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Global existence of a reaction-diffusion-advection system

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Abstract.

A reaction-diffusion-advection system of equations: Mikhailov-Hildebrand-Ertl model [4] in $\mathbb{R}^2$ is treated, and the global solutions are constructed. For the local existence, semigroup method is used, and by constructing a priori estimates the global existence is proved.

§1. Introduction.

In this paper we consider the following reaction-diffusion-advection system which was proposed by Hildebrand et al. [4] (see also [3]):

$$(1.1)\quad \begin{cases} \dfrac{\partial u}{\partial t} = a\Delta u + b\nabla\{u(1-u)\nabla\chi(v)\} \\ \qquad\qquad -ce^{k\chi(v)}u - du + f(1-u) \quad \text{in} \quad \mathbb{R}^2 \times (0,\infty), \\ \dfrac{\partial v}{\partial t} = g\Delta v + hv(u+v-1)(1-v) \quad \text{in} \quad \mathbb{R}^2 \times (0,\infty), \\ u(x,0) = u_0(x),\ v(x,0) = v_0(x) \quad \text{in} \quad \mathbb{R}^2. \end{cases}$$

Here, u and v are unknown functions with $0 \le u \le 1$ and $0 \le v \le 1$. The coefficients a, b, c, d, f, g, h and k are positive constants, and $\chi(v)$ is a real-valued smooth function on $v \in [0,1]$ with $\chi'(v) \le 0$ (in [4], $\chi(v) = \frac{1}{3}v^3 - \frac{1}{2}v^2$ is introduced).

The system (1.1) is a model for a nonequilibrium self-organization process for chemical reaction of microreactors with submicrometer and nanometer sizes on a metallic surface. Typical example of the reaction is the oxidation of CO molecules on a Pt(110) surface (cf. [2]). Then, the functions u and v denote the adsorbate coverage of CO and a continuous

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order parameter of the surface structural state of Pt(110), respectively. The advection term $b\nabla\{u(1-u)\nabla\chi(v)\}$ of the first equation shows that CO molecules flow on the surface by the gradient of local potential $\chi(v)$ at a rate $1-u$. The reaction term $hv(u+v-1)(1-v)$ of the second equation indicates that the system has two stable stationary states $v = 0, 1$ and an unstable stationary state $v = 1-u$.

In [10], Tsujikawa and Yagi treated the system in a bounded domain with C^3 boundary under the Neumann boundary conditions, and proved the existence of global solutions and an exponential attractor (cf. [7] for periodic boundary conditions). Exponential attractor is a compact positively invariant set with finite fractal dimension in an (infinite dimensional) phase space, which includes the global attractor, and attracts every trajectory at an exponential rate. (As for precise definition and examples of exponential attractor, see Eden, Foias, Nicolaenko and Temam [1]). The system shows various spatial-temporal patterns [2] (cf. numerical results [8]).

In [3, 4, 9], the existence and stability of stationary spots and traveling front solutions to (1.1) are discussed, and also the interface equation is introduced in the domain $\mathbb{R}^2$. So, we would like to consider the case $\mathbb{R}^2$ and show the global existence of (1.1) (exponential attractor does not exist in any usual function space such as $L^2(\mathbb{R}^2)$, in fact, a norm of some traveling solution may diverge to infinity). Since $\chi'(v) \leq 0$, there exist only three stationary homogeneous solutions such that stable ones:

$$\text{(S)} \qquad (\tilde{u}, \tilde{v}) = (\tilde{u}_0, 0),\ (\tilde{u}_1, 1),\ \tilde{u}_i = \frac{f}{ce^{k\chi(i)} + d + f},\ i = 0, 1;$$

and an unstable one:

$$\text{(US)} \qquad (\tilde{u}, \tilde{v}) = (\tilde{u}_*, \tilde{v}_*),\ \tilde{u}_* = \frac{f}{ce^{k\chi(1-\tilde{u}_*)} + d + f},\ \tilde{v}_* = 1 - \tilde{u}_*.$$

Since the domain is unbounded, it may be natural to impose on (1.1) a boundary condition of $\lim_{|x|\to\infty}(u, v) = (\tilde{u}, \tilde{v})$. Then, by changing u and v as

$$(1.2) \qquad u = \tilde{u} + u_p \quad \text{and} \quad v = \tilde{v} + v_p$$

an evolution equation for perturbation (u_p, v_p) around $(\tilde{u}, \tilde{v})$ is derived:

$$
\text{(P)} \quad \begin{cases} \dfrac{\partial u_p}{\partial t} = a\Delta u_p + b\nabla\{u(1-u)\nabla\chi(v)\} - (ce^{k\chi(v)} + d + f)u_p \\ \qquad\qquad + c\tilde{u}(e^{k\chi(\tilde{v})} - e^{k\chi(v)}) \quad \text{in} \quad \mathbb{R}^2 \times (0,\infty), \\ \dfrac{\partial v_p}{\partial t} = g\Delta v_p + hv(u+v-1)(1-v) \quad \text{in} \quad \mathbb{R}^2 \times (0,\infty), \\ u_p(x,0) = u_{p,0}(x), \ v_p(x,0) = v_{p,0}(x) \quad \text{in} \quad \mathbb{R}^2, \end{cases}
$$

where u, v, $\tilde{u}$ and $\tilde{v}$ are defined in (S) or (US) and (1.2).

As for the case (S), the author, Takei and Tsujikawa showed the global existence [5]. In this paper, we treat the case (US). Indeed, we prove the following theorem.

Theorem 1.1. *Let $(\tilde{u}, \tilde{v})$ be the homogeneous solution given by* (US), *and $(u_{p,0}, v_{p,0}) \in H^1(\mathbb{R}^2) \times H^2(\mathbb{R}^2)$ with the initial condition:*

$$
(\mathrm{B}_0) \qquad 0 \le \tilde{u} + u_{p,0}(x) \le 1, \quad 0 \le \tilde{v} + v_{p,0}(x) \le 1 \quad \textit{in} \quad \mathbb{R}^2.
$$

Then, there exists a unique global solution (u_p, v_p) to (P) *such that*

$$
(u_p, v_p) \in \mathcal{C}([0,\infty); H^1(\mathbb{R}^2) \times H^2(\mathbb{R}^2)) \cap \mathcal{C}^1((0,\infty); H^1(\mathbb{R}^2) \times H^2(\mathbb{R}^2)) \\ \cap\, \mathcal{C}((0,\infty); H^3(\mathbb{R}^2) \times H^4(\mathbb{R}^2)).
$$

In addition, the global solution satisfies

$$
(\mathrm{B}) \qquad 0 \le \tilde{u} + u_p(x,t) \le 1, \quad 0 \le \tilde{v} + v_p(x,t) \le 1
$$

in $\mathbb{R}^2 \times [0,\infty)$.

The organization of this paper is as follows. In Section 2 we prove the local existence of solutions to (P). Section 3 is devoted to construct a priori estimates of the solutions and show the global existence.

§2. Local Existence.

We show the local existence of solutions to (P). We first formulate (P) as an abstract evolution equation in a suitable function space and use the semigroup method for obtaining local solutions.

Consider an initial value problem for a semilinear abstract evolution equation:

$$
(2.1) \qquad \begin{cases} \dfrac{dU}{dt} + AU = F(U), \quad 0 < t \le T, \\ U(0) = U_0 \end{cases}
$$

in a Banach space X. The function U is an unknown function. The operator A is a closed linear operator in X which satisfies the condition

$$(2.2)\qquad \|(\lambda - A)^{-1}\| \le \frac{M}{|\lambda| + 1}, \quad \lambda \notin \Sigma,$$

with $\Sigma = \{\lambda \in \mathbb{C}; |\arg \lambda| \le \phi\}$, $0 \le \phi < \frac{\pi}{2}$, and $M > 0$ is a constant. The initial value U_0 is in $\mathcal{D}(A^\alpha)$ with the estimate

$$(2.3)\qquad \|A^\alpha U_0\| \le R,$$

where $\alpha \in [0, 1)$ is some exponent and $R > 0$ is a constant. The function $F : \mathcal{D}(A^\eta) \to X$ is a given Lipschitz continuous function satisfying

$$(2.4)\qquad \begin{aligned} &\|F(U) - F(V)\|_X \le p(\|A^\alpha U\|_X + \|A^\alpha V\|_X) \\ &\times\{\|A^\eta(U - V)\|_X + (\|A^\eta U\|_X + \|A^\eta V\|_X)\|A^\alpha(U - V)\|_X\}, \\ &\hspace{20em} U, V \in \mathcal{D}(A^\eta), \end{aligned}$$

where η is some exponent such that $\eta \in [\alpha, 1)$ and $p(\cdot)$ is some increasing continuous function. Then we already know the following local existence result.

Proposition 2.1. ([6, Theorem 3.1]) *Under the conditions* (2.2), (2.3) *and* (2.4), *there exists a unique local solution to* (2.1) *in the space*

$$\begin{cases} U \in \mathcal{C}([0, T_R]; \mathcal{D}(A^\alpha)) \cap \mathcal{C}^1((0, T_R]; X) \cap \mathcal{C}((0, T_R]; \mathcal{D}(A)), \\ t^{1-\alpha} U \in \mathcal{B}((0, T_R]; \mathcal{D}(A)), \end{cases}$$

where $T_R > 0$ is determined by R.

Let us rewrite (P) into an initial value problem for an abstract evolution equation

$$(\widetilde{\mathrm{P}})\qquad \begin{cases} \dfrac{dU}{dt} + AU = F(U), \quad 0 < t < \infty, \\ U(0) = U_0 \end{cases}$$

in X, where

$$X = \left\{ \begin{pmatrix} u \\ v \end{pmatrix} ; \ u \in L^2(\mathbb{R}^2) \ \text{ and } \ v \in H^1(\mathbb{R}^2) \right\}.$$

Here, $U = \begin{pmatrix} u_p \\ v_p \end{pmatrix}$ and $A = \begin{pmatrix} -a\Delta + d + f & 0 \\ 0 & -g\Delta + 1 \end{pmatrix}$ with domain

$$\mathcal{D}(A) = \left\{ \begin{pmatrix} u \\ v \end{pmatrix} ; \ u \in H^2(\mathbb{R}^2) \ \text{ and } \ v \in H^3(\mathbb{R}^2) \right\}.$$

The initial value $U_0 = \begin{pmatrix} u_{p,0} \\ v_{p,0} \end{pmatrix}$ is in

$$\mathcal{D}(A^{\frac{1}{2}}) = \left\{ \begin{pmatrix} u \\ v \end{pmatrix};\ u \in H^1(\mathbb{R}^2) \ \text{ and } \ v \in H^2(\mathbb{R}^2) \right\},$$

and $F(U)$ is a nonlinear operator from $\mathcal{D}(A^{\frac{3}{4}})$ to X, such that

$$\begin{aligned} &F(U) \\ &= \begin{pmatrix} b\nabla\{u(1-u)\nabla\widetilde{\chi}(\operatorname{Re} v)\} - ce^{k\widetilde{\chi}(\operatorname{Re} v)}u_p + c\tilde{v}(e^{k\widetilde{\chi}(\tilde{v})} - e^{k\widetilde{\chi}(\operatorname{Re} v)}) \\ v_p + hv(u+v-1)(1-v) \end{pmatrix}, \\ &\qquad\qquad\qquad\qquad U = \begin{pmatrix} u_p \\ v_p \end{pmatrix} \in \mathcal{D}(A^{\frac{3}{4}}) \end{aligned}$$

with $u = \tilde{u} + u_p$ and $v = \tilde{v} + v_p$, where $\widetilde{\chi}(\operatorname{Re} v)$ is some smooth extension of $\chi(\operatorname{Re} v)$ for $v \in \mathbb{C}$.

Then the following existence theorem of local solutions to $(\widetilde{\mathrm{P}})$ is proved.

Theorem 2.2. *Assume that* $(u_{p,0}, v_{p,0}) \in H^1(\mathbb{R}^2) \times H^2(\mathbb{R}^2)$ *with* $\|(u_{p,0}, v_{p,0})\|_{H^1 \times H^2} \le R$, *where* R *is some number. Then, there exists a unique local solution* (u_p, v_p) *to* $(\widetilde{\mathrm{P}})$ *such that*

$$(u_p, v_p) \in \mathcal{C}([0, T_R];\ \mathcal{D}(A^{\frac{1}{2}})) \cap \mathcal{C}^1((0, T_R];\ X) \cap \mathcal{C}((0, T_R];\ \mathcal{D}(A))$$

with $T_R > 0$ *determined by* R.

Proof. Let us apply Proposition 2.1 to $(\widetilde{\mathrm{P}})$. The operator A from $\mathcal{D}(A)$ to X satisfies (2.2). We choose $\alpha = 1/2$. Then (2.3) is satisfied. For (2.4) we mention only the advection term. Choosing $\eta = 3/4$, that is, $\mathcal{D}(A^\eta) = H^{\frac{3}{2}}(\mathbb{R}^2) \times H^{\frac{5}{2}}(\mathbb{R}^2)$, it is obtained that

$$\begin{aligned} &\|\nabla\left[\{u(1-u) - w(1-w)\}\nabla\widetilde{\chi}(\operatorname{Re} v)\right]\|_{L^2} \\ &\le |1 - 2\tilde{u}|\,\|(u_p - w_p)\nabla\widetilde{\chi}(\operatorname{Re} v)\|_{H^1} + \|(u_p + w_p)(u_p - w_p)\nabla\widetilde{\chi}(\operatorname{Re} v)\|_{H^1} \\ &\qquad \le (\|u_p + w_p\|_{H^{\frac{5}{4}}} + 1)\|u_p - w_p\|_{H^{\frac{5}{4}}}\|\widetilde{\chi}'(\operatorname{Re} v)\nabla(\operatorname{Re} v_p)\|_{H^1} \\ &\le (\|u_p + w_p\|_{H^1}^{\frac{1}{2}}\|u_p + w_p\|_{H^{\frac{3}{2}}}^{\frac{1}{2}} + 1)\|u_p - w_p\|_{H^1}^{\frac{1}{2}}\|u_p - w_p\|_{H^{\frac{3}{2}}}^{\frac{1}{2}} p(\|v_p\|_{H^2}) \\ &\le p(\|v_p\|_{H^2})\{(\|u_p + w_p\|_{H^1} + 1)\|u_p - w_p\|_{H^{\frac{3}{2}}} + \|u_p + w_p\|_{H^{\frac{3}{2}}}\|u_p - w_p\|_{H^1}\} \\ &\qquad\qquad \le p(\|u_p\|_{H^1} + \|w_p\|_{H^1} + \|v_p\|_{H^2}) \\ &\qquad \times \{\|u_p - w_p\|_{H^{\frac{3}{2}}} + (\|u_p\|_{H^{\frac{3}{2}}} + \|w_p\|_{H^{\frac{3}{2}}} + 1)\|u_p - w_p\|_{H^1}\}, \end{aligned}$$

and that

$$\begin{aligned}&\|\nabla\,[u(1-u)\nabla\{\widetilde{\chi}(\operatorname{Re}v)-\widetilde{\chi}(\operatorname{Re}z)\}]\|_{L^2}\le C(\|u_p\|^2_{H^{\frac{5}{4}}}+1)\\&\times[\|\{\widetilde{\chi}'(\operatorname{Re}v)-\widetilde{\chi}'(\operatorname{Re}z)\}\nabla(\operatorname{Re}v_p)\|_{H^1}+\|\widetilde{\chi}'(\operatorname{Re}z)\nabla(v_p-z_p)\|_{H^1}]\\&\qquad\le p(\|u_p\|_{H^1}+\|v_p\|_{H^2}+\|z_p\|_{H^2})(\|u_p\|_{H^{\frac{3}{2}}}+1)\|v_p-z_p\|_{H^2}.\end{aligned}$$

As the other terms are similarly estimated, the condition (2.4) can be verified. Thus, the theorem is proved. Q.E.D.

By considering higher regularity for the local solution, we obtain a regularity theorem of local solution. But, as the arguments are quite similar to the proof of [5, Theorem 3.4.], we omit the proof and give only the statement:

Theorem 2.3. *Let* $(u_{p,0},v_{p,0})\in H^1(\mathbb{R}^2)\times H^2(\mathbb{R}^2)$ *with* (B_0) *and* $\|(u_{p,0},v_{p,0})\|_{H^1\times H^2}\le R$*, where* R *is some number. Then, there exists a unique local solution* (u_p,v_p) *to* (P) *such that*

$$\begin{aligned}(u_p,v_p)\in\mathcal{C}([0,T_R];\,H^1(\mathbb{R}^2)\times H^2(\mathbb{R}^2))\cap\mathcal{C}^1((0,T_R];\,H^1(\mathbb{R}^2)\times H^2(\mathbb{R}^2))\\\cap\,\mathcal{C}((0,T_R];\,H^3(\mathbb{R}^2)\times H^4(\mathbb{R}^2)),\end{aligned}$$

and that (u_p,v_p) *satisfies* (B) *in* $\mathbb{R}^2\times[0,T_R]$*. Here,* $T_R>0$ *is determined by* R*.*

§3. Global Existence.

We construct several a priori estimates for the local solutions and show the global existence of solutions to (P) with (US).

Proposition 3.1. *Let* (u_p,v_p) *be any local solution to* (P) *with* (US) *which belongs to the function space*

$$\begin{aligned}(u_p,v_p)\in\mathcal{C}([0,T);\,H^1(\mathbb{R}^2)\times H^3(\mathbb{R}^2))\cap\mathcal{C}^1((0,T);\,H^1(\mathbb{R}^2)\times H^3(\mathbb{R}^2))\\\cap\,\mathcal{C}((0,T);\,H^2(\mathbb{R}^2)\times H^4(\mathbb{R}^2)).\end{aligned}$$

Then, there exists some increasing continuous function $p(\cdot)$ *independent of* u_p *and* v_p*, such that*

$$\|(u_p(t),v_p(t))\|_{H^1\times H^3}\le p(t+\|(u_{p,0},v_{p,0})\|_{H^1\times H^3}),\quad 0\le t<T. \tag{3.1}$$

Proof. In the proof, we use another expression of the second equation of (P) with (US):

$$
(3.2) \qquad \begin{aligned}
\frac{\partial v_p}{\partial t} &= g\Delta v_p - v_p + P(u_p, v_p), \\
P(u_p, v_p) &= v_p + hv(u+v-1)(1-v) \\
&= h\left[v(1-v)u_p + \{1+v(1-v)\}v_p\right].
\end{aligned}
$$

Step 1. Multiply (3.2) by v_p and integrate the product in $\mathbb{R}^2$. From (B), we obtain

$$
(3.3) \quad \frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^2} v_p^2 dx + g\int_{\mathbb{R}^2} |\nabla v_p|^2\, dx \le C\left(\int_{\mathbb{R}^2} u_p^2\, dx + \int_{\mathbb{R}^2} v_p^2\, dx\right).
$$

Multiply next (3.2) by Δv_p and integrate the product in $\mathbb{R}^2$. Then,

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^2} |\nabla v_p|^2\, dx + g\int_{\mathbb{R}^2} |\Delta v_p|^2\, dx + \int_{\mathbb{R}^2} |\nabla v_p|^2\, dx& \\
\le \frac{g}{2}\int_{\mathbb{R}^2} |\Delta v_p|^2\, dx + C\left(\int_{\mathbb{R}^2} u_p^2\, dx + \int_{\mathbb{R}^2} v_p^2\, dx\right)&.
\end{aligned}
$$

It follows that

$$
(3.4) \quad \begin{aligned}
\frac{d}{dt}\int_{\mathbb{R}^2} |\nabla v_p|^2\, dx + g\int_{\mathbb{R}^2} |\Delta v_p|^2\, dx + 2\int_{\mathbb{R}^2} |\nabla v_p|^2\, dx& \\
\le C\left(\int_{\mathbb{R}^2} u_p^2\, dx + \int_{\mathbb{R}^2} v_p^2\, dx\right)&.
\end{aligned}
$$

Multiply again (3.2) by $\Delta^2 v_p$ and integrate the product in $\mathbb{R}^2$. Then,

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^2} |\Delta v_p|^2\, dx + g\int_{\mathbb{R}^2} |\nabla\Delta v_p|^2\, dx + \int_{\mathbb{R}^2} |\Delta v_p|^2\, dx& \\
\le \frac{g}{2}\int_{\mathbb{R}^2} |\nabla\Delta v_p|^2\, dx + \frac{1}{2g}\int_{\mathbb{R}^2} |\nabla P|^2 dx&.
\end{aligned}
$$

By (B) it is easy to see that $|\nabla P|^2 \le C\left(|\nabla u_p|^2 + |\nabla v_p|^2\right)$. Therefore, we obtain

$$
(3.5) \quad \begin{aligned}
\frac{d}{dt}\int_{\mathbb{R}^2} |\Delta v_p|^2\, dx + g\int_{\mathbb{R}^2} |\nabla\Delta v_p|^2\, dx + 2\int_{\mathbb{R}^2} |\Delta v_p|^2\, dx& \\
\le C\left(\int_{\mathbb{R}^2} |\nabla u_p|^2\, dx + \int_{\mathbb{R}^2} |\nabla v_p|^2\, dx\right)&.
\end{aligned}
$$

Meanwhile, multiply the first equation of (P) by u_p and integrate the product in $\mathbb{R}^2$. Then,

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^2} u_p^2 dx + a\int_{\mathbb{R}^2} |\nabla u_p|^2 dx + (ce^{k\chi(1)} + d + f)\int_{\mathbb{R}^2} u_p^2 dx \\ \leq -b\int_{\mathbb{R}^2} u(1-u)\chi'(v)\nabla u_p \nabla v_p dx + c\tilde{v}\int_{\mathbb{R}^2} (e^{k\chi(\tilde{v})} - e^{k\chi(v)})u_p dx \\ \leq \frac{a}{2}\int_{\mathbb{R}^2} |\nabla u_p|^2 dx + C\left(\int_{\mathbb{R}^2} u_p^2 dx + \int_{\mathbb{R}^2} |\nabla v_p|^2 dx + \int_{\mathbb{R}^2} v_p^2 dx\right).\end{aligned}$$

So, we have

$$\begin{aligned}(3.6)\quad \frac{d}{dt}\int_{\mathbb{R}^2} u_p^2 dx + a\int_{\mathbb{R}^2} |\nabla u_p|^2 dx \\ \leq C\left(\int_{\mathbb{R}^2} u_p^2 dx + \int_{\mathbb{R}^2} |\nabla v_p|^2 dx + \int_{\mathbb{R}^2} v_p^2 dx\right).\end{aligned}$$

By adding (3.3), (3.4) and (3.5) to (3.6) multiplied by a large constant, we obtain that with some constant $\delta > 0$,

$$\begin{aligned}(3.7)\quad \|u_p(t)\|_{L^2}^2 + \|v_p(t)\|_{H^2}^2 \\ \leq Ce^{-\delta t}\|\Delta v_{p,0}\|_{L^2}^2 + p(t + \|u_{p,0}\|_{L^2} + \|v_{p,0}\|_{H^1}), \quad 0 \leq t < T.\end{aligned}$$

Step 2. Operate $\nabla\Delta$ to (3.2), take the inner product with $\nabla\Delta v_p$ and integrate the product in $\mathbb{R}^2$. Then,

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^2} |\nabla\Delta v_p|^2 dx + g\int_{\mathbb{R}^2} |\Delta^2 v_p|^2 dx + \int_{\mathbb{R}^2} |\nabla\Delta v_p|^2 dx \\ \leq \frac{g}{2}\int_{\mathbb{R}^2} |\Delta^2 v_p|^2 dx + \frac{1}{2g}\int_{\mathbb{R}^2} |\Delta P|^2 dx.\end{aligned}$$

Thanks to (B) it is easily verified that

$$|\Delta P|^2 \leq C\left(|\Delta u_p|^2 + |\Delta v_p|^2 + |\nabla v_p|^4 + |\nabla u_p|^2 |\nabla v_p|^2\right).$$

Moreover, we obtain that

$$(3.8)\quad \int_{\mathbb{R}^2} \left(|\Delta v_p|^2 + |\nabla v_p|^4\right) dx \leq \|v_p\|_{H^2}^2 + \|\nabla v_p\|_{L^4}^4 \leq p(\|v_p\|_{H^2}).$$

In addition, by noting that $\int_{\mathbb{R}^2} |\nabla u_p|^2 \, dx \le \frac{1}{2} \int_{\mathbb{R}^2} \left(|\Delta u_p|^2 + |u_p|^2 \right) dx$, we see that

$$
(3.9) \quad \int_{\mathbb{R}^2} |\nabla u_p|^2 |\nabla v_p|^2 \, dx \le \|\nabla u_p\|_{L^3}^2 \|\nabla v_p\|_{L^6}^2
$$
$$
\le p(\|u_p\|_{L^2} + \|v_p\|_{H^2}) \|u_p\|_{H^2}^{\frac{4}{3}} \le C \|\Delta u_p\|_{L^2}^2 + p(\|u_p\|_{L^2} + \|v_p\|_{H^2}).
$$

Therefore, it follows that

$$
(3.10) \quad \frac{d}{dt} \int_{\mathbb{R}^2} |\nabla \Delta v_p|^2 \, dx + g \int_{\mathbb{R}^2} |\Delta^2 v_p|^2 \, dx + 2 \int_{\mathbb{R}^2} |\nabla \Delta v_p|^2 \, dx
$$
$$
\le C \|\Delta u_p\|_{L^2}^2 + p(\|u_p\|_{L^2} + \|v_p\|_{H^2}).
$$

Meanwhile, multiply the first equation of (P) by Δu_p and integrate the product in $\mathbb{R}^2$. Then,

$$
\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^2} |\nabla u_p|^2 \, dx + a \int_{\mathbb{R}^2} |\Delta u_p|^2 \, dx + (c e^{k\chi(1)} + d + f) \int_{\mathbb{R}^2} |\nabla u_p|^2 \, dx
$$
$$
\le -ck \int_{\mathbb{R}^2} e^{k\chi(v)} \chi'(v) u_p \nabla u_p \nabla v_p dx - b \int_{\mathbb{R}^2} \Delta u_p \nabla \{u(1-u)\nabla \chi(v)\} \, dx
$$
$$
- c\tilde{u} \int_{\mathbb{R}^2} \{e^{k\chi(\tilde{v})} - e^{k\chi(v)}\} \Delta u_p dx
$$
$$
\le \frac{a}{4} \int_{\mathbb{R}^2} |\Delta u_p|^2 \, dx + \frac{b^2}{a} \int_{\mathbb{R}^2} |\nabla \{u(1-u)\nabla \chi(v)\}|^2 \, dx
$$
$$
+ \frac{(c e^{k\chi(1)} + d + f)}{2} \int_{\mathbb{R}^2} |\nabla u_p|^2 \, dx + p(\|v_p\|_{H^1}).
$$

In addition, by similar estimates to (3.8) and (3.9) we have

$$
\int_{\mathbb{R}^2} |\nabla \{u(1-u)\nabla \chi(v)\}|^2 \, dx
$$
$$
\le \frac{a^2}{4b^2} \int_{\mathbb{R}^2} |\Delta u_p|^2 \, dx + p(\|u_p\|_{L^2} + \|v_p\|_{H^2}).
$$

It then follows that

$$
\frac{d}{dt} \int_{\mathbb{R}^2} |\nabla u_p|^2 \, dx + a \int_{\mathbb{R}^2} |\Delta u_p|^2 \, dx + (c e^{k\chi(1)} + d + f) \int_{\mathbb{R}^2} |\nabla u_p|^2 \, dx
$$
$$
\le p(\|u_p\|_{L^2} + \|v_p\|_{H^2}).
$$

We add this to (3.10) multiplied a small positive constant. Then, it is deduced from (3.7) that with some constant $\delta > 0$,

$$\begin{aligned} &\|u_p(t)\|_{H^1}^2 + \|v_p(t)\|_{H^3}^2 \\ &\le Ce^{-\delta t}\{\|\nabla u_{p,0}\|_{L^2}^2 + \|\nabla \Delta v_{p,0}\|_{L^2}^2\} + p(t + \|u_{p,0}\|_{L^2} + \|v_{p,0}\|_{H^2}), \\ &\hspace{20em} 0 \le t < T. \end{aligned}$$

Hence, we have proved the estimate (3.1). Q.E.D.

By Proposition 3.1, we can give the proof of global existence, Theorem 1.1. But, since quite similar arguments as [5, Proof of Theorem 1.1.] assure our global existence, we omit the proof.

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