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Perturbation of structures of radial solutions to elliptic equations

Yoshitsugu Kabeya

Abstract.

The change of structures of radial solutions to elliptic equations is discussed. Especially, by some perturbation on a "potential function", the non-existence of rapidly decaying solutions fails and the existence of that one is ensured. Also, in another case, the uniqueness of rapidly decaying solutions breaks and multiple rapidly decaying ones appear. Some insights to the number of solutions are also presented.

§1. Introduction

In this paper, as a survey, we consider the equation

$$(1.1)\qquad (r^{n-1}u_r)_r + r^{n-1}K(r)u^{(n+2)/(n-2)} = 0,\ r > 0$$

with the condition

$$(1.2)\qquad \begin{cases} u \in C^2((0,\infty)) \cap C([0,\infty)), \\ u > 0, r > 0, \\ \lim_{r\to\infty} r^{n-2}u(r) < \infty, \end{cases}$$

where the dimension $n \geq 3$, $K(r) > 0$ in $(0,\infty)$ and belongs to $C^1(0,\infty)$. A solution to (1.1) which satisfies (1.2) is called a rapidly decaying solution. We will see that how rapidly decaying solutions are created by suitable perturbations of K. Note that (1.1) is uniquely globally solvable for $K \in C([0,\infty))$ with $K > 0$ in $[0,\infty)$ if we impose the initial value $u(0) = \alpha > 0$. We denote the solution u to (1.1) with $u(0) = \alpha$ by $u(r;\alpha)$.

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Besides adding a small "forcing term" in the right-hand side of (1.1) or singular perturbations, we are concerned with perturbations of K.

By suitable ways, we can obtain at least one rapidly decaying solution even though the original problem does not have a rapidly decaying solution, and similarly, we obtain at least two even though the original one has a unique rapidly decaying one. Moreover, we can obtain more rapidly decaying solutions for other perturbations.

We decompose $K = K_\varepsilon$ as

$$K_\varepsilon(r) = K_0(r) + \tilde{K}_\varepsilon(r) \tag{1.3}$$

with small parameter $\varepsilon > 0$, and consider perturbations for various types of K_0. First, we consider the case where at least one rapidly decaying solution is generated. We assume that K_0 and $\tilde{K}_\varepsilon$ fulfill the hypotheses

$$(K_0) \qquad \begin{cases} K_0(r) \in C^1((0,\infty)) \cap C([0,\infty)), \\ K_0'(r) \le 0 \text{ on } (0,a), \\ K_0'(r) \equiv 0 \text{ on } [a,\infty), \\ K_0(0) > K_0(a) = c_0 > 0, \end{cases}$$

and

$$(K_\varepsilon) \qquad \begin{cases} \tilde{K}_\varepsilon(r) \in C^1((0,\infty)) \cap C([0,\infty)), \\ \tilde{K}_\varepsilon(r) \ge 0 \text{ on } [0,\infty), \\ \tilde{K}_\varepsilon'(r) > 0 \text{ on } (r_\varepsilon, \tilde{r}_\varepsilon), \\ \tilde{K}_\varepsilon(r) \equiv 0 \text{ on } [0, r_\varepsilon] \cup [\tilde{r}_\varepsilon, \infty), \\ \lim_{\varepsilon \downarrow 0} \tilde{K}_\varepsilon(\tilde{r}_\varepsilon) = 0, \\ \int_0^{\tilde{r}_\varepsilon} r^n \tilde{K}_\varepsilon'(r)\, dr > -\int_0^{\tilde{r}_\varepsilon} r^n K_0'(r)\, dr, \end{cases}$$

where $0 < a < r_\varepsilon < \tilde{r}_\varepsilon$ and $r_\varepsilon \to \infty$ as $\varepsilon \downarrow 0$. We agree that $\tilde{K}_\varepsilon \equiv 0$ if $\varepsilon = 0$. One example for the pair of K_0 and $\tilde{K}_\varepsilon$ is

$$K_0(r) = \begin{cases} 3 - r^2, & 0 \le r \le 1, \\ 1 + (r-2)^2, & 1 < r \le 2, \\ 1, & 2 \le r, \end{cases}$$

with $a = 2$ and

$$\tilde{K}_\varepsilon(r) = \begin{cases} 0, & 0 \le r \le \varepsilon^{-1}, \\ \varepsilon^3(r-\varepsilon^{-1})^2, & \varepsilon^{-1} \le r \le 2\varepsilon^{-1}, \\ -\varepsilon^3(r-3\varepsilon^{-1})^2 + 2\varepsilon, & 2\varepsilon^{-1} < r < 3\varepsilon^{-1}, \\ 2\varepsilon, & 3\varepsilon^{-1} \le r, \end{cases}$$

with $r_\varepsilon = \varepsilon^{-1}$ and $\tilde{r}_\varepsilon = 3\varepsilon^{-1}$. If $\varepsilon > 0$ is very small, then $\tilde{K}_\varepsilon$ satisfies (K_ε).

If $\varepsilon = 0$ under (K_0), then by Theorem 3 of Kawano, Yanagida and Yotsutani [2], there exists no rapidly decaying solution for (1.1). However, we can find at least one rapidly decaying solution to (1.1) and the behavior of the solution as $\varepsilon \downarrow 0$ is made clear.

Theorem 1.1. *Under the decomposition* (1.3), *suppose that* K_0 *and* $\tilde{K}_\varepsilon$ *satisfies* (K_0) *and* (K_ε), *respectively. Then for any sufficiently small* $\varepsilon > 0$, *there exists at least one rapidly decaying solution* u_ε *to* (1.1) *such that* $\|u_\varepsilon\|_\infty \to 0$ *as* $\varepsilon \downarrow 0$.

We remark that the existence of a rapidly decaying solution is ensured by Theorem 1 of Yanagida and Yotsutani [6]. Sasahara and Tanaka [4] also proved the existence of a rapidly dcaying solution for K with exactly one local minimum point. Theorem 1.1 emphasizes on the behavior of the solution as $\varepsilon \downarrow 0$.

Next, we assume that K_0 and $\tilde{K}_\varepsilon$ fulfill the hypotheses

$$(K_0^1) \quad \begin{cases} K_0(r) \in C^1((0,\infty)), \\ K_0(r) > 0 \text{ on } [0,\infty), \\ K_0'(r) = 0 \text{ on } r \in [0,a] \cup \{b\} \cup [c,\infty), \\ K_0'(r) > 0 \text{ on } (a,b), \\ K_0'(r) < 0 \text{ on } (b,c), \\ K_0(0) = K_0(c) = A > 0, \end{cases}$$

and

$$(K_\varepsilon^1)\quad \begin{cases} \tilde{K}_\varepsilon(r) \in C^1((0,\infty)), \\ \tilde{K}_\varepsilon(r) \le 0 \text{ on } [0,\infty), \\ \tilde{K}'_\varepsilon(r) = 0 \text{ on } [0,\rho_\varepsilon] \cup \{\rho_{1,\varepsilon}\} \cup [\rho_{2,\varepsilon},\infty), \\ \tilde{K}'_\varepsilon(r) < 0 \text{ on } (\rho_\varepsilon, \rho_{1,\varepsilon}), \\ \tilde{K}'_\varepsilon(r) > 0 \text{ on } (\rho_{1,\varepsilon}, \rho_{2,\varepsilon}), \\ \tilde{K}_\varepsilon(0) = \tilde{K}_\varepsilon(\rho_{2,\varepsilon}) = 0, \\ \lim_{\varepsilon\downarrow 0} \tilde{K}_\varepsilon(\rho_{1,\varepsilon}) = 0, \\ \int_0^{\rho_{2,\varepsilon}} r^n \tilde{K}'_\varepsilon(r)\,dr > -\int_0^{\rho_{2,\varepsilon}} r^n K'_0(r)\,dr, \end{cases}$$

where $A > 0$, $0 \le a < b < c < \rho_\varepsilon < \rho_{1,\varepsilon} < \rho_{2,\varepsilon}$ and $\rho_\varepsilon \to \infty$ as $\varepsilon \downarrow 0$.

An example for such K_0 and $\tilde{K}_\varepsilon$ is

$$K_0(r) = \begin{cases} 1 + r^2, & 0 \le r \le 1, \\ 3 - (r-2)^2, & 1 < r \le 3, \\ 1 + (r-4)^2, & 3 < r \le 4, \\ 1, & 4 < r, \end{cases} \tag{1.4}$$

with $A = 1$, $a = 0$, $b = 2$, $c = 4$ and

$$\tilde{K}_\varepsilon(r) = \begin{cases} 0, & 0 \le r \le \varepsilon^{-1}, \\ -\varepsilon^3(r - \varepsilon^{-1})^2, & \varepsilon^{-1} < r \le 2\varepsilon^{-1}, \\ -2\varepsilon + \varepsilon^3(r - 3\varepsilon^{-1})^2, & 2\varepsilon^{-1} < r \le 4\varepsilon^{-1}, \\ -\varepsilon^3(r - 5\varepsilon^{-1})^2, & 4\varepsilon^{-1} < r \le 5\varepsilon^{-1}, \\ 0, & 5\varepsilon^{-1} < r \end{cases} \tag{1.5}$$

with $\rho_\varepsilon = \varepsilon^{-1}$, $\rho_{1,\varepsilon} = 3\varepsilon^{-1}$ and $\rho_{2,\varepsilon} = 5\varepsilon^{-1}$. If $\varepsilon > 0$ is sufficiently small, then all the assumptions in (K_0^1) and (K_ε^1) are satisfied.

Under the assumptions (K_ε^1) and (K_0^1), we see that $K_\varepsilon(r)$ converges to a limiting function K_0 uniformly on $[0,\infty)$ as $\varepsilon \downarrow 0$. In the case where $\varepsilon = 0$, due to Theorem 3 of Yanagida and Yotsutani [5], the uniqueness of positive radial solutions is ensured.

Theorem 1.2. *Under the decomposition* (1.3)*, suppose that* K_0 *and* $\tilde{K}_\varepsilon$ *satisfy* (K_0^1) *and* (K_ε^1)*, respectively. If* $\varepsilon > 0$ *is sufficiently small, then* (1.1) *has at least two positive rapidly decaying solutions. Moreover, one of them converges to* 0 *uniformly on* $[0,\infty)$ *and the other does to the unique solution for* (1.1) with $\varepsilon = 0$ as $\varepsilon \downarrow 0$.

In general, the number of the change of the sign of K'_ε must have a strong relation to the number of rapidly decaying solutions. We can expect the existence of n rapidly decaying solutions if K_ε' changes its sign n times in a suitable way. We hope that the number of the rapidly decaying solutions is exactly one in Theorem 1.1 and exactly two in Thereom 1.2.

In Section 2, we present key tools for proofs of Theorems 1.1 and 1.2. Proofs are given in Section 3 as well as concluding remarks.

§2. Key Tools

In this section, we give key lemmas to prove Theorems 1.1 and 1.2. There is a useful tool, the Pohozaev value, to classify solutions to (1.1). First, we classify a solution u to (1.1) into one of the following three types. We say

(i): u is a *crossing* solution if u has a finite zero.
(ii): u is a *slowly decaying* solution if $u > 0$ on $[0, \infty)$ and the limit satisfies $\lim_{r\to\infty} r^{n-2}u(r) = \infty$.
(iii): u is a *rapidly decaying* solution if $u > 0$ on $[0, \infty)$ and the limit satisfies $\lim_{r\to\infty} r^{n-2}u(r) < \infty$.

To characterize the types of solutions above, we introduce the *Pohozaev* value $P(r; u)$ by

$$P(r;u) := \frac{1}{2}r^{n-1}u_r\{ru_r + (n-2)u_+\} + \frac{n-2}{2n}r^n K_\varepsilon(r)(u_+)^{\frac{2n}{n-2}}.$$

We have the following characterizations of solutions in terms of the Pohozaev value (see e.g., Kawano, Yanagida and Yotsutani [4]).

Lemma 2.1. *Suppose that $K_\varepsilon > 0$ and $K_\varepsilon \in C^1((0,\infty)) \cap C([0,\infty))$.*

(i): *If u is a crossing solution, then $P(r;u) = P(r_0; u(r_0)) = r_0^n u_r(r_0)^2/2 > 0$ for $r \geq r_0$, where r_0 is a zero of u.*
(ii): *If u is a slowly decaying solution, then there exists a sequence $\{r_j\}$ ($r_j \to \infty$ as $j \to \infty$) such that $P(r_j; u(r_j)) < 0$ for any j.*
(iii): *If u is a rapidly decaying solution, then there exists a sequence $\{\hat{r}_j\}$ ($\hat{r}_j \to \infty$ as $j \to \infty$) such that $P(\hat{r}_j; u(\hat{r}_j)) \to 0$ as $j \to \infty$.*

We note that these characterizations in general indicate the necessary conditions for each type of solutions. However, in our cases, we will see later that the sign of $\lim_{r\to\infty} P(r;u)$ can distinguish each type of solution from the other.

We also note that the Pohozaev value has a relationship with K_ε.

Lemma 2.2. *For any solution u to (1.1), there holds*

$$\frac{d}{dr}P(r;u) = \frac{n-2}{2n}r^n K_\varepsilon'(r)(u_+)^{\frac{2n}{n-2}}. \tag{2.1}$$

If we define a function $G_\varepsilon(r)$ by

$$G_\varepsilon(r) = \frac{n-2}{2n}r^n K_\varepsilon(r) - \frac{n-2}{2}\int_0^r s^{n-1}K_\varepsilon(s)\,ds,$$

then we see that

$$G_\varepsilon'(r) = \frac{n-2}{2n}r^n K_\varepsilon'(r) \quad \text{and} \quad \frac{d}{dr}P(r;u) = G_\varepsilon'(r)(u_+)^{\frac{2n}{n-2}}.$$

Hence, the behavior of P is governed by that of K_ε.

If $K_\varepsilon' > 0$ near $r = \infty$, then there might exist a slowly decaying solution u with $P(r;u) < 0$ near $r = \infty$ and $\lim_{r\to\infty} P(r;u) = 0$. However, since K_ε' has a compact support, in view of (2.1), the Pohozaev value $P(r;u)$ is always constant near $r = \infty$. That is, there exists no slowly decaying solution u such that

$$P(r_j; u(r_j)) < 0 \ \text{ and } P(r_j; u(r_j)) \to 0$$

as $j \to \infty$.

In our cases, we can say that if u is a *slowly decaying* solution, then $P(r;u(r)) < 0$ and is constant near $r = \infty$. Hence, we can classify solutions into one of the three types according to the sign of $\lim_{r\to\infty} P(r;u)$.

Lemma 2.3. *Under the assumptions (K_0) with (K_ε) or (K_0^1) with (K_ε^1), the following hold.*

(i): *If $\lim_{r\to\infty} P(r;u) > 0$, then u is a crossing solution.*
(ii): *If $\lim_{r\to\infty} P(r;u) < 0$, then u is a slowly decaying solution.*
(iii): *If $\lim_{r\to\infty} P(r;u) = 0$, then u is a rapidly decaying solution.*

Another useful relation is the asymptotic behavior of $P(r;u)$ as $\alpha = u(0) \downarrow 0$.

Lemma 2.4 (Lemma 2.5 of [6]). *For any solution u to (1.1) with $u(0) = \alpha > 0$, there holds*

$$\lim_{\alpha\downarrow 0} \alpha^{-2n/(n-2)}P(r;u) = G_\varepsilon(r)$$

on any compact set of $[0,\infty)$.

Thus, if $u(0)$ is sufficiently small, then the Pohozaev value is governed by $G_\varepsilon(r)$. Thus, the sign of $G_\varepsilon(r)$ near $r = \infty$ determines the behavior of the solution u to (1.1).

Lemma 2.5 (Theorem 3 of [6]). *If there holds*

$$\liminf_{r\to\infty} G_\varepsilon(r) > 0,$$

then there exists $\delta > 0$ such that a solution $u(r;\alpha)$ to (1.1) has a finite zero for any $\alpha \in (0,\delta)$.

Under (K_ε) or (K^1_ε), we see that

$$\int_0^{\tilde{r}_\varepsilon} s^n K'_\varepsilon(s)\,ds > 0,$$

and this ensures the assumption of Lemma 2.5. Indeed, since $G_\varepsilon(0) = 0$, we have

$$G_\varepsilon(r) = \frac{n-2}{2n}\int_0^{\tilde{r}_\varepsilon} s^n K'_\varepsilon(s)\,ds > 0$$

for any $r \geq \tilde{r}_\varepsilon$.

If the sign is opposite, then u becomes a slowly decaying solution.

Lemma 2.6 (Theorem 2 of [6]). *If there holds*

$$\limsup_{r\to\infty} G_\varepsilon(r) < 0,$$

then there exists $\tilde{\delta} > 0$ such that a solution $u(r;\alpha)$ to (1.1) is a slowly decaying solution for any $\alpha \in (0,\tilde{\delta})$.

Under (K_0), $G_0(r) \leq 0$ on $[0,\infty)$. Then we have a stronger assertion.

Lemma 2.7 (Theorem 3 of [2]). *In case of $\varepsilon = 0$ with (K_0), $u(r;\alpha)$ is a slowly decaying solution for any $\alpha > 0$.*

The following is crucial to prove Theorems 1.1 and 1.2. Here we denote a solution to (1.1) with $u(0) = \alpha$ under (K_0) with (K_ε) or (K^1_0) with (K^1_ε) by $u(r;\alpha;\varepsilon)$.

Lemma 2.8. *For any sufficiently small $\varepsilon_* > 0$, define two dimensional sets by*

$$S_{\varepsilon_*} := \{(\alpha,\varepsilon)\,|\,u(r;\alpha;\varepsilon) \text{ is a slowly decaying solution and } 0 \leq \varepsilon < \varepsilon_*.\}$$

and

$$C_{\varepsilon_*} := \{(\alpha,\varepsilon)\,|\,u(r;\alpha;\varepsilon) \text{ has a finite zero and } 0 \leq \varepsilon < \varepsilon_*.\}.$$

Suppose that K_ε satisfies (K_0) with (K_ε) or (K_0^1) with (K_ε^1). If $(\alpha_0, \varepsilon_0) \in S_{\varepsilon_*}$, then there exists $\delta(\alpha_0, \varepsilon_0) > 0$ such that $(\alpha, \varepsilon) \in S_{\varepsilon_*}$ for any $(\alpha, \varepsilon) \in \{(\alpha, \varepsilon) \mid (\alpha - \alpha_0)^2 + (\varepsilon - \varepsilon_0)^2 < \delta(\alpha_0, \varepsilon_0),\ \varepsilon \geq 0\}$. Similarly, If $(\alpha_1, \varepsilon_1) \in C_{\varepsilon_*}$, then there exists $\tilde{\delta}(\alpha_1, \varepsilon_1) > 0$ such that $(\alpha, \varepsilon) \in C_{\varepsilon_*}$ for any $(\alpha, \varepsilon) \in \{(\alpha, \varepsilon) \mid (\alpha - \alpha_1)^2 + (\varepsilon - \varepsilon_1)^2 < \tilde{\delta}(\alpha_1, \varepsilon_1),\ \varepsilon \geq 0\}$.

This Lemma indicates, in one sense, the "openness" of S_{ε_*} and C_{ε_*}. To prove the openness of S_{ε_*}, we use the a priori estimate of the decay-rate of slowly decaying solutions due to Ni [3]. Due to Theorem 1.10 of [3], we see that

$$u(r) \leq C r^{-(n-2)/2}$$

near $r = \infty$ with some constant $C > 0$ which may depend on α continuously. Integration of the Pohozaev identity near infinity yields the openness of S_{ε_*}. Indeed, we have

$$\begin{aligned} P(r; u) &= P(r_1; u) + \frac{n-2}{2n} \int_{r_1}^{r} s^n K_\varepsilon'(s)(u_+)^{2n/(n-2)}\, ds \\ &\leq P(r_1; u) + \frac{(n-2)C^{2n/(n-2)}}{2n} \int_{r_1}^{r} |K_\varepsilon'(s)|\, ds \end{aligned}$$

for fixed $r_1 > 0$. If $\varepsilon > 0$ is sufficiently small, the second term is also sufficiently small and the openness follows.

The openness of C_{ε_*} follows from the continuity of solutions with respect to the initial value and ε.

For $\varepsilon > 0$, if we see $S_{\varepsilon_*} \cap (0, \infty) \times \{\varepsilon\} \neq \emptyset$ and $C_{\varepsilon_*} \cap (0, \infty) \times \{\varepsilon\} \neq \emptyset$, then on the boundary of these sets, we can find an initial value α so that $u(r; \alpha; \varepsilon)$ is a rapidly decaying solution. This is a key idea for proofs of Theorems 1.1 and 1.2.

§3. Sketch of Proofs of Theorems

In this section, we give sketchy proofs of Theorems 1.1 and 1.2.

Proof of Theorem 1.1. By Lemma 2.7, $u(r; \alpha; 0)$ is a slowly decaying solution for any $\alpha > 0$. For fixed $\alpha_* > 0$, in view of Lemma 2.8, by choosing $\varepsilon^\dagger > 0$ suitably, we see that $u(r; \alpha_*; \varepsilon)$ is also a slowly decaying solution for $0 \leq \varepsilon < \varepsilon^\dagger$.

On the other hand, by Lemma 2.5, for each $\varepsilon \in (0, \varepsilon^\dagger)$, there exists $\alpha(\varepsilon) (< \alpha_*)$ such that $u(r; \alpha; \varepsilon)$ is a crossing solution for any $\alpha \in (0, \alpha(\varepsilon))$. In view of Lemma 2.8, there exists $\alpha_\sharp(\varepsilon) \in [\alpha(\varepsilon), \alpha_*)$ such that $u(r; \alpha_\sharp(\varepsilon); \varepsilon)$ is a rapidly decaying solution. Since $\alpha_* > 0$ is fixed arbitrarily, we see that $\alpha_\sharp(\varepsilon) \downarrow 0$ as $\varepsilon \downarrow 0$. □

Before proving Theorem 1.2, we recall Theorem 3 of [6].

Proposition 3.1. *Under the assumption* (K_0^1) *for* (1.1) *with* $\varepsilon = 0$, *there exists* $\alpha_f > 0$ *such that* $u(r;\alpha;0)$ *is a slwoly decaying solution for* $\alpha \in (0, \alpha_f)$, $u(r;\alpha_f;0)$ *is a rapildy decaying solution, and* $u(r;\alpha;0)$ *is a crossing solution for* $\alpha \in (\alpha_f, \infty)$.

Using Proposition 3.1, we give a proof of Theorem 1.2.

Proof of Theorem 1.2. By Proposition 3.1, $u(r;\alpha;0)$ is a slowly decaying solution for any $\alpha \in (0, \alpha_f)$. For fixed $\alpha^* \in (0, \alpha_f)$, by Lemma 2.8, by choosing $\varepsilon^\ddagger > 0$ suitably, we see that $u(r;\alpha^*;\varepsilon)$ is also a slowly decaying solution for $0 \le \varepsilon < \varepsilon^\ddagger$.

On the other hand, by choosing $\varepsilon^\ddagger$ smaller if necessary, in view of Lemma 2.5, for each $\varepsilon \in (0, \varepsilon^\ddagger)$, there exists $\alpha(\varepsilon) < \alpha^*$ such that $u(r;\alpha;\varepsilon)$ is a crossing solution for any $\alpha \in (0, \alpha(\varepsilon))$. Again, by Lemma 2.8, there exists $\hat{\alpha}_\sharp(\varepsilon) \in [\alpha(\varepsilon), \alpha^*)$ such that $u(r;\hat{\alpha}_\sharp(\varepsilon);\varepsilon)$ is a rapidly decaying solution. Since $\alpha^* > 0$ is fixed arbitrarily, we see that $\hat{\alpha}_\sharp(\varepsilon) \downarrow 0$ as $\varepsilon \downarrow 0$.

Finally, for another rapidly decaying solution, we use Lemma 2.8 near $\alpha = \alpha_f$ to obtain $\alpha_\natural(\varepsilon) > 0$ such that $u(r;\alpha_\natural(\varepsilon);\varepsilon)$ is a rapidly decaying solution. We also see that $\alpha_\natural(\varepsilon) \to \alpha_f$ as $\varepsilon \downarrow 0$. □

As concluding remarks, we enumerate the following.

- **(i):** Can we find K_ε so that (1.1) has at least three (in general, n) rapidly decaying solutions?
- **(ii):** For other exponent $p > 1$, what conditions should be imposed?
- **(iii):** In Theorem 1.2, is the number of rapidly decaying solutions exactly two?

For (i), first, we take K_ε as in Theorem 1.2. Next, we perturb K_ε so that exactly one local maximum point is added to K_ε and that the assumption in Lemma 2.6 is satisfied. Then we will have three rapidly decaying solutions. For example, take $\varepsilon_1 > 0$ sufficiently small and take $K_{\varepsilon_1} = K_0 + \tilde{K}_{\varepsilon_1}$ as in (1.4) and (1.5). Let

$$K_\varepsilon(r) = K_{\varepsilon_1}(r) + K_\varepsilon^*(r)$$

with

$$\tilde{K}_\varepsilon^*(r) = \begin{cases} 0, & 0 \le r \le \varepsilon^{-2}, \\ \varepsilon^6(r-\varepsilon^{-2})^2, & \varepsilon^{-2} < r \le 2\varepsilon^{-2}, \\ 2\varepsilon^2 - \varepsilon^6(r-3\varepsilon^{-2})^2, & 2\varepsilon^{-2} < r \le 4\varepsilon^{-2}, \\ \varepsilon^6(r-5\varepsilon^{-2})^2, & 4\varepsilon^{-2} < r \le 5\varepsilon^{-2}, \\ 0, & 5\varepsilon^{-2} < r \end{cases}$$

Then, $\varepsilon > 0$ is sufficiently small, by Lemma 2.6, an open interval of the form $(0, \delta)$ which is for the set of initial values of slowly decaying solutions appears. Above this open interval, there is still an open interval for crossing solutions. Thus a new rapidly decaying solution is generated.

For (ii), conditions should be imposed on

$$g(r) := rK' - \frac{(n-2)p-(n+2)}{2}K.$$

In the Sobolev critical case, the second term disappears. This makes us easy to treat (1.1). If the exponent is not the Sobolev critical one, then crucial conditions should be imposed on the shape of g, more precisely, on the number of the change of its sign and on the sign of

$$\int_0^r s^{n-1}\{sK' - \frac{(n-2)p-(n+2)}{2}\}K(s)\,ds.$$

Finally, as for (iii), we suspect that the number of rapidly decaying solutions is exactly two. To prove this, at least we need to show that $\lim_{r\to\infty} P(r;u)$ changes its sign exactly twice. However, we do not succeed in proving this conjecture.

These changes of structures of solutions are also found in Kabeya [1] for the scalar-filed equation with the Robin condition.

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Department of Mathematical Scineces
Osaka Prefecture University
1-1 Gakuencho, Sakai, 599-8531
Japan