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On the role of the source terms in an activator-inhibitor system proposed by Gierer and Meinhardt

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Abstract.

Considered is a reaction-diffusion system consisting of an activator and an inhibitor which was proposed by Gierer and Meinhardt to model biological pattern formation. We prove that the initial-boundary value problem for the activator-inhibitor system has a unique solution for all $t > 0$ if the production rate of the activator is well-controlled by the inhibitor. Moreover, we prove that the solution stays in a bounded region if the source term for the activator becomes positive somewhere. We consider also how the source term for the activator affects the shape of stationary solutions in one spatial dimension.

§1. Introduction and Statement of Results

In the celebrated paper [14], A. M. Turing found that the reaction between two chemicals with different diffusion rates may cause the destabilization of the spatially homogeneous state, thus leading to the formation of nontrivial spatial structure. Developing Turing's idea, A. Gierer and H. Meinhardt ([2]) proposed a system consisting of a slowly diffusing activator and a rapidly diffusing inhibitor. They assumed that a change in cells or tissue takes place in the region where the activator concentration is high. Suppose that the activator and the inhibitor fill a bounded domain Ω in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$ and that there is no flux through the boundary. Let $A(x,t)$ and $H(x,t)$ denote the respective concentrations of the activator and the inhibitor at position $x \in \overline{\Omega}$ and time $t \geqslant 0$. Let ν denote the unit outer normal vector to $\partial\Omega$

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and $\Delta = \sum_{j=1}^{N} \partial^2/\partial x_j^2$ be the Laplace operator in $\mathbb{R}^N$. In this paper we consider the following activator-inhibitor system proposed by Gierer and Meinhardt:

$$\frac{\partial A}{\partial t} = \varepsilon^2 \Delta A - A + \frac{A^p}{H^q} + \sigma_a(x) \quad \text{for } x \in \Omega,\ t > 0, \tag{1.1}$$

$$\tau \frac{\partial H}{\partial t} = D\Delta H - H + \frac{A^r}{H^s} + \sigma_h(x) \quad \text{for } x \in \Omega,\ t > 0, \tag{1.2}$$

$$\frac{\partial A}{\partial \nu} = \frac{\partial H}{\partial \nu} = 0 \quad \text{for } x \in \partial\Omega,\ t > 0, \tag{1.3}$$

$$A(x,0) = A_0(x), \quad H(x,0) = H_0(x) \quad \text{for } x \in \Omega. \tag{1.4}$$

Here, ε, D and τ are positive constants;

$$\sigma_a, \sigma_h \in C^\beta(\overline{\Omega}), \quad \text{and} \quad \sigma_a(x) \geqslant 0,\ \sigma_h(x) \geqslant 0 \text{ on } \overline{\Omega}; \tag{1.5}$$

and concerning the initial data we assume

$$\begin{aligned} &A_0, H_0 \in C^{2+\beta}(\overline{\Omega}), \ \left.\frac{\partial A_0}{\partial \nu}\right|_{\partial\Omega} = \left.\frac{\partial H_0}{\partial \nu}\right|_{\partial\Omega} = 0, \\ &\text{and} \quad A_0(x) > 0,\ H_0(x) > 0 \text{ on } \overline{\Omega}, \end{aligned} \tag{1.6}$$

where $0 < \beta < 1$. Moreover, the exponents (p, q, r, s) are assumed to satisfy

$$p > 1,\ q > 0,\ r > 0,\ s \geqslant 0, \ \text{and} \ 0 < \frac{p-1}{q} < \frac{r}{s+1}. \tag{1.7}$$

From a mathematical point of view, one of the fundamental questions is whether the initial-boundary value problem has a solution for all $t > 0$ or not. There have appeared several results on this question (see, e.g., [11], [6], [15], [5]). In particular, under the assumption that $\min_{x\in\overline{\Omega}} \sigma_a(x) > 0$ and $(p-1)/r < 2/(N+2)$, Masuda and Takahashi [6] proved not only that the solution exists for all $t > 0$ but also that, as $t \to +\infty$, the set $\{(A(x,t), H(x,t)) \in \mathbb{R}^2 \mid x \in \Omega\}$ is confined in a fixed rectangle which is independent of the initial data. On the other hand, Li, Chen and Qin [5] proved that the solution exists for all $t > 0$ if $\min_{x\in\overline{\Omega}} \sigma_a(x) > 0$ and $p - 1 < r$.

This paper has two purposes. One is to study the initial-boundary value problem (1.1)–(1.4) in the case $\min_{x\in\overline{\Omega}} \sigma_a(x) = 0$. The following Theorems 1.1–1.3 complement the results by [6] and [5], and give us a complete understanding of the global existence and the boundedness of solutions in the case $p - 1 < r$. The other is to study the effect of $\sigma_a(x)$ upon the steady-state patterns. We shall consider this problem in the simplest situation, i.e., the one-dimensional shadow system (1.14)–(1.16) below.

We now state our results on the global existence of solutions of the initial-boundary value problem.

Theorem 1.1. *Assume, in addition to (1.7), that*

$$p - 1 < r \tag{1.8}$$

and, in addition to (1.5), that

$$\max_{x\in\overline{\Omega}} \sigma_a(x) > 0. \tag{1.9}$$

Then the initial-boundary value problem (1.1)–(1.4) has a unique solution for all $t > 0$. *Moreover, there exist positive constants* m_a, M_a, m_h, M_h, *independent of the initial data* $(A_0(x), H_0(x))$, *such that*

$$\begin{cases} m_a \leqslant \liminf\limits_{t\to\infty} \min\limits_{x\in\overline{\Omega}} A(x,t) \leqslant \limsup\limits_{t\to\infty} \max\limits_{x\in\overline{\Omega}} A(x,t) \leqslant M_a, \\ m_h \leqslant \liminf\limits_{t\to\infty} \min\limits_{x\in\overline{\Omega}} H(x,t) \leqslant \limsup\limits_{t\to\infty} \max\limits_{x\in\overline{\Omega}} H(x,t) \leqslant M_h. \end{cases} \tag{1.10}$$

Theorem 1.2. *Assume that (1.8) is satisfied in addition to (1.7). Moreover, suppose that*

$$\sigma_a(x) \equiv 0 \quad \text{and} \quad \max_{x\in\overline{\Omega}} \sigma_h(x) > 0. \tag{1.11}$$

Then the initial-boundary value problem (1.1)–(1.4) has a unique solution for all $t > 0$. *Moreover, there are positive constants* M_a, m_h *and* M_h *which are independent of the initial data* $(A_0(x), H_0(x))$ *such that*

$$\begin{cases} e^{-t} \min\limits_{x\in\overline{\Omega}} A_0(x) \leqslant A(x,t) \quad \text{for all} \quad x \in \overline{\Omega},\ t > 0, \\ \qquad \text{and} \quad \limsup\limits_{t\to+\infty} \max\limits_{x\in\overline{\Omega}} A(x,t) \leqslant M_a, \\ m_h \leqslant \liminf\limits_{t\to+\infty} \min\limits_{x\in\overline{\Omega}} H(x,t) \leqslant \limsup\limits_{t\to+\infty} \max\limits_{x\in\overline{\Omega}} H(x,t) \leqslant M_h. \end{cases} \tag{1.12}$$

Theorem 1.3. *Assume, in addition to (1.7), that (1.8) is satisfied. Moreover, let*

$$\sigma_a(x) \equiv 0 \quad \text{and} \quad \sigma_h(x) \equiv 0.$$

Then the initial-boundary value problem (1.1)–(1.4) has a unique solution for all $t > 0$. *Moreover, there are positive constants* λ *and* μ *which are dependent only on* p, q, r, s *and* τ, *and a positive constant* C *depending on* $A_0(x)$ *and* $H_0(x)$ *such that*

$$
(1.13) \qquad \begin{cases} e^{-t} \min\limits_{x\in\overline{\Omega}} A_0(x) \leqslant A(x,t) \leqslant Ce^{\lambda t}, \\ e^{-t/\tau} \min\limits_{x\in\overline{\Omega}} H_0(x) \leqslant H(x,t) \leqslant Ce^{\mu\tau} \end{cases}
$$

for all $t > 0$ *and* $x \in \overline{\Omega}$.

Some remarks on the theorems above are in order. First, we call the ratio $\rho_A = (p-1)/r$ the *net self-activation index* since it compares how strongly the activator activates the production of itself with how strongly it activates that of the inhibitor. On the other hand, we call $\rho_I = q/(s+1)$ the *net cross-inhibition index.* All the three theorems above assume that ρ_A be less than one, which is important to rule out the occurrence of finite time blow-up of solutions. Indeed, it has been shown in [5] and [7] that if $\rho_A > 1$, then there exist solutions of (1.1)–(1.4) with $\sigma_a(x) \equiv \sigma_h(x) \equiv 0$ which blow up in finite time.

Second, Wu and Li [15] proved that if $\sigma_a(x) \equiv 0$ and $\sigma_h(x) \equiv 0$ and if $\tau > q/(p-1)$, then there are solutions of (1.1)–(1.4) such that $(A(x,t), H(x,t)) \to (0,0)$ uniformly on $\overline{\Omega}$ as $t \to +\infty$. We call such a phenomenon the *collapse of patterns.* Theorem 1.1 implies patterns never collapse as long as $\sigma_a(x)$ is nontrivial.

Third, in [7] it is proved that if $\rho_A \leqslant 1$ and $\rho_I \geqslant 1$ then some solutions of (1.1)–(1.4) with $\sigma_a(x) \equiv \sigma_h(x) \equiv 0$ exist for all $t > 0$, but they are unbounded. By virtue of Theorem 1.1 all solutions are bounded if $\sigma_a(x)$ is nontrivial.

Next we study the effect of $\sigma_a(x)$ on the shape of stationary solutions in one space dimension. Here we restrict ourselves to the simplest situation, i.e., the limiting system obtained by letting $D \to +\infty$, which is called the *shadow system* for (1.1)–(1.4). By dividing both sides of (1.2) by D and then letting $D \to +\infty$, we see that $\Delta H \to 0$ as a formal limit. Thanks to the boundary condition this implies that in the limit H is independent of x. To determine this constant we integrate (1.2) over Ω, which leads to $\tau \frac{d}{dt} \int_\Omega H\,dx = -\int_\Omega H\,dx + \int_\Omega A^r/H^s\,dx + \int_\Omega \sigma_h(x)\,dx$. Letting $H(x,t) \to \xi(t)$, we obtain the equation for $\xi(t)$. Therefore, assuming that $\sigma_h(x) \equiv 0$, we are led to the shadow system in the case of spatial dimension one:

$$\frac{\partial A}{\partial t} = \varepsilon^2 \frac{\partial^2 A}{\partial x^2} - A + \frac{A^p}{\xi^q} + \sigma_a \quad \text{for } 0 < x < \ell,\ t > 0, \tag{1.14}$$

$$\tau \frac{d\xi}{dt} = -\xi + \frac{1}{\ell \xi^s} \int_0^\ell A^r dx \quad \text{for } t > 0, \tag{1.15}$$

$$\frac{\partial A}{\partial x}(0,t) = \frac{\partial A}{\partial x}(\ell, t) = 0 \qquad \text{for } t > 0. \tag{1.16}$$

In order to state our results we need some preparation. Let $w(y)$ be the solution of the following boundary value problem:

$$\begin{cases} w'' - w + w^p = 0, \quad \text{and} \quad w > 0 \quad \text{for } 0 < y < +\infty, \\ w'(0) = 0, \quad \lim_{y \to +\infty} w(y) = 0. \end{cases} \tag{1.17}$$

It is well-known that the solution w is unique and decays exponentially as $y \to +\infty$: $\sup_{0<y<\infty} w(y)e^y < +\infty$. Let $\phi_1(y)$ be a solution of

$$\begin{cases} \phi_1'' - \phi_1 + pw^{p-1}\phi_1 + pw^{p-1} = 0 \quad \text{for } 0 < y < +\infty, \\ \phi_1'(0) = 0, \quad \lim_{y \to +\infty} \phi_1(y) = 0. \end{cases} \tag{1.18}$$

This problem is known to have a unique solution (see, e.g., pp. 330–331 of [8]). The following theorem generalizes Theorem 1 of [13] which treats the case where σ_a is a constant, and tells us how the term $\sigma_a(x)$ affects the shape of stationary solutions.

Theorem 1.4. *There exists an $\varepsilon_0 > 0$ such that for each $\varepsilon \in (0, \varepsilon_0)$ the shadow system has a pair of stationary solutions $(A_{l,\varepsilon}(x), \xi_{l,\varepsilon})$ and $(A_{r,\varepsilon}(x), \xi_{r,\varepsilon})$ satisfying*

$$\begin{aligned} A_{l,\varepsilon}(x) = {} & \xi_{l,\varepsilon}^{q/[qr-(p-1)(s+1)]}\{w(x/\varepsilon) + o(1)\} \\ & + \sigma_a(x) + \sigma_a(0)\phi_1(x/\varepsilon) + o(1), \end{aligned} \tag{1.19}$$

$$\xi_{l,\varepsilon} = \left\{ \varepsilon \left(\frac{1}{l} \int_0^\infty w(z)^r dz + o(1) \right) \right\}^{(p-1)/[qr-(p-1)(s+1)]}, \tag{1.20}$$

$$\begin{aligned} A_{r,\varepsilon}(x) = {} & \xi_{r,\varepsilon}^{q/[qr-(p-1)(s+1)]}\{w((\ell - x)/\varepsilon) + o(1)\} \\ & + \sigma_a(x) + \sigma_a(l)\phi_1((\ell - x)/\varepsilon) + o(1), \end{aligned} \tag{1.21}$$

$$\xi_{r,\varepsilon} = \left\{ \varepsilon \left(\frac{1}{l} \int_0^\infty w(z)^r dz + o(1) \right) \right\}^{(p-1)/[qr-(p-1)(s+1)]}, \tag{1.22}$$

as $\varepsilon \downarrow 0$. Here, the terms $o(1)$ in (1.19) and (1.21) are uniform in $x \in [0, \ell]$.

Precisely speaking, we should notice that $\sigma_a(x)$ is not smooth in general, and hence the term $\sigma_a(x)$ in (1.19) and (1.21) should be replaced with $\Sigma_\varepsilon(x)$ which is a unique solution of a boundary value problem (3.14) below. We can prove that $\Sigma_\varepsilon(x)$ converges to $\sigma_a(x)$ uniformly on $[0, \ell]$ as $\varepsilon \downarrow 0$ (see Lemma 3.3). We now turn to the question of the stability of $(A_{l,\varepsilon}(x), \xi_{l,\varepsilon})$ and $(A_{r,\varepsilon}(x), \xi_{r,\varepsilon})$. A stationary solution $(A_*(x), \xi_*)$ of the shadow system (1.14)–(1.16) is said to be *stable* if, given any neighborhood $\mathcal{U}$, one can find a neighborhood $\mathcal{V}$ of $(A_*(x), \xi_*)$ such that the solution $(A(x,t), \xi(t))$ of the initial-boundary value problem stays in $\mathcal{U}$ for all $t \geqslant 0$ whenever the initial data $(A_0(x), \xi_0)$ is chosen in $\mathcal{V}$. If it is not stable, we call the stationary solution $(A_*(x), \xi_*)$ *unstable*. Moreover, a stationary solution $(A_*(x), \xi_*)$ of (1.14)–(1.16) is said to be *asymptotically stable* if the solution of the initial-boundary value problem tends to $(A_*(x), \xi_*)$ as $t \to +\infty$ provided that the initial data is sufficiently close to $(A_*(x), \xi_*)$. As for the stability of these stationary solutions we have the following two results. Let $\alpha = qr/(p-1) - (s+1)$. Note that $\alpha > 0$ by (1.7).

Theorem 1.5. *Let $r = 2$ and $1 < p < 5$. For each $\alpha \in (0, \alpha_0)$ where α_0 is a sufficiently small number, one can choose an $\varepsilon_1 > 0$ so that for $0 < \varepsilon < \varepsilon_1$, there exist $\tau_r > 0$ and $\tau_l > 0$ such that*

(i) *if $0 < \tau < \tau_l$, then $(A_{l,\varepsilon}(x), \xi_{l,\varepsilon})$ is asymptotically stable; and if $0 < \tau < \tau_r$, then $(A_{r,\varepsilon}(x), \xi_{r,\varepsilon})$ is asymptotically stable;*

(ii) *$(A_{l,\varepsilon}(x), \xi_{l,\varepsilon})$ is unstable if $\tau > \tau_l$ and $(A_{r,\varepsilon}(x), \xi_{r,\varepsilon})$ is unstable if $\tau > \tau_r$.*

Theorem 1.6. *Let $1 < p < 5$ and $r = p + 1$. Assume that α is sufficiently small. Then for each $\varepsilon > 0$ sufficiently small there exist positive constants $0 < \tau_{2,r} < \tau_{1,r}$ and $0 < \tau_{2,l} < \tau_{1,l}$, depending on (p, q, s) and ε, such that*

(i) *if $\tau_{2,l} < \tau < \tau_{1,l}$, then $(A_{l,\varepsilon}(x), \xi_{l,\varepsilon})$ is asymptotically stable; and if $\tau_{2,r} < \tau < \tau_{1,r}$, then $(A_{r,\varepsilon}(x), \xi_{r,\varepsilon})$ is asymptotically stable;*

(ii) *$(A_{l,\varepsilon}(x), \xi_{l,\varepsilon})$ is unstable if $\tau > \tau_{1,l}$ and $(A_{r,\varepsilon}(x), \xi_{r,\varepsilon})$ is unstable if $\tau > \tau_{1,r}$.*

To sum up, stable steady-state solutions has a very large spike at one of the end points of the interval superimposed onto the distribution of the basic production term $\sigma_a(x)$. In the case of the shadow system, therefore, the boundary spike is the major part and the contribution of $\sigma_a(x)$ is relatively small as far as resulting patterns are concerned. Its principal role seems to stabilize the system and avoid the collapse of patterns.

We close this section by making a few remarks. (i) While preparing the manuscript we learned that Jiang [4] obtained independently some results similar to ours on the global existence and boundedness of solutions of the initial-boundary value problem. (ii) Zhang and Li [16] considered global-in-time solutions and blow-up solutions of the initial-boundary value problem for (1.1)–(1.2) under the Robin boundary condition. (iii) The assumptions $r = 2$ and $r = p + 1$ in Theorems 1.5 and 1.6 have been made for technical reasons (see, e.g., [9] and [10]) and should be relaxed considerably. We suspect that in Theorem 1.6, $(A_{l,\varepsilon}(x), \xi_{l,\varepsilon})$ is stable whenever $0 < \tau < \tau_{l,1}$ and $(A_{r,\varepsilon}(x), \xi_{r,\varepsilon})$ is stable if $0 < \tau < \tau_{r,1}$. (iv) Our results on the steady-state solutions of the shadow system imply that the distribution of the source term $\sigma_a(x)$ does not affect very much the patterns generated by (1.14)–(1.16); in particular, the major concentration occurs at one of the end points of the interval. This is an important observation from a biological view point. However, the situation changes drastically when we replace the term A^p/H^q in (1.1) with $\rho_a(x)A^p/H^q$, i.e., if we consider the spatially inhomogeneous reaction rate. The details of the results will appear in a forthcoming paper.

§2. Initial-Boundary Value Problem.

2.1. Proof of Theorem 1.1.

In this section we sketch the proof of Theorem 1.1, which follows closely the approach developed by Li, Chen and Qin [5].

Step 1. We derive lower bounds for $A(x,t)$ and $H(x,t)$. For this purpose, the following observation is crucial.

Lemma 2.1. *Assume that (1.9) is satisfied. Then, for each positive number δ, there is a positive constant $m(\delta)$ such that the unique solution $a(x,t)$ of the initial-boundary value problem*

$$\frac{\partial a}{\partial t} = \varepsilon^2 \Delta a - a + \sigma_a(x) \quad \textit{for } (x,t) \in \Omega \times (0,\infty) \tag{2.1}$$

$$\frac{\partial a}{\partial \nu} = 0 \quad \textit{for } (x,t) \in \partial\Omega \times (0,\infty) \tag{2.2}$$

$$a(x,0) = 0 \quad \textit{for } \ x \in \Omega \tag{2.3}$$

satisfies

$$a(x,t) \geqslant m(\delta) \quad \textit{for all } x \in \overline{\Omega}, \ t \geqslant \delta. \tag{2.4}$$

The constant $m(\delta)$ depends only on δ, ε, Ω and $\sigma_a(x)$.

Proof. Let $G(x,y,t)$ be the fundamental solution of the following linear parabolic equation subject to the homogeneous Neumann boundary conditions:

$$\begin{cases} \dfrac{\partial u}{\partial t} = \varepsilon^2 \Delta u - u \quad \text{for } (x,t) \in \Omega \times (0,\infty), \\ \dfrac{\partial u}{\partial \nu} = 0 \quad \text{for } (x,t) \in \partial\Omega \times (0,\infty). \end{cases}$$

Then the solution a of the problem (2.1)–(2.3) is expressed by the formula

$$a(x,t) = \int_0^t ds \int_\Omega G(x,y,t-s)\sigma_a(y)dy. \tag{2.5}$$

Since $\sigma_a(x) \geqslant 0$ and (1.9) is satisfied, we see that $a(x,t) > 0$ for all $x \in \overline{\Omega}$ whenever $t > 0$ because of the positivity of $G(x,y,t)$.

Moreover, by the standard theory of linear parabolic equations (see, e.g., §18 of [3] for a classical approach or Theorem 13.1 of [1] for an abstract setting), we know that $a(x,t)$ converges to a stationary solution of (2.1)–(2.3) uniformly on $\overline{\Omega}$ as $t \to \infty$. By invoking the maximum principle one can prove easily that the stationary solution is unique and positive on $\overline{\Omega}$. Therefore, the assertion of the lemma is an immediate consequence of the continuity of the solution $a(x,t)$. q.e.d.

As a corollary to this lemma one obtains the following

Lemma 2.2. *Assume that (1.9) is satisfied. Then, there is a positive constant m_1 such that the unique solution $a_l(x,t)$ of the initial-boundary value problem*

$$\frac{\partial a_l}{\partial t} = \varepsilon^2 \Delta a_l - a_l + \sigma_a(x) \quad \textit{for } (x,t) \in \Omega \times (0,\infty), \tag{2.6}$$

$$\frac{\partial a_l}{\partial \nu} = 0 \quad \textit{for } (x,t) \in \partial\Omega \times (0,\infty), \tag{2.7}$$

$$a_l(x,0) = A_0(x) \quad \textit{for } x \in \Omega \tag{2.8}$$

satisfies

$$a_l(x,t) \geqslant m_1 \quad \textit{for all } \; x \in \overline{\Omega},\ t \geqslant 0. \tag{2.9}$$

The constant m_1 depends only on $A_0(x)$, ε, Ω and $\sigma_a(x)$. Moreover,

$$\liminf_{t\to\infty} \min_{x\in\overline{\Omega}} a_l(x,t) \geqslant m_2 \tag{2.10}$$

for some positive constant m_2 independent of the initial data A_0.

Proof. Indeed, $a_l(x,t)$ is given by the following formula:

$$\begin{aligned} a_l(x,t) &= \int_\Omega G(x,y,t)A_0(y)dy + \int_0^t ds \int_\Omega G(x,y,t-s)\sigma_a(y)dy \\ &= \int_\Omega G(x,y,t)A_0(y)dy + a(x,t), \end{aligned}$$

where $a(x,t)$ is the solution of (2.1)–(2.3). By making use of the maximum principle it is easy to see that

$$\int_\Omega G(x,y,t)A_0(y)dy \geqslant e^{-t} \min_{x\in\overline{\Omega}} A_0(x).$$

Hence, $a_l(x,t) \geqslant e^{-t}\min_{x\in\overline{\Omega}} A_0(x) + a(x,t) \geqslant e^{-t}\min_{x\in\overline{\Omega}} A_0(x)$ for, say, $0 \leqslant t \leqslant 1$ (note that $\min_{x\in\overline{\Omega}} A_0(x) > 0$ by (1.6)). Now (2.9) follows from this estimate and (2.4). Recall that $a_l(x,t)$ converges to the stationary solution of (2.6)–(2.7) uniformly on $\overline{\Omega}$ as $t \to +\infty$ (see the final paragraph of the proof of Lemma 2.1), which implies (2.10). q.e.d.

Put

$$A_l(t) = \min_{x\in\overline{\Omega}} a_l(x,t) \tag{2.11}$$

and let $h_l(t)$ be the solution of the initial value problem for the following ordinary differential equation:

$$\tau\frac{dh_l}{dt} = -h_l + \frac{A_l(t)^r}{h_l^s} \quad \text{for} \quad t > 0, \tag{2.12}$$

$$h_l(0) = \min_{x\in\overline{\Omega}} H_0(x). \tag{2.13}$$

By the standard comparison theorem, we obtain the following

Lemma 2.3. *Let $(A(x,t), H(x,t))$ solve the initial-boundary value problem (1.1)–(1.4). Then*

$$A(x,t) \geqslant a_l(x,t), \tag{2.14}$$

$$H(x,t) \geqslant h_l(t), \tag{2.15}$$

for all $x \in \overline{\Omega}$ and $t \geqslant 0$. Here, $a_l(x,t)$ is the solution of (2.6)–(2.8) and $h_l(t)$ is the solution of (2.12)–(2.13).

To bound a_l and h_l from below, the following observation is useful.

Lemma 2.4. *Suppose that (1.9) is satisfied. Let a_l and h_l be the solutions of (2.6)–(2.8) and (2.12)–(2.13), respectively. Then*

$$a_l(x,t) \to a_*(x) \quad \text{uniformly on } \overline{\Omega},$$
$$h_l(t) \to h_*$$

as $t \to +\infty$, where $a_*(x)$ is the unique stationary solution of (2.6)–(2.7) and $h_* = (\min_{x\in\overline{\Omega}} a_*(x))^{r/(s+1)}$ is a positive number.

Proof. As was pointed out in the proof of Lemma 2.1, the assertion for a_l is a standard fact. To prove the assertion for h_l, we observe that $A_l(t) \to \min_{x\in\overline{\Omega}} a_*(x)$ as $t \to +\infty$ and this is sufficient to conclude that $h_l(t)^{s+1} \to \min_{x\in\overline{\Omega}} a_*(x)^r$. Hence the proof is complete. q.e.d.

Therefore, there exists a positive constant m_3 such that

$$H(x,t) \geqslant h_l(t) \geqslant m_3 \quad \text{for all } x \in \overline{\Omega},\ t \geqslant 0, \tag{2.16}$$

and an $m_4 > 0$ independent of $(A_0(x), H_0(x))$ such that

$$\liminf_{t\to+\infty} \min_{x\in\overline{\Omega}} H(x,t) \geqslant m_4. \tag{2.17}$$

Step 2. The following lemma is due to Li, Chen and Qin [5]. We state the assertion in a slightly different way to accomodate our formulation.

Lemma 2.5. *Suppose that (1.8) is satisfied. Let (A(x,t), H(x,t)) be a solution of the initial-boundary value problem (1.1)–(1.4). Let $M > 2, m > 0$ be constants satisfying*

$$\frac{\tau m M(\varepsilon^2 + D/\tau)^2}{4D(m+1)} \leqslant \varepsilon^2(M-1) \quad \text{and} \quad M - \frac{2m}{\tau} > 0. \tag{2.18}$$

Then for each $\theta \in (0,1)$ satisfying both the conditions

$$\frac{p-1}{r} < \theta < \frac{p}{r+1} \tag{2.19}$$
$$\text{and} \quad \kappa := \frac{Mq - m(p-1) + [mr - M(s+1)]\theta}{\theta r - (p-1)} > 0,$$

the following estimate holds:

$$\begin{aligned}\int_\Omega \frac{A^M}{H^m} dx \leqslant \int_\Omega \frac{A_0^M}{H_0^m} dx e^{-(M-2m/\tau)t} \\ + C_1 \int_0^t e^{-(M-2m/\tau)(t-t')} \int_\Omega \frac{dx}{H(x,t')^\kappa} dt' \\ + C_2 \int_0^t e^{-(M-2m/\tau)(t-t')} \int_\Omega \frac{dx}{H(x,t')^m} dt',\end{aligned} \tag{2.20}$$

where C_1 *and* C_2 *are positive constants defined as follows:*

$$\begin{aligned} C_0 =& \{C_a M(\tau/(c_h m))^\theta\}^{1/(1-\theta)}, \\ C_1 =& \{C_0(2\tau/m)^{1-\theta_1}\}^{1/\theta_1} \quad \textit{with} \ \ \theta_1 = (\theta r - (p-1))/(M(1-\theta)), \\ C_2 =& M^M(2\tau/m)^{M-1}\|\sigma_a\|_{L^\infty(\Omega)}. \end{aligned}$$

We note that for each $M > 2$ and $\gamma > 0$ there is an $0 < m_0 \leqslant \infty$ such that $m/(m+1) < \gamma(M-1)/M$ if $0 < m < m_0$. Hence, we can always choose a pair (M, m) satisfying (2.18).

Step 3. We are now ready to derive an upper bound on $A(x,t)$. Let $\mathcal{A}_\varepsilon = I - \varepsilon^2 \Delta$ be a closed linear operator on $L^P(\Omega)$ with domain $\mathcal{D}(\mathcal{A}_\varepsilon) = \{u \in W^{2,P}(\Omega) \mid \partial u/\partial\nu = 0 \text{ on } \partial\Omega\}$, where we assume $P > N$. Then the fractional power ${\mathcal{A}_\varepsilon}^{1/2}$ is defined and it is known that $\mathcal{D}\left({\mathcal{A}_\varepsilon}^{1/2}\right) \subset W^{1,P}(\Omega) \subset C^\beta(\overline{\Omega}), 0 < \beta < 1$. Let $e^{-t\mathcal{A}_\varepsilon}$ denote the analytic semigroup generated by $-\mathcal{A}_\varepsilon$. Recall that the following estimate holds:

$$\|{\mathcal{A}_\varepsilon}^{1/2} e^{-t\mathcal{A}_\varepsilon}\|_{\mathcal{B}(L^P\Omega))} \leqslant C t^{-1/2} e^{-\delta t} \quad \text{for } t > 0, \tag{2.21}$$

where C and δ are positive constant depending only on Ω, ε, and P; $\mathcal{B}(X)$ stands for the Banach space of all bounded linear operators on a Banach space X equipped with the operator norm.

With these preparations, we see that (1.1) is converted into the integral equation for $A(t) = A(\cdot, t)$ in $L^P(\Omega)$

$$A(t) = e^{-t\mathcal{A}_\varepsilon} A_0 + \int_0^t e^{-(t-t')\mathcal{A}_\varepsilon}\left(\frac{A^p(t')}{H^q(t')} + \sigma_a\right) dt'. \tag{2.22}$$

From this we have

(2.23)

$$\begin{aligned} \|A(t)\|_{L^\infty(\Omega)} &\leqslant C\|{\mathcal{A}_\varepsilon}^{1/2} A(t)\|_{L^P(\Omega)} \\ &\leqslant C\|e^{-t\mathcal{A}_\varepsilon}{\mathcal{A}_\varepsilon}^{1/2} A_0\|_{L^P(\Omega)} \\ &\quad + C\int_0^t (t-t')^{-1/2} e^{-\delta(t-t')}\left\|\frac{A^p(t')}{H^q(t')} + \sigma_a\right\|_{L^P(\Omega)} dt'. \end{aligned}$$

Let M and m be positive numbers satisfying $M > Pp$, $Mq > mp$ and (2.18). Since

$$\int_\Omega \left(\frac{A^p}{H^q}\right)^P dx \leqslant \int_\Omega \frac{A^M}{H^m} dx + \int_\Omega \frac{dx}{H^\eta}$$

with $\eta = P(Mq - mp)/(M - pP)$, it follows from Lemma 2.5 and (2.17) that

$$\limsup_{t\to+\infty} \int_\Omega \left(\frac{A^p}{H^q}\right)^P dx \leqslant K$$

for some positive constant K independent of the initial data (A_0, H_0). By virtue of (2.23) this in turn yields that

$$\limsup_{t\to+\infty} \|A(t)\|_{L^\infty(\Omega)} \leqslant K_1 \tag{2.24}$$

for some positive constant K_1 independent of (A_0, H_0).

Next, let $\mathcal{A}_D = I - D\Delta$ be a closed linear operator on $L^P(\Omega)$ with domain $\mathcal{D}(\mathcal{A}_D) = \{u \in W^{2,P}(\Omega) \mid \partial u/\partial\nu = 0 \text{ on } \partial\Omega\}$. Then

$$H(t) = e^{-t\tau^{-1}\mathcal{A}_D} H_0 + \frac{1}{\tau}\int_0^t e^{-(t-t')\tau^{-1}\mathcal{A}_D} \left\{\frac{A^r(t')}{H^s(t')} + \sigma_h\right\} dt'. \tag{2.25}$$

Since $\mathcal{A}_D$ satisfies an estimate similar to (2.21) we obtain easily by making use of (2.17) and (2.24) that

$$\limsup_{t\to+\infty} \|H(t)\|_{L^\infty(\Omega)} \leqslant K_2 \tag{2.26}$$

for some positive constant K_2 also independent of (A_0, H_0). Therefore, the proof of the theorem is now complete. q.e.d.

2.2. Proof of Theorem 1.2.

The proof is carried out along the same line as that of Theorem 1.1.

Step 1. First we give a lower bound on $A(x, t)$.

Lemma 2.6. *Suppose that $(A(x,t), H(x,t))$ is a solution of the initial-boundary value problem (1.1)–(1.4). Then*

$$A(x,t) \geqslant \min_{x\in\overline{\Omega}} A_0(x) e^{-t}$$

for all $x \in \overline{\Omega}$ and $t \geqslant 0$.

Proof. By the maximum principle, it is easy to see that $A(x,t) \geqslant a_*(t)$, where $a_*(t)$ is the solution of the initial value problem

$$\begin{cases} \dfrac{da_*}{dt} = -a_*, \\ a_*(0) = \min\limits_{x\in\overline{\Omega}} A_0(x). \end{cases} \tag{2.27}$$

Since $a_*(t) = e^{-t} \min_{x\in\overline{\Omega}} A_0(x)$, we have the conclution of the lemma.

q.e.d.

Step 2. Next, we derive a lower bound on $H(x,t)$.

Lemma 2.7. *Assume that $(A(x,t), H(x,t))$ is a solution of the initial-boundary value problem (1.1)–(1.4). Then there are positive constants m_5 and m_6 such that*

$$H(x,t) \geqslant m_5 \quad \text{for all } x \in \overline{\Omega},\ t \geq 0, \tag{2.28}$$

$$\liminf_{t\to+\infty} \min_{x\in\overline{\Omega}} H(x,t) \geqslant m_6. \tag{2.29}$$

Here, m_5 depends on (A_0, H_0), while m_6 is independent of (A_0, H_0).

Proof. Let $h(x,t)$ be a solution of the initial-boundary value problem

$$\begin{cases} \tau \dfrac{\partial h}{\partial t} = D\Delta h - h + \sigma_h \quad \text{for } (x,t) \in \Omega \times (0,\infty), \\ \dfrac{\partial h}{\partial \nu} = 0 \quad \text{for } (x,t) \in \partial\Omega \times (0,\infty), \\ h(x,0) = H_0(x) \quad \text{for } x \in \Omega. \end{cases} \tag{2.30}$$

Then

$$H(x,t) \geqslant h(x,t) \quad \text{for all } x \in \overline{\Omega},\ t \geqslant 0.$$

On the other hand, by Lemma 2.2 we see that

$$h(x,t) \geqslant m_5 \ \text{ for all } x \in \overline{\Omega},\ t \geqslant 0$$

and

$$\liminf_{t\to+\infty} \min_{x\in\overline{\Omega}} h(x,t) \geqslant m_6.$$

These inequalities yield the conclusion of the lemma. q.e.d.

Step 3. Once we have the lower bounds (2.28) and (2.29) on $H(x,t)$, we can apply the same arguments as in Step 3 of the proof of Theorem 1.1 to obtain (2.24) and then (2.26). (Recall that Lemma 2.5 holds true even when $\sigma_a(x) \equiv 0$.) Thus the proof of the theorem is complete. q.e.d.

2.3. Proof of Theorem 1.3.

Step 1. We have the following estimates by comparing $A(x,t)$ with $a_*(t)$ and $H(x,t)$ with $h_*(t)$, where a_* is the solution of (2.27) and $h_*(t)$ is the solution of the initial value problem

$$\begin{cases} \tau \dfrac{dh_*}{dt} = -h_*, \\ h_*(0) = \min\limits_{x\in\overline{\Omega}} H_0(x). \end{cases} \tag{2.31}$$

Lemma 2.8. *Let $(A(x,t), H(x,t))$ solve the initial-boundary value problem (1.1)-(1.4). Then*

$$A(x,t) \geqslant \min_{x\in\overline{\Omega}} A_0(x)e^{-t} \quad \text{and} \quad H(x,t) \geqslant \min_{x\in\overline{\Omega}} H_0(x)e^{-t/\tau}$$

for all $x \in \overline{\Omega}$, $t \geqslant 0$.

Step 2. We apply the arguments in Step 3 of the proof of Theorem 1.1 to obtain an upper bound on $A(x,t)$ and then an upper bound on $H(x,t)$ by making use of Lemma 2.5 and Lemma 2.8. This finishes the proof of Theorem 1.3. q.e.d.

§3. Stationary Problem for Shadow System.

3.1. Proof of Theorem 1.4.

Let $(A(x), \xi)$ be a stationary solution of the shadow system (1.14)–(1.16). If we put

$$A(x) = \xi^{q/(p-1)} u(x) \tag{3.1}$$

then $(u(x), \xi)$ satisfies

$$\varepsilon^2 u'' - u + u^p + \xi^{-q/(p-1)} \sigma_a(x) = 0 \quad \text{for } 0 < x < \ell, \tag{3.2}$$

$$\xi^{-\alpha} = \frac{1}{\ell} \int_0^\ell u^r \, dx, \tag{3.3}$$

$$u'(0) = u'(\ell) = 0. \tag{3.4}$$

It is well-known that there is an $\varepsilon_0 > 0$ such that the Neumann problem

$$\varepsilon^2 u_0'' - u_0 + u_0^p = 0 \quad \text{and} \quad u_0 > 0 \quad \text{for } 0 < x < \ell, \tag{3.5}$$

$$u_0'(0) = u_0'(\ell) = 0 \tag{3.6}$$

has a unique monotone decreasing solution $u_0(x) = u_0(x;\varepsilon)$ for $0 < \varepsilon < \varepsilon_0$ and that

$$u_0(\varepsilon) \to w(y) \qquad \text{uniformly on } [0, \ell/\varepsilon) \tag{3.7}$$

as $\varepsilon \downarrow 0$, where $w(y)$ is the solution of (1.17). It is crucial to observe that the linearized operator

$$L_\varepsilon = \varepsilon^2 \frac{d^2}{dx^2} - 1 + pu_0^{p-1} \tag{3.8}$$

under homogeneous Neumann boundary conditions is invertible and there exists a positive constant C_1 independent of $\varepsilon \in (0, \varepsilon_0)$ such that

$$\|L_\varepsilon^{-1}\|_{\mathcal{B}(C^0[0,\ell])} \leqslant C_1. \tag{3.9}$$

(See, e.g, [9].) We begin with the perturbation theory for the boundary value problem (3.5)–(3.6).

Lemma 3.1. *There is a $\delta_0 > 0$ such that if $0 < \delta < \delta_0$, then the boundary value problem*

$$\varepsilon^2 v'' - v + v^p + \delta\sigma_a(x) = 0 \quad \textit{and} \quad v > 0 \quad \textit{on } 0 < x < \ell, \tag{3.10}$$

$$v'(0) = v'(\ell) = 0 \tag{3.11}$$

has a solution $v(x;\varepsilon,\delta)$ and it satisfies

$$\max_{0\le x\le \ell} \left|v(x;\varepsilon,\delta) - \big(u_0(x) - \delta[L_\varepsilon^{-1}\sigma_a](x)\big)\right| \le C\delta^{1+\kappa_p} \tag{3.12}$$

for some positive constant independent of δ and ε, where $\kappa_p = \min\{1, p-1\}$.

For a proof, see [8] (pp. 348–349) in which the case $\sigma_a(x) \equiv \sigma_0$, a positive constant, is considered. The method works also for nonconstant $\sigma_a(x)$. Therefore, we put $\phi_0 = -L_\varepsilon^{-1}\sigma_a$ and study its behavior as $\epsilon \downarrow 0$. Since ϕ_0 solves the boundary value problem

$$\begin{cases} \varepsilon^2\phi_0'' - \phi_0 + pu_0^{p-1}\phi_0 + \sigma_a(x) = 0 & \text{for } 0 < x < \ell, \\ \phi_0'(0) = \phi_0'(\ell) = 0 \end{cases} \tag{3.13}$$

it is convenient to introduce a function $\Sigma_\varepsilon(x)$ as the solution of

$$\begin{cases} \varepsilon^2\Sigma_\varepsilon'' - \Sigma_\varepsilon + \sigma_a(x) = 0 & \text{for } 0 < x < \ell, \\ \Sigma_\varepsilon'(0) = \Sigma_\varepsilon'(\ell) = 0. \end{cases} \tag{3.14}$$

Let

$$\phi_0(x) = \Sigma_\varepsilon(x) + \psi_\varepsilon(x).$$

As to the asymptotic behavior of ψ_ε and Σ_ε, we have the following two lemmas.

Lemma 3.2. *Let $\tilde{\psi}_\varepsilon(y) = \psi_\varepsilon(\varepsilon y)$ for $0 \leqslant y \leqslant \ell/\varepsilon$. Then as $\varepsilon \downarrow 0$, $\tilde{\psi}_\varepsilon$ converges to $\psi_1(y)$, locally uniformly on $[0, +\infty)$, which satisfies*

$$\text{(3.15)} \quad \begin{cases} \psi_1'' - \psi_1 + pw^{p-1}\psi_1 + pw^{p-1}\sigma_a(0) = 0 & \text{for } 0 < y < \infty, \\ \psi_1'(0) = 0, \qquad \lim_{y\to+\infty} \psi_1(y) = 0. \end{cases}$$

For the proof, see [8] (pp. 349–350) for the case where σ_a is a constant. The argument works also for the case of nonconstant $\sigma_a(x)$.

Lemma 3.3. *As $\varepsilon \downarrow 0$, $\Sigma_\varepsilon(x)$ converges to $\sigma_a(x)$ uniformly on the interval $[0, \ell]$.*

We emphasize that, although (3.14) is a singular perturbation problem, neither a boundary layer nor an interior layer appears in this case. For the proof, see [12].

Now we are ready to construct a solution of (3.2)–(3.4). Put

$$\xi_0 = \left(\frac{1}{\ell}\int_0^\ell u_0^r\,dx\right)^{-1/\alpha}, \quad u_1(x) = v(x; \varepsilon, \xi_0^{-q/(p-1)}),$$

$$\text{and} \qquad \xi_1 = \left(\frac{1}{\ell}\int_0^\ell u_1^r\,dx\right)^{-1/\alpha}.$$

Note that

$$\xi_0^{-\alpha} = \frac{\varepsilon}{\ell}\left(\int_0^\infty w(y)^r\,dy + o(1)\right) \qquad \text{as } \varepsilon \downarrow 0$$

by virtue of (3.7). Therefore, $\xi_0 \to +\infty$ as $\varepsilon \downarrow 0$, which allows us to define $u_1(x)$ by Lemma 2.8. Moreover, we can prove that $\lim_{\varepsilon\downarrow 0} \xi_1/\xi_0 = 1$ (see Lemma 3.3 of [8]).

We put

$$\text{(3.16)} \qquad u(x) = u_1(x) + \xi_1^{-q/(p-1)}\phi(x), \qquad \xi = \xi_1(1+\eta)$$

and reduce the problem to that of seeking $(\phi(x), \eta)$. As in pp. 354–360 of [8], we can find $(\phi(x), \eta)$ by the contraction mapping principle, provided that ε is sufficiently small, say, $0 < \varepsilon < \varepsilon_0$; and it satisfies the estimate

$$\text{(3.17)} \qquad \max_{0\leqslant x\leqslant \ell} |\phi(x)| \le C_2 \qquad \text{and} \qquad |\eta| \leqslant C_3\xi_0^{-(s+1)/r_*}$$

for $0 < \varepsilon < \varepsilon_0$, where C_2, C_3 are independent of ε, and $r_* = \max\{r, 1\}$.

Therefore, we obtain a steady-state solution of the shadow system $(A_{l,\varepsilon}(x), \xi_{l,\varepsilon})$ of the following form:

$$A_{l,\varepsilon}(x) = \varepsilon^{q/[qr-(p-1)(s+1)]}\{w(x/\varepsilon) + o(1)\} + \Sigma_\varepsilon(x) + \sigma_a(0)\phi_1(x/\varepsilon) + o(1) \tag{3.18}$$

and

(3.19)

$$\xi_{l,\varepsilon} = \varepsilon^{(p-1)/[qr-(p-1)(s+1)]}\left(\int_0^\infty w(z)^r dz + o(1)\right)^{(p-1)/[qr-(p-1)(s+1)]}.$$

Similarly, we obtain a one-parameter family of stationary solutions $\{(A_{r,\varepsilon}(x), \xi_{r,\varepsilon})\}_{0<\varepsilon<\varepsilon_0}$ which has a spike at $x = \ell$. Hence the proof of Theorem 1.4 is now complete. q.e.d.

3.2. Proofs of Theorems 1.5 and 1.6.

By (3.16) and (3.17), we obtain an upper bound on $u(x)$:

$$u(x) \leq C_0\left(e^{-x/\varepsilon} + \xi_0^{-q/(p-1)}\right). \tag{3.20}$$

Therefore, the assertions of Theorems 1.5 and 1.6 can be proved in exactly the same way as in Sections 3 and 4 of [9]. Note that if $r \geqslant 1$ then we do not need lower bounds of the solution u for the proof of Proposition 3.4 of [9] which is the key to the stability analysis. We omit the detail.

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