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Stability analysis for a stripe solution in the Gierer-Meinhardt system

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Abstract.

We consider the Gierer-Meinhardt system on a rectangle in two-dimensional space. This system is considered to generate some spiky pattern for a wide range of parameters. On a rectangle, this system also has a stripe solution, which is known to be unstable from numerical results. In this paper, we show the instability of the stripe pattern in a mathematically rigorous manner by using the SLEP method.

§1. Introduction

In this paper, we consider the following Gierer-Meinhardt system:

$$
\text{(GM)} \quad \begin{cases}
\dfrac{\partial A}{\partial t} = \epsilon^2 \Delta A - A + \dfrac{A^p}{H^q}, & (x,y) \in (-1,1) \times (-L,L), t > 0, \\
\tau \dfrac{\partial H}{\partial t} = d\Delta H - \mu H + \dfrac{1}{\epsilon}\dfrac{A^r}{H^s}, & (x,y) \in (-1,1) \times (-L,L), t > 0, \\
\dfrac{\partial A}{\partial x} = \dfrac{\partial H}{\partial x} = 0, & x = \pm 1, y \in (-L,L),\ t > 0, \\
\dfrac{\partial A}{\partial y} = \dfrac{\partial H}{\partial y} = 0, & y = \pm L, x \in (-1,1),\ t > 0.
\end{cases}
$$

This system was proposed by A.Gierer and H.Meinhardt in 1972 as a mathematical model for biological morphogenesis [3]. In the system (GM), A and H represent the scaled activator concentration and inhibitor concentration, respectively, ϵ, d, μ and τ are positive parameters, and the exponents p, q, r and s satisfy $p > 1, q > 0, r > 0, s \geq 0$, and $D \equiv qr - (p-1)(s+1) > 0$. The conditions for these parameters imply that the Gierer-Meinhardt system exhibits the Turing instability so that a homogeneous steady state becomes unstable by spatially inhomogeneous disturbance (see [15]). Therefore we expect that the spatially

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inhomogeneous state (or a spatial pattern) will appear in the Gierer-Meinhardt system. Indeed, some mathematicians succeeded in proving the existence of a stable spiky pattern. See, e.g., [1], [2], [5], [10], [16], [17], [18].

On the other hand, it is known that stripe patterns are not observed in numerical simulations for the Gierer-Meinhardt system, though other reaction diffusion models may generate stable stripe patterns. The authors of the paper [9] considered how the choice of reaction terms affects the tendency to generate either striped, spotted (spotted is equal to spiky), or reversed spotted pattern, and this consideration was confirmed in [9], by numerically simulation. Our aim here is to verify the consideration in a mathematically rigorous manner. To this aim, we will prove that some stationary solution with a stripe pattern is necessarily unstable in (GM).

We formulate our problem as follows. Let us first consider the following one-dimensional steady state problem:

$$
\begin{cases} \epsilon^2 a'' - a + \dfrac{a^p}{h^q} = 0, & x \in (-1,1), \\ dh'' - \mu h + \dfrac{1}{\epsilon}\dfrac{a^r}{h^s} = 0, & x \in (-1,1), \\ a' = h' = 0, & x = \pm 1. \end{cases} \tag{1.1}
$$

In [5], [10], [17], it is shown that there exists a stationary solution with a single spike at the origin, which will be denoted by $(a(x), h(x))$. Since (a, h) is independent of y-variable and the region of (GM) is a rectangle, it is clear that (a, h) also satisfies (GM). Since a has its peak over y-axis, we call (a, h) *stripe solution* of (GM). See Lemma 1 below for a more precise existence result.

Our purpose in this paper is to analyze the stability of (a, h) in (GM). In order to study the stability of the stripe solution, it suffices to consider an eigenvalue problem associated with the linearized system of (GM). Due to the lack of y-dependency of (a, h), without loss of generality, we may assume that the eigenfunction (ϕ, η) satisfies $(\phi(x, y), \eta(x, y)) = (\phi(x)\psi_k(y), \eta(x)\psi_k(y))$, where ψ_k is defined by $\psi_k(y) = \cos k\pi y/2L$ (resp. $\sin k\pi y/2L$) if k is even (resp. odd). Then we are led to the following eigenvalue problem on a bounded interval:

$$
\text{(P)} \quad \begin{cases} \lambda\phi = \epsilon^2 \phi'' - (1+\epsilon^2 l^2)\phi + p\dfrac{a^{p-1}}{h^q}\phi - q\dfrac{a^p}{h^{q+1}}\eta & \textit{in } (-1,1), \\ \tau\lambda\eta = d\eta'' - (\mu + dl^2)\eta + \dfrac{1}{\epsilon}\left(r\dfrac{a^{r-1}}{h^s}\phi - s\dfrac{a^r}{h^{s+1}}\eta\right) & \textit{in } (-1,1), \\ \phi'(\pm 1) = \eta'(\pm 1) = 0, \end{cases}
$$

where $l = k\pi/2L$.

We now describe our main result, which shows that (P) has exactly one real and positive eigenvalue.

Theorem 1. *Let (a, h) be a stripe solution of (1.1). Fix $l \neq 0$. Then there exist $d_0 > 0$ and $\epsilon_0 > 0$ such that for each $d > d_0$ and $\epsilon < \epsilon_0$, (P) has exactly one real eigenvalue λ satisfying $\lim_{\epsilon \to 0} \lambda > 0$. In particular, the stationary solution (a, h) is unstable.*

Here we remark two related results on the instability of stripe patterns in the Gierer-Meinhardt system. One is the work of Doelman and van der Ploeg [2], and another is that of Kolokolnikov et al. [6]. In both of these works, stationary solutions with the stripe pattern depending on only one spatial variable are considered for 2-dimensional Gierer-Meinhardt system. Hence, as described as above, eigenvalue problems on one dimensional space with a parameter l arises naturally. In [2], the eigenvalue problem was studied on the whole line by using the theory of Evans functions, which does not seem to be powerful in the case of bounded interval. On the other hand, in [6], the eigenvalue problem was considered on a bounded interval as our formulation. In order to analyze the eigenvalue problem, Kolokolnikov et al. adopt the NLEP (Non-Local Eigenvalue Problem) method by assuming that the exponents satisfy some extra conditions; p, r satisfy either $p = r = 2$, or $r = p + 1$ and $1 < p \leq 5$. In our work, we will use the SLEP (Singular Limit Eigenvalue Problem) method without assuming any extra conditions on the exponents.

SLEP method introduced in [8] has been used in many papers (see [7], [11]-[14]). The authors of [13], [14] considered the stability of planar interfaces of a reaction-diffusion system in two-dimensional space. Since each of their equilibrium solutions, denoted by $(\overline{u}_\epsilon, \overline{v}_\epsilon)$, has only one thin layer, and $\overline{u}_\epsilon$ approaches a step function as $\epsilon \to 0$, their eigenvalue problems are essentially the same as in the case of [8]. On the contrary, our eigenvalue problem is essentially different from theirs because (a, h) has spiky pattern in one-dimensional space. This paper is the first case that SLEP method is used to analyze the stability of a solution with spiky pattern.

In order to explain how we use the SLEP method, we first carry out formal calculations in Section 2. Then in Section 3, we give some key lemmas, in particular for the stripe solution (a, h). By using the lemmas, we describe an outline of a proof for our theorem in Section 4.

We use the following useful notations throughout our paper. First we define several Banach spaces for open intervals and notations associated with their Banach spaces. We denote by $L^2(-R, R)$, $H^1(-R, R)$,

$H^2(-R,R)$ the usual Lebesgue space and Sobolev spaces. Let $H^{-1}(-R,R)$ be the dual space of $H^1(-R,R)$. Let $\|\cdot\|_{L^2(-R,R)}$, $\|\cdot\|_{H^1(-R,R)}$, $\|\cdot\|_{H^2(-R,R)}$, and $(\cdot,\cdot)_{L^2(-R,R)}$ be the usual norms and inner product in each Banach space. We denote by $<\cdot,\cdot>$ the pairing between $H^{-1}(-1,1)$ and $H^1(-1,1)$.

Secondly, let us define a streched-coordinate y by $y = x/\epsilon$ for $x \in (-1,1)$. Note that $-1/\epsilon < y < 1/\epsilon$. For a function $\psi(x)$, we denote the streched function of $\tilde{\psi}(y)$ by $\tilde{\psi}(y) = \psi(\epsilon y)$. Furthermore we define $\hat{\psi}(y) = \sqrt{\epsilon}\tilde{\psi}(y)$ if $\psi \in L^2(-1,1)$ satisfies $\|\psi\|_{L^2(-1,1)} = 1$. Then we have $\|\hat{\psi}\|_{L^2(-1/\epsilon,1/\epsilon)} = 1$.

Thirdly we prepare some definitions and notations for simplicity. By a constant c, we mean a generic constant independent of ϵ. For each function $\phi(y)$ on $(-1/\epsilon, 1/\epsilon)$, we extend it to the whole line by a natural way, that is, $\phi \equiv 0$ for sufficiently large $|y|$. Then we do not need to distinguish the extended function from the original function, so we use the same symbol ϕ to denote the extended function.

§2. Formal calculations

In this section, we carry out formal calculations to give the idea for the proof of our theorem. We rewrite $(\cdot,\cdot)_{L^2(-1,1)}$ as $(\cdot,\cdot)$ for simplicity throughout this section. At first, we reformulate the eigenvalue problem (P) to the following form:

$$\lambda \begin{pmatrix} \phi \\ \eta \end{pmatrix} = \begin{pmatrix} L_\epsilon & f_h^\epsilon \\ g_a^\epsilon & M_\epsilon \end{pmatrix} \begin{pmatrix} \phi \\ \eta \end{pmatrix},$$

where L_ϵ and M_ϵ are differential operators defined by

$$L_\epsilon \equiv \epsilon^2 \frac{d^2}{dx^2} + f_a^\epsilon , \quad M_\epsilon \equiv \frac{d}{\tau}\frac{d^2}{dx^2} + g_h^\epsilon,$$

and f_a^ϵ, f_h^ϵ, g_a^ϵ and g_h^ϵ are given by

$$f_a^\epsilon = -(1+\epsilon^2 l^2) + p\frac{a^{p-1}}{h^q}, \quad f_h^\epsilon = -q\frac{a^p}{h^{q+1}},$$
$$g_a^\epsilon = \frac{r}{\tau\epsilon}\frac{a^{r-1}}{h^s}, \qquad g_h^\epsilon = -\frac{dl^2+\mu}{\tau} - \frac{s}{\tau\epsilon}\frac{a^r}{h^{s+1}}.$$

Since we are interested in an unstable eigenvalue of (P), it suffices to restrict λ to the right half-plane of $\mathbb{C}$ denoted by

$$\Lambda_+ = \{\lambda \in \mathbb{C} \mid Re\lambda \geq 0\}.$$

Furthermore, we can also show that any eigenvalue of (P) in Λ_+ satisfying $\lim_{\epsilon\to 0} Re\lambda > 0$ cannot be the eigenvalues of L_ϵ. Hence we can solve the first equation of (P) with respect to ϕ to obtain

$$\phi = (L_\epsilon - \lambda)^{-1}(-f_h^\epsilon \eta). \tag{2.1}$$

We can decompose $(L_\epsilon - \lambda)^{-1}$ into three parts by using pairs of eigenvalues and eigenfunctions of L_ϵ, denoted by $\{\xi_i^\epsilon, \varphi_i^\epsilon\}_{i=0}^\infty$, satisfying $\xi_0^\epsilon > \xi_1^\epsilon > \xi_2^\epsilon > \cdots$ (see (3.1)), and $\|\varphi_i^\epsilon\|_{L^2(-1,1)} = 1$ for each $i \geq 0$ as follows:

$$(L_\epsilon - \lambda)^{-1} = \frac{(\cdot, \varphi_0^\epsilon)}{\xi_0^\epsilon - \lambda}\varphi_0^\epsilon + \frac{(\cdot, \varphi_1^\epsilon)}{\xi_1^\epsilon - \lambda}\varphi_1^\epsilon + R_{\epsilon,\lambda},$$

where $R_{\epsilon,\lambda}$ is defined by

$$R_{\epsilon,\lambda} \equiv \sum_{i=2}^{\infty} \frac{(\cdot, \varphi_i^\epsilon)}{\xi_i^\epsilon - \lambda}\varphi_i^\epsilon.$$

Substituting (2.1) into the second equation of (P), we obtain

$$(-M_\epsilon - g_a^\epsilon R_{\epsilon,\lambda}(-f_h^\epsilon \cdot) + \lambda)\eta = \frac{(-f_h^\epsilon \eta, \varphi_0^\epsilon)}{\xi_0^\epsilon - \lambda} g_a^\epsilon \varphi_0^\epsilon + \frac{(-f_h^\epsilon \eta, \varphi_1^\epsilon)}{\xi_1^\epsilon - \lambda} g_a^\epsilon \varphi_1^\epsilon. \tag{2.2}$$

We can show that $(-M_\epsilon - g_a^\epsilon R_{\epsilon,\lambda}(-f_h^\epsilon \cdot) + \lambda)$ is invertible, and its inverse operator is denoted by $K_{\epsilon,\lambda} : H^{-1}(-1,1) \to H^1(-1,1)$ (see Lemma 5). Applying $K_{\epsilon,\lambda}$ to the both sides of (2.2), we have

$$\eta = \frac{(-f_h^\epsilon \eta, \varphi_0^\epsilon)}{\xi_0^\epsilon - \lambda} K_{\epsilon,\lambda}(g_a^\epsilon \varphi_0^\epsilon) + \frac{(-f_h^\epsilon \eta, \varphi_1^\epsilon)}{\xi_1^\epsilon - \lambda} K_{\epsilon,\lambda}(g_a^\epsilon \varphi_1^\epsilon). \tag{2.3}$$

This implies that η must be written as

$$\eta = \alpha K_{\epsilon,\lambda}(g_a^\epsilon \varphi_0^\epsilon) + \beta K_{\epsilon,\lambda}(g_a^\epsilon \varphi_1^\epsilon), \tag{2.4}$$

where α, β are some constants. Substituting (2.4) into (2.3), we have

$$\begin{aligned}\alpha K_{\epsilon,\lambda}(g_a^\epsilon \varphi_0^\epsilon) + \beta K_{\epsilon,\lambda}(g_a^\epsilon \varphi_1^\epsilon) &= \alpha A_{00} K_{\epsilon,\lambda}(g_a^\epsilon \varphi_0^\epsilon) + \beta A_{01} K_{\epsilon,\lambda}(g_a^\epsilon \varphi_0^\epsilon)\\ &+ \alpha A_{10} K_{\epsilon,\lambda}(g_a^\epsilon \varphi_1^\epsilon) + \beta A_{11} K_{\epsilon,\lambda}(g_a^\epsilon \varphi_1^\epsilon),\end{aligned}$$

where

$$A_{ij} = (K_{\epsilon,\lambda}(g_a^\epsilon \varphi_j^\epsilon), -f_h^\epsilon \varphi_i^\epsilon)/(\xi_i^\epsilon - \lambda), \quad i, j = 0, 1.$$

Since $K_{\epsilon,\lambda}(g_a^\epsilon \varphi_0^\epsilon)$ and $K_{\epsilon,\lambda}(g_a^\epsilon \varphi_1^\epsilon)$ are linearly independent, it follows that

$$\begin{pmatrix}\alpha\\ \beta\end{pmatrix} = \begin{pmatrix}A_{00} & A_{01}\\ A_{10} & A_{11}\end{pmatrix}\begin{pmatrix}\alpha\\ \beta\end{pmatrix}.$$

From $(\alpha,\beta)^t \neq 0$, we have $(A_{00}-1)(A_{11}-1)-A_{10}A_{01}=0$. Hence we obtain

(2.5)
$$\begin{aligned}((K_{\epsilon,\lambda}(g_a^\epsilon\varphi_0^\epsilon), -f_h^\epsilon\varphi_0^\epsilon) - \xi_0^\epsilon + \lambda)((K_{\epsilon,\lambda}(g_a^\epsilon\varphi_1^\epsilon), -f_h^\epsilon\varphi_1^\epsilon) - \xi_1^\epsilon + \lambda) \\ = (K_{\epsilon,\lambda}(g_a^\epsilon\varphi_0^\epsilon), -f_h^\epsilon\varphi_1^\epsilon)(K_{\epsilon,\lambda}(g_a^\epsilon\varphi_1^\epsilon), -f_h^\epsilon\varphi_0^\epsilon).\end{aligned}$$

Here we take the limit of $\epsilon \to 0$ in (2.5). First we have $\xi_0^\epsilon \to \xi_0^* > 0$ and $\xi_1^\epsilon \to 0$ as $\epsilon \to 0$, as described in Section 3. Moreover, in Section 4, we shall see that

$$-\frac{1}{\sqrt{\epsilon}} f_h^\epsilon \varphi_0^\epsilon \to c_1\delta, \quad -\frac{1}{\sqrt{\epsilon}} f_h^\epsilon \varphi_1^\epsilon \to 0, \quad \sqrt{\epsilon} g_a^\epsilon \varphi_0^\epsilon \to c_2\delta, \quad \sqrt{\epsilon} g_a^\epsilon \varphi_1^\epsilon \to 0$$

in $H^{-1}(-1,1)$ as $\epsilon \to 0$, where c_1, c_2 are positive constants uniformly bounded in large $d > 0$, and δ is Dirac's δ-function at the origin (see Lemma 4). Furthermore we see that there exists an operator $K_{*,\lambda}$: $H^{-1}(-1,1) \to H^1(-1,1)$ such that $K_{\epsilon,\lambda} \to K_{*,\lambda}$ as $\epsilon \to 0$ in a certain sense (see Lemma 9). Finally we have $\lambda(\lambda - \xi_0^* + c_1c_2 < \delta, K_{*,\lambda}\delta >$ $) = 0$ as $\epsilon \to 0$. Since we consider an unstable eigenvalue satisfying $\lim_{\epsilon\to 0} Re\lambda > 0$, λ must satisfy $\lambda = \xi_0^* - c_1c_2 < \delta, K_{*,\lambda}\delta >$. Since $< \delta, K_{*,\lambda}\delta >$ is small if d is large, as shown in Section 4, we see that there exists an unstable eigenvalue satisfying (2.5) by using the implicit function theorem.

§3. Known results and key lemmas

To prove Theorem 1, we need some properties of the stationary solution (a,h) of (1.1) with a single spike at the origin. We describe them according to [5], [17]. The following lemma gives the asymptotic behavior of (a,h) as $\epsilon \to 0$. Here we define $(\tilde{a}(y), \tilde{h}(y)) = (a(\epsilon y), h(\epsilon y))$ by using the streched-coordinate.

Lemma 1 ([5],[17]). *If ϵ is sufficiently small, there exists a solution (a,h) of (1.1) with the following property: There exists a constant $c > 0$ independent of ϵ such that $\|\tilde{a} - \zeta^{q/(p-1)} w\|_{H^2(-1/\epsilon,1/\epsilon)} \leq c\epsilon$, where w, ζ are defined by*

$$w(y) = \left(\frac{p+1}{2\cosh^2(p-1)y/2}\right)^{1/(p-1)}, \ \zeta = \left(\frac{2\sqrt{d\mu}}{\int_{\mathbb{R}} w^r dy} \tanh\sqrt{\frac{\mu}{d}}\right)^{(p-1)/D},$$

respectively. Moreover, for each $R > 0$, one has $\sup_{-R\leq y\leq R} |\tilde{h}(y) - \zeta| \to 0$ as $\epsilon \to 0$.

This lemma implies that $(\tilde{a}, \tilde{h})$ approaches $(\zeta^{q/(p-1)}w, \zeta)$ as $\epsilon \to 0$. Next we find that $\tilde{a}$ has the exponentially decaying property similar to w. Here we note that $ce^{-|y|} \le w \le c' e^{-|y|}$ for $y \in \mathbb{R}$ with some constants c, c'.

Lemma 2. *For every sufficiently large $R > 0$, there exists $\epsilon_0 > 0$ such that $ce^{-|y|} \le \tilde{a} \le c' e^{-|y|}$ for $\epsilon < \epsilon_0$ and $R \le |y| \le 1/\epsilon$, where c, c' are positive constants independent of ϵ, R, d.*

Next we consider pairs of the eigenvalues and eigenfunctions of L_ϵ, denoted by $\{\xi_i^\epsilon, \varphi_i^\epsilon\}_{i=0}^\infty$. In particular, we shall study the properties of $\{\xi_0^\epsilon, \varphi_0^\epsilon\}$, $\{\xi_1^\epsilon, \varphi_1^\epsilon\}$. Without loss of generality, we may assume that $\{\varphi_i^\epsilon\}_{i=0}^\infty$ is an orthonormal system in $L^2(-1,1)$. It can be shown in the same way as in the proof of Lemma 3.10 in [7] that there exists a constant $\gamma > 0$ independent of ϵ such that

$$\xi_0^\epsilon > \gamma > \xi_1^\epsilon > -\gamma > \xi_2^\epsilon > \xi_3^\epsilon > \cdots . \tag{3.1}$$

Furthermore, $\xi_0^\epsilon \to \xi_0^*$ and $\xi_1^\epsilon \to 0$ as $\epsilon \to 0$, where ξ_0^* is a unique positive eigenvalue of the following eigenvalue problem:

$$\begin{cases} \lambda\varphi = \varphi'' - \varphi + pw^{p-1}\varphi, & y \in \mathbb{R}, \\ \varphi \to 0, & |y| \to \infty. \end{cases} \tag{3.2}$$

We denote the eigenfunctions corresponding to the eigenvalue $\xi_0^*, 0$ of (3.2) by $\hat{\varphi}_0^*$, $\hat{\varphi}_1^*$, which are normalized as $\|\hat{\varphi}_0^*\|_{L^2(\mathbb{R})} = 1$ and $\|\hat{\varphi}_1^*\|_{L^2(\mathbb{R})} = 1$, respectively. In the same way as in the proof of Lemma 3.10 in [7], it follows that $\hat{\varphi}_i^\epsilon$ converges to $\hat{\varphi}_i^*$ as $\epsilon \to 0$ in $C^2(\mathbb{R})$ for $i = 0, 1$ and has the exponentially decaying property similar to $\hat{\varphi}_i^*$ for $i = 0, 1$.

Lemma 3. *Set $\theta = \max\{1/2, 1 - r/2\}$. Then there exist a constant $c > 0$ independent of ϵ and $\epsilon_0 > 0$ such that*

$$\left| \frac{d^k}{dy^k} \hat{\varphi}_0^\epsilon \right| \le ce^{-\sqrt{1+\gamma}|y|}, \quad \left| \frac{d^k}{dy^k} \hat{\varphi}_1^\epsilon \right| \le ce^{-\theta|y|} \tag{3.3}$$

for $\epsilon < \epsilon_0$, $y \in (-1/\epsilon, 1/\epsilon)$ and $k = 0, 1, 2$. Furthermore, $\hat{\varphi}_i^\epsilon$ converges to $\hat{\varphi}_i^*$ in $C^2(\mathbb{R})$ as $\epsilon \to 0$ for $i = 0, 1$.

§4. Outline of the proof of Theorem 1

In order to prove Theorem 1, we will consider the convergence of each term in (2.5). As the first step, we study the asymptotic behavior of $f_h^\epsilon \varphi_0^\epsilon/\sqrt{\epsilon}$, $f_h^\epsilon \varphi_1^\epsilon/\sqrt{\epsilon}$, $\sqrt{\epsilon} g_a^\epsilon \varphi_0^\epsilon$ and $\sqrt{\epsilon} g_a^\epsilon \varphi_1^\epsilon$ in $H^{-1}(-1,1)$ as $\epsilon \to 0$ and determine constants c_1, c_2.

Lemma 4. *In the limit of* $\epsilon \to 0$*, one has*

$$(4.1)\quad -\frac{1}{\sqrt{\epsilon}} f_h^\epsilon \varphi_0^\epsilon \to c_1\delta,\ -\frac{1}{\sqrt{\epsilon}} f_h^\epsilon \varphi_1^\epsilon \to 0,\ \sqrt{\epsilon} g_a^\epsilon \varphi_0^\epsilon \to c_2\delta,\ \sqrt{\epsilon} g_a^\epsilon \varphi_1^\epsilon \to 0$$

in $H^{-1}(-1,1)$*, where* c_1 *and* c_2 *are positive constants defined by*

$$c_1 \equiv q\zeta^{(q-p+1)/(p-1)} \int_{-\infty}^{\infty} w^p \hat{\varphi}_0^* dy,$$
$$c_2 \equiv \frac{r}{\tau}\zeta^{(q(r-1)-s(p-1))/(p-1)} \int_{-\infty}^{\infty} w^{r-1} \hat{\varphi}_0^* dy.$$

Proof. For each $z \in H^1(-1,1)$, we have

$$-\int_{-1}^{1} \frac{1}{\sqrt{\epsilon}} f_h^\epsilon \varphi_0^\epsilon z dx = \int_{-1/\epsilon}^{1/\epsilon} q\frac{\tilde{a}^p}{\tilde{h}^{q+1}} \hat{\varphi}_0^\epsilon \tilde{z} dy$$
$$\to z(0) q\zeta^{(q-p+1)/(p-1)} \int_{-\infty}^{\infty} w^p \hat{\varphi}_0^* dy.$$

Hence we obtain $-f_h^\epsilon \varphi_0^\epsilon/\sqrt{\epsilon} \to c_1\delta$ in $H^{-1}(-1,1)$ as $\epsilon \to 0$. The remainder of the lemma is shown in a similar way. So we omit the details. Q.E.D.

In Section 1, we used $K_{\epsilon,\lambda}$ which is the invertible operator and $K_{*,\lambda}$ which is the limiting operator of $K_{\epsilon,\lambda}$ as $\epsilon \to 0$ in a certain sense. In what follows, we give precise definition of these operators. To do so, we consider the following bilinear form:

$$(4.2)\qquad B_{\epsilon,\lambda}(z_1, z_2) = \frac{d}{\tau}(z_1', z_2') - (\{g_a^\epsilon R_{\epsilon,\lambda}(-f_h^\epsilon \cdot) + g_h^\epsilon - \lambda\} z_1, z_2),$$

where $\lambda \in \Lambda_+$, $z_1, z_2 \in H^1(-1,1)$. Here we abbreviate $(\cdot,\cdot)_{L^2(-1,1)}$ as $(\cdot,\cdot)$ for simplicity. Applying Lax-Milgram's theorem to this bilinear form, we define $K_{\epsilon,\lambda}$ as follows.

Lemma 5. *For each* $T \in H^{-1}(-1,1)$*, there exists a unique* $\phi \in H^1(-1,1)$ *such that*

$$(4.3)\qquad B_{\epsilon,\lambda}(\phi, \psi) = < T, \psi > \quad for\ \psi \in H^1(-1,1),$$

where ϵ *is sufficiently small,* d *is sufficiently large, and* $\lambda \in \Lambda_+$*. Moreover, the operator*

$$K_{\epsilon,\lambda} : H^{-1}(-1,1) \to H^1(-1,1)$$

is well-defined by $\phi = K_{\epsilon,\lambda} T$*. In addition,* $K_{\epsilon,\lambda}$ *is analytic with respect to* λ *and continuous with respect to* ϵ*.*

In order to prove Lemma 5, we first study the asymptotic behavior of $g_a^\epsilon R_{\epsilon,\lambda}(-f_h^\epsilon \cdot)$ contained in the right-hand side of (4.2).

Lemma 6. *For each $z \in H^1(-1,1)$ and $\lambda \in \Lambda_+$, it holds that*

$$g_a^\epsilon R_{\epsilon,\lambda}(-f_h^\epsilon z) \to z(0)F\delta \quad in\ H^{-1}(-1,1),$$
$$g_a^\epsilon \frac{d}{d\lambda} R_{\epsilon,\lambda}(-f_h^\epsilon z) \to z(0)\frac{dF}{d\lambda}\delta \quad in\ H^{-1}(-1,1) \tag{4.4}$$

as $\epsilon \to 0$. Here $F = F(\lambda)$ is an analytic function in Λ_+, which is real-valued for real λ, defined by

$$F(\lambda) = \frac{qr}{\tau}\zeta^{D/(p-1)}(\phi_0, w^{r-1})_{L^2(\mathbb{R})},$$

where ϕ_0 is some function belonging to $H^1(\mathbb{R})$, is independent of ϵ, z and has the exponentially decaying property such as $|\phi_0| \le ce^{-|y|}$ for $y \in \mathbb{R}$, with a constant $c > 0$ independent of $\lambda \in \Lambda_+$. Furthermore, in the limit of $\epsilon \to 0$, one has

$$\|g_a^\epsilon R_{\epsilon,\lambda}(-f_h^\epsilon z) - z(0)F\delta\|_{H^{-1}(-1,1)} \to 0$$

uniformly in $\|z\|_{H^1(-1,1)} \le M$.

Proof. Since the proof needs length argument, we only describe the outline of the proof.

Setting $\phi_\epsilon = R_{\epsilon,\lambda}(-f_h^\epsilon z)$ and using the streched-coordinate $y = x/\epsilon$, we can show that $\tilde{\phi}_\epsilon$ is uniformly bounded in $H^1(\mathbb{R})$. Hence there exists $\phi_0 \in H^1(\mathbb{R})$ independent of ϵ, z such that $\tilde{\phi}_\epsilon \to z(0)q\zeta^{(q-p+1)/(p-1)}\phi_0$ weakly in $H^1(-1,1)$ as $\epsilon \to 0$. Here we do not need to choose a subsequence of ϵ. In addition, we find that ϕ_0 is a unique solution of the following problem:

$$\phi_0'' - (1+\lambda)\phi_0 + pw^{p-1}\phi_0 = w^p - (w^p, \hat{\varphi}_0^*)\hat{\varphi}_0^* \tag{4.5}$$

for $y \in \mathbb{R}$. Using this equation, we obtain the exponentially decaying property of ϕ_0. Finally, by this property of ϕ_0, we have (4.4).

Next we show that the last part of the lemma by contradiction. Suppose that there exist a constant $\kappa > 0$ independent of $\epsilon > 0$, ϵ_n and $z_n \in H^1(-1,1)$ such that $\epsilon_n \to 0$ as $n \to \infty$, $\|z_n\|_{H^1(-1,1)} \le M$, and

$$\|g_a^{\epsilon_n} R_{\epsilon_n,\lambda}(-f_h^{\epsilon_n} z_n) - z_n(0)F\delta\|_{H^{-1}(-1,1)} \ge \kappa.$$

By $\|z_n\|_{H^1(-1,1)} \le M$, there exists $z_0 \in H^1(-1,1)$ such that $z_n \to z_0$ weakly in $H^1(-1,1)$. Here we may replace z_n with an appropriate subsequence if needed, but we use the same notation. Since $H^1(-1,1)$ is compactly embedded in a Hölder space, we may assume that z_n converges to z_0 uniformly on $[-1,1]$. Then we have

$$\|g_a^{\epsilon_n} R_{\epsilon_n,\lambda}(-f_h^{\epsilon_n} z_n) - z_n(0)F\delta\| \leq \|g_a^{\epsilon_n} R_{\epsilon_n,\lambda}(-f_h^{\epsilon_n}(z_n - z_0))\| \\ +\|g_a^{\epsilon_n} R_{\epsilon_n,\lambda}(-f_h^{\epsilon_n} z_0) - z_0(0)F\delta\| + \|(z_0(0) - z_n(0))F\delta\|,$$

where we abbreviate $\|\cdot\|_{H^{-1}(-1,1)}$ as $\|\cdot\|$ for simplicity. All terms of the right side of the above inequality converge to 0 as $n \to 0$, which is a contradiction. Q.E.D.

Next we consider the convergence of $a^r z/\epsilon h^{s+1}$ in $H^{-1}(-1,1)$ as $\epsilon \to 0$, contained in the term $g_h^\epsilon z$, where $z \in H^1(-1,1)$.

Lemma 7. *For any $z \in H^1(-1,1)$, in the limit of $\epsilon \to 0$, it holds that*

$$\frac{a^r}{\epsilon h^{s+1}} z \to z(0)\xi^{D/(p-1)} \int_{\mathbb{R}} w^r dy\delta \ in\ H^{-1}(-1,1). \tag{4.6}$$

Furthermore, for each $M > 0$, one has as $\epsilon \to 0$

$$\|\frac{a^r}{\epsilon h^{s+1}} z - z(0)\xi^{D/(p-1)} \int_{\mathbb{R}} w^r dy\delta\|_{H^{-1}(-1,1)} \to 0$$

uniformly in $\|z\|_{H^1(-1,1)} \leq M$.

Proof. It is easy to show (4.6). The last part of the lemma can be shown in the same argument as in the proof of Lemma 6. We omit the details. Q.E.D.

We can prove Lemma 5 by applying Lemmas 6 and 7. In what follows, we abbreviate $\|\cdot\|_{H^1(-1,1)}$ as $\|\cdot\|_{H^1}$.

Proof. For any $z_1, z_2 \in H^1(-1,1)$, we have

$$|B_{\epsilon,\lambda}(z_1, z_2)| \leq \frac{2}{\tau} \max\{d, dl^2 + \mu + \tau|\lambda|\} \|z_1\|_{H^1} \|z_2\|_{H^1},$$

and for any $z \in H^1(\mathbb{R})$, we have

$$|B_{\epsilon,\lambda}(z,z)| \geq \frac{\min\{d, dl^2 + \mu\}}{2\tau} \|z\|_{H^1}^2. \tag{4.7}$$

By these two inequalities, we can apply Lax-Milgram's theorem to $B_{\epsilon,\lambda}$, that is, for each $T \in H^{-1}(-1,1)$, there exists a unique $\phi \in H^1(-1,1)$ such that

$$B_{\epsilon,\lambda}(\phi, \psi) =< T, \psi > \quad for\ any\ \psi \in H^1(-1,1)$$

for $\epsilon > 0$ and $\lambda \in \Lambda_+$. Then we define the operator $K_{\epsilon,\lambda} : H^{-1}(-1,1) \to H^1(-1,1)$ by $K_{\epsilon,\lambda} T = \phi$, which implies that $K_{\epsilon,\lambda}$ is well-defined.

Secondly we shall show that $K_{\epsilon,\lambda}$ is continuous with respect to ϵ and analytic with respect to $\lambda \in \Lambda_+$ in a certain sense. We take any $T \in H^{-1}(-1,1)$. Then, for each $\psi \in H^1(-1,1)$, $\epsilon, \epsilon' > 0$, and $\lambda, \lambda' \in \Lambda_+$, we have

$$B_{\epsilon,\lambda}(K_{\epsilon,\lambda} T, \psi) = < T, \psi > = B_{\epsilon',\lambda'}(K_{\epsilon',\lambda'} T, \psi).$$

Using this, we obtain

$$B_{\epsilon',\lambda'}((K_{\epsilon,\lambda} - K_{\epsilon',\lambda'})T, \psi) = B_{\epsilon',\lambda'}(K_{\epsilon,\lambda} T, \psi) - B_{\epsilon,\lambda}(K_{\epsilon,\lambda} T, \psi). \tag{4.8}$$

Substituting $(K_{\epsilon,\lambda} - K_{\epsilon',\lambda'})T$ into ψ and using the inequality (4.7), we obtain

$$\begin{aligned} &|B_{\epsilon',\lambda'}((K_{\epsilon,\lambda} - K_{\epsilon',\lambda'})T, (K_{\epsilon,\lambda} - K_{\epsilon',\lambda'})T)| \\ &\qquad \geq \frac{\min\{d, dl^2 + \mu\}}{2\tau} \|(K_{\epsilon,\lambda} - K_{\epsilon',\lambda'})T\|_{H^1}^2. \end{aligned}$$

On the other hand, the right side of (4.8) is estimated from above as

$$\begin{aligned} &|B_{\epsilon',\lambda'}(K_{\epsilon,\lambda} T, \psi) - B_{\epsilon,\lambda}(K_{\epsilon,\lambda} T, \psi)| \\ &\leq \Bigg(\|(g_a^{\epsilon'} R_{\epsilon',\lambda'}(-f_h^{\epsilon'} \cdot) - g_a^{\epsilon} R_{\epsilon,\lambda}(-f_h^{\epsilon} \cdot)) K_{\epsilon,\lambda} T\|_{H^{-1}} \\ &\quad + \left\| \left(\frac{s}{\tau\epsilon'} \frac{a_{\epsilon'}^r}{h_{\epsilon'}^{s+1}} - \frac{s}{\tau\epsilon} \frac{a_\epsilon^r}{h_\epsilon^{s+1}} \right) K_{\epsilon,\lambda} T \right\|_{H^{-1}} + |\lambda' - \lambda| \|K_{\epsilon,\lambda} T\|_{H^1} \Bigg) \|\psi\|_{H^1}, \end{aligned}$$

where we write a, h as a_ϵ, h_ϵ for ϵ to distinguish a_ϵ, h_ϵ from $a_{\epsilon'}, h_{\epsilon'}$. Hence we take $\psi = (K_{\epsilon,\lambda} - K_{\epsilon',\lambda'})T$ in (4.8) and find from the above two inequalities that

(4.9)

$$\begin{aligned} \|(K_{\epsilon,\lambda} - K_{\epsilon',\lambda'})T\|_{H^1} &\leq \|((g_a^{\epsilon'} R_{\epsilon',\lambda'}(-f_h^{\epsilon'} \cdot) - g_a^{\epsilon} R_{\epsilon,\lambda}(-f_h^{\epsilon} \cdot)) K_{\epsilon,\lambda} T\|_{H^{-1}} \\ &+ \left\| \left(\frac{s}{\tau\epsilon'} \frac{a_{\epsilon'}^r}{h_{\epsilon'}^{s+1}} - \frac{s}{\tau\epsilon} \frac{a_\epsilon^r}{h_\epsilon^{s+1}} \right) K_{\epsilon,\lambda} T \right\|_{H^{-1}} + |\lambda' - \lambda| \, \|K_{\epsilon,\lambda} T\|_{H^1}. \end{aligned}$$

Here it follows that $\|K_{\epsilon,\lambda} T\|_{H^1} \leq 2\tau \|T\|_{H^{-1}} / \min\{d, dl^2 + \mu\}$ so that $\|K_{\epsilon,\lambda} T\|_{H^1}$ is uniformly bounded in $\epsilon > 0$ and $\lambda \in \Lambda_+$. By using this and Lemmas 6, 7, all terms in the right-hand side of (4.9) tend to 0 as $\epsilon' \to \epsilon$ and $\lambda' \to \lambda$. Hence $K_{\epsilon,\lambda}$ is continuous with respect to ϵ and analytic with respect to λ. Q.E.D.

Next let us define $K_{*,\lambda}$. From Lemmas 6, 7 we see that

$$(g_a^\epsilon R_{\epsilon,\lambda}(-f_h^\epsilon z_1), z_2)_{L^2(-1,1)} \to F(\lambda) z_1(0)\overline{z_2}(0),$$
$$(\frac{1}{\epsilon}\frac{a^r}{h^{s+1}} z_1, z_2)_{L^2(-1,1)} \to \xi^{D/(p-1)} \int_{\mathbb{R}} w^r dy z_1(0)\overline{z_2}(0)$$

as $\epsilon \to 0$. Hence we define $B_{*,\lambda}$, which is the limiting form of $B_{\epsilon,\lambda}$ as $\epsilon \to 0$, by

$$B_{*,\lambda}(z_1, z_2) = \frac{d}{\tau}(z_1', z_2') + \frac{dl^2 + \mu + \tau\lambda}{\tau}(z_1, z_2) + (E - F(\lambda)) z_1(0)\overline{z_2}(0),$$

where $\lambda \in \Lambda_+$ and $z_1, z_2 \in H^1(-1,1)$. Here we set

$$E = \frac{s}{\tau}\xi^{D/(p-1)} \int_{\mathbb{R}} w^r dy$$

and abbreviate $(\cdot,\cdot)_{L^2(-1,1)}$ as $(\cdot,\cdot)$. We apply Lax-Milgram's theorem to $B_{*,\lambda}$ as well as $B_{\epsilon,\lambda}$ to define $K_{*,\lambda}$.

Lemma 8. *For each $T \in H^{-1}(-1,1)$, there exists a unique $\phi \in H^1(-1,1)$ such that*

$$B_{*,\lambda}(\phi, \psi) = < T, \psi > \qquad \text{for } \psi \in H^1(-1,1), \tag{4.10}$$

where d is sufficiently large and $\lambda \in \Lambda_+$. Moreover, the operator

$$K_{*,\lambda} : H^{-1}(-1,1) \to H^1(-1,1)$$

is well-defined by $\phi = K_{*,\lambda} T$ and is analytic with respect to λ.

Proof. This lemma can be shown in the same way as in the proof of Lemma 5. So, we omit the details of the proof. Q.E.D.

As described previously, $B_{*,\lambda}$ is the limiting form of $B_{\epsilon,\lambda}$ as $\epsilon \to 0$. Hence we expect that $K_{\epsilon,\lambda}$ converges to $K_{*,\lambda}$ as $\epsilon \to 0$ in a certain sense. Indeed, the following lemma holds.

Lemma 9. *Fix $\lambda \in \Lambda_+$ and $T \in H^{-1}(-1,1)$. Let $\epsilon_n > 0$, $\lambda_n \in \Lambda_+$, and $T_n \in H^{-1}(-1,1)$ satisfy $\epsilon_n \to 0$, $\lambda_n \to \lambda$, and $T_n \to T$ in $H^{-1}(-1,1)$ as $n \to \infty$, respectively. Then $K_{\epsilon_n,\lambda_n} T_n$ converges to $K_{*,\lambda} T$ in $H^1(-1,1)$ as $n \to \infty$.*

Proof. We can show this lemma by a similar inequality to (4.9). So, we omit the details of the proof. Q.E.D.

Now we are in a position to prove Theorem 1. Since $K_{*,\lambda}\delta$ must satisfy $B_{*,\lambda}(K_{*,\lambda}\delta, \psi) =< \delta, \psi >$ for any $\psi \in H^1(-1,1)$, $K_{*,\lambda}\delta$ is a solution of

$$-\frac{d}{\tau}(K_{*,\lambda}\delta)'' + \frac{dl^2 + \mu + \tau\lambda}{\tau} K_{*,\lambda}\delta = \{1 + (F - E)(K_{*,\lambda}\delta)(0)\}\delta.$$

Finally, we have

$$(K_{*,\lambda}\delta)(x) = \frac{\tau}{dl^2 + \mu + \tau\lambda}\{1 + (F - E)(K_{*,\lambda}\delta)(0)\}G_{d,\lambda}(x, 0), \tag{4.11}$$

where $G_{d,\lambda}$ is a Green's function with Neumann boundary condition for the following equation:

$$-\frac{d}{dl^2 + \mu + \tau\lambda}G(\cdot, z)'' + G(\cdot, z) = \delta_z \quad \text{on} \quad (-1, 1),$$

where $z \in (-1, 1)$ and δ_z is Dirac's δ-function at z. Here we see from (4.11) that $(K_{*,\lambda}\delta)(0)$ satisfies a compatibility condition so that we can solve (4.11) with respect to $(K_{*,\lambda}\delta)(0)$ to obtain

$$(K_{*,\lambda}\delta)(0) = \frac{\tau G_{d,\lambda}(0, 0)}{dl^2 + \mu + \tau\lambda + \tau G_{d,\lambda}(0, 0)(E - F)},$$

where $G_{d,\lambda}(0, 0)$ is explicitly given by

$$G_{d,\lambda}(0, 0) = \frac{\sqrt{dl^2 + \mu + \tau\lambda}}{2\sqrt{d}\tanh(\sqrt{dl^2 + \mu + \tau\lambda}/\sqrt{d})}.$$

If d is sufficiently large, $(K_{*,\lambda}\delta)(0)$ is much smaller than ξ_0^*. Furthermore $d(K_{*,\lambda}\delta)/d\lambda$ is also small if d is sufficiently large. Therefore we obtain an unstable eigenvalue λ by the implicit function theorem.

It is not difficult to show that (P) has exactly one unstable eigenvalue and it is a real number. More precise proof will be given in a forthcoming paper [4].

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