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A connectedness principle in positively curved Finsler manifolds

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Abstract.

We study a connectedness principle in positively curved Finsler manifolds. Some results from positively curved Riemannian manifolds are generalized to Finsler spaces, and we also emphasize differences between the Riemann and Finsler settings.

§1. Introduction

In algebraic and in Riemannian geometry connectedness principles give uniform formulation for many classical results, for example Synge's theorem, Frankel theorem and Wilking theorem for totally geodesic submanifolds.

Here we develop this tool in the Finsler setting. Here the situation is much more complicated than in the Riemannian context. The variation of the energy applied to a geodesic with the ends on submanifolds gives rise naturally to a second fundamental form (see [17]). The statement that a submanifold is totally geodesic (that is geodesics of the submanifold are also geodesics for the ambient manifold) is equivalent to the statement that the second fundamental form vanishes only for Berwald spaces. That is because the reference vector of the second fundamental form (which appear in the connection coefficients) is not tangent to the submanifold. The asymptotic index is defined via the second fundamental form and a connectedness principle is developed using the asymptotic index. But the results concerning totally geodesic submanifolds are true for Berwald spaces (in these spaces the asymptotic index is equal to the dimension of submanifold iff the submanifold is totally geodesic).

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In the Riemannian case such kind of results are obtained in terms of asymptotic index, totally geodesic submanifolds or extrinsic curvature of a submanifold (see [5, 8, 9] in the last notion). In the Finsler setting, using our variational approach, the connectedness results involving the asymptotic index can be extended to Finsler spaces, the results concerning totally geodesic submanifolds are proved for Berwald spaces and the results involving the extrinsic curvature are not treated because even in Berwald spaces, where the reference vector is irrelevant for the connection coefficients, and further for the curvature tensor, the inner products which appear in the flag curvature have a dependence of the reference vector.

§2. Preliminaries

Let M be a real manifold of dimension m and (TM, π, M) the tangent bundle of M. The vertical bundle of the manifold M is the vector bundle $(\mathcal{V}, \overline{\pi}, M)$ given by $\mathcal{V} = \operatorname{Ker} \pi \subset T(TM)$. (x^i) will denote the local coordinates on an open subset U of M, and (x^i, y^i) are the induced coordinates on $\pi^{-1}(U) \subset TM$. The radial vector field ι is locally given by $\iota(x, y) = y^i \dfrac{\partial}{\partial x^i}$.

A Finsler metric on M is a function $F : TM \to \mathbb{R}_+$ satisfying the following properties:

(1) F is smooth on $\widetilde{M}$ where $\widetilde{M} = TM \setminus 0$
(2) $F(x, y) > 0$ for all $(x, y) \in \widetilde{M}$
(3) $F(x, \lambda y) = \lambda F(x, y)$ for all $x \in M, y \in T_x M$, $\lambda \in \mathbb{R}^+$
(4) the quantities $g_{ij}(x, y) = \dfrac{1}{2} \dfrac{\partial^2 F^2(x, y)}{\partial y^i \partial y^j}$ form a positive definite matrix.

A manifold M endowed with a Finsler metric F is called a Finsler manifold (M, F).

Condition 4 is equivalent to the fact that that for any $x \in M$ the indicatrix $I_x = \{y \in T_p M |\ F(x, y) < 1\}$ is strongly convex and also implies that the quantities
$g_{ij}(x, y) = \dfrac{1}{2} \dfrac{\partial^2 F^2(x, y)}{\partial y^i \partial y^j}$ induce a Riemannian metric $\langle \cdot, \cdot \rangle$ on the vertical bundle $(\mathcal{V}, \overline{\pi}, TM)$.

On a Finsler manifold there is not, in general, a linear metrical connection. The analogue of the Levi-Civita connection lives just in the vertical bundle, however, there are several of them.

In this paper we use the Cartan connection ([1]) which is a good vertical connection on $\mathcal{V}$, i.e. an $\mathbb{R}$-linear map

$$\nabla^v : \mathfrak{X}(\widetilde{M}) \times \mathfrak{X}(\mathcal{V}) \to \mathfrak{X}(\mathcal{V})$$

having the usual properties of a covariant derivation, metrical with respect to $\langle \cdot, \cdot \rangle$, and 'good' in the sense that the bundle map $\Lambda : T\widetilde{M} \to \mathcal{V}$ defined by $\Lambda(Z) = \nabla^v_Z \iota$ is a bundle isomorphism when ∇^v restricted to $\mathcal{V}$. The latter property induces the horizontal subspaces $H_u = \operatorname{Ker} \Lambda$ for all $u \in \widetilde{M}$ which are direct summands of the vertical subspaces $V_u = \operatorname{Ker}(d\pi)_u$:

$$T\widetilde{M} = \mathcal{H} \oplus \mathcal{V}.$$

For a tangent vector field X on M we have its vertical lift X^V and its horizontal lift X^H to $T\widetilde{M}$.

$\Theta : \mathcal{V} \to \mathcal{H}$ denotes the horizontal map associated to the horizontal bundle $\mathcal{H}$. Using Θ, first we get the radial horizontal vector field $\chi = \Theta \circ \iota$. In our case $\sigma^H = \chi(\dot{\sigma})$. Secondly we can extend the covariant derivation ∇^v of the vertical bundle to the whole tangent bundle of $\widetilde{M}$. Denoting it with ∇, for horizontal vector fields we have

$$\nabla_Z H = \Theta(\nabla^v_Z(\Theta^{-1}(H))), \ \forall \ Z \in \mathfrak{X}(\widetilde{M}).$$

An arbitrary vector field $Y \in \mathfrak{X}(\widetilde{M})$ is decomposed into vertical and horizontal parts $Y = Y^V + Y^H$, then we obtain

$$\nabla_Z Y = \nabla_Z Y^V + \nabla_Z Y^H.$$

Thus $\nabla : \mathfrak{X}(\widetilde{M}) \times \mathfrak{X}(T\widetilde{M}) \to \mathfrak{X}(T\widetilde{M})$ is a linear connection on $\widetilde{M}$ induced by a good vertical connection. The connection coefficients depends on $(x, y) \in \widetilde{M} = TM \setminus 0$, and y will be called reference vector. Its torsion θ and curvature R are defined as usual:

$$\nabla_X Y - \nabla_Y X = [X, Y] + \theta(X, Y)$$
$$R_Z(X, Y) = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

and the torsion has the property that for horizontal vectors $\theta(X, Y)$ is a vertical vector [1]. Specially the horizontal flag curvature of ∇ along a curve σ is given as follows:

$$R_{\dot{\sigma}}(U^H, U^H) = \langle \Omega(\dot{\sigma}^H, U^H) U^H, \dot{\sigma}^H \rangle$$

for any $U \in \mathfrak{X}(M)$. This is called the horizontal flag curvature in [1] and gives the flag curvature of [4, 18] when $\dot{\sigma}^H, U^H$ are orthonormal with respect to $\langle , \rangle_{\dot{\sigma}}$.

The metrical property of the Cartan connection is also important [1]:

$$X\langle Y, Z\rangle = \langle\nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle.$$

Let N be a submanifold of M of dimension $p < m$. We consider the set

$$A = \{(x, v) | x \in N, v \in T_x M\} = \{\widetilde{x} \in \widetilde{M} | \pi(\widetilde{x}) \in N\}.$$

We consider $H_{\widetilde{x}} T_x M$ and $H_{\widetilde{x}} T_x N$ be the horizontal liftings of $T_x M$ and $T_x N$ respectively along $\widetilde{x}$ and

$$H_N TM = \bigcup_{\widetilde{x} \in A} H_{\widetilde{x}} T_x M$$

and

$$H_N TN = \bigcup_{\widetilde{x} \in A} H_{\widetilde{x}} T_x N.$$

Let $N_{\widetilde{x}}^{\perp}$ be the $\langle\cdot,\cdot\rangle_{\widetilde{x}}$ orthogonal complement of $H_{\widetilde{x}} TN$ in $H_{\widetilde{x}} TM$. Let $X, Y \in H_N TN$ and let X^*, Y^* be their prolongations to $H_N TM$ (that is if $X, Y \in H_{\widetilde{x}} T_x N$ for some $\tilde{x} \in TM$ it follows that $X^*, Y^* \in H_{\widetilde{x}} T_x M$). The restriction of $\nabla_{X^*} Y^*$ to $\widetilde{N}$ does not depend on the choice of the prolongation. By the orthogonal decomposition

$$H_{\widetilde{x}} T_x M = H_{\widetilde{x}} T_x N \oplus N_{\widetilde{x}}^{\perp}$$

we obtain that

$$\nabla_{X^*} Y^* = \nabla_X^* Y + \mathbb{I}(X, Y).$$

We will call $\mathbb{I}(X, Y)$ the second fundamental form at X and Y. Note that for $\widetilde{x} = (x, v) \in A$ with $v \in T_x M \setminus T_x N$ we have

$$\langle\nabla_{X^*} Y^*, v^H\rangle_v = \mathbb{I}_v(X, Y)$$

and we call it the second fundamental form of X and Y in the direction of v (note that v is also the reference vector in the covariant derivative). We will be interested mostly in the sign of the second fundamental form.

Let $f : N \to M$ be an immersion. The asymptotic index of the immersion f in the direction is defined by

$$\nu_f = \min_{x \in N} \nu_f(x)$$

where $\nu_f(x)$ is the maximal dimension of a subspace of $T_x N$ on which the second fundamental form vanishes in every direction $v \in T_x M \setminus T_x N$. The submanifold N will be called a totally geodesic submanifold (in the analytic sense) if and only if $\nu(f) = \dim N$.

Generally, for the notions and facts from algebraic topology we used the books of A. Hatcher [11] and G. W. Whitehead [20].

§3. Morse Theory on path space

At this points we are ready to apply Morse theory to obtain several connectedness principles on positively curved manifolds. First we present some technical constructions.

Let M be a connected Finsler manifold and $P(M)$ denote the path space of the manifold with the topology induced by the metric

$$d(\alpha_0, \alpha_1) = (\int_0^1 (F(\dot{\alpha_0}) - F(\dot{\alpha_1}))^2 dt)^{\frac{1}{2}} + \max_{t\in[0,1]} \mathrm{d}_M(\alpha_0(t), \alpha_1(t)),$$

which is well defined even if α_0 and α_1 have finitely many cusps (here d_M denote the metric induced on M by the Finsler metric).

Consider the projection map $p : P(M) \to M \times M$ given by $p(\alpha) = (\alpha(0), \alpha(1))$ which defines a Serre fibration given by $\Omega(M) \to P(M) \to M \times M$ where the fiber $\Omega(M)$ is the loop space of M with a fixed basepoint.

For a manifold N and a smooth map $f : N \to M\times M$ we consider the pullback fibration by f, $\Omega(M) \to P(M, f) \to N$. $P(M, f) \subset N \times P(M)$ consists of (x, α) such that $f(x) = (\alpha(0), \alpha(1))$ and has the induced topology.

3.1. Morse theory on $P(M, f)$.

We will study in this section the space $P(M, f)$ from the Morse theory of the energy functional $E(x, \alpha) = \frac{1}{2}\int_0^1 F^2(\dot{\alpha}(t))dt$. By the results from [12] any critical point (x, α) of the energy functional E is a geodesic for which $(\dot{\alpha}(0), -\dot{\alpha}(1))^H$ is $\langle\ ,\ \rangle_{\dot{\alpha}}$ orthogonal to $(f_*(T_x(N)))^H$. We will restrict in this section to compact manifolds.

Theorem 1. *Let M and N be compact Finsler manifolds, and $f : N \to M \times M$ an isometric immersion, and let $\Delta \subset M \times M$ be the diagonal. Assume that every nontrivial critical point (x, α) of E has index $I_\alpha \geq \lambda_0$. Then the following assertions are true.*

(1) *If $\lambda_0 \geq 1$, then $f^{-1}(\Delta) \neq \emptyset$.*
(2) *If $\lambda_0 \geq 2$ and M is simply connected, then $f^{-1}(\Delta)$ is connected. If in addition $f = f_1 \times f_1 : N = N_1 \times N_1 \to M \times M$, where f is an embedding, then*
(3) *$\pi_i(P(M, f), f^{-1}(\Delta)) = 0$ for all $i < \lambda_0$.*
(4) *For $\lambda_0 \geq i$, then there is an exact sequence of homotopy groups,*

$$\pi_i(f^{-1}(\Delta)) \longrightarrow \pi_i(N) \xrightarrow{(p_1 f)_* - (p_2 f)_*} \pi_i(M) \longrightarrow$$
$$\longrightarrow \pi_{i-1}(f^{-1}(\Delta)) \longrightarrow \dots,$$

where p_1 and p_2 are the projections to the factors.

Proof. (1) $f^{-1}(\Delta) = \emptyset$ implies that E has an absolute minimum at some non-trivial critical point (x, α), hence its index should be zero, this is a contradiction.

(2) At this step we use the finite approximation of the path space proved for the Finsler setting by Dazord ([6] p. 129-134).

Let $P_c(M, f) = E^{-1}([0, c))$ be an open subset of $P(M, f)$ and $B_c^k(M, F) \subset P_c(M, f)$ the space of piecewise smooth geodesics with k-cusps (with each piece of length less than the injectivity radius). For k sufficiently large $B_c^k(M, f)$ and $P_c(M, f)$ are homotopy equivalent. Being the space $B_c^k(M, f)$ formed by k-broken geodesic it can be identified with an open submanifold of the product $N \times M \times \ldots \times M$ (k copies of M). E is a proper function when restricted to $B_c^k(M, f)$ and furthermore $E|_{B_c^k(M,f)}$ and $E|_{P_c(M,f)}$ have the same critical points with identical indices ([6] p. 129-134).

Suppose, by contrary, that $f^{-1}(\Delta)$ is not connected. In this case there exist disjoint non-empty compact subsets A and B such that $A \cup B = f^{-1}(\Delta)$. We can think at $f^{-1}(\Delta)$ as the set of constant paths in $P(M, f)$. Let $p \in A$ and $q \in B$.

The manifold M being simply connected it follows that the loop space $\Omega(M)$ is path connected, and it implies that $P(M, f)$ is also path connected, and, furthermore, there exists a path α_0 in $P(M, f)$ joining p and q. By the previous observations related to $B_c^k(M, f)$ we can choose a path $\alpha_0 \in B_{c_0}^k(M, f)$ joining p and q for some constant $c_0 > 0$ and some $k \in \mathbb{N}$, enough large such that $B_{c_0}^k(M, f)$ has the same homotopy type as $P_c(M, f)$.

Let $X = B_{c_0}(M, f)$ with the induced product metric [15] from $N \times M \times \ldots M$ (k times of M) and consider $g = E|_X$. By an above consideration we identify $f^{-1}(\Delta)$ with $g^{-1}(0)$. We will prove (3) under the assumption that there exists a sequence of connected paths $\alpha_k : [0, 1] \to g^{-1}[0, \frac{1}{k}]$, $(k \geq 1)$ in X with $\alpha_k(0) = p$ and $\alpha_k(1) = q$, in the homotopy class of $[\alpha_0]$, keeping the endpoints fixed. By the compactness of A and B the distance $d_X(A, B) > 0$. Consider the function $a(x) = d(x, A) - d(x, B)$. We can see that $a|_{\alpha_k}$ satisfies $a(p) < 0$ and $a(q) > 0$, so there exists a point $x_k \in \alpha_k$ with $a(x_k) = 0$. Now $(x_k)_{k \in \mathbb{N}}$ contains a convergent subsequence, so we can assume that (x_k) itself is convergent to a point x, and $\lim_{k \to \infty} g(x_k) = 0$. Thus we obtain a contradiction $x \in g^{-1}(0)$ and $a(x) = 0$, since $A \cap B \neq \emptyset$.

By Corollary 6.8 in [16] and from the fact that $\lambda_0 \geq 2$, there exists a Morse function h on X such that $|g - h| \leq \frac{1}{10k}$ on the sublevel set $X^{\leq c_0 + \frac{1}{2}} = \{x \in X, h(x) \leq c_0 + \frac{1}{2}\}$, such that the critical points of h

in the set $h^{-1}([\frac{1}{2k}, c_0 + \frac{1}{2}])$ have Morse index greater than 1, so from Morse Theory it follows that $h^{-1}(-\infty, c_0 + \frac{1}{2}]$ is homotopy equivalent to $h^{-1}(-\infty, \frac{1}{2k}]$ by gluing cells of dimensions at least 2. But this implies that the relative homotopy group $\pi_1((-\infty, c_0 + \frac{1}{2}], (-\infty, \frac{1}{2k}]) = 0$, so α_0 is homotopic to a path α_k in $h^{-1}(-\infty, \frac{1}{2k}]$ with the same endpoints p and q fixed.

(3) It is enough to show that $E^{-1}(0) = f^{-1}(\Delta) \subset B_c(M, f)$ has an open neighborhood $U \in B_c(M, f)$ which is a deformation retract of $f^{-1}(\Delta)$. The existence of such a neighborhood will be given in Proposition 2.

(4) We have that $\pi_i(\Omega M) \cong \pi_{i+1}(M)$. Further we have the diagram

$$\begin{array}{ccccccc}
\ldots\pi_i(P(M,f)) & \longrightarrow & \pi_i(N) & \xrightarrow{\phi} & \pi_i(M) & \longrightarrow & \pi_{i-1}(P(M,f))\ldots \\
\tilde{f}_* \downarrow & & f_* \downarrow & & = \downarrow & & \tilde{f}_* \downarrow \\
\ldots\pi_i(P(M)) & \xrightarrow{p_*} & \pi_i(M \times M) & \xrightarrow{\delta_*} & \pi_i(M) & \longrightarrow & \pi_{i-1}(P(M))\ldots
\end{array}$$

because $\Omega(M) \to P(M, f) \to N$ is the pullback of the Serre fibration $\Omega(M) \to P(M) \to M \times M$ via the immersion $f : N \to M \times M$. Being $P(M)$ homotopic to M it follows that $p : M \to M \times M$ is homotopic to the diagonal map. Denoting p_i the projection of $M \times M$ to the i-th factor, then $\delta_* = (p_1)_* - (p_2)_*$. ¿From the diagram we have the homomorphism $\phi = (p_1 f)_* - (p_2 f)_*$. Q.E.D.

The next proposition states the existence of the neighborhood used in the proof Theorem 1 point (3). Let X be a complete Finsler manifold and let $f : N \to X \times X$ be a isometric immersion, where N is a compact manifold. Let S be the subset

$$S = \{(x, f(x), \ldots, f(x)) \in N \times (X \times X) \times \cdots \times (X \times X) | x \in f^{-1}(\Delta)\},$$

k times $X \times X$.

Proposition 2. *Let X be a complete Finsler manifold and $f : N \to X \times X$ be an isometric immersion as above. The subset S in $N \times (X \times X) \times \ldots (X \times X)$, ($k$-copies of $(X \times X)$) is a deformation retract of an open neighborhood U if one of the following conditions holds:*

- *f is a totally geodesic map*
- *$N = N_1 \times N_1$ and $f = f_1 \times f_1$, where f_1 is an embedding.*

Proof. In the second case S is diffeomorphic to N, so S is closed. In the first case S is diffeomorphic to $f^{-1}(\Delta)$, and it follows that it is closed. Take U any open tubular neighborhood of S and this is done. We present for convenience the argument that for f totally geodesic S is closed.

Take any open disk $D \subset N$ such that $f(D)$ is imbedded in $X \times X$. $f(D) \cap \Delta$ is a totally geodesic submanifold since it is an intersection of two totally geodesic submanifolds. Furthermore $f : f^{-1}(\Delta) \cap D \to \Delta \cup f(D)$ is a diffeomorphism, which implies that $f^{-1}(\Delta)$ is a manifold.

Q.E.D.

3.2. Index estimates in the case of positively curved Finsler manifolds

Consider as in the previous subsection the energy E of the Finsler metric on $P(M, f)$. At a point $(x, \alpha) \in P(M, f)$ the tangent space consists of vectors (v, W), with $v \in T_xM$ and W piecewise smooth vector field along α such that $f_*(v) = (W(0), W(1))$. Being f an immersion the tangent space can be identified with the space $(W(0), W(1))$. For a parallel vector field along α, by the second variation formula of Finsler energy, the Hessian of the energy function satisfies

$$\begin{aligned} E_{**} &= \int_0^1 -\langle \Omega(\dot{\alpha}^H, W^H)W^H, \dot{\alpha}^H\rangle_{\dot{\alpha}} dt \\ &\quad + \langle \mathbb{I}_{\dot{\alpha}}(f_*(v), f_*(v))^H, (-\dot{\alpha}(0), \dot{\alpha}(1))^H\rangle_{\dot{\alpha}} \end{aligned}$$

Theorem 3. *Let M be a compact Finsler manifold of positive flag curvature and let $f : N \to M \times M$ be an isometric immersion with asymptotic index ν_f. Let (x, α) be a non-trivial critical point of E, with Morse Index I_α. Then,*

(1) $I_\alpha \geq \nu_f - m$

(2) *If $f = (f_1, f_2) : N = N_1 \times N_2 \to M \times M$ such that $f_i : N_i \to M$ is an immersion, $i = 1, 2$, then $I_\alpha \geq \nu_f - m + 1$.*

Proof. (1) Consider the set V of vector fields along α such that v^H is parallel along and $\langle\ ,\ \rangle_{\dot{\alpha}}$ orthogonal to $\dot{\alpha}^H$. It is clear that $\dim V = m - 1$ and we can identify $V = \{v(0), v(1)\}$.

Consider further $\mathcal{N}_x$ the maximal subspace of T_xN such that the second fundamental form in the direction $\dot{\alpha}$ is zero, so $\dim \mathcal{N}_x \geq \nu_f$. Being both V^h and $f_*(\mathcal{N}_x)^H$ $\langle , \rangle_{\dot{\alpha}}$ orthogonal to $(\dot{\alpha}(0), -\dot{\alpha}(1))$ we have by the dimension theorem that

$$\dim((f_*(\mathcal{N}_x))^H \cap V) \geq \nu_f + \dim V - 2m + 1 = \nu_f - m.$$

(2) α is a geodesic in M such that $\dot{\alpha}^h(0)$ is $\langle\ ,\ \rangle_{\dot{\alpha}(0)}$ orthogonal to $((f_1)_*(T_{x_1}N_1))^H$ and $\dot{\alpha}^H(1)$ is $\langle\ ,\ \rangle_{\dot{\alpha}(1)}$ orthogonal to $((f_2)_*(T_{x_2}N_2))^H$,

where $x = (x_1, x_2)$. Now both $(\dot{\alpha}(0), 0)^H$ and $(0, \dot{\alpha}(1))^H$ are normal to $f_*(T_xN)^H$ and V, so we have for the dimension of their intersection

$$\dim((f_*(\mathcal{N}_x))^H \cap V) \geq \nu_f + \dim V - 2m + 2 = \nu_f - m + 1.$$

Q.E.D.

§4. Main Results

Theorem 4. *Let M be an m-dimensional compact Finsler manifold of positive flag curvature and Δ the diagonal of $M \times M$. Consider an isometric immersion $f : N \to M \times M$ of a closed manifold with asymptotic index ν_f. The following statements hold:*

(1) *If $\nu_f > m$, then $f^{-1}(\Delta) \neq \emptyset$*
(2) *If $\nu_f > m + 1$ and M is simply connected, then $f^{-1}(\Delta)$ is connected.*
(3) *For $\nu_f > m + i$ the following sequence of homotopy groups*

$$\pi_i(f^{-1}(\Delta)) \longrightarrow \pi_1(N) \xrightarrow{(p_1f)_*-(p_2f)_*} \pi_i(M) \longrightarrow \pi_{i-1}(f^{-1}(\Delta)) \longrightarrow \cdots$$

is exact.

Proof. The three assertions follow from Theorems 1 and 3. Q.E.D.

In the case where f is not a correspondence but a pair of immersions we have the following stronger result:

Theorem 5. *Under the assumptions of Theorem 4 if in addition $N = N_1 \times N_2$ and $f = (f_1, f_2)$ with asymptotic index ν_f, then*

(1) *If $\nu_f \geq m$ then $f^{-1}(\Delta) \neq \emptyset$*
(2) *If $\nu_f \geq m + 1$ and M is simply connected, then it follows that $f^{-1}(\Delta)$ is connected.*
If $f = (f_1, f_1)$ with f_1 embedding then
(3) *For $\nu_f \geq m + i$ the following sequence of homotopy groups*

$$\pi_i(f^{-1}(\Delta)) \longrightarrow \pi_i(N) \xrightarrow{(p_1f)_*-(p_2f)_*} \pi_i(M) \longrightarrow \pi_{i-1}(f^{-1}(\Delta))$$

is exact.
(4) *We have the natural isomorphism*

$$\pi_i(N_1, f^{-1}(\Delta)) \to \pi_i(M, N_1)$$

for $i \leq \nu_f - m$ and a surjection for $i = \nu_f - m + 1$. Here $\pi_i(N_j, f^{-1}(\Delta))$ is understood as the i-th homotopy group of the composition map $f^{-1}(\Delta) \hookrightarrow N \xrightarrow{p_j} N_j$.

Proof. The assertions come from Theorems 1 and 3. Q.E.D.

§5. Some consequences of the main results

Theorem 6. *Let M be a compact simply connected Finsler manifold of positive flag curvature. Let $f_i : N_i \to M$ be a compact isometric immersion with asymptotic index $\nu_f, i = 1, 2$. For $\nu_{f_1} + \nu_{f_2} \geq m + 1$ then both $f_1^{-1}(f_2(N_2))$ and $f_2^{-1}(f_1(N_1))$ are connected.*

Proof. Let us consider the immersion $(f_1, f_2) : N_1 \times N_2 \to M \times M$. By the Theorem 5, $(f_1, f_2)^{-1}(\Delta)$ is connected and it follows that $f_1^{-1}(f_2(N_2)) = p_1((f_1, f_2)^{-1}(\Delta))$ and $f_2^{-1}(f_1(N_2)) = p_2((f_1, f_2)^{-1}(\Delta))$ are connected. Q.E.D.

The next theorem is a Frankel type result about intersection of submanifolds, see [7, 13, 14] for different versions in the Riemann and Finsler setting.

The original Frankel theorem states that for a complete connected Riemannian manifold M of positive sectional curvature, two totally geodesic submanifolds V and W have nonempty intersection, $V \cap W \neq \emptyset$, provided that $\dim V + \dim W \geq \dim M$.

Theorem 7. *Let M be an m dimensional connected Finsler manifold of positive flag curvature and let $f_1 : N_i \to M$ be an isometric immersion of a compact submanifold with asymptotic index ν_{f_i}. If $\nu_{f_1} + \nu_{f_2} \geq m$, then $f_1(N_1) \cap f_2(N_2) \neq \emptyset$.*

Proof. Consider $f = (f_1, f_2) : N_1 \times N_2 \to M \times M$. Now $\nu_f = \nu_{f_1} + \nu_{f_2} \geq m$, so from Theorem 4 it follows that $f^{-1}(\Delta) \neq \emptyset$, that is $f_1(N_1) \cap f_2(N_2) \neq \emptyset$.

Q.E.D.

A map $f : N \to M$ is said to be $(i + 1)$- connected if it induces an isomorphism up to the i-th homotopy group and a surjective homomorphism on the $(i + 1)$-th homotopy group.

In what follows we prove some results related to the above defined connectedness notion.

Theorem 8. *Let M be a compact simply connected Finsler manifold of positive flag curvature and let $f : N \to M$ be an immersion of a compact manifold. If the asymptotic index $\nu_f > \frac{\dim M}{2}$, then f is an embedding.*

Proof. It is enough to show that f is one-to-one map since f is an immersion. We have that

$$f^{-1}(\Delta) = \{(x, x), x \in N\} \cup \{(x, y)|f(x) = f(y), x \neq y\}.$$

If f is not injective it follows that $f^{-1}(\Delta)$ is not connected, this is a contradiction. Q.E.D.

Theorem 9. *Let M be a compact Finsler manifold of positive flag curvature and N be a compact embedded submanifold with asymptotic index ν. We have $\pi_i(M, N) = 0$ for $i \leqslant 2\nu - \dim M$.*

Proof. In Theorem 5 consider $N_1 = N_2$ and $f_i = \imath : N \hookrightarrow M$ the inclusion. Now $f^{(-1)}(\Delta) = N = N_1 \cap N_2$ and the result follows. Q.E.D.

The next theorem is a result related to embeddings, similar to the results of B. Wilking [21], which states that the for a positively curved n-dimensional manifold M of positive sectional curvature and a $(n-k)$-dimensional totally geodesic compact submanifold N the inclusion $\iota : N \to M$ is $n - 2k + 1$ connected.

Theorem 10. *Let M be a compact simply connected Finsler manifold of positive flag curvature and let $f : N \to M$ be an embedding of a compact manifold with asymptotic index ν_f. Then f is $(2\nu_f - \dim M + 1)$-connected.*

Proof. The result is a consequence of Theorems 5 and 9. Q.E.D.

Theorem 11. *Let M be a compact simply connected manifold of positive flag curvature. If $f : N \to M$ is an isometric immersion of a compact manifold with asymptotic index ν_f then f is $(2\nu_f - \dim M + 1)$ connected.*

Proof. We have that $(2\nu_f - \dim M + 1) > 1$, so $2\nu_f > \dim M$. Unsing Theorems 8 and 10 the proof is concluded. Q.E.D.

§6. Applications for totally geodesic submanifolds of Berwald manifolds

In Riemannian geometry the fact that the second fundamental form of a submanifold is zero is equivalent to the property that the submanifold is totally geodesic.

In our case the situation is more subtle. The second fundamental form, defined from the variational approach (second variation of the energy of a geodesic where the ends of the geodesics are in two submanifolds) is rather an analytical definition. The reference vector of the second fundamental form is not tangent to the submanifold, so it is an extrinsic characteristic of the submanifold (it is related to the way the submanifold lies in the ambient manifold). The condition that the second fundamental form vanishes is not equivalent to the geometric property of a submanifold being totally geodesic. For Berwald spaces the above equivalence is true, due to the fact that the reference vector of the covariant derivative is irrelevant. So, we have two notions of a totally geodesic submanifold. One of them is analytical, that is, the second fundamental form in a non-tangent direction vanishes, and the other is geometrical, that is, the geodesics of the submanifold are geodesics in the ambient manifold.

In the case when the manifold M is of Berwald type, the connection of M lives on the tangent level (the reference vector is irrelevant) and is linear. In this case, for a submanifold, the condition that the second fundamental form defined in Section 2 vanishes is equivalent to the property that the submanifold is totally geodesic, that is the geodesic of the submanifold are geodesics for the ambient manifold. Furthermore in this case a submanifold N of the manifold M is totally geodesic in both senses, analytic and geometric iff $\nu_f = \dim N$ (in the Berwald category the reference vector is irrelevant).

In the case of Berwald spaces, the previous characterization of totally geodesic submanifolds is also implied by Szabó's structure theorems on Berwald spaces (see [19]). One of Szabó's results says that if (M, L, ∇) is a manifold endowed with a Berwald metric L and ∇ is the Berwald connection, then the connection is Riemann metrizable, i.e., there exists a non-unique Riemannian metric g on M such that ∇ is the Riemannian connection of g. This implies that the geodesics of the Berwald metric L and the non-unique Riemannian metric g coincide. It follows now that a submanifold of M is totally geodesic with respect to g iff it is totally geodesic with respect to L, i.e., the totally geodesic submanifolds of M, with respect to the Berwald metric L, coincide with the totally geodesic submanifolds with respect to the non-unique Riemannian metric g whose existence is guaranteed by Szabo's results.

The second fundamental form defined in the Section 2 has the reference vector non tangent to the submanifold. But this second fundamental form appears naturally in the study of geodesics joining two submanifolds. This shows us an important difference between the Finsler and Riemann cases, and as expected, this difference comes up from the fact that the Cartan connection (and any other connection used in Finsler geometry) has a directional dependence. In the Finsler category, there is no such strong relationship between the asymptotic index and the property of a submanifold to be totally geodesic as in the Riemannnian case.

All the results concerning asymptotic index can be restated for totally geodesic submanifolds N of Berwald manifolds M, with $f : N \to M$ totally isometric immersion and $\nu_f = \dim N$.

Finally we present some problems in Finsler geometry where we expect that these tools can be applied. Grove and Searle ([10]) introduced the symmetry rank of a Riemannian manifold (M, g), to be the rank of the isometry group of (M, g). Grove also proposed to classify those manifolds with a large isometry group. One of the aims of this program is to find general obstructions to the existence of Riemannian metrics of positive curvature (taking benefit from the obstructions for manifolds with a large amount of symmetries). We expect that these tools can be applied to the study of manifolds with Finsler metrics of positive flag curvature.

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References

[1] M. Abate and G. Patrizio, Finsler metrics–a global approach. With applications to geometric function theory, Lecture Notes in Math., **1591**, Springer-Verlag, Berlin, Heidelberg, 1994.

[2] L. Auslander, On curvature in Finsler geometry, Trans. Amer. Math. Soc., **79** (1955), 378–388.

[3] D. Bao and S. S. Chern, On a notable connection in Finsler geometry, Houston J. Math., **19** (1993), 135–180.

[4] D. Bao, S. S. Chern and Z. Shen, An Introduction to Riemann–Finsler Geometry, Springer-Verlag, 2000.

[5] S. S. Chern and N. H. Kuiper, Some threorems on the isometric embedding of compact Riemannian manifolds in Euclidean space, Ann. of Math., **56** (1952), 422–430.

[6] P. Dazord, Propriétés globales des géodesique des Espaces de Finsler, Thesis, Univ. Lyon, 1969.

[7] T. Frankel, Manifolds of positive curvature, Pacific J. Math., **11** (1961), 165–174.

[8] F. Fang, S. Mendoca and X. Rong, A connectedness principle in the manifolds of positive curvature, Comm. Anal. Geom., **13** (2005), 671–696.

[9] L. Florit, On submanifolds with nonpositive extrinsec curvature, Math. Ann., **298** (1994), 187–192.

[10] K. Grove and C. Searle, Positively curved manifolds of maximal symmetry rank, J. Pure Appl. Algebra, **91** (1994), 137–142.

[11] A. Hatcher, Algebraic Topology, Cambridge Univ. Press, 2002, `http://www.math.cornell.edu/ hatcher/` .

[12] L. Kozma, A. Kristály and C. Varga, Critical point theorems on Finsler manifolds, Beiträge Algebra Geom., **45** (2004), 47–59.

[13] L. Kozma and I. R. Peter, Intersections theorems for Finsler manifolds, Publ. Math. Debrecen, **57** (2000), 193–201.

[14] L. Kozma and I. R. Peter, Coincidence of geodesic correspondences in real Finsler manifolds, An. Ştiinţ. Univ. Al. I. Cuza Iaşi. Mat. (N.S.), **48** (2002), 363–369.

[15] L. Kozma and I. R. Peter, Warped product of Finsler manifolds, Ann. Univ. Sci. Budapest. Eötvös Sect. Math., **44** (2001), 155–168.

[16] J. Milnor, Morse Theory, Ann. of Math. Studies, **51**, Princeton Univ. Press, 1963.

[17] I. R. Peter, The Morse Index Theorem where the ends are submanifolds in Finsler geometry, Houston J. Math., to appear, **32** (2006).

[18] Z. Shen, Lecture Notes on Finsler Geometry, Springer-Verlag, 2001.

[19] Z. I. Szabó, Positive definite Berwald spaces, Tensor (N. S.), **35** (1981), 25–39.

[20] G. W. Whitehead, Elements of Homotopy Theory, Springer-Verlag, **61**, 1979.

[21] B. Wilking, Torus actions on manifolds of positive sectional curvature, Acta Math., **191** (2003), 259–297.

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