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Blowup solutions to some systems related to biology

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Abstract.

In this paper, we describe blowup solutions to so called Keller-Segel system and its simplified system. Keller-Segel system was introduced to describe the aggregation of cellular slime molds.

We want to investigate blowup solutions to Keller-Segel system. However, it is difficult for us. Then, we investigate solutions to a simplified system of Keller-Segel system, since we can expect that the structure of solutions to the simplified system is similar to the one to Keller-Segel system.

In this paper, we will describe some results for solutions to the simplified system, our conjecture for solutions to Keller-Segel system and the relation between those results and our conjecture.

§1. Introduction

We consider the following system

$$(KS)\begin{cases} u_t = \nabla \cdot (\nabla u - u\nabla v) \text{ in } \Omega \times [0,\infty), \\ v_t = \Delta v - v + u \text{ in } \Omega \times [0,\infty), \\ \dfrac{\partial u}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = 0 \text{ on } \partial\Omega \times [0,\infty), \\ u(\cdot,0) = u_0, v(\cdot,0) = v_0 \text{ in } \Omega. \end{cases}$$

We refer to this system as Keller-Segel system.

Here $u(x,t)$ and $v(x,t)$ represent the density of cells and the chemical concentration at (x,t), respectively. Ω is a domain in $\mathbf{R}^N$ ($N = 1,2,3,\cdots$). $u_0(\not\equiv 0)$ and v_0 are smooth and nonnegative in $\overline{\Omega}$.

Keller and Segel [3] introduced a system to describe the aggregation of cellular slime molds.

Nanjundiah [4] introduced (KS) as a simplified system of the one introduced by Keller and Segel.

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In this paper, we describe our conjecture for blowup solutions to Keller-Segel system, related results, and the relation between our conjecture and those results.

§2. Our Conjecture and Related Results

We shall define blowup of solutions.

If $\lim_{t\to T}(\sup_{x\in\Omega} u(x,t)) = \infty$, we say that the solution **blows up** and that T is the **blowup time**. If there exist two sequences $\{q_n\} \subset \overline{\Omega}$ and $\{t_n\} \subset (0,T)$ such that $\lim_{n\to\infty}(q_n, t_n) = (q, T)$ and that $\lim_{n\to\infty} u(q_n, t_n) = +\infty$, we say that the point $q \in \overline{\Omega}$ is a **blowup point.**

We can easily show the following theorem.

Theorem 1. *Let $\Omega = (a,b)$ and $-\infty < a < b < \infty$, solutions (u,v) to (KS) do not blow up. Moreover, the solution (u,v) satisfies that*

$$\sup_{(x,t)\in(a,b)\times(0,\infty)} (|u(x,t)| + |v(x,t)|) < \infty.$$

Then, we obtain that blowup can not occur in one dimensional case. Since we consider blowup solutions to Keller-Segel system, then we consider two or more dimensional case.

Concerning blowup solutions to Keller-Segel system in two or more dimensional case, we shall describe our conjecture.

Our conjecture. *In the case where $N = 2$, a delta function appears at each blowup point, if the solution to (KS) blows up in finite time.*

In the case where $N \geq 3$, there exist two or more kinds of singularities of finite time blowup solutions to (KS) and there exist singularities which are different from a delta function.

We think that the following result shown in [1] is an evidence that our conjecture is true.

Theorem 2. *Let $\Omega = \{x \in \mathbf{R}^2 | |x| < L\}$ and $L \in (0,\infty)$. Then, there exists a radial solution to (KS) satisfying*

$$u(\cdot,t) \rightharpoonup 8\pi\delta_0 + f \ \text{in}\ \mathcal{M}(\overline{\Omega}) \ \text{as}\ t \to T$$

for some $T \in (0,\infty)$.

Here δ_0 is a delta function whose support is the origin, $f \in L^1(\Omega) \cap C(\overline{\Omega}\backslash\{0\})$, and $\mathcal{M}(\overline{\Omega})$ is the dual space of $C(\overline{\Omega})$.

The following result shown in [5] is also an evidence that our conjecture is true.

Theorem 3. *Let Ω be a bounded domain with smooth boundary in $\mathbf{R}^2$. Suppose that a solution (u, v) to (KS) blows up at $T \in (0, \infty)$ and that blowup points are finite. Then, the solution (u, v) satisfies that*

$$u(\cdot, t) \rightharpoonup \sum_{q \in \mathcal{B}} m(q)\delta_q + f \text{ in } \mathcal{M}(\overline{\Omega}) \text{ as } t \to T,$$

where $\mathcal{B}$ is a set of blowup points, δ_q is a delta function whose support is the point $q \in \overline{\Omega}$, $f \in L^1(\Omega) \cap C(\overline{\Omega} \setminus \mathcal{B})$ and

$$m(q) \geq \begin{cases} 8\pi & \text{if } q \in \Omega, \\ 4\pi & \text{if } q \in \partial\Omega. \end{cases}$$

Next, we shall consider three or more dimensional case. Since it is difficult for us to investigate solutions to Keller-Segel system in three or more dimensional case. Then, we shall consider solutions to the following system.

$$(HMV) \begin{cases} u_t = \nabla \cdot (\nabla u - u\nabla v) & \text{in } \mathbf{R}^N \times (0, T), \\ 0 = \Delta v + u & \text{in } \mathbf{R}^N \times (0, T), \\ u(\cdot, 0) = u_0 & \text{in } \mathbf{R}^N. \end{cases}$$

This system was introduced as a simplified system of Keller-Segel system by Herrero, Medina and Velázquez [2]. The second equation and the domain of (HMV) are different from those of (KS). Then, the initial condition of v and the boundary condition are not necessary. However, we expect that the structure of blowup solutions to (HMV) is similar to the one to (KS) in a neighbourhood of each blowup point.

Henceforth, we consider only radial solutions (u, v) to (HMV). Then, it holds that

$$v(x, t) = C - \int_0^{|x|} \frac{1}{\omega_N \xi^{N-1}} \int_{|\tilde{x}| < \xi} u(\tilde{x}, t) d\tilde{x} d\xi,$$

where C is an arbitrary constant and ω_N is the area of a unite sphere in $\mathbf{R}^N$. Therefore, we regard that (HMV) is a system with respect to u.

We can find blowup solutions to (HMV). Those blowup solutions are so called backward self-similar solutions. The definition is as follows.

We say that u is a **(backward) self-similar solution** to (HMV), if there exists a function $\overline{u}$ in $\mathbf{R}^N$ such that

$$u(x, t) = \frac{1}{T - t} \overline{u}\left(\frac{x}{\sqrt{T - t}}\right)$$

is a solution to (HMV) in $\mathbf{R}^N \times (0,T)$ for any $T \in (0,\infty)$. Then, we say that $\overline{u}$ is a **profile function**.

This solution u blows up at $(x,t) = (0,T)$, if the corresponding profile function $\overline{u}$ is positive and bounded.

The following theorem is concerned with the existence of profile functions.

Theorem 4. *In the case where $3 \le N \le 9$, there exist infinitely many, radial, positive and bounded profile functions having limits*

$$C_S = \lim_{|x|\to\infty} |x|^2 \overline{u}(x) \in (0,\infty).$$

In the case where $N \ge 10$, there exists a radial, positive and bounded profile function having a limit

$$C_S = \lim_{|x|\to\infty} |x|^2 \overline{u}(x) \in (0,\infty).$$

Concerning Theorem 4, Herrero and Velázquez [1] considered three dimensional case and the author [6] considered three or more dimensional case.

For $T > 0$ and a profile function $\overline{u}$ in Theorem 4, the self-similar solution

$$u(x,t) = \frac{1}{(T-t)} \overline{u}\left(\frac{x}{\sqrt{T-t}}\right)$$

blows up at $(x,t) = (0,T)$ and satisfies that

$$\lim_{t\to T} u(x,t) = \lim_{t\to T} \frac{1}{|x|^2} \cdot \left(\frac{|x|^2}{T-t} \overline{u}\left(\frac{x}{\sqrt{T-t}}\right)\right) = \frac{C_S}{|x|^2}$$

locally uniform in $\mathbf{R}^N \setminus \{0\}$.

Then, we obtain that the solution u has a $1/|x|^2$ type singularity at $(x,t) = (0,T)$, and that the singularity is different from a delta function.

Then, we think that Theorem 4 is an evidence that our conjecture is true.

§3. Sketch of Proof of Theorem 4

In this section, we explain the sketch of proof of Theorem 4. We define $(\overline{V}, \overline{H})$, a variable $s \in \mathbf{R}$ and a parameter $\tau \in \mathbf{R}$ as

$$e^{s+\tau} = \frac{r}{\sqrt{T-t}}, \quad \overline{V}(s;\tau) = \frac{1}{r^{N-2}} \int_0^{r/\sqrt{T-t}} \frac{\tilde{u}(\xi)}{T-t} \xi^{N-1} d\xi$$

and

$$\overline{H}(s;\tau) = \frac{d\overline{V}}{ds}(s;\tau),$$

respectively. Here, $\tilde{u}(|x|) = \overline{u}(x)$ for $x \in \mathbf{R}^N$ and $\overline{u}$ is a radial profile function.

Therefore, if we find $(\overline{V}, \overline{H})$ satisfying

$$(1)\qquad \begin{cases} \dfrac{d\overline{H}}{ds} = -(N-4)\overline{H} + 2(N-2)\overline{V} \\ \qquad\qquad + \dfrac{1}{2}e^{2(s+\tau)}\overline{H} - \overline{V}(\overline{H} + (N-2)\overline{V}) \quad \text{in } \mathbf{R}, \\ \overline{H}(s;\tau) = \dfrac{d\overline{V}}{ds}(s;\tau) \quad \text{in } \mathbf{R}, \\ \lim_{s\to-\infty}(\overline{V}(s;\tau), \overline{H}(s;\tau)) = (0,0), \\ \lim\limits_{s\to\infty}(\overline{V}(s;\tau), \overline{H}(s;\tau)) = \left(\dfrac{C_S}{N-2}, 0\right), \end{cases}$$

we can find a radial profile function $\overline{u}$ having a limit $C_S = \lim_{|x|\to\infty}|x|^2\overline{u}(x) \in (0,\infty)$. Moreover, we obtain that the profile function is positive and bounded in $\mathbf{R}^N$.

In order to find $(\overline{V}, \overline{H})$, for any sufficiently small $\varepsilon > 0$, we consider the following problem.

$$(2)\qquad \begin{cases} \dfrac{dH}{ds} = -(N-4)H + 2(N-2)V \\ \qquad\qquad + \dfrac{1}{2}e^{2(s+\tau)}H - V(H + (N-2)V) \quad \text{in } \mathbf{R}, \\ H(s;\tau) = \dfrac{dV}{ds}(s;\tau) \quad \text{in } \mathbf{R}, \\ \lim_{s\to-\infty}(V(s;\tau), H(s;\tau)) = (0,0), \quad V(0;\tau) = \varepsilon. \end{cases}$$

For any sufficiently small $\varepsilon > 0$, we can show the existence and uniqueness of a solution (V, H) to (2).

Let τ_0 be a constant satisfying

$$(3)\qquad \varepsilon = \frac{4e^{2\tau_0}}{2(N-2) + e^{2\tau_0}}.$$

Then,

$$(V(s;\tau_0), H(s;\tau_0)) = \left(\frac{4e^{2(s+\tau_0)}}{2(N-2) + e^{2(s+\tau_0)}}, \frac{16(N-2)e^{2(s+\tau_0)}}{\left[2(N-2) + e^{2(s+\tau_0)}\right]^2}\right)$$

is a solution to (2) and satisfying

$$\lim_{s\to\infty}(V(s;\tau_0), H(s;\tau_0)) = (4,0).$$

Then, we find a radial, positive and bounded profile function $\overline{u}$ having a limit $4(N-2) = \lim_{|x|\to\infty} |x|^2\overline{u}(x)$. That is to say, we get Theorem 4 in the case where $N \geq 10$.

Henceforth, we consider only the case where $3 \leq N \leq 9$.

Let $\hat{u}$ be a radial and positive stationary solution to (HMV). Put $r = e^s$ and

$$\hat{V}(s) = \frac{1}{\omega_N r^{N-2}} \int_{|x|<r} \hat{u}(x)dx.$$

For $N \geq 3$, $(\hat{V}, \hat{H})$ satisfies that

$$\begin{cases} \dfrac{d\hat{V}}{ds} = \hat{H} \quad \text{in } \mathbf{R}, \\ \dfrac{d\hat{H}}{ds} = -(N-4)\hat{H} + 2(N-2)\hat{V} - \hat{V}(\hat{H} + (N-2)\hat{V}) \quad \text{in } \mathbf{R} \end{cases}$$

and that

$$\begin{cases} \lim_{s\to-\infty} (\hat{V}(s), \hat{H}(s)) = (0,0), \\ \lim_{s\to\infty} (\hat{V}(s), \hat{H}(s)) = (2,0). \end{cases}$$

In the case where $3 \leq N \leq 9$, the orbit $(\hat{V}, \hat{H})$ starts from the origin, moves round the point $(2,0)$ infinitely many times, and converges to the point $(2,0)$ as s tends to ∞. Moreover, we can show that

$$\lim_{\tau\to-\infty} V(\cdot;\tau) = \hat{V} \quad \text{and} \quad \lim_{\tau\to-\infty} H(\cdot;\tau) = \hat{H} \tag{4}$$

uniformly in $(-\infty, S)$, for any $S \in \mathbf{R}$.

Let us put

$$\tilde{\Lambda}_1 = \{\tau < \tau_0 \mid (V(\cdot;\tau), H(\cdot;\tau)) \text{ moves round the point } (2,0) \text{ one time or more}\},$$

where τ_0 is the constant in (3).

Since we obtain that $\tilde{\Lambda}_1$ is unbounded from below by (4), then we can define Λ_1 as the unbounded connected component of $\tilde{\Lambda}_1$.

Putting $\tau_1 = \sup \Lambda_1$, we can show that $\tau_1 < \tau_0$, $(V(\cdot;\tau_1), H(\cdot;\tau_1))$ moves round $(2,0)$ just one time, and that there exists a limit $\lim_{s\to\infty}(V(s;\tau_1), H(s;\tau_1)) = (C_{S1}/(N-2), 0)$ with some $C_{S1} \in (0,\infty)$.

Next, putting

$$\tilde{\Lambda}_2 = \{\tau < \tau_1 \mid (V(\cdot;\tau), H(\cdot;\tau)) \text{ moves round the point } (2,0) \text{ two times or more}\}$$

and using an argument similar to the above, we can find the unbounded connected component Λ_2 of $\tilde{\Lambda}_2$. Putting $\tau_2 = \sup \Lambda_2$, we can show that $\tau_2 < \tau_1$, $(V(\cdot;\tau_2), H(\cdot;\tau_2))$ moves round the point $(2,0)$ just two times, and that there exists a limit $\lim_{s\to\infty}(V(s;\tau_2), H(s;\tau_2)) = (C_{S2}/(N-2), 0)$ with some $C_{S2} \in (0,\infty)$.

Repeating the above argument, for any positive integer m, we get the solutions $(V(\cdot;\tau_m), H(\cdot;\tau_m))$ to (2) moving round the point $(2,0)$ just m times and satisfying a limit $\lim_{s\to\infty}(V(s;\tau_m), H(s;\tau_m)) = (C_{Sm}/(N-2), 0)$ with some $C_{Sm} \in (0,\infty)$.

Therefore, $(V(s;\tau_m), H(s;\tau_m))$ is not equal to $(V(s;\tau_{m'}), H(s;\tau_{m'}))$, if m is not equal to m'. Moreover, for each $(V(s;\tau_m), H(s;\tau_m))$, we can find a radial, positive and bounded profile function $\overline{u}_m$ having a limit $C_{Sm} = \lim_{|x|\to\infty} |x|^2 \overline{u}_m(x)$.

Then, we can find infinitely many profile functions in the case where $3 \le N \le 9$. Thus, we get Theorem 4.

References

[1] M. A. Herrero and J. J. L. Velázquez, A blow-up mechanism for a chemotaxis model, Ann. Scoula Norm. Sup. Pisa IV, **35** (1997), 633–683.

[2] M. A. Herrero, E. Medina and J. J. L. Velázquez, Self-similar blowup for a reaction-diffusion system, J. Comput. Appl. Math., **97** (1998), 99–119.

[3] E. F. Keller and L. G. Segel, Initiation of slime mold aggregation viewed as an instability, J. Theor. Biol., **26** (1970), 399–415.

[4] V. Nanjundiah, Chemotaxis, signal relaying and aggregation morphology, J. Theor. Biol., **42** (1973), 63–105.

[5] T. Nagai, T. Senba and T. Suzuki, Chemotaxis collapse in a parabolic system of mathematical biology, Hiroshima Math. J., **30** (2000), 463–497.

[6] T. Senba, Blowup behavior of radial solutions to Jäger-Luckhaus system in high dimensional domains, Funckcial. Ekvac., **48** (2005), 247–271.

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