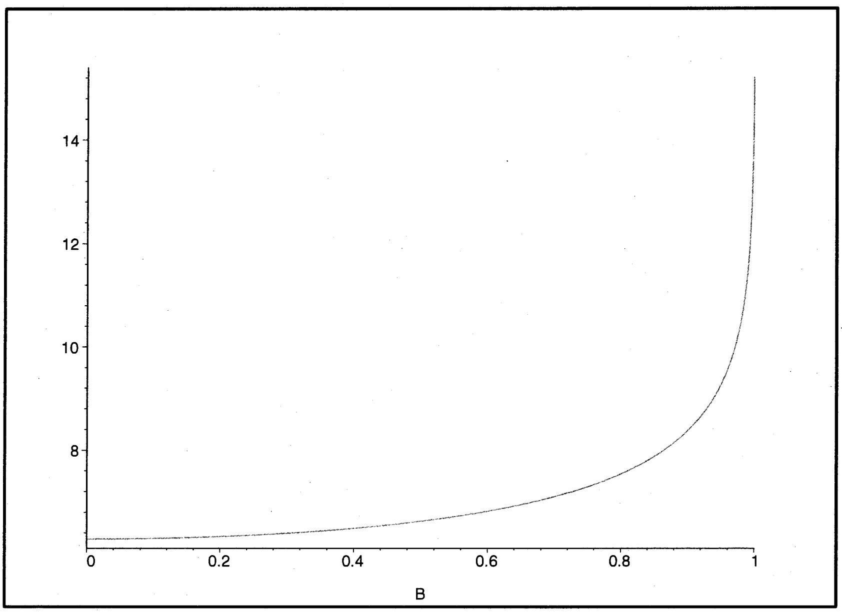

Figure 1. The variation of Riemannian length of the indicatrix for the metric given in Example 1.

Using polar coordinates

$$y^1 = r\cos\varphi, \quad y^2 = r\sin\varphi,$$

the polar equation of the indicatrix is

$$r = \frac{1}{1 + B\cos\varphi},$$

and the indicatrix length is given by the elliptic integral

$$L = \frac{4}{\sqrt{1+B}} \int_0^{\frac{\pi}{2}} \frac{d\mu}{\sqrt{1 - k^2 \sin^2 \mu}}, \tag{3.5}$$

where $\varphi = 2\mu$, and $k := \sqrt{\dfrac{2B}{1+B}}$.

We remark that for $B \equiv 0$, i.e. an Euclidean norm, one obtains $L = 2\pi$, as expected. The result of numerical estimation of the integral in (3.5) gives the graph in Figure 1.

One can see that unlike the Euclidean case, L increases to infinity when B approaches 1.

Example 2: ([BS1994]).
Consider the Minkowski norm

$$F(y) = \sqrt{(y^1)^2 + (y^2)^2 + \lambda\sqrt{(y^1)^4 + (y^2)^4}}, \quad \lambda \geq 0$$

in $\mathbf{R}^2$.

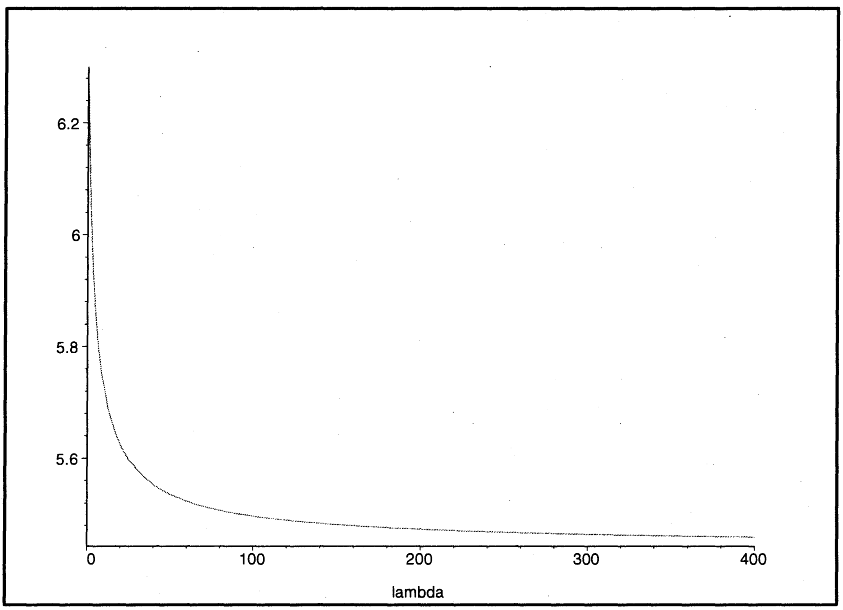

Figure 2. The variation of Riemannian length of the indicatrix for the metric given in Example 2.

With the substitution $u := \dfrac{y^2}{y^1}$ one obtains the indicatrix length

$$L = 8\int_0^1 \frac{\sqrt{1+\lambda\dfrac{(1+u^2)^3}{(1+u^4)^{3/2}}+\lambda^2\dfrac{3u^2}{1+u^4}}}{1+u^2+\lambda\sqrt{1+u^4}}du.$$

The numerical integration results are presented in Figure 2.

In this case L decreases from 2π to $\sqrt{3}\pi$ as λ increases from 0 to ∞. Indeed, one can see that

$$\lim_{\lambda\to\infty} L = \sqrt{3}\pi.$$

§4. Finsler surfaces

This section follows the exposition in [BCS2000].

Let us consider, in the following, Finsler metrics defined on an orientable surface M. Recall that a **Finsler surface** is the pair (M, F) where $F : TM \to [0,\infty)$ is $\mathbf{C}^\infty$ on $\widetilde{TM} := TM\backslash\{0\}$ and whose restriction to each tangent plane T_xM is a Minkowski norm.

For each $x \in M$ the quadratic form $ds^2 := g_{ij}(x,y)dy^i \otimes dy^j$ gives a Riemannian metric on the punctured tangent space $\widetilde{T_xM}$. Using the Finslerian fundamental function F we define the *indicatrix bundle* (or *unit sphere bundle*) $IM := \cup_{x\in M} I_xM$, where $I_xM := \{y \in T_xM : F(x,y) = 1\}$. Topologically, I_xM is diffeomorphic with the Euclidean unit sphere S^2 in $\mathbf{R}^3$. Moreover, the above ds^2 induces a Riemannian metric h_x on each I_xM.

On the other hand, let $SM := TM/\sim$ be the *projective sphere bundle*, where the equivalence relation "$\sim$" is given by $y \sim y'$ if and only if there exists $\lambda > 0$ such that $y = \lambda y'$. The natural projection $\pi : SM \to M$ pulls back the tangent bundle TM to a 2-dimensional vector bundle π^*TM over the 3-dimensional manifold SM. Local coordinates x^1, x^2 on M induce global coordinates y^1, y^2 on each fiber T_xM by $y = y^i\dfrac{\partial}{\partial x^i}$. Therefore (x^i, y^i) is a coordinate system on SM (y^i regarded as homogeneous coordinates).

It is known ([BS1994]) that for each $x \in M$, the canonical map $i : I_xM \to S_xM$, $y \mapsto [y]$ is a diffeomorphism, and $i : (I_xM, h_x) \to (S_xM, \dot{g}_x)$ is a Riemannian isometry, where $\dot{g}_x$ is the induced Riemannian metric on each projective sphere S_xM.

Let us also remark that since the Finslerian fundamental tensor $g_{ij}(x,y)$ is invariant under the rescaling $y \mapsto \lambda y$, $\lambda > 0$, the inner products in the fibers T_xM are actually identical. This redundancy is removed by working with the pull-back bundle π^*TM over SM.

Using the global section $l := \dfrac{y^i}{F(y)}\dfrac{\partial}{\partial x^i}$ of π^*TM, one can construct a positively oriented g-orthonormal frame $\{e_1, e_2\}$ for π^*TM, where $g = g_{ij}(x,y)dx^i \otimes dx^j$ is the induced Riemannian metric on the fibers of π^*TM.

Namely,

$$\begin{aligned} e_1 &:= \frac{1}{\sqrt{g}}\left(\frac{\partial F}{\partial y^2}\frac{\partial}{\partial x^1} - \frac{\partial F}{\partial y^1}\frac{\partial}{\partial x^2}\right) = m^1\frac{\partial}{\partial x^1} + m^2\frac{\partial}{\partial x^2} \quad , \\ e_2 &:= \frac{y^1}{F}\frac{\partial}{\partial x^1} + \frac{y^2}{F}\frac{\partial}{\partial x^2} = l^1\frac{\partial}{\partial x^1} + l^2\frac{\partial}{\partial x^2}. \end{aligned} \tag{4.1}$$

The frame $\{e_1, e_2\}$ is a globally defined g-orthonormal frame field for π^*TM called the **Berwald frame**. There are authors who consider the frame $(l, -m)$ as the Berwald frame [M1986].

The corresponding dual coframe is

$$\begin{aligned} \omega^1 &= \frac{\sqrt{g}}{F}(y^2dx^1 - y^1dx^2) = m_1dx^1 + m_2dx^2 \\ \omega^2 &= \frac{\partial F}{\partial y^1}dx^1 + \frac{\partial F}{\partial y^2}dx^2 = l_1dx^1 + l_2dx^2. \end{aligned} \tag{4.2}$$

The sphere bundle SM is a 3-dimensional Riemannian manifold with the **Sasaki** (*type*) **metric**

$$\omega^1 \otimes \omega^1 + \omega^2 \otimes \omega^2 + \omega^3 \otimes \omega^3, \tag{4.3}$$

where

$$\omega^3 := \frac{\sqrt{g}}{F}\left(y^2\frac{\delta y^1}{F} - y^1\frac{\delta y^2}{F}\right) = m_1\frac{\delta y^1}{F} + m_2\frac{\delta y^2}{F}. \tag{4.4}$$

Here, $\left\{\dfrac{\delta}{\delta x^i}, F\dfrac{\partial}{\partial y^i}\right\}$ is a local adapted basis for $T(\widetilde{TM})$, where

$$\frac{\delta}{\delta x^i} := \frac{\partial}{\partial x^i} - N_i^j\frac{\partial}{\partial y^j}, \tag{4.5}$$

with the dual coframe $\{dx^i, \dfrac{\delta y^i}{F}\}$, where

$$\delta y^i := dy^i + N_j^i dx^j. \tag{4.6}$$

The N^i_j are the local coefficients of the Finslerian nonlinear connection (for details see [BCS2000], [M1986]).

Remark 4.1. Remark that the above basis components $F\dfrac{\partial}{\partial y^i}$, or $\dfrac{\delta y^i}{F}$ are "adjusted" such that they become an 0-homogeneous vector field, and an 0-homogeneous 1-form, respectively. In other words, the adapted basis components are invariant under the action of $y \mapsto \lambda y$, for any $\lambda > 0$.

The globally defined orthonormal coframe $\{\omega^1, \omega^2, \omega^3\}$ on SM has the dual

$$
\begin{aligned}
\hat{e}_1 &= \frac{1}{\sqrt{g}}\left(\frac{\partial F}{\partial y^2}\frac{\delta}{\delta x^1} - \frac{\partial F}{\partial y^1}\frac{\delta}{\delta x^2}\right) = m^1\frac{\delta}{\delta x^1} + m^2\frac{\delta}{\delta x^2}\\
\hat{e}_2 &= \frac{y^1}{F}\frac{\delta}{\delta x^1} + \frac{y^2}{F}\frac{\delta}{\delta x^2} = l^1\frac{\delta}{\delta x^1} + l^2\frac{\delta}{\delta x^2}\\
\hat{e}_3 &= \frac{F}{\sqrt{g}}\left(\frac{\partial F}{\partial y^2}\frac{\partial}{\partial y^1} - \frac{\partial F}{\partial y^1}\frac{\partial}{\partial y^2}\right) = F\left(m^1\frac{\partial}{\partial y^1} + m^2\frac{\partial}{\partial y^2}\right).
\end{aligned}
\tag{4.7}
$$

Remark 4.2. (1) $(dF)(\hat{e}_1) = (dF)(\hat{e}_2) = (dF)(\hat{e}_3) = 0$

Indeed, from $\dfrac{\delta F}{\delta x^i} = 0$ (for a proof of this relation see for example D. Bao's paper in the present volume) it follows $dF = \dfrac{\delta F}{\delta x^i}dx^i + \dfrac{\partial F}{\partial y^i}\delta y^i = \dfrac{\partial F}{\partial y^i}\delta y^i$. Applying now this to $\hat{e}_1, \hat{e}_2, \hat{e}_3$ the result follows immediately.

(2) The vector $\hat{e}_3$ is tangent to each indicatrix.

(3) Using the indicatrix arc length form $ds = \dfrac{\sqrt{g}}{F}(y^1dy^2 - y^2dy^1)$ one sees that $ds(\hat{e}_3) = -1$, i.e. $\hat{e}_3$ points opposite to the direction indicated as positive by ds.

Remark 4.3. Remark that the orthonormal coframe $\{\omega^1, \omega^2, \omega^3\}$ on SM can be completed to a g- orthonormal coframe $\{\omega^1, \omega^2, \omega^3, \omega^4\}$ on the 4-dimensional manifold TM, where

$$\omega^4 = d(\log F) = l_i\frac{\delta y^i}{F}$$

(see [M1986] or [BCS2000] for details.)

Remark 4.4. Recall that on a Riemann-Finsler manifold, one have to deal with two kinds of metrics. This is easy to understand through a comparison with the Riemannian case.

(1) **The Riemannian case.** On a Riemannian manifold (M, g) the metric

$$g = g_{ij}(x)dx^i \otimes dx^j \tag{4.8}$$

is a specific inner product in each tangent space T_xM viewed as a vector space. Moreover,

$$\hat{g} = g_{ij}(x)dy^i \otimes dy^j \tag{4.9}$$

is an isotropic and constant Riemannian metric on T_xM viewed as a differentiable manifold.

(2) **The Riemann-Finslerian case**. On a Riemann-Finsler manifold (M, F) the metric

$$g = g_{ij}(x, y)dx^i \otimes dx^j \tag{4.10}$$

is a family of inner products in each tangent space T_xM viewed as a vector space, parametrized by rays ty, $(t > 0)$ which emanate from origin. This is actually a Riemannian metric on π^*TM. Moreover,

$$\hat{g} = g_{ij}(x, y)dy^i \otimes dy^j \tag{4.11}$$

is a non-isotropic Riemannian metric on T_xM viewed as a differentiable manifold, and which is invariant along each ray and possibly singular at the origin.

§5. The Chern connection on Finslerian surfaces

This section summarizes some exposition in [BCS2000].

The vector bundle π^*TM has a torsion-free and almost g-compatible connection $D : C^\infty(TSM) \otimes C^\infty(\pi^*TM) \to C^\infty(\pi^*TM)$, where

$$D_{\hat{X}}Z := \{\hat{X}(z^i) + z^j \omega_j{}^i(\hat{X})\}e_i, \tag{5.1}$$

where $\hat{X}$ is a vector field on SM, $Z := z^i e_i$ is a section of π^*TM, and $\{e_i\}$ is the g-orthonormal frame field on π^*TM.

Indeed, there is a unique set of connection 1-forms $\{\omega_j{}^i\}$ on SM such that

$$dx^j \wedge \omega_j{}^i = 0 \tag{5.2}$$

$$dg_{ij} - g_{kj}\omega_i{}^k - g_{ik}\omega_j{}^k = 2A_{ijk}\frac{\delta y^k}{F}, \quad i, j, k \in \{1, 2\}, \tag{5.3}$$

where δy^k is given in (4.6), and A_{ijk} is the **Cartan tensor** of a Finsler metric given by

$$A_{ijk}(x,y) = \frac{F(x,y)}{4}\frac{\partial^3 F^2(x,y)}{\partial y^i \partial y^j \partial y^k}. \tag{5.4}$$

The connection forms $\{\omega_i^{\ j}\}$ define the well-known **Chern connection** of the Finsler manifold (M,F).

Remark 5.1. The torsion freeness condition (5.2) is equivalent to

$$\omega_j^{\ i} = \Gamma^i_{jk} dx^k \tag{5.5}$$

together with

$$\Gamma^i_{jk} = \Gamma^i_{kj}, \tag{5.6}$$

where

$$\Gamma^i_{jk} = \frac{1}{2} g^{is}\Big(\frac{\delta g_{js}}{\delta x^k} + \frac{\delta g_{ks}}{\delta x^j} - \frac{\delta g_{jk}}{\delta x^s}\Big) \tag{5.7}$$

are the local coefficients of the Chern connection, and $\frac{\delta}{\delta x^k}$ is given in (4.5).

Remark 5.2. A straightforward computation shows that the structure equations (5.2), (5.3) of the Chern connection can be written also as

$$d\omega^j = \omega^k \wedge \omega_k^{\ j} \tag{5.8}$$

$$\omega_{ij} + \omega_{ji} = -2A_{ijk}\omega^{2+k}, \quad i,j,k \in \{1,2\}, \tag{5.9}$$

where $\{\omega^1, \omega^2, \omega^3\}$ is the g-orthonormal coframe on SM.

The following lemma provides a very useful formula for computations.

Lemma 5.1. *With the above notations one has*

$$\frac{\partial g_{ij}}{\partial x^k} = g_{sj}\Gamma^s_{ik} + g_{si}\Gamma^s_{jk} + 2A_{ijs}\frac{N^s_k}{F}. \tag{5.10}$$

Indeed, by plugging (5.5) and (4.6) into the structure equation (5.3) one obtains (5.10).

The connection matrix $(\omega_j^{\ i})$ of the Chern connection for Finsler surfaces with respect to the g-orthonormal frame $\{e_1, e_2\}$ of π^*TM is given by

$$\begin{pmatrix} \omega_1^{\ 1} & \omega_1^{\ 2} \\ \omega_2^{\ 1} & \omega_2^{\ 2} \end{pmatrix} = \begin{pmatrix} -I\omega^3 & -\omega^3 \\ \omega^3 & 0 \end{pmatrix} \tag{5.11}$$

where $I := A_{111} = A(e_1, e_1, e_1)$ is the **Cartan scalar** for Finsler surfaces (for details see [BCS2000]). Remark that $I = 0$ is equivalent to the Finsler structure to be Riemannian.

We remark that if the Chern connection were g-compatible, then the connection matrix (5.11) would have been skew-symmetric. It is known ([M1986]) that on a Finsler manifold there is no connection which possesses both torsion-freeness and g-compatibility, except the case when the Finsler structure is actually Riemannian. One has to drop the torsion-freeness or the g-compatibility in order to work with Finsler metrics that are not Riemannian.

Cartan connection (see [M1986] for definition and details), for example, it is g-compatible, but has some surviving torsion. On the other hand, the Chern connection used in the present paper claims for torsion-freeness but it can afford only almost g-compatibility (see (5.2) and (5.3)). In the case of the Chern connection, the connection matrix (5.11) is only "almost" skew-symmetric.

A regular piecewise C^∞ curve $\sigma : [0, r] \to M$ with velocity vector T is called a **Finslerian geodesic** if it satisfies the geodesic equation

$$D_T\Big[\frac{T}{F(T)}\Big] = 0. \tag{5.12}$$

Remark 5.3. Observe that in natural coordinates one has

$${\omega_1}^2 = \frac{\sqrt{g}}{F}\left(y^1\frac{\delta y^2}{F} - y^2\frac{\delta y^1}{F}\right). \tag{5.13}$$

By taking the exterior derivatives and using torsion-free condition (5.2) one obtains the **structure equations** of a Finsler surface

$$\begin{aligned} d\omega^1 &= -I\omega^1 \wedge \omega^3 + \omega^2 \wedge \omega^3 \\ d\omega^2 &= -\omega^1 \wedge \omega^3 \\ d\omega^3 &= K\omega^1 \wedge \omega^2 - J\omega^1 \wedge \omega^3. \end{aligned} \tag{5.14}$$

Remark 5.4. (1) For comparison, recall the structure equations of a Riemannian surface. They are obtained from (5.14) by setting $I = J = 0$.

(2) The scalar K is called the **Gauss curvature** of a Finsler surface. In the case when F is Riemannian, K coincides with the usual Gauss curvature of a Riemannian surface.

Differentiating again (5.14) one obtains the **Bianchi identities**

$$J = I_2 = \frac{1}{F}\Big(y^1\frac{\delta I}{\delta x^1} + y^2\frac{\delta I}{\delta x^2}\Big) \tag{5.15}$$
$$K_3 + KI + J_2 = 0.$$

Here indices indicate directional derivatives along $\hat{e}_1$, $\hat{e}_2$, $\hat{e}_3$, respectively. For example $dK = K_1\omega^1 + K_2\omega^2 + K_3\omega^3$. The scalars K_1, K_2, K_3 are the directional derivatives of K.

Nevertheless, observe that the scalars $I = I(x,y)$, $J = J(x,y)$, $K = K(x,y)$ and their derivatives lives on SM, not on M as in the Riemannian case!

More generally, given any function $f : SM \to \mathcal{R}$, one can write its differential in the form

$$df = f_1\omega_1 + f_2\omega_2 + f_3\omega_3.$$

Taking one more exterior differentiation of this formula, one obtains the following **Ricci identities**:

$$f_{21} - f_{12} = -Kf_3 \tag{5.16}$$
$$f_{32} - f_{23} = -f_1 \tag{5.17}$$
$$f_{31} - f_{13} = If_1 + f_2 + Jf_3. \tag{5.18}$$

The curvature 2-form of the Chern connection

$$\Omega_j^{\ i} := d\omega_j^{\ i} - \omega_j^{\ k}\wedge\omega_k^{\ i} \tag{5.19}$$

can be written by means of the g-orthogonal coframe as

$$\Omega_j^{\ i} = \frac{1}{2}R_j^{\ i}{}_{kl}\omega^k\wedge\omega^l + P_j^{\ i}{}_{kl}\omega^k\wedge\omega^{2+l}, \quad i,j,k,l\in\{1,2\}. \tag{5.20}$$

Indeed, let us remark first that (5.19) expands as follows

$$\Omega_j^{\ i} = \frac{1}{2}R_j^{\ i}{}_{kl}\omega^k\wedge\omega^l + P_j^{\ i}{}_{kl}\omega^k\wedge\omega^{l+n} + \frac{1}{2}Q_j^{\ i}{}_{kl}\omega^{k+n}\wedge\omega^{l+n}, \tag{5.21}$$

where the coefficients R, P, Q are written in the cobasis ω^i, and $n = 2$. The $\frac{1}{2}$ coefficient appears because of the skew-symmetry of R and Q, in other words, we have

$$R_j^{\ i}{}_{kl} = -R_j^{\ i}{}_{lk}, \quad Q_j^{\ i}{}_{kl} = -Q_j^{\ i}{}_{lk}. \tag{5.22}$$

Next, taking the exterior derivative of the torsion freeness condition (5.8) it follows

$$\Omega_k^{\ j} \wedge \omega^k = 0,$$

and using this, (5.21) implies

(5.23)

$$0 = \frac{1}{2} R_j^{\ i}{}_{kl} \omega^k \wedge \omega^l \wedge \omega^j + P_j^{\ i}{}_{kl} \omega^k \wedge \omega^{l+n} \wedge \omega^j + \frac{1}{2} Q_j^{\ i}{}_{kl} \omega^{k+n} \wedge \omega^{l+n} \wedge \omega^j.$$

It follows that all three coefficients in the right member of this equality have to vanish identically. Therefore, the symmetric part of R and the skew-symmetric parts of P and Q has to vanish, respectively, i.e.

$$\begin{aligned} &R_j^{\ i}{}_{kl} + R_k^{\ i}{}_{lj} + R_l^{\ i}{}_{jk} = 0, \\ &P_j^{\ i}{}_{kl} - P_k^{\ i}{}_{jl} = 0, \quad Q_j^{\ i}{}_{kl} - Q_j^{\ i}{}_{lk} = 0. \end{aligned} \tag{5.24}$$

Now, from (5.22) and (5.24) it follows that $Q_j^{\ i}{}_{kl}$ has to vanish, and therefore (5.21) implies (5.20).

Moreover, this simplifies to

$$\Omega_1^{\ 2} = d\omega_1^{\ 2} = R_1^{\ 2}{}_{12} \omega^1 \wedge \omega^2 + P_{2111} \omega^1 \wedge \omega_1^{\ 2}. \tag{5.25}$$

Indeed, for a Finsler surface, from (5.11) follows the following important relation

$$\Omega_j^{\ i} = d\omega_j^{\ i}, \tag{5.26}$$

i.e. the curvature form is an exact form, and therefore a closed one. Even though this formula is familiar in the Riemannian case, in the Finslerian setting it is a peculiarity of Finsler surfaces.

We rewrite (5.20) as

$$\Omega_j^{\ i} = R_j^{\ i}{}_{12} \omega^1 \wedge \omega^2 + P_j^{\ i}{}_{k1} \omega^k \wedge \omega^3 + P_j^{\ i}{}_{12} \omega^k \wedge \omega^4 \tag{5.27}$$

using the coframe $\{\omega^1, \omega^2, \omega^3, \omega^4\}$ on TM.

Remark now that from the 0-homogeneity of P it follows

$$P_j^{\ i}{}_{12} = 0 \tag{5.28}$$

in the g-orthonormal coframe.

Therefore, we obtain

$$\Omega_j^{\ i} = R_j^{\ i}{}_{12} \omega^1 \wedge \omega^2 + P_j^{\ i}{}_{k1} \omega^k \wedge \omega^3. \tag{5.29}$$

In the case $i = 2$, $j = 1$ we obtain

$$\Omega_1^{\ 2} = R_1^{\ 2}{}_{12} \omega^1 \wedge \omega^2 + P_1^{\ 2}{}_{11} \omega^1 \wedge \omega^2 + P_1^{\ 2}{}_{21} \omega^2 \wedge \omega^3. \tag{5.30}$$

Due to the properties of the tensor P, we have

$$P_1{}^2{}_{21} = P_2{}^2{}_{11} = P_{2211} = 0$$
$$P_1{}^2{}_{11} = P_{1211} = -P_{2111}. \tag{5.31}$$

Let us remark that in the natural frame, the tensor P is not skew-symmetric in its first and second indices. However, the above formula is written in the g-orthonormal frame, and in this setting P becomes skew-symmetric in first and second indices if they contain an index equal to 2. The reason is that an index equal to 2 means contraction with l^i.

Using now (5.11) and (5.30) we obtain (5.25).

Moreover, from (5.11) and (5.25) we obtain

$$d\omega^3 = -R_1{}^2{}_{12}\omega^1 \wedge \omega^2 + P_{2111}\omega^1 \wedge \omega^3 \tag{5.32}$$

and comparing this with (5.14) it follows

$$K = R_2{}^1{}_{12} = -R_{1212}, \quad J = -P_2{}^1{}_{11}. \tag{5.33}$$

In the case $i = 1$, $j = 1$, (5.29) implies

$$\Omega_1{}^1 = d\omega_1{}^1 = R_1{}^1{}_{12}\omega^1 \wedge \omega^2 + P_1{}^1{}_{11}\omega^1 \wedge \omega^3 + P_1{}^1{}_{21}\omega^2 \wedge \omega^3. \tag{5.34}$$

On the other hand, from (5.11) and (5.14) we have

$$\begin{aligned} d\omega_1{}^1 &= d(-I\omega^3) = -dI \wedge \omega^3 - I d\omega^3 \\ &= -(I_1\omega^1 + I_2\omega^2 + I_3\omega^3) \wedge \omega^3 - I d\omega^3 \\ &= (I\omega^1 + I\omega^2) \wedge \omega^3 - I(K\omega^1 \wedge \omega^2 - J\omega^1 \wedge \omega^3) \\ &= -IK\omega^1 \wedge \omega^2 - (I_1 - IJ)\omega^1 \wedge \omega^3 - I_2\omega^2 \wedge \omega^3. \end{aligned} \tag{5.35}$$

It follows,

$$R_{1112} = -IK, \quad P_{1111} = IJ - I_1. \tag{5.36}$$

Remark 5.5. The Gauss curvature $K = R_2{}^1{}_{12}$ is an important geometrical quantity because its sign decides whether geodesic rays emanating from a common point $x \in M$ are going to focus or diverge (see [BCS2000]).

The Gauss curvature is the particularization to the case $n = 2$ of the flag curvature of an arbitrary dimension Finsler manifold. Indeed, one constructs a flag (x, y, V) at $x \in M$ using

- a base point $x \in M$,
- a flag pole $y \in \widetilde{T_xM}$,

- an edge $V \in T_xM$ which is transversal to the flag pole.

Then

$$
\begin{aligned}
K(x,y,V) &= \frac{g_y(R(V,y)y,V)}{g_y(y,y)g_y(V,V)-[g_y(y,V)]^2} \\
&= \frac{V^i(y^j R_{jikl} y^l)V^k}{g_y(y,y)g_y(V,V)-[g_y(y,V)]^2}
\end{aligned}
$$

is called the **flag curvature** of (M,F) and it is the Finslerian analog of the sectional curvature of a Riemannian manifold.

In arbitrary dimension, a Finsler metric F is said to be of **scalar curvature** if $K(x,y,V)$ does not depend on V. This can be rewritten as

$$R_{ik} = K(x,y)(g_{ik} - l_i l_k),$$

where $R_{ik} = l^j R_{jikl} l^l$.

If $K = K(x,y,V)$ is a constant, then F is said to be of **constant flag curvature**.

There is an essential difference between the Gauss curvature on Riemannian and Finslerian surfaces. If base point x and the flag pole y are chosen, the tangent plane T_xM is actually spanned by y and V, i.e. every Finsler surface is of **scalar curvature** $K(x,y)$. By contrast, when the surface is Riemannian the scalar curvature $K(x,y)$ reduces to the Riemannian Gauss curvature $K(x)$, which does not depend on y.

There is another important difference between curvatures in Riemannian and Finslerian setting.

In Riemannian geometry the sectional curvature completely determine the curvature tensor (see for example [L1997] or another standard textbook of Riemannian geometry). This is not the case anymore for a Finsler manifold (arbitrary dimension). The curvature form of the Chern connection $\Omega^i{}_j$ contains two curvature tensors: the so called (hh)-curvature tensor $R_j{}^i{}_{kl}$ and the (hv)-curvature tensor $P_j{}^i{}_{kl}$ (see (5.20)). The former, also called the **Riemannian curvature tensor of a Finsler manifold**, is used for constructing the flag curvature and Jacobi equations.

§6. Berwald and Landsberg surfaces

A Finsler surface is said to be of **Landsberg type** if $J = 0$, or equivalently, $I_2 = 0$. Both Riemannian surfaces and (locally) Minkowski surfaces belong here. Recall that a Finsler manifold (M,F) is called **locally Minkowski** if there exists certain privileged local coordinates

(x^i) on M which, together with the coordinates on TM induced by $y = y^i \frac{\partial}{\partial x^i}$, make F dependent only on y and not x ([M1986], [BCS2000]). Minkowski planes are always locally Minkowskian.

A stronger condition on a Finsler surface is that the Chern connection local coefficients depend only on x, i.e. $\Gamma^i{}_{jk}(x,y) = \Gamma^i{}_{jk}(x)$. These Finsler surfaces are said to be of **Berwald type**.

The geometrical meaning of these special Finsler spaces can be described by means of **parallel translation.**

Let us consider an arbitrary curve $\sigma : [a,b] \to M$ with velocity vector $T(t) = \dot{\sigma}(t)$, and an arbitrary vector field $W(t) = W^i(t)\frac{\partial}{\partial x^i}_{|\sigma(t)}$ along σ.

The **linear covariant derivative** of W along σ is defined by

$$D_T W := \left[\frac{dW^i}{dt} + W^j T^k \Gamma^i_{jk}(\sigma(t), T(t))\right] \frac{\partial}{\partial x^i}_{|\sigma(t)}, \tag{6.1}$$

where Γ^i_{jk} are the coefficients of the Chern connection.

The vector field $W(t)$ is said to be **linearly parallel** along $\sigma(t)$ if $D_T W = 0$.

Remark 6.1. Remark that in this case the Chern connection coefficients Γ^i_{jk} are evaluated along σ, i.e. at the points $(\sigma(t), T(t)) \in T_{\sigma(t)}M$. In other words **the linear covariant derivative $D_T W$ is with reference vector T.**

Remark also that if one deals with Berwald spaces, then the Chern connection coefficients satisfy $\Gamma^i_{jk}(x,y) = \Gamma^i_{jk}(x)$, therefore the reference vector is irrelevant in this case.

Remark 6.2. Recall that from the torsion-freeness and almost compatibility conditions of the Chern connection (5.2), (5.3) it follows

$$\frac{d}{dt}\Big[g_T(U,V)\Big] = g_T(D_T U, V) + g_T(U, D_T V) + 2A(U, V, D_T T), \tag{6.2}$$

for any two vector fields U, V along σ.

Moreover, if one of the following three conditions holds

(1) U or V is proportional to T,
(2) σ is geodesic,
(3) A vanishes along σ,

then

$$\frac{d}{dt}\Big[g_T(U,V)\Big] = g_T(D_T U, V) + g_T(U, D_T V), \tag{6.3}$$

for any two vector fields U, V along σ (see [BCS2000] for details).

Let us consider now the case when the curve $\sigma : [a, b] \to M$ is a **geodesic** of the Finsler space (M, F). The **linear parallel translation** along $\sigma(t)$ is given by the map

$$(6.4) \qquad P_\sigma : T_{\sigma(a)}M \to T_{\sigma(b)}M, \qquad P_\sigma(v) = w,$$

where $V(t)$ is a linearly parallel vector field along σ with $V(a) = v$, $V(b) = w$.

An immediate property of the linear parallel transport on Finsler manifolds follows ([CS2005]).

Lemma 6.1. *Let (M, F) be a Finsler manifold, $\sigma : [a, b] \to M$ a geodesic of F with velocity vector $T(t) = \dfrac{d\sigma}{dt}$, and $U(t)$, $V(t)$ linearly parallel vector fields along σ. Then we have*

$$(6.5) \qquad g_{T(t)}\Big(U(t), V(t)\Big) = 0.$$

Proof. Since σ is geodesic, and U, V are two parallel vector fields along σ, using Remark 6.2 it follows

$$\frac{d}{dt}\Big[g_T(U, V)\Big] = g_T(D_T U, V) + g_T(U, D_T V) = 0.$$

Therefore $g_T(U, V)$ is constant along σ.

Q.E.D.

Let $\sigma : [a, b] \to M$ be again an arbitrary C^∞ piecewise curve on M with velocity vector $T(t)$, and let $W(t)$ be an arbitrary vector field along σ.

Another way of defining the covariant derivative of W is with reference vector W. Indeed, the **nonlinear covariant derivative** of W along σ is defined by

$$(6.6) \qquad D_T^{(W)} W := \left[\frac{dW^i}{dt} + W^j T^k \Gamma^i_{jk}(\sigma(t), W(t))\right] \frac{\partial}{\partial x^i}_{|\sigma(t)},$$

where Γ^i_{jk} are the coefficients of the Chern connection. The top letter indicates the reference vector. If it is absent it means that the reference vector is T.

The vector field $W(t)$ is said to be **nonlinearly parallel** along $\sigma(t)$ if $D_T^W W = 0$.

Using this covariant derivative one can define another kind of parallel translation of W along σ. The **nonlinear parallel translation** along $\sigma(t)$ is given by the map

$$P^n_\sigma : T_{\sigma(a)}M \to T_{\sigma(b)}M, \qquad P_\sigma(v) = w, \tag{6.7}$$

where $V(t)$ is a nonlinearly parallel vector field along σ with $V(a) = v$, $V(b) = w$. The top letter n means "nonlinear". If it is missing, then it means the linear parallel translation. This is the canonical parallel translation on a Finsler manifold.

Let us also remark that for a C^∞ piecewise curve σ on a Finsler manifold (M, F), the nonlinear parallel translation preserves the Finslerian norm, i.e. if $W(t)$ is nonlinearly parallel along σ, then $F(\sigma(t), W(t)) =$ *constant* (for a proof of this fact see for example D. Bao's paper in the present volume).

In the case of Berwald manifolds, the nonlinear parallel translation has the following important property.

Lemma 6.2. *([I1978]) Let (M, F) be a Berwald manifold and $\sigma : [a, b] \to M$ a C^∞ curve on M with velocity vector field $T(t) = \dfrac{d\sigma}{dt}$. Then, the nonlinear parallel translation $P^n_\sigma : T_{\sigma(a)}M \to T_{\sigma(b)}M$ is a linear isomorphism.*

Proof. For a vector field $V = V^i(t)\dfrac{\partial}{\partial x^i}$ that is nonlinearly parallel along $\sigma(t)$, it follows

$$\left[\frac{dV^i}{dt} + V^j T^k \Gamma^i_{jk}(\sigma)\right] \frac{\partial}{\partial x^i}_{|\sigma(t)} = 0, \tag{6.8}$$

where $\Gamma^i_{jk}(\sigma, T) = \Gamma^i_{jk}(\sigma)$ are the coefficients of the Chern connection of the Berwald manifold (M, F).

One can see that (6.8) is a linear ODE in V. Therefore it induces a linear map between $T_{\sigma(a)}M$ and $T_{\sigma(b)}M$. The details follows immediately.

Q.E.D.

Geometrically, Lemma 6.2 states that for a Berwald manifold, parallel translation is an isometry of linear spaces.

Remark 6.3. In the proof of Lemma 6.2, the condition that the Finsler manifold (M, F) is a Berwald one is essential. Indeed, we used

$$V^j T^k \Gamma^i_{jk}(\sigma, V) = V^j T^k \Gamma^i_{jk}(\sigma). \tag{6.9}$$

This relation is not true for other Finsler manifolds that are not of Berwald type.

Remark 6.4. In a Berwald space (arbitrary dimension) the Chern connection defines a linear connection directly on the underlying manifold M.

Z. I. Szabo ([Sz1981]) proved that this linear connection is in fact the Levi–Civita connection of a (non-unique) Riemannian metric on M. In this way any Berwald space is **metrizable** by such a Riemannian metric. Of course the Finsler metric and induced Riemannian metric on M have the same geodesics as curves on M.

For a Landsberg surface, it is remarkable that the Gauss curvature K at any point $y(t)$ of the indicatrix S_xM is determined by the Cartan scalar I by

$$K(t) = K(0)e^{-\left[\int_0^t I(\tau)d\tau\right]}. \tag{6.10}$$

Indeed, from the Bianchi identities (5.15) on a Landsberg surface, i.e. when $J = 0$, one obtains the following ODE:

$$\dot{K}(t) + I(t)K(t) = 0.$$

Taking into account that $I(t)$ is continuous, the result follows directly by integration ([BCS2000]).

Obviously, any Berwald space (arbitrary dimension) is a Landsberg space. Even though there are many examples of Berwald spaces, unfortunately concrete example of Landsberg spaces that are not Berwald are not studied yet enough. Especially the construction of such examples in the two dimensional case remains to be considered in the future. See D. Bao's and Z. Shen's papers in the present volume, as well as Asanov's recent results [A2006].

For Berwald surfaces there exists the following rigidity result (see [Sz1981]).

Theorem 6.1. Rigidity theorem for Berwald surfaces

Let (M, F) be a connected Berwald surface for which the Finsler structure F is smooth and strongly convex on all of $\widetilde{TM}$.

(1) *If $K \equiv 0$, then F is locally Minkowski everywhere.*
(2) *If $K \not\equiv 0$, then F is Riemannian everywhere.*

Proof. We give here the idea of the proof.

If on a surface $K = 0$, then (5.33), (5.36) imply $R_j{}^i{}_{kl} = 0$.

On the other hand, the surface being Berwald it follows that the Chern connection coefficients depend on the position only, in other words $\Gamma^i_{jl}(x,y) = \Gamma^i_{jl}(x)$, therefore these coefficients determine a torsion free linear connection D on M having the Christoffel coefficients precisely $\Gamma^i_{jl}(x)$. It follows immediately that D must be flat on M. Therefore there is a preferential coordinate chart $x = (x^i)$ on M where $\Gamma^i_{jl}(x) = 0$, or equivalently, $\dfrac{\partial g_{ij}}{\partial x^k} = 0$. This is of course equivalent with $F(x,y) = F(y)$, i.e. F is locally Minkowski.

Conversely, if one assumes now that $K \not\equiv 0$, then there exists an indicatrix S_pM, at a certain $p \in M$, such that $K \neq 0$ at some point of it, and therefore nonzero at all points of this S_pM (because of (6.10)).

On a Berwald surface, the main scalar I is horizontally constant ([M1986], [BCS2000]), i.e. $I_1 = I_2 = 0$, where the subscripts mean directional derivatives with respect to the vectors of the g-orthonormal frame on SM.

Ricci identity (5.16) written for the Cartan scalar I on a Berwald surface implies $KI_3 = 0$, in other words on S_pM we have $K\dot{I} = 0$, and therefore I =constant on this S_pM.

Taking into account the fact that the average value of the Cartan scalar I over the indicatrix T_pM is zero ([BCS2000]), i.e. $\int_0^L I(t)dt = 0$, one sees that this constant has to be in fact zero. Therefore, $F(p,\tilde{y})$ has to be Riemannian for any $\tilde{y} \in S_pM$.

On the other hand, recall from Lemma 6.2 that on a connected Berwald surface the Minkowski plane $(T_xM, F(x,\cdot))$ is linearly isometric to $(T_pM, F(p,\cdot))$ for any $x \in M$.

Since M is connected there exists a smooth path σ from x to p. If we denote by (x,y) and $(p,\tilde{y})$ the corresponding points in TM, respectively, and taking into account that $\tilde{y}$ is related to y by a linear transformation depending on x and σ, it follows that $F(x,y)$ is equal to $F(p,\tilde{y})$ and therefore F is Riemannian everywhere.

Q.E.D.

The geometrical meaning of a Landsberg space is that the induced Riemannian tangent spaces $(\widetilde{T_xM}, g_x)$ are **isometric** to each other by nonlinear parallel translations.

Indeed, one has:

Lemma 6.3. ([I1978]) *Let (M,F) be a Landsberg manifold, and let $\sigma : [a,b] \to M$, $\sigma(a) = p \in M$, $\sigma(b) = q \in M$, be a piecewise C^∞ curve. Then, the nonlinear parallel translation*

$$P^n_\sigma : (\widetilde{T_pM}, g_p) \to (\widetilde{T_qM}, g_q) \tag{6.11}$$

is an isometry, where g_x is the induced Riemannian metric in T_xM, for any $x \in M$.

For a proof see the initial paper [I1978] or D. Bao's paper in the present volume for an alternative proof.

On Landsberg surfaces, the following interesting properties hold.

Theorem 6.2. *([BS1994])*
The indicatrix length of a Landsberg surface is constant.

Let us consider the Riemannian length of the indicatrix

$$S_x := \{y \in T_xM, \quad F(x,y) = 1, \quad x \in M\}.$$

Recall from §2 that the indicatrix length is computed by integrating the Landsberg angle.

Therefore, we have

$$L(x) = \int_{\mathbf{S}^1} \text{the pure } dy \text{ part of } {\omega_1}^2. \tag{6.12}$$

Some computations lead to the following lemma.

Lemma 6.4. *([BCS2000]) The Riemannian length of the indicatrix satisfies*

$$\frac{\partial L}{\partial x^i} = (-1)^i \int_{\mathbf{S}^1} J(l^{3-i}\sqrt{g})d\theta, \tag{6.13}$$

where $i \in \{1,2\}$, $d\theta$ is the 1-form given in (3.3), g is the determinant of the matrix g_{ij}, and J is the scalar in (5.14).

Therefore the constantly of indicatrix length follows. However, typically this is NOT 2π as already shown in §2.

§7. Curves on a Finsler surface

Let us consider the unit speed curve $\gamma : [a,b] \to M$ on a Finsler surface (M,F), given by $x^i = x^i(t)$, $T(t) = \dot{\gamma}(t)$, $F(\gamma(t), T(t)) = 1$, for any $t \in [a,b]$.

Using some ideas from [Sh2001], we are going to construct a Finslerian unit normal vector field N of γ such that the pair $\{T(t), N(t)\}$ is an oriented g_N-orthogonal basis in the fiber of π^*TM over the point $(\gamma(t), N(t))$.

Proposition 7.1. *For each fixed point $x(t)$, $t \in [a,b]$, on the curve γ, there exists a Finslerian unit length vector field $N(t) \in \widetilde{T_{x(t)}M}$ such that*

$$g_{N(t)}(N(t), T(t)) = 0. \tag{7.1}$$

Proof. For each $x = x(t) \in M$ fixed, let us regard $\widetilde{T_xM}$ as a Minkowski plane with the Minkowski norm $F(x,y) = F(y)$ induced by the Finslerian structure F of M.

For the sake of simplicity we omit writing the dependency on the parameter t of the curve. All the vectors are assumed to be taken along the curve γ.

First of all, we are going to construct the vector N. Consider an arbitrary vector V in the plane T_xM, other than T. Then the function

$$f : \mathbf{R} \to [0,\infty), \quad f(\lambda) := F(V - \lambda T)$$

attains its minimum for a unique value λ_0, i.e.

$$\min_{\lambda} F(V - \lambda T) = F(V - \lambda_0 T) = m. \tag{7.2}$$

The vector $\lambda_0 T$ is in fact the "projection" of V on T (see Figure 3). We define now

$$N := \frac{V - \lambda_0 T}{F(V - \lambda_0 T)} = \frac{V - \lambda_0 T}{m}. \tag{7.3}$$

Obviously, $F(N) = 1$.

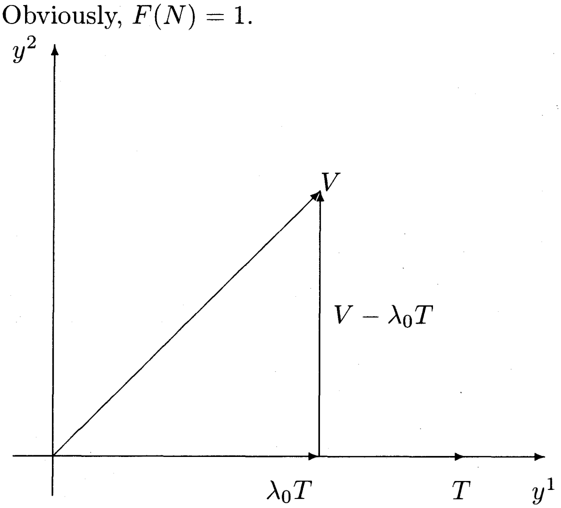

Figure 3. The construction of the normal vector field.

Next, we are going to show that this N **is independent on the choice of** V.

Let us consider another arbitrary vector $\widetilde{V} \in T_xM$ in the same upper half plane (determined by the direction of the vector T) as V. This vector can be written as a linear combination of V and T, for example

$$\widetilde{V} = \mu_1 V + \mu_2 T, \qquad \mu_1 > 0 \ . \tag{7.4}$$

Consider now the function

$$g : \mathbf{R} \to [0, \infty), \quad g(\widetilde{\lambda}) := F(\widetilde{V} - \widetilde{\lambda} T)$$

that attains its minimum for a unique value $\widetilde{\lambda}_0$, i.e.

$$\min_{\widetilde{\lambda}} F(\widetilde{V} - \widetilde{\lambda} T) = F(\widetilde{V} - \widetilde{\lambda}_0 T) = \widetilde{m}.$$

We can write now

$$\begin{aligned} \widetilde{m} &= \min_{\widetilde{\lambda}} F(\widetilde{V} - \widetilde{\lambda} T) = \min_{\widetilde{\lambda}} F(\mu_1 V + \mu_2 T - \widetilde{\lambda} T) \\ &= \mu_1 \min_{\widetilde{\lambda}} F\big(V - \frac{\widetilde{\lambda} - \mu_2}{\mu_1} T\big) = \mu_1 m. \end{aligned} \tag{7.5}$$

Since $F(V - \lambda T)$ attains its minimum at λ_0 (see (7.2)), it follows

$$\lambda_0 = \frac{\widetilde{\lambda}_0 - \mu_2}{\mu_1}. \tag{7.6}$$

Therefore, starting with $\widetilde{V}$ we can construct the normal vector

$$\widetilde{N} := \frac{\widetilde{V} - \widetilde{\lambda}_0 T}{\widetilde{m}}. \tag{7.7}$$

Using now (7.4), (7.5) and (7.6) in (7.7), a simple calculation shows that $\widetilde{N} = N$, therefore N does not depend on the choice of the direction V.

Finally, we are going to prove that this N **is** g_N **orthogonal to** T. Recall that from the homogeneity condition of F it follows

$$g_{ij}(y) y^i = \frac{1}{2} \frac{\partial F^2(y)}{\partial y^j}. \tag{7.8}$$

Indeed, F^2 is a 2-positive homogeneous function in y, therefore the Euler theorem for homogeneous functions gives $\dfrac{\partial F^2(y)}{\partial y^i} \cdot y^i = 2F^2(y)$. Taking now the derivative of this relation with respect to y^j, the relation (7.8) follows immediately.

For any V In the fiber of π^*TM over $(\gamma(t), N(t))$, relation (7.8) implies

$$g_N(N, W) = g_{ij}(N)N^iW^j = \frac{1}{2}\frac{\partial F^2(N)}{\partial y^j}W^j. \tag{7.9}$$

Consider now the function

$$h : \mathbf{R} \to [0, \infty), \quad h(\lambda) := \frac{1}{2}F^2(N + \lambda T). \tag{7.10}$$

We have

$$\begin{aligned} h(\lambda) &= \frac{1}{2}F^2\Big(\frac{V - \lambda_0 T}{m} + \lambda T\Big) = \frac{1}{2m^2}\, F^2\big(V - (\lambda_0 - m\lambda)T\big) \\ &\geq \frac{1}{2m^2}\, F^2(V - \lambda_0 T) = \frac{1}{2m^2}\, F^2(mN) = \frac{1}{2m^2}\, m^2 F^2(N) = \frac{1}{2}, \end{aligned}$$

for any λ, where we have used (7.2).

In other words, the minimum of h is $h(0) = \frac{1}{2}$.

On the other hand,

$$\begin{aligned} h'(0) &= \frac{1}{2}\frac{d}{d\lambda}\big[F^2(N + \lambda T)\big]_{|\lambda=0} \\ &= \frac{1}{2}\Big[\frac{\partial F^2}{\partial y^j}(N + \lambda T)\ \frac{d(N + \lambda T)^j}{d\lambda}\Big]_{|\lambda=0}. \end{aligned}$$

Taking now into account the minimum of the function h computed before, it follows

$$h'(0) = \frac{1}{2}\frac{\partial F^2}{\partial y^j}(N)T^j = g_N(N, T) = 0.$$

Q. E. D.

Remark 7.1. One can see that there exists exactly two normal vectors, one in the upper half plane determined by the direction of T, and the other one in the opposite half plane. However, remark that these two normals are not parallel unless F is absolute homogeneous.

We have therefore constructed a Finslerian unit normal vector field $N = N(t)$ along the curve γ with the following properties

(1) N is g_N-orthogonal to T,
(2) $F(N) = 1$.

We have to remark however, that in the fiber over $(\gamma(t), N(t))$ we have $F^2(T) \neq g_N(T,T)$. We denote this quantity by $\sigma^2 := g_N(T,T)$. Therefore, the frame $\{T(t), N(t)\}$ is not a g_N-orthonormal basis, but only a g_N-orthogonal one. In the particular case when M is a Riemannian surface with the Riemannian metric g, the pair $\{T(t), N(t)\}$ is a g-orthonormal basis.

Next, we will define the geodesic curvature for the curve γ on the Finsler surface (M, F).

Let us recall that if M is a Riemannian surface with the Riemannian metric g, then

$$\mathbf{K}(t) := D_T T = \Big(\frac{d^2x^i}{dt^2} + \gamma^i_{jk}(x)\frac{dx^j}{dt}\frac{dx^k}{dt}\Big)\frac{\partial}{\partial x^i}_{|\gamma(t)} \tag{7.11}$$

is called the **geodesic curvature vector** of the curve γ,

$$k(t) := \big[g(\mathbf{K}(t), \mathbf{K}(\mathbf{t}))\big]^{\frac{1}{2}} \tag{7.12}$$

is called the **geodesic curvature** of γ, and

$$k_N(t) := g(\mathbf{K}(t), N(t)) \tag{7.13}$$

is called the **signed curvature** of γ. Here, D is the Levi-Civita connection of g and $\gamma^i_{jk}(x)$ the Christoffel coefficients of D. Recall also that if γ is a geodesic of the Riemannian metric g, then $\mathbf{K}(t)$, $k(t)$ and $k_N(t)$ all vanish. In other words the geodesic curvature vector $\mathbf{K}(\mathbf{t})$ measures the failure of the curve γ from being a geodesic.

Now we return to the more general case of a Finsler surface. We would like to define a geodesic curvature vector and a geodesic curvature with similar properties as in the Riemannian case. Keeping in mind the geometrical meaning these quantities should have, one might define

$$\mathbf{K}^{(T)}(t) := D^{(T)}_T T = \Big(\frac{dT^i}{dt} + \Gamma^i_{jk}(x,T)T^jT^k\Big)\frac{\partial}{\partial x^i}_{|\gamma(t)} \tag{7.14}$$

and

$$k^{(T)}(t) := \big[g_T(\mathbf{K}^{(T)}(t), \mathbf{K}^{(T)}(t))\big]^{\frac{1}{2}}. \tag{7.15}$$

These quantities might be called the **geodesic curvature vector over** T, and the **geodesic curvature over** T of the curve γ, respectively.

Remark two things about the quantities defined in (7.14), (7.15):

(1) they are defined in the fiber of π^*TM over $(\gamma(t), T(t))$,

(2) if γ is a geodesic of the Finsler structure, in other words the tangent vector field is autoparallel along γ, i.e. $D_T^{(T)}T = 0$, then both $\mathbf{K}^{(T)}(t)$ and $k^{(T)}(t)$ vanish like in the Riemannian case (see [BCS2000] for more on Finslerian geodesics).

These definitions are used in [Sh2001].

However, if one wants to work with the normal vector field $N(t)$ constructed in Proposition 7.1., then the definitions (7.14), (7.15) are not satisfactory simply because our N lives in the fiber of π^*TM over $(\gamma(t), N(t))$.

We are led in this way to the following definitions. Let

$$\mathbf{K}^{(N)}(t) := D_T^{(N)}N = \Big(\frac{dN^i}{dt} + \Gamma^i_{jk}(x, N)N^j T^k\Big)\frac{\partial}{\partial x^i}\Big|_{\gamma(t)} \tag{7.16}$$

and

$$k^{(N)}(t) := \big[g_N(\mathbf{K}^{(N)}(t), \mathbf{K}^{(N)}(t))\big]^{\frac{1}{2}} \tag{7.17}$$

be the **geodesic curvature vector over** N, and the **geodesic curvature over** N of the curve γ, respectively.

We remark that the quantities defined in (7.16), (7.17)

(1) are defined in the fiber of π^*TM over $(\gamma(t), N(t))$,

(2) if γ is a geodesic, i.e. $D_T^{(T)}T = 0$, then $\mathbf{K}^{(N)}(t)$ and $k^{(N)}(t)$ do not vanish anymore as in the Riemannian case. Instead, if γ is a curve along which N is parallely displaced, i.e. $D_T^{(N)}N = 0$, then they will vanish. The reason behind this strange fact is that in the Riemannian case along a geodesic γ the conditions $D_T T = 0$ and $D_T N = 0$ are equivalent, but for a Finslerian surface, $D_T^{(T)}T = 0$ and $D_T^{(N)}N = 0$ are not.

Let us assume that γ is a smooth closed curve in the plane. Then we can regard γ as the boundary of a bounded open set $\Omega \subset M$. Recall that if γ is parametrized so that $T(t)$ is consistent with the induced orientation on $\gamma = \partial\Omega$ in the sense of Stokes' theorem, then γ is called **positively oriented**. Intuitively, this means that γ is parametrized in the counterclockwise direction, or that Ω is always to the left of γ ([L1997]).

We can regard now $\{T(t), N(t)\}$ as an oriented g_N-orthogonal basis for $T_{\gamma(t)}M$ for each t. If γ is positively oriented as the boundary of Ω, this is equivalent to N being the inward-pointing normal to $\partial\Omega$.

Using now (7.16) we define the **signed curvature over N** of γ by

$$k_N^{(N)}(t) := -g_N(\mathbf{K}^{(N)}(t), T(t)), \tag{7.18}$$

where $T(t)$ is considered now as a vector field in the fiber of π^*TM over $(\gamma(t), N(t))$.

Remark 7.2. If (M, F) is an absolute homogeneous Finsler metric, or a Berwald metric, then the sign of $k_N^{(N)}$ is positive if γ is curving towards Ω, and negative if it is curving away. This is the meaning of the word "signed" here. In the more general case of a Finsler metric (only positive homogeneous), the sign of $k_N^{(N)}$ is positive if γ is curving towards Ω, but it is difficult to predict how this changes if γ is curving away (see (7.16)).

We will see in the next section why we prefer our definitions (7.16), (7.17) to Shen's definitions (7.14), (7.15).

Lemma 7.1. *Let $U(t)$ and $V(t)$ be two vector fields along the curve γ. Then we have:*

(7.19)
$$\frac{d}{dt}\, g_N(U, V) = g_N(D_T^{(N)}U, V) + g_N(U, D_T^{(N)}V) + 2A(U, V, D_T^{(N)}N).$$

Proof. By straightforward computation and use of Lemma 5.1 we have:

(7.20)
$$\begin{aligned}\frac{d}{dt}\, g_N(U, V) &= \frac{d}{dt}\, g_{ij}(x, N)U^iV^j + g_{ij}(x, N)\Big(\frac{dU^i}{dt}\, V^j + U^i\frac{dV^j}{dt}\Big)\\ &= \Big(g_{sj}\Gamma^s_{ik} + g_{si}\Gamma^s_{jk} + 2A_{ijs}\frac{N^s_k}{F}\Big)(x, N)T^kU^iV^j\\ &+ 2\frac{A_{ijk}}{F}(x, N)\frac{dN^k}{dt}U^iV^j\\ &+ g_{ij}(x, N)\Big(\frac{dU^i}{dt}\, V^j + U^i\frac{dV^j}{dt}\Big),\end{aligned}$$

where $N^s_k(x, N) = N^j\Gamma^s_{jk}(x, N)$ is the nonlinear connection of F. Grouping now these terms, and taking into account of (6.6), relation (7.19) follows easily.

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Remark 7.3. *If one of the following conditions holds*
(1) U or V is proportional to N,
(2) N is parallel transported along γ,
(3) A vanishes along γ,
then,

$$\frac{d}{dt} g_N(U,V) = g_N(D_T^{(N)}U, V) + g_N(U, D_T^{(N)}V). \tag{7.21}$$

One might remark also that Lemma 7.1. is in fact the "with reference vector N" version of the Remark 6.2.

From Remark 7.3. it also follows that the signed curvature $k_N^{(N)}$ of γ defined in (7.18) can be also written as

$$k_N^{(N)}(t) = g_N(D_T^{(N)}T, N).$$

One can remark that in the Riemannian case this coincides with the usual signed curvature of γ defined in (7.13).

Proposition 7.2. *If γ is a smooth curve on the Finsler surface (M, F), then the following relations hold good*

$$\mathbf{K}^{(N)}(t) = -\frac{1}{\sigma^2(t)} k_N^{(N)}(t)T(t), \tag{7.22}$$

$$k^{(N)}(t) = \frac{1}{\sigma(t)} |k_N^{(N)}(t)|, \tag{7.23}$$

where $\sigma^2(t) = g_N(T,T)$, and $\mathbf{K}^{(N)}(t)$, $k^{(N)}(t)$, $k_N^{(N)}(t)$ are defined in (7.16), (7.17), (7.18), respectively.

Proof. Using Remark 7.3., in the case $U = V = N$, it follows $g_N(D_T^{(N)}N, N) = 0$. In other words, the geodesic curvature tensor $\mathbf{K}^{(N)}(t) = D_T^{(N)}N$ is g_N-orthogonal to N. This implies that there exists a smooth function f such that

$$\mathbf{K}^{(N)}(t) = D_T^{(N)}N = f(t)T(t), \tag{7.24}$$

where T is the tangent vector to γ in the fiber of π^*TM over (γ, N).

We compute now

$$g_N(\mathbf{K}^{(N)}, T) = g_N(fT, T) = fg_N(T,T) = f\sigma^2.$$

It follows

$$k_N^{(N)}(t) = -f(t)\sigma^2(t), \tag{7.25}$$

and therefore from (7.24), (7.25) we obtain (7.22).

Next, we compute

$$
\begin{aligned}
(k^{(N)}(t))^2 = g_N(\mathbf{K}^{(N)}(t), \mathbf{K}^{(N)}(t)) &= \frac{1}{\sigma^4(t)}(k_N^{(N)}(t))^2 g_N(T,T) \\
&= \frac{1}{\sigma^2(t)}(k_N^{(N)}(t))^2,
\end{aligned}
$$

and (7.23) follows immediately.

Q. E. D.

§8. The Gauss-Bonnet Theorem for Finsler surfaces

The proof of the Gauss-Bonnet theorem for Finsler manifolds without boundary was given by D. Bao and S. S. Chern in [BC1996] using the transgression method. Using their method we will extend the result to Finsler surfaces with smooth boundary.

Let (M, F) a compact Finsler surface with smooth boundary $\partial M = \gamma : [a, b] \mapsto M$, given by $x^i = x^i(t)$. We assume γ to be unit speed, i.e. $F(T) = 1$, where $T = \dot{\gamma}(t)$.

For an arbitrary vector field $V : M \to TM$, $x \longmapsto V(x) \in T_xM$, denote its zeros by $x_1, \cdots, x_k \in M$, and denote by i_α the index of V at x_α, for all $\alpha = 1, 2, \cdots, k$. By removing from M the interiors of the geodesic circles S_α^ε (centered at x_α of radius $\varepsilon > 0$), one obtains the manifold with boundary M_ε. Remark that in this case, the boundary of M_ε consists of the boundaries of the geodesic circles S_α^ε and the boundary of M.

Assuming that V has all zeros in $M \setminus \partial M$, it follows that V has no zeros on M_ε and therefore we can normalize it obtaining in this way the application

$$
X = \frac{V}{F(V)} : M_\varepsilon \to SM, \quad x \mapsto \frac{V(x)}{F(V(x))}. \tag{8.1}
$$

Using X we can lift M_ε to SM constructing in this way the 2-dimensional submanifold $X(M_\varepsilon)$ of SM.

Recall that on the 3-dimensional manifold SM one has the exterior forms, ${\Omega_1}^2$ and ${\omega_1}^2$, defined in (5.19) and (5.11), respectively.

Proposition 8.1. *Let (M, F) be a compact oriented Landsberg surface with boundary ∂M. And let $N : \partial M \to SM$ be the inward pointing Finslerian unit normal on ∂M.*

Then, we have

$$\frac{1}{L}\int_M K\sqrt{g}\, dx^1 \wedge dx^2 + \frac{1}{L}\int_{N(\partial M)} {\omega_1}^2 = \mathcal{X}(M), \tag{8.2}$$

where L is the Riemannian length of the indicatrix, K and g the Gauss curvature and the determinant of the fundamental tensor g_{ij} of the Finsler metric, respectively, N the inward pointing normal to the boundary ∂M, and $\mathcal{X}(M)$ the Euler characteristic of M.

The proof follows [BCS2000]. Indeed, remark first that we can extend the normal vector field N on γ to a vector field X on M with only finitely many zeros $x_1, x_2, \dots x_k \in M \setminus \partial M$. By considering the lift $X(M_\varepsilon)$ constructed above we integrate formula (5.25) over the two dimensional manifold $X(M_\varepsilon)$. Applying Stokes' theorem and taking the limit $\varepsilon \to 0$, we obtain

(8.3)

$$\int_M X^*({R_1}^2{}_{12}\omega^1 \wedge \omega^2 + P_{2111}\omega^1 \wedge {\omega_1}^2) = \sum_{\alpha=1}^{k} \lim_{\varepsilon\to 0}\int_{X(S_\alpha^\varepsilon)} {\omega_1}^2 + \int_{N(\partial M)} {\omega_1}^2,$$

where the lift $X(S_\alpha^\varepsilon)$ of each boundary cycle is traced out in the clockwise direction, and N is the inward pointing normal to the boundary ∂M.

From the degree theory (see for example [Mil1965]) it results that

$$\int_{X(S_\alpha^\varepsilon)} {\omega_1}^2 \longrightarrow -i_\alpha \int_{S_{x_\alpha}M} {\omega_1}^2$$

as $\varepsilon \to 0$. Here the indicatrix $S_{x_\alpha}M$ is given in the counterclockwise orientation.

Recall now that

$${\omega_1}^2 = \frac{\sqrt{g}}{F}\Big(y^1\frac{\delta y^2}{F} - y^2\frac{\delta y^1}{F}\Big),$$

where $\delta y^i = dy^i + N^i{}_j dx^j$. Taking the limit of the integral ${\omega_1}^2$ implies that actually the dx terms do not contribute anymore because the metric radius continuously shrinks. It follows that

$$\lim_{\varepsilon\to 0}\int_{S_{x_\alpha}M} {\omega_1}^2 = \lim_{\varepsilon\to 0}\int_{\mathbf{S}^1} \frac{\sqrt{g}}{F^2}\Big(y^1 dy^2 - y^2 dy^1\Big) = L(x_\alpha), \tag{8.4}$$

where $L(x_\alpha)$ is the Riemannian length of the indicatrix at $x_\alpha \in M$.

Let us consider now the case of Landsberg surfaces. There are three important results that make these surfaces special.

(1) On a Landsberg surface $P_{2111} = -J = 0$, therefore the integrand in the left hand side of (8.3) reads now ${\Omega_1}^2 = d{\omega_1}^2 = {R_1}^2{}_{12}\, \omega^1 \wedge \omega^2$. In natural coordinates we have

$$
{\Omega_1}^2 = d{\omega_1}^2 = -K\sqrt{g}\; dx^1 \wedge dx^2.
$$

(2) Moreover, even though $K = K(x,y)$ and $\sqrt{g} = \sqrt{g}(x,y)$ are functions on SM, their product $K\sqrt{g} = (K\sqrt{g})(x)$ lives on M. Indeed, taking exterior derivative of $d{\omega_1}^2 = -K\sqrt{g}\; dx^1 \wedge dx^2$ it follows

$$
d(K\sqrt{g}) \wedge dx^1 \wedge dx^2 = 0,
$$

or equivalently,

$$
\left(\frac{\partial (K\sqrt{g})}{\partial x^i}\, dx^i + \frac{\partial (K\sqrt{g})}{\partial y^i}\, dy^i \right) \wedge dx^1 \wedge dx^2 = 0.
$$

Taking into account the properties of exterior differential d, it follows $\dfrac{\partial (K\sqrt{g})}{\partial y^i} = 0$. This was remarked by S. S. Chern for the first time (see [BC1996] for details).

Therefore, the first term in the left hand side of (8.3) simplifies to

$$
\int_M -K\sqrt{g}\; dx^1 \wedge dx^2. \tag{8.5}
$$

(3) Finally, recall that on a Landsberg surface, the indicatrix length $L(x)$ is a constant, but this constant typically is not equal to 2π as in Riemannian case. From (8.4) it follows that on a Landsberg surface we have

$$
\sum_{\alpha=1}^{k} \lim_{\varepsilon \to 0} \int_{X(S_\alpha^\varepsilon)} {\omega_1}^2 = -\sum_{\alpha=1}^{k} i_\alpha L(x_\alpha) = -L \sum_{\alpha=1}^{k} i_\alpha = -L\mathcal{X}(M). \tag{8.6}
$$

Using now (8.5) and (8.6) in (8.3) we obtain two of the three terms in (8.2).

We are going to deduce now the remainig term on the left hand side of (8.2).

In order to do this, let us remark first that the normal $N(t)$ introduced in Proposition 7.1. can be regarded as the mapping

$$
N : \partial M = \gamma \to SM \quad \gamma(t) \mapsto (\gamma(t), N(t)), \tag{8.7}
$$

where SM is the projective sphere bundle of F.

This gives the lift $\hat{\gamma} = N(\partial M)$ of the curve γ to SM given by

$$\hat{\gamma} : [0,1] \to SM, \quad \hat{\gamma}(t) = (\gamma(t), N(t)). \tag{8.8}$$

Remark that $\hat{\gamma}$ is different from the canonical lift $(\gamma(t), \dot{\gamma}(t))$ of γ used in [BCS2000].

The tangent vector $\hat{T}$ to $\hat{\gamma}$ in a point $u = (\gamma(t), N(t)) \in \hat{\gamma}$ is given by

$$\begin{aligned}\hat{T} &= \frac{d}{dt}\hat{\gamma} = \dot{\gamma}^i(t)\frac{\partial}{\partial x^i}|_{(\gamma,N)} + \frac{dN^i(t)}{dt}\frac{\partial}{\partial y^i}|_{(\gamma,N)} \\ &= \dot{\gamma}^i(t)\frac{\delta}{\delta x^i}|_{(\gamma,N)} + \left[\frac{dN^i(t)}{dt} + N^j(t)\dot{\gamma}^k(t)\Gamma^i_{jk}(\gamma,N)\right]\frac{\partial}{\partial y^i}|_{(\gamma,N)} \\ &= \dot{\gamma}^i(t)\frac{\delta}{\delta x^i}|_{(\gamma,N)} + (D_T^{(N)}N)^i\frac{\partial}{\partial y^i}|_{(\gamma,N)},\end{aligned} \tag{8.9}$$

where the functions Γ^i_{jk} are the local coefficients of the Chern connection, and $(D_T^{(N)}N)^i$ are the components in the natural basis of the fiber of the covariant derivative along γ with reference vector N defined in (6.6).

One can easily see that this $\hat{T}$ gives the derivative map of the mapping $N : \partial M \to SM$. Indeed, (8.7) implies

$$N_{*,\gamma(t)} : T_{\gamma(t)}\partial M \to T_{(\gamma(t),N(t))}SM, \quad N_{*,\gamma(t)}\left(\frac{d}{dt}|_{\gamma(t)}\right) = \hat{T}(t), \tag{8.10}$$

where $\frac{d}{dt}|_{\gamma(t)}$ is the natural basis of $T_{\gamma(t)}\partial M$.

Now we can prove the following important result.

Proposition 8.2. *On the Finsler manifold (M, F) with smooth boundary $\partial M = \gamma : [a,b] \to M$ we have*

$$\int_{N(\partial M)} {\omega_1}^2 = \int_\gamma \frac{1}{\sigma(t)}\, k_N^{(N)}(t)dt, \tag{8.11}$$

where N is the inward pointing normal on γ, ${\omega_1}^2$ the Chern connection form defined in (5.13), and $\sigma = \sqrt{g_N(T,T)}$.

Proof.

Using (8.9) and (8.10) we compute

(8.12)

$$\begin{aligned}N^*({\omega_1}^2)\frac{d}{dt}_{|\gamma(t)} &= {\omega_1}^2(N_*\frac{d}{dt})\\&= {\omega_1}^2\Big[\dot\gamma^i(t)\frac{\delta}{\delta x^i}_{|(\gamma,N)} + (D_T^{(N)}N)^i\frac{\partial}{\partial y^i}_{|(\gamma,N)}\Big]\\&= \dot\gamma^i(t){\omega_1}^2\Big(\frac{\delta}{\delta x^i}_{|(\gamma,N)}\Big) + (D_T^{(N)}N)^i{\omega_1}^2\Big(\frac{\partial}{\partial y^i}_{|(\gamma,N)}\Big).\end{aligned}$$

On the other hand, recall the local expression (5.13) of the connection form ${\omega_1}^2$, and taking into account the duality of the adapted basis $\{\frac{\delta}{\delta x^i}, F\frac{\partial}{\partial y^i}\}$ and cobasis $\{dx^i, \frac{\delta y^i}{F}\}$ it follows that the first term in the last equality of (8.12) vanishes and therefore we obtain

(8.13)

$$\begin{aligned}N^*({\omega_1}^2)\frac{d}{dt}_{|\gamma(t)} &= (D_T^{(N)}N)^i{\omega_1}^2\Big(\frac{\partial}{\partial y^i}_{|(\gamma,N)}\Big)\\&= (D_T^{(N)}N)^1{\omega_1}^2\Big(\frac{\partial}{\partial y^1}_{|(\gamma,N)}\Big) + (D_T^{(N)}N)^2{\omega_1}^2\Big(\frac{\partial}{\partial y^2}_{|(\gamma,N)}\Big)\\&= \sqrt{g}(\gamma,N)\Big[-y^2(D_T^{(N)}N)^1 + y^1(D_T^{(N)}N)^2\Big]_{|(\gamma,N)}\\&= \sqrt{g}(\gamma,N)\Big[-N^2(D_T^{(N)}N)^1 + N^1(D_T^{(N)}N)^2\Big].\end{aligned}$$

We used here implicitly that $F(\gamma, N) = 1$.

We would like now to express this relation using the signed geodesic curvature $k_N^{(N)}$ defined in (7.18). In order to do this, remark that writing (7.21) for $U = V = N$, and taking into account $F(N) = 1$, it follows $g_N(D_T^{(N)}N, N) = 0$. We write this last relation and (7.18) in the components with respect of the natural basis of the fiber of π^*TM over (γ, N). We have

$$\begin{aligned}&(D_T^{(N)}N)^1(N^1g_{11} + N^2g_{12}) + (D_T^{(N)}N)^2(N^1g_{12} + N^2g_{22}) = 0\\&(D_T^{(N)}N)^1(\dot\gamma^1g_{11} + \dot\gamma^2g_{12}) + (D_T^{(N)}N)^2(\dot\gamma^1g_{12} + \dot\gamma^2g_{22}) = -k_N^{(N)},\end{aligned}$$

where g_{ij}, $i, j \in \{1, 2\}$ are the components of the Riemannian metric g_N of the fiber of π^*TM over the point (γ, N) with respect to the natural basis of the fiber.

This is a system of linear equations with the solution

$$
(8.14)\qquad
\begin{aligned}
(D_T^{(N)}N)^1 &= -k_N^{(N)}\frac{N^1g_{12}+N^2g_{22}}{g(N^2\dot\gamma^1-N^1\dot\gamma^2)}\\
(D_T^{(N)}N)^2 &= k_N^{(N)}\frac{N^1g_{11}+N^2g_{12}}{g(N^2\dot\gamma^1-N^1\dot\gamma^2)}.
\end{aligned}
$$

Substituting now (8.14) in the last equality of (8.13) we have

(8.15)

$$
\begin{aligned}
N^*({\omega_1}^2)\frac{d}{dt}_{|\gamma(t)} &= k_N^{(N)}\frac{N^2(N^1g_{12}+N^2g_{22})+N^1(N^1g_{11}+N^2g_{12})}{\sqrt{g}(N^2\dot\gamma^1-N^1\dot\gamma^2)}\\
&= k_N^{(N)}\frac{g_N(N,N)}{\sqrt{g}(N^2\dot\gamma^1-N^1\dot\gamma^2)}=\frac{k_N^{(N)}}{\sqrt{g}(N^2\dot\gamma^1-N^1\dot\gamma^2)}.
\end{aligned}
$$

On the other hand, we remark that

$$
(8.16)\qquad \sqrt{g}(N^2\dot\gamma^1-N^1\dot\gamma^2)=\sqrt{g}dx^1\wedge dx^2(T,N)=\sqrt{\bar g}\theta^1\wedge\theta^2(T,N),
$$

where θ^1 and θ^2 are the dual 1-forms of T and N, respectively, and $\bar g$ is the determinant of the Riemannian metric in the fiber of π^*TM over (γ, N) with respect to the g_N orthogonal basis $\{T, N\}$, i.e.

$$
(8.17)\qquad \bar g_{ij}(\gamma,N)=\begin{pmatrix}1&0\\0&\sigma^2\end{pmatrix}
$$

and therefore its determinant is $\bar g=\sigma^2$.

It follows

$$
(8.18)\qquad \sqrt{g}(N^2\dot\gamma^1-N^1\dot\gamma^2)=\sigma
$$

and from here we obtain

$$
(8.19)\qquad N^*({\omega_1}^2)\frac{d}{dt}_{|\gamma(t)}=\frac{k_N^{(N)}}{\sigma(t)}.
$$

Finally, we have

$$
(8.20)\qquad \int_{N(\partial M)}{\omega_1}^2=\int_{\partial M}N^*({\omega_1}^2)=\int_\gamma N^*({\omega_1}^2)\frac{d}{dt}_{|\gamma(t)}dt=\int_\gamma\frac{k_N^{(N)}}{\sigma(t)}dt.
$$

Q. E. D.

From Propositions 8.1. and 8.2. we obtain the main result of this section.

Theorem 8.1. Gauss-Bonnet theorem for Landsberg surfaces with smooth boundary.

Let (M, F) be a compact, connected Landsberg surface with unit velocity smooth boundary γ. Then

$$\frac{1}{L}\int_M K\sqrt{g}\, dx^1 \wedge dx^2 + \frac{1}{L}\int_\gamma \frac{1}{\sigma(t)}\, k_N^{(N)}(t)dt = \mathcal{X}(M), \tag{8.21}$$

where L is the Riemannian length of the indicatrix of (M, F), $k_N^{(N)}$ the signed curvature over N of γ, the scalar $\sigma = \sqrt{g_N(T, T)}$, and $\mathcal{X}(M)$ the Euler characteristic of M.

Remark 8.1. (1) On a Berwald surface without boundary an alternative Gauss-Bonnet formula can be given using the Riemannian metrizability of the Berwald surface ([BCS1996]). Indeed, let us denote by h the (non-unique) Riemannian metric on M having the Levi-Civita connection coefficients identical with the Chern connection coefficients of (M, F). Denote by ${}^hK$ its Riemannian Gauss curvature. Therefore the following formula also holds:

$$\int_M {}^hK\sqrt{h}dx^1 \wedge dx^2 = 2\pi \cdot \mathcal{X}(M). \tag{8.22}$$

(2) In the case of a Landsberg surface without boundary Theorem 8.1 reduces to the Gauss-Bonnet theorem for Landsberg surfaces without boundary [BC1996].

(3) In the case of a Riemannian surface with smooth boundary, since the Riemannian metric has no directional dependence, and the indicatrix length of a Riemannian manifold is equal to 2π, Theorem 8.1 reduces to the usual Gauss-Bonnet theorem for Riemannian manifolds with smooth boundary [Spiv1979], [L1997].

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