Advanced Studies in Pure Mathematics 47-2, 2007
Asymptotic Analysis and Singularities
pp. 709–728

Asymptotic form of solutions of the Tadjbakhsh-Odeh variational problem

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Abstract.

We consider a variational problem posed by Tadjbakhsh and Odeh to describe the shape of an elastic ring in the plane under uniform pressure. Regarding the ring as a smooth closed curve, the Euler-Lagrange equation reduces to a second order ordinary differential equation for the curvature with the periodic boundary condition. The asymptotic form of solutions is presented as the external pressure tends to infinity. This is done by studying a singular perturbation problem for the Euler-Lagrange equation.

§1. Introduction

Since M. Levy's work in 1884 [3] the buckling behavior of a circular elastic ring under uniform pressure has been studied by many people, for example, G. F. Carrier [1], I. Tadjbakhsh and F. Odeh [4], J. E. Flaherty, J. B. Keller and S. I. Rubinow [2], K. Watanabe [5]. It was Tadjbakhsh and Odeh who formulated the problem as a variational problem, see Problem 1.1 below. The purpose of this paper is to give a precise information on the shape of critical points of this variational problem when the external pressure is very high.

Let us consider an elastic circle wire immersed in the plane under uniform pressure p_i and p_o, which act on the wire from the inner domain enclosed by the wire and from its exterior, respectively. Set $p = p_o - p_i$. We consider the case where p is positive, i.e., the external pressure p_o is higher than the internal pressure p_i. If $p > 0$ is small, then the circle is stable. However, when $p > 0$ exceeds a certain critical value, which is called the buckling load, the circle becomes unstable and the wire is deformed to a buckled state. We are interested in the equilibrium states of such a wire. We regard the wire as a closed plane curve $\gamma \in S$, where S

Received October 31, 2005.
Revised June 2, 2006.

denotes the set of all smooth closed curves in the plane whose perimeter is equal to L and the rotation number is one. Let γ be parametrized by the arc-length s, and κ denote the curvature of γ. Tadjbakhsh and Odeh ([4]) posed the following variational problem to determine the shape of the elastic ring in an equilibrium state:

Problem 1.1. *Minimize*

$$E(\gamma) = \frac{1}{2}\int_0^L \kappa^2(s)\,ds + p\mathcal{A}(\gamma) \quad \textit{over } \gamma \in S, \tag{1.1}$$

where $p > 0$ *is a given constant and* $\mathcal{A}$ *denotes the area enclosed by* γ*.*

The first term of the right-hand side of (1.1) represents the elastic energy and is also known as the total squared curvature. The second term represents the work done on the wire by the external pressure p. We notice that the requirement of having the length L is the only constraint in this variational problem. For, the rotation number is invariant under continuous deformation of a curve. In [4] it was proved that (i) the variational problem has a minimizer for any $p > 0$, (ii) the circle is always a critical point of this problem, regardless of the size of p, and (iii) the circle is an unstable equilibrium point if p is sufficiently large.

The Euler-Lagrange equation for the variational problem 1.1 can be written as a second order ordinary differential equation for the curvature κ subject to the periodic boundary condition. By the uniqueness of solution of the initial value problem, it is not difficult to see that any critical point of the variational problem 1.1 is obtained from a solution of the boundary value problem

$$\text{(P)} \quad \begin{cases} \kappa''(s) + \dfrac{1}{2}\kappa^3(s) - \mu\kappa(s) - p = 0 \qquad \text{for} \quad s \in [0, L/(2n)], \\ \displaystyle\int_0^{L/(2n)} \kappa(s)\,ds = \frac{\pi}{n}, \\ \kappa'(0) = \kappa'(L/(2n)) = 0, \end{cases}$$

for some integer $n \geq 2$, where the prime $'$ denotes the derivative with respect to the arc-length parameter s. The first equation of (P) is the Euler-Lagrange equation and contains a Lagrange multiplier μ to be determined together with κ. The second integral condition means that the rotation number of γ is one. For a given constant $p > 0$, we find a pair $(\kappa(s), \mu)$ which satisfies (P).

A critical point $\gamma \in S$ of the energy $E(\gamma)$ is said to be of *mode* n if it has exactly n axes of symmetry. Observe that a strictly monotone solution of the boundary value problem (P) extends to a solution of the Euler-Lagrange equation over the entire interval $[0, L]$ satisfying the periodic boundary condition, and hence it gives rise to a critical point of $E(\gamma)$ of mode n. Therefore, for simplicity, we call a strictly monotone solution of (P) an n-mode solution.

In this paper, we consider the asymptotic behavior of an n-mode solution of (P) as $p \to \infty$ by a singular perturbation method. Our main results are stated as follows:

Theorem 1.1. *For each $n \geq 2$, there is a positive number P_0 such that* (P) *has a one-parameter family of solutions $\{(\kappa(s;p), \mu(p))\}_{p>P_0}$ of mode n with the following properties:* (i) *$\kappa(s;p)$ is a strictly decreasing in $s \in [0, L/(2n)]$, and* (ii) *as $p \to \infty$,*

$$\max_{s\in[0,L/(2n)]} \kappa(s;p) = \kappa(0;p) = A^* - \left(\frac{1}{4\sqrt{M_n}\sqrt{p}} + O(1/p)\right)\delta^2 + O(\delta^3), \tag{1.2}$$

$$\min_{s\in[0,L/(2n)]} \kappa(s;p) = \kappa(L/(2n);p) = B^* + \delta + O(\delta^2), \tag{1.3}$$

$$\mu(p) = M_n p - \frac{4n\sqrt{M_n}}{L}\sqrt{p} + O(p^{3/8}), \tag{1.4}$$

$$s_0(p) = \frac{1}{2\sqrt{M_n}}\frac{\log p}{\sqrt{p}} + \frac{\log(4{M_n}^{3/2})}{\sqrt{M_n}}\frac{1}{\sqrt{p}} + O(1/p^{5/8}). \tag{1.5}$$

Here, $s_0(p)$ is the point at which $\kappa(s;p)$ vanishes, M_n is a positive constant given by

$$M_n = \frac{L}{2(n-1)\pi}, \tag{1.6}$$

and $A^* = A^*(p)$, $B^* = B^*(p)$, $\delta = \delta(p)$ are expanded as follows:

$$A^* = 2\sqrt{M_n\sqrt{p}} - \frac{4n}{L} + \frac{1}{M_n} + O(1/p^{1/8}), \tag{1.7}$$

$$B^* = -\frac{1}{M_n} - \frac{4n}{L{M_n}^{3/2}}\frac{1}{\sqrt{p}} + O(1/p^{5/8}), \tag{1.8}$$

$$\delta = 8e\sqrt{M_n\sqrt{p}}\exp\left[-\frac{L\sqrt{M_n\sqrt{p}}}{2n}\right](1 + O(1/p^{1/8})) \tag{1.9}$$

as $p \to \infty$.

Remark 1.1. *Theorem* 1.1 *gives us the asymptotic form of an n-mode solution of* (P) *as $p \to \infty$ for each $n \geq 2$. However there might* *exist an n-mode solution whose asymptotic form as* $p \to \infty$ *is different from the form* (1.2)–(1.9).

The closed curve corresponding to an n-mode solution of (P) has n axes of symmetry, as we stated above. When p is sufficiently large, Theorem 1.1 gives the shape of the closed curve. In Theorem 1.1, we prove that (i) the curvature of the closed curve is very large at $s = jL/n$, and is positive only on a neighborhood U_j of $s = jL/n$, where $j = 0, 1, \cdots, n$, and (ii) the curvature of the closed curve is very close to $-2(n-1)\pi/L$ on the complement of $\bigcup_{j=0}^{n-1} U_j$. $-2(n-1)\pi/L$ is the curvature of a circle with the perimeter $L/(n-1)$ and negative enclosed area. Therefore we see that the closed curve has a small circular part on each U_j and the other parts are close to an arc with radius $L/(2(n-1)\pi)$.

We mention here a result which is related to this paper. Recently K. Watanabe ([5]) has given explicit representations of the n-mode solutions of (P) for any $n \geq 2$ and $p > 0$. When p is sufficiently large, the n-mode solution κ_n is expressed as

$$\kappa_n(s) = \frac{1}{a\,\mathrm{dn}\left(\frac{2nK(m)s}{L}\right) + b} + c, \tag{1.10}$$

where $\mathrm{dn}(\cdot)$ is the Jacobian elliptic function and $K(m)$ is the complete elliptic integral of the first kind. The quadruplet (a, b, c, m) is the solution of the following system of equations:

$$\int_0^{L/(2n)} \kappa_n(s)\,ds = \frac{\pi}{n}, \tag{1.11}$$

$$a = -\sqrt{\frac{1 + 4b^2h^2(2-m) - \sqrt{1 + 8b^2h^2(2-m) + 16b^4h^4m^2}}{8h^2(1-m)}}, \tag{1.12}$$

$$c = -\frac{1 + \sqrt{1 + 8b^2h^2(2-m) + 16b^4h^4m^2}}{2b}, \tag{1.13}$$

$$2bh^4m^2\sqrt{1 + 8b^2h^2(2-m) + 16b^4h^4m^2} = p, \tag{1.14}$$

where $h = 2nK(m)/L$. Since the quantities a, b, c, m and p are related in a very complicated fashion, it is difficult to obtain the asymptotic form of n-mode solutions of (P) as $p \to \infty$ from the representations (1.10)–(1.14). We shall prove Theorem 1.1 by a singular perturbation method.

Acknowledgements. The author would like to express his gratitude to Professor I. Takagi who gave him many useful suggestions.

§2. Formulation

In order to prove Theorem 1.1, we look for a family of solutions $(\kappa(s;p), \mu(p))$ of (P) with the property that $\lim_{p\to\infty} \mu(p)/p = \mu^*$ for some positive constant μ^*. Thus, putting $\varepsilon = 1/\sqrt{p}$ and defining

$$\mu_0(\varepsilon) = \varepsilon^2 \mu(1/\varepsilon^2), \tag{2.1}$$

we see that $(\kappa(s), \mu_0) = (\kappa(s; 1/\varepsilon^2), \varepsilon^2\mu(1/\varepsilon^2))$ satisfies

$$(\mathrm{P}_\varepsilon)\quad \begin{cases} \varepsilon^2\kappa''(s) + \dfrac{\varepsilon^2}{2}\kappa^3(s) - \mu_0(\varepsilon)\kappa(s) - 1 = 0 \quad \text{for} \quad s \in [0, L/(2n)], \\ \displaystyle\int_0^{L/(2n)} \kappa(s)\,ds = \frac{\pi}{n}, \\ \kappa'(0) = \kappa'(L/(2n)) = 0. \end{cases}$$

We shall construct a strictly decreasing solution $(\kappa(s;\varepsilon), \mu_0(\varepsilon))$ of (P_ε) for ε sufficiently small. Since $\kappa'(0) = 0$, by multiplying the first equation of (P_ε) by $\kappa'(s)$ and integrating from 0 to s, we have

$$\frac{d\kappa}{ds} = -\frac{1}{\varepsilon}\sqrt{2\,(F_\varepsilon(A) - F_\varepsilon(\kappa))}, \tag{2.2}$$

where $A = \kappa(0)$ and

$$F_\varepsilon(\kappa) = \frac{\varepsilon^2}{8}\kappa^4 - \frac{\mu_0}{2}\kappa^2 - \kappa. \tag{2.3}$$

Since $\kappa(s)$ is strictly monotone decreasing and satisfies the second integral equation of (P_ε), $A = \kappa(0)$ must be positive. From (2.2), it holds that

$$\int_\kappa^A \frac{\varepsilon\,d\kappa}{\sqrt{2(F_\varepsilon(A) - F_\varepsilon(\kappa))}} = s. \tag{2.4}$$

The inverse function of (2.4) is the solution of the first equation of (P_ε). Set $B = \kappa(L/(2n))$. Then it follows from $\kappa'(L/(2n)) = 0$ and (2.2) that

$$F_\varepsilon(A) = F_\varepsilon(B). \tag{C1$_\varepsilon$}$$

Moreover, by (2.4), we find

$$\int_B^A \frac{\varepsilon\,d\kappa}{\sqrt{2(F_\varepsilon(A) - F_\varepsilon(\kappa))}} = \frac{L}{2n}. \tag{I1$_\varepsilon$}$$

On the other hand, by the second integral condition of (P_ε) and (2.2), we obtain

$$(I2_\varepsilon) \qquad \int_B^A \frac{\varepsilon\kappa\, d\kappa}{\sqrt{2(F_\varepsilon(A) - F_\varepsilon(\kappa))}} = \frac{\pi}{n}.$$

Therefore, our job is to find a triplet $(A(\varepsilon), B(\varepsilon), \mu_0(\varepsilon))$ satisfying $(I1_\varepsilon)$, $(I2_\varepsilon)$ and $(C1_\varepsilon)$.

Remark 2.1. It has not been verified that the representation (1.10)–(1.14) implies $\mu/p \to \mu^*$ as $p \to \infty$, for μ depends on a, b, c, and m in a very complicated way.

§3. Proof of Theorem 1.1

We carry out the proof of Theorem 1.1 in the following way: To begin with, we define a function $F(\kappa, \varepsilon, \nu)$ by

$$F(\kappa, \varepsilon, \nu) = \frac{\varepsilon^2}{8}\kappa^4 - \frac{\nu}{2}\kappa^2 - \kappa,$$

where $\varepsilon > 0$ and $\nu > 0$ are constants. In what follows, when there is no fear of confusion, we write $F(\kappa)$ instead of $F(\kappa, \varepsilon, \nu)$. Let two numbers A and B satisfy

$$(C1) \qquad F(A, \varepsilon, \nu) = F(B, \varepsilon, \nu).$$

Let λ be an arbitrarily fixed number in the interval $0 < \lambda < 1$. First, for each $B \in (B^*, 0)$ and $\nu \in [\lambda, 1/\lambda]$, we consider the asymptotic behavior of the integrals on the left-hand sides of

$$(I1) \qquad \int_B^A \frac{\varepsilon\, d\kappa}{\sqrt{2(F(B, \varepsilon, \nu) - F(\kappa, \varepsilon, \nu))}} = \frac{L}{2n}$$

and

$$(I2) \qquad \int_B^A \frac{\varepsilon\kappa\, d\kappa}{\sqrt{2(F(B, \varepsilon, \nu) - F(\kappa, \varepsilon, \nu))}} = \frac{\pi}{n}$$

as $\varepsilon \downarrow 0$, where B^* is the point at which $F(\kappa) = F(\kappa, \varepsilon, \nu)$ attains the local maximum (see Lemma 3.2). Second, setting $B = B^* + \exp[(-d/\varepsilon)]$, we derive a system of equations for (d, ν) and find the first approximation of solutions by computing the limit of the integrals as $\varepsilon \downarrow 0$. Third, we prove that there exists a family of solutions $(d(\varepsilon), \nu(\varepsilon))$ for sufficiently small ε by invoking the implicit function theorem. Finally, by a careful look at asymptotic expansions which is given by Lemma 3.2, we shall prove Theorem 1.1.

Let us set

$$f(\kappa) = \frac{\varepsilon^2}{2}\kappa^3 - \nu\kappa - 1,$$

which satisfies $dF/d\kappa = f$. Henceforth we denote the derivatives of F and f with respect to κ as the prime $'$ for short.

We begin by computing five important values of κ.

Lemma 3.1. *Fix a constant λ arbitrarily in the interval $0 < \lambda < 1$ and let ν be an arbitrarily fixed number in $[\lambda, 1/\lambda]$. Let B^* be the point where F attains its local maximum and $A^* > 0$ satisfy $F(A^*) = F(B^*)$. Let A_0, A_1 and A_2 be the positive numbers at which F vanishes, attains the local minimum and f' vanishes, respectively. Then, as $\varepsilon \downarrow 0$,*

$$A^* = \frac{2\sqrt{\nu}}{\varepsilon} + \frac{1}{\nu} - \frac{1}{2\nu^2\sqrt{\nu}}\varepsilon + O(\varepsilon^2), \tag{3.1}$$

$$A_0 = \frac{2\sqrt{\nu}}{\varepsilon} + \frac{1}{\nu} - \frac{3}{4\nu^2\sqrt{\nu}}\varepsilon + O(\varepsilon^2), \tag{3.2}$$

$$A_1 = \frac{\sqrt{2\nu}}{\varepsilon} + \frac{1}{2\nu} - \frac{3\sqrt{2}}{16\nu^2\sqrt{\nu}}\varepsilon + O(\varepsilon^2), \tag{3.3}$$

$$A_2 = \frac{\sqrt{2\nu}}{\sqrt{3}}\frac{1}{\varepsilon}, \tag{3.4}$$

$$B^* = -\frac{1}{\nu} - \frac{1}{2\nu^4}\varepsilon^2 + O(\varepsilon^4), \tag{3.5}$$

where the terms $O(\varepsilon^\beta)$ are uniform with respect to $\nu \in [\lambda, 1/\lambda]$. In particular, B^* is expanded as a power series in ε^2.

Proof. In what follows, the terms $O(\varepsilon^\beta)$ are uniform with respect to $\nu \in [\lambda, 1/\lambda]$. First, we derive the value B^*. Let us set $B^* = b_0 + b_1\varepsilon + b_2\varepsilon^2 + O(\varepsilon^3)$. Then it is clear that $F''(B^*) < 0$ holds. Since $\nu \in [\lambda, 1/\lambda]$ is fixed and independent of ε, it follows from $F'(B^*) = f(B^*) = 0$ that $b_0 = -1/\nu$, $b_1 = 0$ and $b_2 = -1/(2\nu^4)$. Let b_j be a coefficient of ε^j of B^*. We claim that $b_{2l-1} = 0$ holds for any positive integer l. Suppose that $b_{2l-1} = 0$ holds for $1 \le l \le m$. By this assumption, the coefficient of ε^{2m-1} of B^{*3} is equal to 0. Thus the coefficient of ε^{2m+1} of $f(B^*)$ is equal to $-\nu b_{2m+1}$. Hence we get $b_{2m+1} = 0$. Therefore we obtain (3.5). By (3.5), it holds that $F(B^*) = 1/(2\nu) + O(\varepsilon^2) > 0$. Since $F(A^*) = F(B^*)$, we obtain (3.1) along the same line as above. Next, set $A_0 = a_0/\varepsilon + a_1 + a_2\varepsilon + O(\varepsilon^2)$. It follows from the equation $F(A_0) = 0$

that $a_0 = 2\sqrt{\nu}$, $a_1 = 1/\nu$ and $a_2 = -3/(4\nu^2\sqrt{\nu})$. Thus we obtain (3.2). By the same arguments, we find (3.3) and (3.4). Q.E.D.

By the condition (C1), the point A must satisfy $A_1 < A$. Moreover, by simple calculations, we see that A satisfies $A_0 < A < A^*$. Indeed, we can prove that (i) if $A \in (A_1, A_0]$, then (I1) does not hold, (ii) if A satisfies $A^* < A$ and either $F(A) - F(B^*) = O(1)$ or $F(A) - F(B^*) \to \infty$ as $\varepsilon \downarrow 0$, then (I1) does not hold, (iii) if A satisfies $A^* < A$ and $F(A) - F(B^*) \to 0$ as $\varepsilon \downarrow 0$, then (I2) does not hold, and (iv) if $A = A^*$, then (I1) and (I2) do not hold. Then, by the condition (C1), it holds that $B \in (B^*, 0)$, for $0 = F(A_0) < F(A) < F(A^*)$ implies $F(0) < F(B) < F(B^*)$. Moreover, from the fact of (ii) and (iii), we surmise that B must be close to B^*. Therefore, henceforth suppose that B is close to B^* and $B^* < B < 0$, i.e. $B = B^* + \delta$, where $\delta > 0$ is sufficiently small and to be determined later.

We observe that $f'(\kappa)$ vanishes at $\kappa = \pm A_2$ and is negative on $[-A_2, A_2]$. Since $f(B^*) = 0$ and $(B^*, 0) \subset [-A_2, A_2]$, we see that $f(B) < 0$ and $f'(B) < 0$ hold for each $B \in (B^*, 0)$. We use these facts in the proof of Lemmas below.

Next we study the asymptotic behavior of the integrals on the left-hand sides of (I1) and (I2) as $\varepsilon \downarrow 0$ for each $B^* < B < 0$ and $\lambda \le \nu \le 1/\lambda$. Note that the inequality $A_0 < A < A^*$ follows from the inequality $B^* < B < 0$.

Lemma 3.2. *Let λ be an arbitrarily fixed number in $(0,1)$. For each $B \in (B^*, 0)$ and $\nu \in [\lambda, 1/\lambda]$, it holds that, as $\varepsilon \downarrow 0$,*

$$(3.6) \quad \int_B^A \frac{\varepsilon\, d\kappa}{\sqrt{2(F(A) - F(\kappa))}} = \frac{\varepsilon \log \varphi(B)}{\sqrt{-f'(B)}}(1 + R_1(\varepsilon, \nu, B)) + \frac{1}{\sqrt{\nu}}\varepsilon \log\left(\psi_1\sqrt{\psi_2}\right) + R_2(\varepsilon, \nu, B),$$

$$(3.7) \quad \int_B^A \frac{\varepsilon\kappa\, d\kappa}{\sqrt{2(F(A) - F(\kappa))}} = \frac{\varepsilon \log \varphi(B)}{\sqrt{-f'(B)}}\left(B - \frac{f(B)}{f'(B)}\right)(1 + R_3(\varepsilon, \nu, B)) + \pi + 2\left(\frac{A\nu^{1/4}}{f(A)} - \frac{1}{\nu^{3/4}}\right)\sqrt{\varepsilon} - \frac{1}{\nu^{3/2}}\varepsilon \log\left(\psi_1\sqrt{\psi_2}\right) + \frac{2}{\nu^{3/2}}\varepsilon + R_4(\varepsilon, \nu, B),$$

where $R_1 = O(\varepsilon^2)$, $R_2 = O(\varepsilon^{5/4})$, $R_3 = O(\varepsilon^2)$ and $R_4 = O(\varepsilon^{5/4})$ as $\varepsilon \downarrow 0$, uniformly in $B \in (B^*, 0)$ and $\nu \in [\lambda, 1/\lambda]$. Here, ψ_1, ψ_2, and

$\varphi(B)$ are expressed as follows:

$$\psi_1 = \frac{2\nu\sqrt{\nu} + \sqrt{\varepsilon} + \sqrt{4\nu^3 + 4\nu\sqrt{\nu}\sqrt{\varepsilon} + 2\nu F(B)\varepsilon}}{\sqrt{\varepsilon}\left(1 + \sqrt{2\nu F(B)}\right)}, \tag{3.8}$$

$$\psi_2 = \left(\frac{2}{\varepsilon} + \frac{2\sqrt{1-\varepsilon}}{\varepsilon} - 1\right)\frac{1 - \sqrt{2\sqrt{\varepsilon} - \varepsilon}}{1 + \sqrt{2\sqrt{\varepsilon} - \varepsilon}}, \tag{3.9}$$

$$\varphi(B) = -\frac{\sqrt{-2F(B)f'(B)}}{f(B)} + \sqrt{-\frac{2F(B)f'(B)}{f(B)^2} + 1}. \tag{3.10}$$

Proof. In what follows, we fix a number B in the interval $(B^*, 0)$ and $\nu \in [\lambda, 1/\lambda]$, where $\lambda \in (0, 1)$ is an arbitrarily number. First we consider (I1). Henceforth, the terms $O(\varepsilon^\alpha)$ are uniform with respect to $B \in (B^*, 0)$ and $\nu \in [\lambda, 1/\lambda]$. To begin with, we calculate the integral over the interval $[2\sqrt{\nu}/\varepsilon, A]$. Note that $A - 2\sqrt{\nu}/\varepsilon$ is positive and is equal to $O(1)$ as $\varepsilon \downarrow 0$ because of Lemma 3.1 and the inequality $A_0 < A < A^*$. Set $\zeta = \sqrt{2(F(A) - F(\kappa))}$. Then we find that

$$\int_{2\sqrt{\nu}/\varepsilon}^{A} \frac{\varepsilon\, d\kappa}{\sqrt{2(F(A) - F(\kappa))}} = \int_0^{\tilde{\sigma}} \frac{\varepsilon\, d\zeta}{f(\kappa)}, \tag{3.11}$$

where $\tilde{\sigma} = \sqrt{2(F(A) - F(2\sqrt{\nu}/\varepsilon))} = O(1/\sqrt{\varepsilon})$ as $\varepsilon \downarrow 0$. Set $x = \kappa - A$. It follows from the Taylor expansion of $F(\kappa)$ at $\kappa = A$ that

$$\zeta^2 = -2f(A)x - f'(A)x^2 - \varepsilon^2 A x^3 - \frac{\varepsilon^2}{4}x^4.$$

Since it holds that $f(A) = O(1/\varepsilon)$, $f'(A) = O(1)$ and $x = O(1)$ as $\varepsilon \downarrow 0$, we have

$$x = x_0(\zeta)\left(1 + r_1(\varepsilon, \nu, B, \zeta)\right), \tag{3.12}$$

where $r_1 = O(\varepsilon^2)$ as $\varepsilon \downarrow 0$ uniformly in B, ν and $\zeta \in [0, \tilde{\sigma}]$, and x_0 is given by

$$x_0(\zeta) = \frac{-f(A) + \sqrt{f(A)^2 - f'(A)\zeta^2}}{f'(A)}. \tag{3.13}$$

By virtue of (3.12) and (3.13), we find

$$\begin{aligned} f(\kappa) &= f(A + x_0(\zeta)(1 + r_1)) \\ &= f(A) + f'(A)x_0(\zeta)(1 + r_1) + f''(A + \theta x_0(\zeta)(1 + r_1)) \\ &= -\sqrt{f(A)^2 - f'(A)\zeta^2} + r_2(\varepsilon, \nu, B, \zeta), \end{aligned}$$

where $r_2 = O(\varepsilon^2)$ as $\varepsilon \downarrow 0$ uniformly in B, ν and ζ. Hence we reduce (3.11) to

$$\int_0^{\tilde{\sigma}} \frac{\varepsilon\, d\zeta}{f(\kappa)} = \int_0^{\tilde{\sigma}} \frac{\varepsilon(1 + r_3(\varepsilon, \nu, B, \zeta))\, d\zeta}{\sqrt{f(A)^2 - f'(A)\zeta^2}}$$
$$= \frac{\varepsilon}{\sqrt{f'(A)}} \arcsin\left(\frac{\sqrt{f'(A)}\tilde{\sigma}}{f(A)}\right)\{1 + \rho_1(\varepsilon, \nu, B)\},$$

where $r_3 = O(\varepsilon^2)$ as $\varepsilon \downarrow 0$ uniformly in B, ν and ζ and $\rho_1 = O(\varepsilon^2)$ as $\varepsilon \downarrow 0$ uniformly in B and ν. Since $\tilde{\sigma}\sqrt{f'(A)}/f(A) = O(\sqrt{\varepsilon})$ as $\varepsilon \downarrow 0$, the integral over the interval $[2\sqrt{\nu}/\varepsilon, A]$ is equal to $O(\varepsilon^{3/2})$ as $\varepsilon \downarrow 0$ uniformly in B and ν. Let I_0 denote the integral over $[0, 2\sqrt{\nu}/\varepsilon]$. By the change of variable $\kappa = k/\varepsilon$, we see that

$$I_0 = \int_0^{2\sqrt{\nu}} \frac{\varepsilon\, dk}{\sqrt{-\frac{k^4}{4} + \nu k^2 + 2\varepsilon k + 2\varepsilon^2 F(A)}}. \tag{3.14}$$

We decompose I_0 as $I_0 = I_1 + I_2 + I_3$:

$$I_1 = \int_0^{2\sqrt{\nu}\sqrt{\varepsilon}} \frac{\varepsilon\, dk}{\sqrt{-\frac{k^4}{4} + \nu k^2 + 2\varepsilon k + 2\varepsilon^2 F(A)}},$$
$$I_2 = \int_{2\sqrt{\nu}\sqrt{\varepsilon}}^{2\sqrt{\nu}(1-\sqrt{\varepsilon})} \frac{\varepsilon\, dk}{\sqrt{-\frac{k^4}{4} + \nu k^2 + 2\varepsilon k + 2\varepsilon^2 F(A)}},$$
$$I_3 = \int_{2\sqrt{\nu}(1-\sqrt{\varepsilon})}^{2\sqrt{\nu}} \frac{\varepsilon\, dk}{\sqrt{-\frac{k^4}{4} + \nu k^2 + 2\varepsilon k + 2\varepsilon^2 F(A)}}.$$

First we consider I_1. For $k \in [0, 2\sqrt{\nu}\sqrt{\varepsilon}]$, it holds that

$$\left|\frac{k^4/4}{\nu k^2 + 2\varepsilon k + 2\varepsilon^2 F(A)}\right| \le \varepsilon.$$

Thus we have

$$I_1 = \int_0^{2\sqrt{\nu}\sqrt{\varepsilon}} \frac{\varepsilon(1 + r_4(\varepsilon, \nu, B, \kappa))\, dk}{\sqrt{\nu k^2 + 2\varepsilon k + 2\varepsilon^2 F(A)}} = \frac{\varepsilon}{\sqrt{\nu}} \log \psi_1 + \rho_2(\varepsilon, \nu, B),$$

where $r_4 = O(\varepsilon)$ as $\varepsilon \downarrow 0$ uniformly in B, ν and κ and $\rho_2 = O(\varepsilon^2 \log \varepsilon)$ as $\varepsilon \downarrow 0$ uniformly in B and ν. Next we turn to the integral I_2. For

$k \in [2\sqrt{\nu}\sqrt{\varepsilon}, 2\sqrt{\nu}(1-\sqrt{\varepsilon})]$, we can verify that there exists a positive constant C_1 being independent of B and ν such that

$$\left| \frac{2\varepsilon k + 2\varepsilon^2 F(A)}{\frac{k^2}{4}(4\nu - k^2)} \right| \le C_1\sqrt{\varepsilon}.$$

By the change of variable $\sqrt{4\nu - k^2} = \xi$, we have

$$I_2 = \int_{2\sqrt{\nu}\sqrt{\varepsilon}}^{2\sqrt{\nu}(1-\sqrt{\varepsilon})} \frac{2\varepsilon(1 + r_5(\varepsilon, \nu, B, \kappa))\, dk}{k\sqrt{4\nu - k^2}} = \frac{\varepsilon}{2\sqrt{\nu}} \log \psi_2 + \rho_3(\varepsilon, \nu, B),$$

where $r_5 = O(\sqrt{\varepsilon})$ as $\varepsilon \downarrow 0$ uniformly in B, ν and κ and $\rho_3 = O(\varepsilon^{3/2} \log \varepsilon)$ as $\varepsilon \downarrow 0$ uniformly in B and ν. Next we calculate I_3. Using $k = 2\sqrt{\nu}(1-\eta)$, we obtain

$$I_3 = \frac{1}{\sqrt{\nu}} \int_0^{\sqrt{\varepsilon}} \frac{\varepsilon\, d\eta}{\sqrt{g(\eta)}}, \tag{3.15}$$

where

$$\begin{aligned} g(\eta) &= -\eta^4 + 4\eta^3 + h(\eta), \\ h(\eta) &= -5\eta^2 + \left(2 - \frac{\varepsilon}{\nu\sqrt{\nu}}\right)\eta + \frac{\varepsilon}{\nu\sqrt{\nu}} + \frac{F(A)\varepsilon^2}{2\nu^2}. \end{aligned}$$

For $\eta \in [0, \sqrt{\varepsilon}\,]$, it holds that

$$\left| \frac{-\eta^4 + 4\eta^3}{h(\eta)} \right| \le C_2\varepsilon,$$

where the constant C_2 is independent of B and ν. Thus we get

$$\int_0^{\sqrt{\varepsilon}} \frac{\varepsilon\, d\eta}{\sqrt{g(\eta)}} = \int_0^{\sqrt{\varepsilon}} \frac{\varepsilon(1 + r_6(\varepsilon, \nu, B, \kappa))\, d\eta}{\sqrt{h(\eta)}},$$

where $r_6 = O(\varepsilon)$ as $\varepsilon \downarrow 0$ uniformly in B, ν and κ. By using the expansion

$$\arcsin(1 - x) = \frac{\pi}{2} - \sqrt{2}\sqrt{x} - \frac{\sqrt{2}}{12} x^{3/2} + O(x^{5/2})$$

as $x \downarrow 0$, we obtain $I_3 = O(\varepsilon^{5/4})$ as $\varepsilon \downarrow 0$ uniformly in B and ν. Therefore, as $\varepsilon \downarrow 0$,

$$\int_0^{2\sqrt{\nu}/\varepsilon} \frac{\varepsilon\, d\kappa}{\sqrt{2(F(A) - F(\kappa))}} = \frac{\varepsilon}{\sqrt{\nu}} \log \psi_1 \sqrt{\psi_2} + \rho_4(\varepsilon, \nu, B),$$

where $\rho_4 = O(\varepsilon^{5/4})$ as $\varepsilon \downarrow 0$ uniformly in B and ν. Finally we calculate the integral over $[B, 0]$. Set $\zeta = \sqrt{2(F(B) - F(\kappa))}$. Then it holds that

$$\int_B^0 \frac{\varepsilon\, d\kappa}{\sqrt{2(F(B) - F(\kappa))}} = -\int_0^{\sigma^*} \frac{\varepsilon\, d\zeta}{f(\kappa)}, \tag{3.16}$$

where $\sigma^* = \sqrt{2F(B)}$. Set $x = \kappa - B$. By the same arguments as in the derivation of (3.12) and (3.13), we have

$$x = x_0(\zeta)\,(1 + r_7(\varepsilon, \nu, B, \zeta)), \tag{3.17}$$

where $r_7 = O(\varepsilon^2)$ as $\varepsilon \downarrow 0$ uniformly in B, ν and $\zeta \in [0, \sigma^*]$, and x_0 is given by

$$x_0(\zeta) = \frac{-f(B) - \sqrt{f(B)^2 - f'(B)\zeta^2}}{f'(B)} \tag{3.18}$$

because of $f(B) < 0$, $f'(B) < 0$ and $x \geq 0$. It follows from (3.17) and (3.18) that

$$f(\kappa) = -\sqrt{f(B)^2 - f'(B)\zeta^2} + x_0(\zeta) r_8(\varepsilon, \nu, B, \zeta),$$

where $r_8 = O(\varepsilon^2)$ as $\varepsilon \downarrow 0$ uniformly in B, ν and ζ. Hence, using the relation

$$\left| \frac{x_0}{\sqrt{f(B)^2 - f'(B)\zeta^2}} \right| \to \begin{cases} -\dfrac{2}{f'(B^*)} & \text{as} \quad B \to B^*, \\ 0 & \text{as} \quad B \to 0, \end{cases}$$

we can reduce the right-hand side of (3.16) to

$$-\int_0^{\sigma^*} \frac{\varepsilon\, d\zeta}{f(\kappa)} = \frac{\varepsilon \log(\varphi(B)}{\sqrt{-f'(B)}} (1 + \rho_5(\varepsilon, \nu, B)),$$

where $\rho_5 = O(\varepsilon^2)$ as $\varepsilon \downarrow 0$ uniformly in B and ν. Therefore we obtain (3.6). Along the same line as in the derivation of (3.6), we obtain (3.7). Q.E.D.

Set $B = B^* + \delta$, as we stated above. Once (ν, δ) is given, $A = A(B) = A(\varepsilon, \nu, \delta)$ is determined by the condition (C1) as follows:

Lemma 3.3. *Fix a sufficiently small constant $\varepsilon > 0$ arbitrarily. Let ν be an arbitrarily fixed number in the interval $\lambda \leq \nu \leq 1/\lambda$, where $\lambda \in (0, 1)$ is an arbitrarily fixed number. Set $B = B^* + \delta$. Then, as $\delta \downarrow 0$, A is expanded as follows:*

$$A = A^* + \frac{f'(B^*)}{2f(A^*)}\delta^2 + O(\delta^3). \tag{3.19}$$

Proof. In what follows, fix a constant ε arbitrarily in the interval $0 < \varepsilon \ll 1$, and let ν be an arbitrarily fixed number in $[\lambda, 1/\lambda]$, where $\lambda \in (0,1)$ is an arbitrarily number. First we prove that the equation

$$F(A^* + \rho) - F(B^* + \delta) = 0 \tag{3.20}$$

has a solution $\rho = \varphi(\delta)$ for sufficiently small δ. Let $\mathcal{F}(\rho, \delta)$ denote the left-hand side of (3.20), i.e., we consider the equation $\mathcal{F}(\rho, \delta) = 0$. Since $F(A^*) = F(B^*)$, it is clear that $\mathcal{F}(0,0) = 0$. Here it holds that

$$\frac{\partial \mathcal{F}}{\partial \rho} = \frac{\partial F}{\partial \kappa}(A^* + \rho)\frac{\partial (A^* + \rho)}{\partial \rho} = f(A^* + \rho),$$
$$\frac{\partial \mathcal{F}}{\partial \delta} = \frac{\partial F}{\partial \kappa}(B^* + \delta)\frac{\partial (B^* + \delta)}{\partial \delta} = f(B^* + \delta).$$

By these equalities, we find

$$\left.\frac{\partial \mathcal{F}}{\partial \rho}\right|_{\rho=\delta=0} = f(A^*) \neq 0, \qquad \left.\frac{\partial \mathcal{F}}{\partial \delta}\right|_{\rho=\delta=0} = f(B^*) = 0.$$

Therefore, by the implicit function theorem, there exist a positive number δ_0 and the C^1 function $\varphi(\delta)$ such that

$$\mathcal{F}(\varphi(\delta), \delta) = 0, \qquad \varphi(0) = 0, \qquad \frac{\partial \varphi}{\partial \delta}(0) = 0$$

for any $0 \le \delta < \delta_0$. Finally we derive the expansion of $\varphi(\delta)$ as $\delta \downarrow 0$ from (3.20). For sufficiently small $\delta > 0$, by use of the Taylor expansion of F, it holds that

$$F(A^* + \varphi) = F(A^*) + f(A^*)\varphi + \frac{1}{2}f'(A^*)\varphi^2 + \frac{1}{2}\varepsilon^2 A^* \varphi^3 + \frac{1}{8}\varepsilon^2\varphi^4,$$
$$F(B^* + \delta) = F(B^*) + \frac{1}{2}f'(B^*)\delta^2 + \frac{1}{2}\varepsilon^2 B^* \delta^3 + \frac{1}{8}\varepsilon^2\delta^4.$$

Thus we find

$$f(A^*)\varphi + \frac{f'(A^*)}{2}\varphi^2 + \frac{\varepsilon^2 A^*}{2}\varphi^3 + \frac{\varepsilon^2}{8}\varphi^4 = \frac{f'(B^*)}{2}\delta^2 + \frac{\varepsilon^2 B^*}{2}\delta^3 + \frac{\varepsilon^2}{8}\delta^4.$$

Since it is easy to check that $\varphi(\delta) = O(\delta^2)$ as $\delta \downarrow 0$, we obtain

$$\varphi(\delta) = \frac{f'(B^*)}{2f(A^*)}\delta^2 + r(\varepsilon, \delta, \nu)$$

as $\delta \downarrow 0$, where $r = O(\delta^3)$ as $\delta \downarrow 0$ uniformly in ε and ν. We observe that $\varphi(\delta)$ is non-positive because of $f'(B^*) < 0$ and $f(A^*) > 0$. This completes the proof. Q.E.D.

In (3.6) and (3.7), setting $B = B^* + \exp[-(d/\varepsilon)]$, i.e., $\delta = \exp[-(d/\varepsilon)]$, we derive the system of equations for (d, ν), where $d > 0$:

Lemma 3.4. *Fix two numbers λ_1 and λ_2 arbitrarily in the interval $(0,1)$, respectively. For each $d \in [\lambda_1, 1/\lambda_1]$ and $\nu \in [\lambda_2, 1/\lambda_2]$, set $B = B^* + \exp[-(d/\varepsilon)]$. Then, for sufficiently small $\varepsilon > 0$, the pair (B, ν) solves* (I1) *and* (I2) *if and only if the pair (d, ν) satisfies the following:*

$$\begin{aligned}(3.21)\quad & \frac{d}{\sqrt{-f'(B^*)}} + \frac{\varepsilon}{\sqrt{\nu}} \log(\psi_1\sqrt{\psi_2}) + \frac{\varepsilon}{\sqrt{-f'(B^*)}} \log \frac{2\sqrt{2F(B^*)}}{\sqrt{-f'(B^*)}} \\ & \qquad + R_5(\varepsilon, d, \nu) = \frac{L}{2n},\end{aligned}$$

$$\begin{aligned}(3.22)\quad & \frac{B^* d}{\sqrt{-f'(B^*)}} - \frac{\varepsilon}{\nu^{3/2}} \log(\psi_1\sqrt{\psi_2}) + \frac{\varepsilon B^*}{\sqrt{-f'(B^*)}} \log \frac{2\sqrt{2F(B^*)}}{\sqrt{-f'(B^*)}} \\ & \qquad + \frac{2}{\nu^{3/2}}\varepsilon + R_6(\varepsilon, d, \nu) = -\frac{(n-1)\pi}{n},\end{aligned}$$

where $R_5(\varepsilon, d, \nu) = O(\varepsilon^{5/4})$ and $R_6(\varepsilon, d, \nu) = O(\varepsilon^{5/4})$ as $\varepsilon \downarrow 0$, uniformly in $d \in [\lambda_1, 1/\lambda_1]$ and $\nu \in [\lambda_2, 1/\lambda_2]$.

Proof. Let $d \in [\lambda_1, 1/\lambda_1]$ and $\nu \in [\lambda_2, 1/\lambda_2]$ be fixed numbers, respectively, and set $B = B^* + \delta = B^* + \exp[-(d/\varepsilon)]$. In what follows, the terms $O(\varepsilon^\alpha)$ and $O(\varepsilon^\alpha \delta^\beta)$ are uniform with respect to $d \in [\lambda_1, 1/\lambda_1]$ and $\nu \in [\lambda_2, 1/\lambda_2]$. First we consider $\log \varphi(B)$. Since $f(B^*) = 0$, it holds that $f(B) \to 0$ as $\delta \downarrow 0$. Recall that $F(B) > 0$, while $f(B) < 0$ and $f'(B) < 0$. From the expansions

$$(3.23)\qquad F(B) = F(B^*) + \frac{1}{2} f'(B^*)\delta^2 + \frac{1}{3!} f''(B^* + \theta_1\delta)\delta^3,$$

$$(3.24)\qquad f(B) = f'(B^*)\delta + \frac{1}{2} f''(B^*)\delta^2 + \frac{1}{3!} f'''(B^* + \theta_2\delta)\delta^3,$$

$$(3.25)\qquad f'(B) = f'(B^*) + f''(B^*)\delta + \frac{1}{2} f'''(B^* + \theta_3\delta)\delta^2,$$

where $\theta_j \in (0,1)$, it holds that

$$\log\left(\frac{\sqrt{-2F(B)f'(B)}}{-f(B)}\right) = \log\left(\frac{\sqrt{2F(B^*)}}{\sqrt{-f'(B^*)}} \frac{1}{\delta}\right) + \rho_1(\varepsilon, \delta, \nu),$$

$$\log\left(1 + \sqrt{1 + \frac{f(B)^2}{-2F(B)f'(B)}}\right) = \log 2 + \rho_2(\varepsilon, \delta, \nu),$$

where $\rho_1 = O(\varepsilon^2\delta)$ and $\rho_2 = O(\delta^2)$ as $\varepsilon \downarrow 0$. Thus we obtain

$$\log\varphi(B) = \log\frac{2\sqrt{2F(B^*)}}{\delta\sqrt{-f'(B^*)}} + \rho_3(\varepsilon,\delta,\nu), \tag{3.26}$$

where $\rho_3 = O(\varepsilon^2\delta)$ as $\varepsilon \downarrow 0$. Moreover, it follows from (3.24) and (3.25) that

$$\frac{1}{\sqrt{-f'(B)}} = \frac{1}{\sqrt{-f'(B^*)}} + \rho_4(\varepsilon,\delta,\nu), \tag{3.27}$$

$$B - \frac{f(B)}{f'(B)} = B^* + \rho_5(\varepsilon,\delta,\nu), \tag{3.28}$$

where $\rho_4 = O(\varepsilon^2\delta)$ and $\rho_5 = O(\varepsilon^2\delta^2)$ as $\varepsilon \downarrow 0$. Finally, by virtue of Lemma 3.3, we get

$$\frac{A\nu^{1/4}}{f(A)} = \frac{A^*\nu^{1/4}}{f(A^*)} + \rho_6(\varepsilon,\delta,\nu) = \frac{1}{\nu^{3/4}} - \frac{3}{2\nu^{9/4}}\varepsilon + \rho_7(\varepsilon,\delta,\nu), \tag{3.29}$$

where $\rho_6 = O(\varepsilon^2\delta^2)$ and $\rho_7 = O(\varepsilon^2)$ as $\varepsilon \downarrow 0$. By using (3.26), (3.27), (3.28), (3.29) and Lemma 3.2, we obtain the equations (3.21) and (3.22).

Q.E.D.

Now we are in a position to prove that the system of equations (3.21)–(3.22) has a solution $(d(\varepsilon),\nu(\varepsilon))$ for sufficiently small $\varepsilon > 0$. For this purpose, we use the following implicit function theorem.

Lemma 3.5. *Let $U \subset \mathbb{R}^N$ be a neighborhood of the origin 0. Suppose that an $\mathbb{R}^N$-valued function $\mathcal{F}(x,y) = (F_1(x,y), F_2(x,y), \cdots, F_N(x,y))$ defined for $(x,y) \in [0,1) \times U$ satisfies the following conditions:*

(i) *$\mathcal{F}(x,y)$ is uniformly continuous on $[0,1) \times U$. Moreover, for each $x \in [0,1)$, $\mathcal{F}(x,y)$ is partially differentiable function with respect to y on U, and $\partial\mathcal{F}_l/\partial y_m$ is continuous on $[0,1) \times U$.*

(ii) *$\mathcal{F}(0,0) = 0$.*

(iii) *An $N \times N$ matrix defined by*

$$\mathcal{F}_y(0,0) = \left(\frac{\partial F_l}{\partial y_m}(0,0)\right)_{1\le l,\, m\le N}$$

is invertible.

Then there exist positive numbers ρ and r and the function $\varphi : [0,1) \to B_r(0) = \{y \in \mathbb{R}^N \mid |y| < r\}$ such that $\mathcal{F}(x,\varphi(x)) \equiv 0$ for any $x \in [0,\rho)$ and $\varphi(0) = 0$ holds. Moreover φ is continuous on $[0,\rho)$.

By Lemma 3.5, we can prove the following lemma.

Lemma 3.6. *There exist a positive number* ε_0 *and the pair of continuous functions* $(d(\varepsilon), \nu(\varepsilon))$ *such that* $(d(\varepsilon), \nu(\varepsilon))$ *satisfies the equations* (3.21) *and* (3.22) *for any* $0 \leq \varepsilon < \varepsilon_0$.

Proof. First, setting $\varepsilon = 0$ in (3.21) and (3.22), we find that

$$\frac{d}{\sqrt{\nu}} = \frac{L}{2n}, \qquad -\frac{d}{\nu^{3/2}} = -\frac{(n-1)\pi}{n} \tag{3.30}$$

because of $\varepsilon \log(\psi_1\sqrt{\psi_2}) = O(\varepsilon \log \varepsilon)$ as $\varepsilon \downarrow 0$ uniformly in $d \in [\lambda_1, 1/\lambda_1]$ and $\nu \in [\lambda_2, 1/\lambda_2]$. (For the precise expansion of $\log(\psi_1\sqrt{\psi_2})$, see the proof of Lemma 3.7.) Therefore, when $\varepsilon = 0$, the equations (3.21) and (3.22) have a solution $(d, \nu) = (d^*, \nu^*)$:

$$d^* = \frac{L}{2n}\sqrt{\frac{L}{2(n-1)\pi}}, \qquad \nu^* = \frac{L}{2(n-1)\pi}. \tag{3.31}$$

In order to prove the existence of a solution of the equations (3.21) and (3.22) for sufficiently small $\varepsilon > 0$, we use the implicit function theorem, which is stated as Lemma 3.5. For this purpose, from (I1) and (I2), let us set

$$P_1(\varepsilon, B, \nu) = \int_B^{A(\varepsilon,B,\nu)} \frac{\varepsilon\, d\kappa}{\sqrt{2(F(\varepsilon,\nu,A(\varepsilon,B,\nu)) - F(\varepsilon,\nu,\kappa))}} - \frac{L}{2n},$$

$$P_2(\varepsilon, B, \nu) = \int_B^{A(\varepsilon,B,\nu)} \frac{\varepsilon\kappa\, d\kappa}{\sqrt{2(F(\varepsilon,\nu,A(\varepsilon,B,\nu)) - F(\varepsilon,\nu,\kappa))}} - \frac{\pi}{n}.$$

Recall that the condition (C1) implies $A = A(\varepsilon, B, \nu)$ (see Lemma 3.3). By virtue of $B(\varepsilon, d, \nu) = B^*(\varepsilon, \nu) + \exp[-(d/\varepsilon)]$ and the relation (3.30), we define $Q_1(\varepsilon, d, \nu)$ and $Q_2(\varepsilon, d, \nu)$ as follows:

$$Q_1(\varepsilon, d, \nu) = \begin{cases} P_1(\varepsilon, B^*(\varepsilon,\nu) + \exp(-d/\varepsilon), \nu) & \text{if} \quad \varepsilon > 0, \\ \dfrac{d}{\sqrt{\nu}} - \dfrac{L}{2n} & \text{if} \quad \varepsilon = 0, \end{cases}$$

$$Q_2(\varepsilon, d, \nu) = \begin{cases} P_2(\varepsilon, B^*(\varepsilon,\nu) + \exp(-d/\varepsilon), \nu) & \text{if} \quad \varepsilon > 0, \\ -\dfrac{d}{\nu^{3/2}} + \dfrac{(n-1)\pi}{n} & \text{if} \quad \varepsilon = 0. \end{cases}$$

By using these Q_1 and Q_2, let us set

$$\mathcal{F}(\varepsilon, d, \nu) = (Q_1(\varepsilon, d, \nu),\ Q_2(\varepsilon, d, \nu)).$$

We shall apply Lemma 3.5 to the equation $\mathcal{F}(\varepsilon, d, \nu) = 0$. Since it holds that $\mathcal{F}(0, d^*, \nu^*) = 0$, we consider the equation $\mathcal{F}(\varepsilon, d, \nu) = 0$ for

$0 \le \varepsilon < \varepsilon_1$, $d^* - \rho_1 \le d \le d^* + \rho_1$ and $\nu^* - \rho_2 \le \nu \le \nu^* + \rho_2$, where $0 < \varepsilon_1 \ll 1$, $0 < \rho_1 \ll d^*$ and $0 < \rho_2 \ll \nu^*$ are some constants. It is easy to check that $\mathcal{F}(\varepsilon, d, \nu)$ is uniformly continuous on $[0, \varepsilon_1) \times [d^* - \rho_1, d^* + \rho_1] \times [\nu^* - \rho_2, \nu^* + \rho_2]$. Moreover we can prove that $Q_1(\varepsilon, d, \nu)$ and $Q_2(\varepsilon, d, \nu)$ are partially differentiable with respect to d and ν on $[0, \varepsilon_1) \times [d^* - \rho_1, d^* + \rho_1] \times [\nu^* - \rho_2, \nu^* + \rho_2]$. And, by simple calculations, we have

$$\frac{\partial Q_1}{\partial d} \longrightarrow \frac{1}{\sqrt{\nu}} \quad \text{as} \quad \varepsilon \downarrow 0, \qquad \frac{\partial Q_1}{\partial \nu} \longrightarrow -\frac{d}{2\nu^{3/2}} \quad \text{as} \quad \varepsilon \downarrow 0,$$
$$\frac{\partial Q_2}{\partial d} \longrightarrow -\frac{1}{\nu^{3/2}} \quad \text{as} \quad \varepsilon \downarrow 0, \qquad \frac{\partial Q_2}{\partial \nu} \longrightarrow \frac{3d}{2\nu^{5/2}} \quad \text{as} \quad \varepsilon \downarrow 0.$$

Thus $\mathcal{F}(\varepsilon, d, \nu)$ satisfies the conditions (i) and (ii) which are stated in Lemma 3.5. We can verify that $\mathcal{F}(\varepsilon, d, \nu)$ satisfies the condition (iii) of Lemma 3.5 by the relation

$$\left\| \begin{matrix} \dfrac{\partial Q_1}{\partial d} & \dfrac{\partial Q_1}{\partial \nu} \\ \dfrac{\partial Q_2}{\partial d} & \dfrac{\partial Q_2}{\partial \nu} \end{matrix} \right\|_{\varepsilon=0} = \frac{d}{\nu^3} \neq 0.$$

Therefore we can apply Lemma 3.5 to the equation $\mathcal{F}(\varepsilon, d, \nu) = 0$ and see that there exist a positive number ε_0 and the pair of continuous functions $(d(\varepsilon), \nu(\varepsilon))$ satisfying the relations $d(0) = d^*$ and $\nu(0) = \nu^*$ and the system of equations (3.21)–(3.22) for any $0 \le \varepsilon < \varepsilon_0$. Q.E.D.

Next, we derive the asymptotic form of $d(\varepsilon)$ and $\nu(\varepsilon)$ as $\varepsilon \downarrow 0$:

Lemma 3.7. *As $\varepsilon \downarrow 0$, the solution $(d(\varepsilon), \nu(\varepsilon))$ of the system of equations* (3.21)–(3.22) *satisfies*

$$d(\varepsilon) = \frac{L\sqrt{M_n}}{2n} + \varepsilon \log \varepsilon - (1 + \log(8\sqrt{M_n}))\varepsilon + O(\varepsilon^{5/4}), \tag{3.32}$$

$$\nu(\varepsilon) = M_n - \frac{4n\sqrt{M_n}}{L}\varepsilon + O(\varepsilon^{5/4}), \tag{3.33}$$

where M_n is a positive constant given by

$$M_n = \frac{L}{2(n-1)\pi}. \tag{3.34}$$

Proof. In what follows, the terms $O(\varepsilon^\alpha)$ are uniform with respect to $d \in [d^* - \rho_1, d^* + \rho_1]$ and $\nu \in [\nu^* - \rho_2, \nu^* + \rho_2]$, where $0 < \rho_1 \ll d^*$

and $0 < \rho_2 \ll \nu^*$ are some constants. First we derive the expansion of $\log(\psi_1\sqrt{\psi_2})$ as $\varepsilon \downarrow 0$. It follows from (3.8) that

$$\log\psi_1 = \log\left(2\nu^{3/2}\right) + \log\left(1 + \frac{\sqrt{\varepsilon}}{2\nu^{3/2}} + \sqrt{1 + \frac{\sqrt{\varepsilon}}{\nu^{3/2}} + \frac{F(B)\varepsilon}{2\nu^2}}\right)$$
$$- \log\sqrt{\varepsilon} - \log\left(1 + \sqrt{2\nu F(B)}\right).$$

By simple calculations, we find

$$\log\left(1 + \frac{\sqrt{\varepsilon}}{2\nu^{3/2}} + \sqrt{1 + \frac{\sqrt{\varepsilon}}{\nu^{3/2}} + \frac{F(B)\varepsilon}{2\nu^2}}\right) = \log 2 + \rho_1(\varepsilon, d, \nu),$$
$$\log\left(1 + \sqrt{2\nu F(B)}\right) = \log 2 + \rho_2(\varepsilon, d, \nu),$$

where $\rho_1 = O(\varepsilon^{1/2})$ and $\rho_2 = O(\varepsilon^2)$ as $\varepsilon \downarrow 0$. Thus, as $\varepsilon \downarrow 0$, we have

$$\log\psi_1 = -\frac{1}{2}\log\varepsilon + \log(2\nu^{3/2}) + \rho_3(\varepsilon, d, \nu), \tag{3.35}$$

where $\rho_3 = O(\sqrt{\varepsilon})$ as $\varepsilon \downarrow 0$. Similarly, as $\varepsilon \downarrow 0$,

$$\log\sqrt{\psi_2} = -\frac{1}{2}\log\varepsilon + \log 2 + \rho_4(\varepsilon, d, \nu), \tag{3.36}$$

where $\rho_4 = O(\varepsilon^{1/4})$ as $\varepsilon \downarrow 0$. Therefore, as $\varepsilon \downarrow 0$, we obtain

$$\log(\psi_1\sqrt{\psi_2}) = -\log\varepsilon + \log(4\nu^{3/2}) + \rho_5(\varepsilon, d, \nu), \tag{3.37}$$

where $\rho_5 = O(\varepsilon^{1/4})$ as $\varepsilon \downarrow 0$. Moreover, as $\varepsilon \downarrow 0$, it holds that

$$\frac{1}{\sqrt{-f'(B^*)}} = \frac{1}{\sqrt{\nu}} + \rho_6(\varepsilon, d, \nu), \qquad \frac{2\sqrt{2F(B^*)}}{\sqrt{-f'(B^*)}} = \frac{2}{\nu} + \rho_7(\varepsilon, d, \nu),$$

where $\rho_6 = O(\varepsilon^2)$ and $\rho_7 = O(\varepsilon^2)$ as $\varepsilon \downarrow 0$. Therefore, for sufficiently small $\varepsilon > 0$, the equations (3.21) and (3.22) become as follows:

$$\frac{d}{\sqrt{\nu}} - \frac{1}{\sqrt{\nu}}\varepsilon\log\varepsilon + \frac{\log 8\sqrt{\nu}}{\sqrt{\nu}}\varepsilon + \rho_8(\varepsilon, d, \nu) = \frac{L}{2n}, \tag{3.38}$$

$$-\frac{d}{\nu^{3/2}} + \frac{1}{\nu^{3/2}}\varepsilon\log\varepsilon - \frac{\log 8\sqrt{\nu}}{\nu^{3/2}}\varepsilon + \frac{2}{\nu^{3/2}}\varepsilon + \rho_9(\varepsilon, d, \nu) = -\frac{(n-1)\pi}{n}, \tag{3.39}$$

where $\rho_8(\varepsilon, d, \nu) = O(\varepsilon^{5/4})$ and $\rho_9(\varepsilon, d, \nu) = O(\varepsilon^{5/4})$ as $\varepsilon \downarrow 0$. Let us set $d(\varepsilon) = d^* + \tilde{d}(\varepsilon)$ and $\nu(\varepsilon) = \nu^* + \tilde{\nu}(\varepsilon)$, where $\tilde{d}(\varepsilon) \to 0$ as $\varepsilon \downarrow 0$ and $\tilde{\nu}(\varepsilon) \to 0$ as $\varepsilon \downarrow 0$. It follows from (3.38) and (3.39) that

$$\frac{2}{\sqrt{\nu}}\varepsilon + \rho_8 + \nu\rho_9 = \frac{L}{2n} - \frac{(n-1)\pi}{n}\nu. \tag{3.40}$$

By the definition of d^* and ν^*, the equation (3.40) reduces to

$$\frac{2}{\sqrt{\nu^* + \tilde{\nu}}}\varepsilon + \rho_8 + (\nu^* + \tilde{\nu})\rho_9 = -\frac{(n-1)\pi}{n}\tilde{\nu}. \tag{3.41}$$

From (3.41), as $\varepsilon \downarrow 0$, we find

$$\tilde{\nu}(\varepsilon) = -\frac{4n\sqrt{\nu^*}}{L}\varepsilon + O(\varepsilon^{5/4}). \tag{3.42}$$

Moreover, by virtue of (3.38), it holds that

$$d = \varepsilon \log \varepsilon - \varepsilon \log(8\sqrt{\nu}) + \frac{L}{2n}\sqrt{\nu} - \sqrt{\nu}\rho_8. \tag{3.43}$$

Since $d^* = L\sqrt{\nu^*}/(2n)$ and

$$\sqrt{\nu} = \sqrt{\nu^*} - \frac{2n}{L}\varepsilon + O(\varepsilon^{5/4})$$

as $\varepsilon \downarrow 0$, we obtain

$$\tilde{d}(\varepsilon) = \varepsilon \log \varepsilon - (1 + \log(8\sqrt{\nu^*}))\varepsilon + O(\varepsilon^{5/4}) \tag{3.44}$$

as $\varepsilon \downarrow 0$. The proof is completed. Q.E.D.

Since $B = B^*(\varepsilon, \nu) + \exp[-(d/\varepsilon)]$, we can derive the asymptotic form of B as $\varepsilon \downarrow 0$. Moreover, by Lemma 3.3, we obtain the asymptotic form of A as $\varepsilon \downarrow 0$.

Finally, we derive the asymptotic form of s_0 which denotes the point such that κ vanishes, as $\varepsilon \downarrow 0$. By the proof of Lemmas 3.2 and 3.7, as $\varepsilon \downarrow 0$, we see that

$$\begin{aligned}\int_0^A \frac{\varepsilon\, d\kappa}{\sqrt{2(F(A) - F(\kappa))}} &= \frac{\varepsilon}{\sqrt{\nu}} \log(\psi_1\sqrt{\psi_2}) + O(\varepsilon^{5/4}) \\ &= \frac{1}{\sqrt{M_n}}\varepsilon \log\frac{1}{\varepsilon} + \frac{\log(4{M_n}^{3/2})}{\sqrt{M_n}}\varepsilon + O(\varepsilon^{5/4}).\end{aligned}$$

As $\varepsilon \downarrow 0$, this equality is equivalent to the following:

$$s_0(\varepsilon) = \frac{1}{\sqrt{M_n}} \varepsilon \log \frac{1}{\varepsilon} + \frac{\log (4{M_n}^{3/2})}{\sqrt{M_n}} \varepsilon + O(\varepsilon^{5/4}). \tag{3.45}$$

We complete the proof of Theorem 1.1.

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