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On the Stokes equation with Robin boundary condition

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§1. Introduction

Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$ be a domain with boundary $\partial\Omega \in C^{2,1}$ and suppose one of the following case; Ω is a bounded domain, an exterior domain, a half-space or a perturbed half-space, *i.e.* there exists $R > 0$ such that $\Omega \setminus B_R = \mathbb{R}^n_+ \setminus B_R$, where $B_R = \{x \in \mathbb{R}^n \,|\, |x| < R\}$, $\mathbb{R}^n_+ = \{x = (x_1, \cdots, x_n) \in \mathbb{R}^n \,|\, x_n > 0\}$. We consider the following Stokes system with Robin boundary condition.

(1.1)
$$\begin{cases} u_t - \operatorname{Div} T(u,\pi) = f, \ \operatorname{div} u = g = \operatorname{div} \tilde{g} & \text{in } \Omega \times (0, T_0) \\ \nu \cdot u|_{\partial\Omega} = 0, \ \alpha u + T(u)\nu - (T(u)\nu, \nu)\nu|_{\partial\Omega} = h & \\ \quad u|_{t=0} = u_0 & \end{cases}$$

where α is a non-negative constant, $u(x,t) = (u_1, \ldots, u_n)$ is a velocity field and $\pi(x,t)$ is a scalar pressure. Here and hereafter ν denotes the unit outer normal to $\partial\Omega$, $T(u,\pi)$ is the stress tensor defined by the formula:

$$T(u,\pi) = D(u) - \pi I \tag{1.2}$$

where $D(u)$ is the deformation tensor of the velocity with elements $D_{ij}(u) = \partial_i u_j + \partial_j u_i$. It is obvious that the boundary condition in (1.1) is equivalent to the following condition

$$\nu \cdot u|_{\partial\Omega} = 0, \ \alpha u + D(u)\nu - \big(D(u)\nu, \nu\big)\nu|_{\partial\Omega} = h. \tag{1.3}$$

So we may consider the condition (1.3) in stead of the boundary condition of (1.1). The cases when $\alpha = \infty$ and $\alpha = 0$ are corresponding to

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the Dirichlet boundary condition and the slip boundary condition, respectively. The latter condition is called the appropriate physical model for flows past chemically reaction walls, and for flows at high angles of attack and high Mach and Reynold's numbers as in the re-entry of a space orbiter case. It is also part of the boundary conditions modelling flow problems with free boundaries as in the coating problem and other examples such as fiber spinning and microfluidics. So, we consider the boundary condition in (1.1) as the generalized condition, and call it the Robin boundary condition.

§2. Resolvent Estimate

In this section, we consider the following system:

$$
\begin{cases} u_t - \operatorname{Div} T(u,\pi) = f, \ \operatorname{div} u = 0 & \text{in}\, \Omega \times (0,\infty) \\ \nu \cdot u|_{\partial\Omega} = 0, \ \alpha u + D(u)\nu - (D(u)\nu,\nu)\nu|_{\partial\Omega} = 0 & \\ u|_{t=0} = u_0 & \end{cases} \tag{2.1}
$$

In order to prove the resolvent estimate, we consider the following resolvent problem which corresponds to (2.1):

$$
\begin{cases} \lambda u - \Delta u + \nabla\pi = f, \qquad \operatorname{div} u = g & \text{in } \Omega \\ \nu \cdot u = 0, \quad \alpha u + D(u)\nu - (D(u)\nu,\nu)\nu = h & \text{on } \partial\Omega \end{cases} \tag{2.2}
$$

where $\lambda \in \Sigma_\epsilon$, $\Sigma_\epsilon = \{\lambda \in \mathbb{C}\setminus\{0\} \mid |\arg\lambda| \le \pi - \epsilon\}$ for some ϵ with $0 < \epsilon < \pi$.

In the bounded domain and the divergence free case, the resolvent estimate is obtained by Giga [2] in the homogeneous boundary condition case, and by Grubb-Solonnikov [3] in the nonhomogeneous boundary condition case. Saal [4] also proved the resolvent estimate in the half-space with the divergence free and homogeneous boundary conditions. In this paper, we shall give a resolvent estimate for the Stokes equation with nonhomogeneous Robin boundary condition. $L_q(\Omega)$ and $W_q^1(\Omega)$ denote the usual Lebesgue space and Sobolev space, respectively. When Ω is bounded we set

$$
W^1_{q,a}(\Omega) = \{g \in W_q^1(\Omega) \mid \int_\Omega g dx = 0\}
$$

When Ω is unbounded we set

$$
\hat{W}_q^1(\Omega) := \{\pi \in L_{q,\,loc}(\Omega) | \nabla\pi \in L_q(\Omega)\}, \qquad \hat{W}_q^{-1}(\Omega) := \hat{W}^1_{q'}(\Omega)^*
$$
$$
W^1_{q,0}(\Omega) = \{g \in W_q^1(\Omega) \mid \text{supp } g \text{ is compact}\}
$$

where X^* denotes the dual space of X, and q is the dual exponent to q' such as $1/q + 1/q' = 1$. The following theorem is our main result concerning the resolvent estimate.

Theorem 2.1. (Resolvent estimate) *Let* $1 < q < \infty$ *and* $0 < \epsilon < \frac{\pi}{2}$. (i) *Let* Ω *be bounded. Assume that* Ω *is not rotationally symmetric. Then, there exists a constant* $\sigma_0 > 0$ *such that for any* $\lambda \in \mathbb{C}\backslash(-\infty, -\sigma_0)$, $f \in L_q(\Omega)^n$, $h \in W_q^1(\Omega)^n$ *with* $\nu \cdot h\,|_{\partial\Omega} = 0$ *and* $g \in W_{q,a}^1(\Omega)$, *then* (2.2) *admits a unique solution* $(u, \pi) \in W_q^2(\Omega)^n \times W_{q,a}^1(\Omega)$ *which enjoys the estimate:*

(2.3)

$$
\begin{aligned}
&|\lambda|\|u\|_{L_q(\Omega)} + |\lambda|^{\frac{1}{2}}\|\nabla u\|_{L_q(\Omega)} + \|u\|_{W_q^2(\Omega)} + \|p\|_{W_q^1(\Omega)} \\
&\leq C\{\|f\|_{L_q(\Omega)} + \alpha(1+|\lambda|)^{-\frac{1}{2}}\|f\|_{L_q(\Omega)} + \|\nabla g\|_{L_q(\Omega)} \\
&\quad + \alpha(1+|\lambda|)^{-\frac{1}{2}}\|\nabla g\|_{L_q(\Omega)} + (1+|\lambda|)\|g\|_{\hat{W}_q^{-1}(\Omega)} + \alpha\|g\|_{L_q(\Omega)} \\
&\qquad + \alpha(1+|\lambda|)^{\frac{1}{2}}\|g\|_{\hat{W}_q^{-1}(\Omega)} + \|\nabla h\|_{L_q(\Omega)} + (1+|\lambda|)^{\frac{1}{2}}\|h\|_{L_q(\Omega)}\}
\end{aligned}
$$

for $\lambda \in \Sigma_\epsilon \cup \{\lambda \in \mathbb{C} \mid |\lambda| \leq \sigma_0\}$ *where* $C = C_{q,\epsilon} > 0$ *is a constant depending on* q *and* ϵ.
(ii) *Let* Ω *be unbounded. For any* $\lambda \in \mathbb{C}\backslash(-\infty, 0]$, $f \in L_q(\Omega)^n$, $h \in W_q^1(\Omega)^n$ *with* $\nu \cdot h\,|_{\partial\Omega} = 0$ *and* $g \in W_{q,0}^1(\Omega) \cap \hat{W}_q^{-1}(\Omega)$, *then* (2.2) *admits a unique solution* $(u, \pi) \in W_q^2(\Omega)^n \times \hat{W}_q^1(\Omega)$ *which enjoys the estimate:*

(2.4)

$$
\begin{aligned}
&|\lambda|\|u\|_{L_q(\Omega)} + |\lambda|^{\frac{1}{2}}\|\nabla u\|_{L_q(\Omega)} + \|\nabla^2 u\|_{L_q(\Omega)} + \|\nabla\pi\|_{L_q(\Omega)} \\
&\leq C\{(1+\alpha|\lambda|^{-\frac{1}{2}})\|f\|_{L_q(\Omega)} + (|\lambda| + \alpha|\lambda|^{\frac{1}{2}})\|g\|_{\hat{W}_q^{-1}(\Omega)} \\
&\quad + \alpha\|g\|_{L_q(\Omega)} + (1+\alpha|\lambda|^{-\frac{1}{2}})\|\nabla g\|_{L_q(\Omega)} + \|\nabla h\|_{L_q(\Omega)} + |\lambda|^{\frac{1}{2}}\|h\|_{L_q(\Omega)}\}
\end{aligned}
$$

for $\lambda \in \Sigma_\epsilon$ *with* $|\lambda| \geq \sigma$ *where* σ *is any positive number and* $C = C_{q,\epsilon,\sigma} > 0$ *is a constant depending on* q ϵ *and* σ.

Now, we introduce the Helmholtz decomposition corresponding to our problem. Set

$$
\begin{aligned}
J_q(\Omega) &:= \overline{C_{0,\sigma}^\infty(\Omega)}^{\|\cdot\|_{L_q(\Omega)}}, \\
C_{0,\sigma}^\infty(\Omega) &:= \{u \in C_0^\infty(\Omega)^n \mid \operatorname{div} u = 0\}, \\
G_q(\Omega) &:= \{\nabla\pi \in L_q(\Omega)^n \mid \pi \in L_{q,loc}(\overline{\Omega})\}.
\end{aligned}
$$

Then, we know that the Helmholtz decomposition:

$$
L_q(\Omega)^n = J_q(\Omega) \oplus G_q(\Omega). \tag{2.5}
$$

Here $\oplus$ denotes the direct sum. It should be noted that since $\partial\Omega$ is $C^{2,1}$-hypersurface, $L_{p,\sigma}(\Omega)$ is characterized as

$$J_p(\Omega) = \{u \in L_p(\Omega) \mid \operatorname{div} u = 0 \text{ in } \Omega,\ \nu \cdot u = 0 \text{ on } \partial\Omega\}.$$

Let P_q denotes the solenoidal projection from $L_q(\Omega)$ onto $J_q(\Omega)$ along $G_q(\Omega)$.

In our case the Stokes operator A_q is defined by following relations:

$$\begin{aligned}
&A_q := P_q(-\Delta),\\
&D(A_q) := \left\{u \in W_q^2(\Omega)^n \cap J_q(\Omega) \mid \alpha u + D(u)\nu - \big(D(u)\nu, \nu\big)\nu|_{\partial\Omega} = 0\right\}.
\end{aligned}$$

Corollary 2.2. *Let $1 < q < \infty$. Then, $-A_q$ generates the analytic semigroup $\{T_q(t)\}_{t\geq 0}$ on $J_q(\Omega)$.*

If Ω is a bounded domain, $\{T_q(t)\}_{t\geq 0}$ is exponentially stable. Moreover we have the following theorem.

Corollary 2.3 (L_q-L_r estimate)**.** *Let Ω be a bounded domain, and $1 \leq q \leq r \leq \infty$ with $q \neq \infty$ and $r \neq 1$. Then there exists constants c and C which depend on only q and r such that*

$$\|T_q(t)u\|_{L_r(\Omega)} \leq C_{q,r}e^{-ct}t^{-\frac{n}{2}\left(\frac{1}{q}-\frac{1}{r}\right)}\|u\|_{L_q(\Omega)} \tag{2.6}$$

$$\|\nabla T_q(t)u\|_{L_r(\Omega)} \leq C_{q,r}e^{-ct}t^{-\frac{1}{2}-\frac{n}{2}\left(\frac{1}{q}-\frac{1}{r}\right)}\|u\|_{L_q(\Omega)} \tag{2.7}$$

for any $t > 0$ and $u \in J_q(\Omega)$.

§3. L_p-L_q Maximal Regularity

In this section, we consider L_p-L_q maximal regularity in the bounded domain case. It was obtained by Shibata-Shimizu [5] in the Neumann boundary condition case. In order to state our main results preciously, first of all we introduce some function spaces.

3.1. Function spaces.

In this subsection, let p and q be exponents $\in [1, \infty]$ and l and m be non-negative integers. Let X be a Banach space with the norm $\|\cdot\|_X$.

For an interval $I \subset \mathbb{R}$ and a domain $D \subset \mathbb{R}^n$, we set

$$
\begin{aligned}
&\|v\|_{L_p(I,X)} = \begin{cases} \left\{\int_I \|v(t)\|_X^p \mathrm{d}x\right\}^{1/p} & 1 \le p < \infty, \\ \mathrm{esssup}_{x\in I}|v(x)| & p = \infty, \end{cases}\\
&W_p^m(I,X) = \{v \mid \partial_t^j v \in L_p(I,X),\ j = 0,1,\cdots,m\},\\
&\|v\|_{W_p^m(I,X)} = \sum_{j=0}^m \|\partial_t^j v\|_{L_p(I,X)};\\
&W_{q,p}^{l,m}(D\times I) = L_p(I, W_q^l(D)) \cap W_p^m(I, L_q(D)),\\
&\|u\|_{W_{q,p}^{l,m}(D\times I)} = \|u\|_{L_p(I,W_q^l(D))} + \|u\|_{W_p^m(I,L_q(D))};\\
&W_q^0(D) = L_q(D),\ W_p^0(I,X) = L_p(I,X),\\
&\dot{W}_p^1([0,T_0),X) = \{u \in W_p^1((-\infty,T_0),X) \mid u = 0 \,\mathrm{for}\, t < 0\}.
\end{aligned}
$$

Given $\theta \in \mathbb{R}$, we set

$$
\begin{aligned}
&\langle D_t\rangle^\theta u(t) = \mathcal{F}^{-1}[(1+s^2)^{\theta/2}\mathcal{F}u(s)](t),\\
&H_p^\theta(\mathbb{R},X) = \{u \in L_p(\mathbb{R},X) \mid \langle D_t\rangle^\theta u \in L_p(\mathbb{R},X)\},\\
&\|u\|_{H_p^\theta(\mathbb{R},X)} = \|u\|_{L_p(\mathbb{R},X)} + \|\langle D_t\rangle^\theta u\|_{L_p(\mathbb{R},X)}.
\end{aligned}
$$

Here and hereafter, $\mathcal{F}$ and $\mathcal{F}^{-1}$ denote the Fourier transform and its inverse formula, respectively. Set

$$
\begin{aligned}
&H_{q,p}^{1,1/2}(D\times\mathbb{R})\\
&\quad = H_p^{1/2}(\mathbb{R},L_q(D)) \cap L_p(\mathbb{R},W_q^1(D)),\\
&\dot{H}_{q,p}^{1,1/2}(D\times[0,\infty))\\
&\quad = \{u \in H_p^{1/2}(\mathbb{R},L_q(D)) \cap L_p(\mathbb{R},W_q^1(D)) \mid u = 0 \quad \mathrm{for}\, t < 0\},\\
&\|u\|_{H_{q,p}^{1,1/2}(D\times\mathbb{R})} = \|u\|_{H_p^{1/2}(\mathbb{R},L_q(D))} + \|u\|_{L_p(\mathbb{R},W_q^1(D))}.
\end{aligned}
$$

Finally, given $0 < T_0 \leq \infty$ we set

$$H^{1,1/2}_{q,p}(D \times (0,T_0)) = \{u \mid {}^\exists v \in H^{1,1/2}_{q,p}(D \times \mathbb{R}),\ u = v \text{ on } D \times (0,T_0)\},$$
$$\|u\|_{H^{1,1/2}_{q,p}(D\times(0,T_0))} = \inf\{\|v\|_{H^{1,1/2}_{q,p}(D\times\mathbb{R})} \mid {}^\forall v \in H^{1,1/2}_{q,p}(D \times \mathbb{R}),\ u = v \text{ on } D \times (0,T_0)\};$$
$$\dot H^{1,1/2}_{q,p}(D \times (0,T_0)) = \{u \mid {}^\exists v \in \dot H^{1,1/2}_{q,p}(D \times [0,\infty)),\ u = v \text{ on } D \times (0,T_0)\},$$
$$\|u\|_{H^{1,1/2}_{q,p}(D\times(0,T_0))} = \inf\{\|v\|_{\dot H^{1,1/2}_{q,p}(D\times[0,\infty))} \mid {}^\forall v \in \dot H^{1,1/2}_{q,p}(D \times [0,\infty)),\ u = v \text{ on } D \times (0,T_0)\}.$$

3.2. Main results

Set $B^{2(1-1/p)}_{q,p}(\Omega) := [L_q(\Omega), W^2_q(\Omega)]_{1-1/p,p}$ and $\mathcal{D}_{q,p} = [J_q(\Omega), D(A_q)]_{1-1/p,p}$ where $[\cdot,\cdot]_{\theta,p}$ denotes the real interpolation functor. By Steiger [6], we can see that $\mathcal{D}_{q,p} \subset B^{2(1-1/p)}_{q,p}(\Omega)$ and the norm of $\mathcal{D}_{q,p}$ and that of $B^{2(1-1/p)}_{q,p}(\Omega)$ are equivarent for p and q with $2(1-1/p) \neq 1/q,\ 1+1/q$. For the solution of (1.1), we have the following theorems.

Theorem 3.1 (L_p-L_q maximal regularity). *Let $1 < p, q < \infty$ and $T_0 > 0$. If $u_0, f, g, \tilde g$ and h of (1.1) satisfy the conditions:*

$$\begin{aligned} &u_0 \in \mathcal{D}_{q,p}, \quad f \in L_p\left((0,T_0), L_q(\Omega)\right)^n, \\ &g \in L_p((0,T_0), W^1_q(\Omega)), \quad \tilde g \in \dot W^1_p\left((0,T_0), L_q(\Omega)\right)^n, \\ &h \in \dot H^{1,1/2}_{q,p}\left(\Omega \times (0,T_0)\right)^n, \quad \nu \cdot h|_{\partial\Omega} = 0, \end{aligned} \tag{3.1}$$

then (1.1) admits a unique solution $(u,\pi) \in W^{2,1}_{q,p}(\Omega\times(0,T_0))^n \times L_p((0,T_0), W^1_q(\Omega))$ which enjoys the estimate:

$$\begin{aligned} &\|u\|_{W^{2,1}_{q,p}(\Omega\times(0,T_0))} + \|\pi\|_{L_p((0,T_0),W^1_q(\Omega))} \leq C\{\|u_0\|_{\mathcal{D}_{q,p}} + \|f\|_{L_p((0,T_0),L_q(\Omega))} \\ &\quad + \|h\|_{\dot H^{1,1/2}_{q,p}(\Omega\times(0,T_0))} + \|g\|_{L_p((0,T_0),W^1_q(\Omega))} + \|\tilde g\|_{W^1_p((0,T_0),L_q(\Omega))}\} \end{aligned} \tag{3.2}$$

where the constant $C > 0$ is independent T, u, π, f, g, $\tilde g$ and h.

Theorem 3.2 (exponential stability). *Let $1 < p, q < \infty$. Then there exists $\gamma_0 > 0$ such that if u_0, f, g, $\tilde g$ and h of (1.1) with $T_0 = \infty$*

satisfy the following conditions:

$$
\begin{aligned}
&u_0 \in \mathcal{D}_{q,p}, \quad e^{\gamma t} f \in L_p\left((0,\infty), L_q(\Omega)\right)^n, \\
(3.3) \qquad & e^{\gamma t} g \in L_p\left((0,\infty), W_q^1(\Omega)\right)^n, \quad e^{\gamma t}\tilde{g} \in \dot{W}_p^1\left((0,\infty), L_q(\Omega)\right)^n, \\
& e^{\gamma t} h \in \dot{H}_{q,p}^{1,1/2}\left(\Omega \times (0,\infty)\right)^n, \quad \nu \cdot h|_{\partial\Omega} = 0
\end{aligned}
$$

for some $\gamma \in [0, \gamma_0]$. *Then* (1.1) *with* $T_0 = \infty$ *admits a unique solution* $(u, \pi) \in W_{q,p}^{2,1}(\Omega \times (0,\infty))^n \times L_p\left((0,\infty), W_q^1(\Omega)\right)$ *which satisfies the estimates:*

$$
\begin{aligned}
&\|e^{\gamma t} u\|_{W_{q,p}^{2,1}(\Omega \times (0,\infty))} + \|e^{\gamma t}\pi\|_{L_p((0,\infty), W_q^1(\Omega))} \\
&\leq C\{\|u_0\|_{\mathcal{D}_{q,p}} + \|e^{\gamma t} f\|_{L_p((0,\infty), L_q(\Omega))} + \|e^{\gamma t} h\|_{\dot{H}_{q,p}^{1,1/2}(\Omega \times (0,\infty))} \\
&+ \|e^{\gamma t} g\|_{L_p((0,\infty), W_q^1(\Omega))} + \|e^{\gamma t}\tilde{g}\|_{W_p^1((0,\infty), L_q(\Omega))}\}
\end{aligned}
$$

where $C > 0$ *is independent of* u, π, f, g, $\tilde{g}$ *and* h.

Outlined proof of theorems 3.1 and 3.2. We consider the maximal L_p-L_q regularity of solutions to (1.1) in the whole space and half-space by using $\mathcal{R}$-boundedness and the operator-valued Fourier multiplier theorem due to Denk, Hieber and Prüss [1] and Weis [7]. The second step is to show maximal L_p-L_q regularity in the bounded domain case. To do this, we reduce (1.1) to the model problem in the half-space and the whole space by using the localization technique. Q.E.D.

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