

On the Cauchy problem for Schrödinger–improved Boussinesq equations

Tohru Ozawa¹ and Kimitoshi Tsutaya¹

*Dedicated to Professor Kôji Kubota on the occasion of his
seventieth birthday*

Abstract.

The Cauchy problem for a coupled system of Schrödinger and improved Boussinesq equations is studied. Local well-posedness is proved in $L^2(\mathbf{R}^n)$ for $n \leq 3$. Global well-posedness is proved in the energy space for $n \leq 2$. Under smallness assumption on the Cauchy data, the local result in L^2 is proved for $n = 4$.

§1. Introduction

We study the Cauchy problem for a coupled system of Schrödinger and improved Boussinesq (S-iB) equations

$$(1) \quad i\partial_t u + \frac{1}{2}\Delta u = vu,$$
$$(2) \quad \partial_t^2 v - \Delta v - \Delta \partial_t^2 v = \Delta |u|^2,$$

where u and v are complex and real-valued functions of $(t, x) \in \mathbf{R} \times \mathbf{R}^n$, respectively, and Δ is the Laplacian in \mathbf{R}^n . The system (S-iB) is regarded as a substitute for the Zakharov (Z) system

$$i\partial_t u + \frac{1}{2}\Delta u = vu,$$
$$\partial_t^2 v - \Delta v = \Delta |u|^2.$$

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See [8] for discussions on this subject. We refer the reader to [1, 4, 6, 10, 11, 12, 16] for the Cauchy problem for (Z). Especially, in [11,12] global well-posedness below energy space is proved for (Z) in one space dimension. For topics related to (S-iB), see [3, 9, 13, 14] and references therein. Function spaces for (S-iB) as well as (Z) are naturally built in the form of product spaces with components $(u, v, \partial_t v)$. In [15] existence and uniqueness of solutions to the Cauchy problem for (S-iB) with $n = 1$ is proved in $H^2 \oplus H^2 \oplus H^2$ for local solutions and in $H^2 \oplus H^2 \oplus (H^2 \cap \dot{H}^{-1})$ for global solutions, where $H^s = (1 - \Delta)^{-s/2} L^2$ is the Sobolev space of order s and $\dot{H}^s = (-\Delta)^{-s/2} L^2$ is the homogeneous Sobolev space of order s .

The purpose in this paper is to prove local well-posedness of the Cauchy problem for (S-iB) in $L^2 \oplus L^2 \oplus L^2$ for $n \leq 3$ and global well-posedness in $H^1 \oplus L^2 \oplus (L^2 \cap \dot{H}^{-1})$ for $n \leq 2$. Moreover, the local result in $L^2 \oplus L^2 \oplus L^2$ is shown below for $n = 4$ under smallness assumption on the Cauchy data.

To state the main result precisely, we introduce basic notation. With Cauchy data $(u_0, v_0, v_1) \in L^2 \oplus L^2 \oplus L^2$ we consider (S-iB) in the form of integral equations

$$(3) \quad u(t) = U(t)u_0 - i \int_0^t U(t-t')(vu)(t')dt',$$

$$(4) \quad v(t) = K(t)v_1 + \dot{K}(t)v_0 + \int_0^t K(t-t')(\omega^2|u|^2)(t')dt',$$

where $U(t) = \exp(i(t/2)\Delta)$, $K(t) = \omega^{-1} \sin t\omega$, $\dot{K}(t) = \cos t\omega$, $\omega = (1 - \Delta)^{-1/2}(-\Delta)^{1/2}$. In connection with Strichartz estimates, we study the integral equations (3) and (4) in the function spaces:

$$X(I) = (L^{8/n}(I; L^4) \cap L^\infty(I; L^2)) \oplus L^\infty(I; L^2) \oplus L^\infty(I; L^2),$$

$$Y(I) = X(I) \cap C(I; L^2 \oplus L^2 \oplus L^2),$$

where $I \subset \mathbf{R}$ is an interval.

Theorem 1. (1) Let $n \leq 3$. For any $(u_0, v_0, v_1) \in L^2 \oplus L^2 \oplus L^2$ there exists $T > 0$ such that (3) and (4) have unique solutions (u, v) such that $(u, v, \partial_t v) \in X(I)$ with $I = [-T, T]$. Moreover, $(u, v, \partial_t v) \in Y(I)$, $\partial_t^2 v \in C(I; L^2)$, and the map $(u_0, v_0, v_1) \mapsto (u, v, \partial_t v)$ is locally Lipschitz from $L^2 \oplus L^2 \oplus L^2$ to $X(I)$.

(2) Let $n = 4$. Then there exists $\varepsilon_0 > 0$ with the following property: For any ε with $0 < \varepsilon \leq \varepsilon_0$ and any $(u_0, v_0, v_1) \in L^2 \oplus L^2 \oplus L^2$ with $\|(u_0, v_0, v_1); L^2 \oplus L^2 \oplus L^2\| \leq \varepsilon$ there exists $T > 0$ such that (3) and

(4) have unique solutions (u, v) such that $(u, v, \partial_t v) \in X(I)$ with $I = [-T, T]$. Moreover, $(u, v, \partial_t v) \in Y(I)$, $\partial_t^2 v \in C(I; L^2)$, and the map $(u_0, v_0, v_1) \mapsto (u, v, \partial_t v)$ is locally Lipschitz from the closed ball of $L^2 \oplus L^2 \oplus L^2$ at the origin with radius ε to $X(I)$.

Remark 1. The local existence time T in Theorem 1 depends only on $\|u_0; L^2\|$, $\|v_0; L^2\|$, $\|v_1; L^2\|$, and n .

To study regularity properties of local solutions given by Theorem 1, we introduce the following function spaces with integer $m \geq 1$:

$$\begin{aligned} X^m(I) &= (L^{8/n}(I; H_4^m) \cap L^\infty(I; H^m)) \oplus L^\infty(I; H^m) \oplus L^\infty(I; H^m), \\ Y^m(I) &= X^m(I) \cap C(I; H^m \oplus H^m \oplus H^m), \end{aligned}$$

where $H_p^m = (1 - \Delta)^{-m/2} L^p$, $H^m = H_2^m$.

Theorem 2. Let $(u_0, v_0, v_1) \in L^2 \oplus L^2 \oplus L^2$. Let (u, v) be solutions with $(u, v, \partial_t v) \in X(I)$ given by Theorem 1.

(1) Let $m \geq 1$ and let $(u_0, v_0, v_1) \in H^m \oplus H^m \oplus H^m$. Then $(u, v, \partial_t v) \in Y^m(I)$.

(2) Let $m \geq 1$ and let $\{(u_0^{(k)}, v_0^{(k)}, v_1^{(k)})\} \subset H^m \oplus H^m \oplus H^m$ satisfy $(u_0^{(k)}, v_0^{(k)}, v_1^{(k)}) \rightarrow (u_0, v_0, v_1)$ in $L^2 \oplus L^2 \oplus L^2$ as $k \rightarrow \infty$. Let $(u^{(k)}, v^{(k)})$ be the corresponding solutions of (3) and (4). Then $(u^{(k)}, v^{(k)}, \partial_t v^{(k)}) \in Y^m(I)$ and $(u^{(k)}, v^{(k)}, \partial_t v^{(k)}) \rightarrow (u, v, \partial_t v)$ in $X(I)$ as $k \rightarrow \infty$.

Remark 2. Theorem 2 ensures that existence time of local solutions in $X(I)$ can be taken independent of order of Sobolev space.

Concerning the global existence of finite energy solutions, we have the following result.

Theorem 3. Let $n \leq 2$. Let $(u_0, v_0, v_1) \in H^1 \oplus L^2 \oplus (L^2 \cap \dot{H}^{-1})$ and let (u, v) be solutions such that $(u, v, \partial_t v) \in Y(I)$ given by Theorem 1. Then the local solutions extend to the whole time interval and satisfy

$$\begin{aligned} u &\in L_{loc}^{8/n}(\mathbf{R}; H_4^1) \cap C(\mathbf{R}; H^1), \\ v &\in C^2(\mathbf{R}; L^2), \\ \partial_t v &\in C(\mathbf{R}; L^2 \cap \dot{H}^{-1}). \end{aligned}$$

Moreover, if $n = 1$,

$$(u, v, \partial_t v) \in L^\infty(\mathbf{R}; H^1 \oplus L^2 \oplus (L^2 \cap \dot{H}^{-1})).$$

Remark 3. The space $H^1 \oplus L^2 \oplus (L^2 \cap \dot{H}^{-1})$ is the natural energy space for (S-iB).

Remark 4. Smallness conditions are not necessary for $n = 2$. This is a significant difference in view of related equations such as the Zakharov and nonlinear Schrödinger equations [1,10,16,17].

We prove Theorem 1 in Section 2. The method of the proof depends on a direct use of the Strichartz estimates for construction of local solutions to (3) and (4). We do not use an equivalent system of equations as in [4,10,15]. We prove Theorem 2 in Section 3, following [10]. We prove Theorem 3 in Section 4. The method of the proof depends on a compactness argument and on a priori estimates.

§2. Proof of Theorem 1

In this section we prove Theorem 1. For simplicity, we consider the Cauchy problem for positive times since the other case is treated similarly.

Let $(u_0, v_0, v_1) \in L^2 \oplus L^2 \oplus L^2$. For $(u, v, \partial_t v) \in X(I)$ with $I = [0, T]$, $T > 0$, we define

$$N(u, v) = (N_1(u, v), N_2(u)),$$

where

$$\begin{aligned} N_1(u, v) &= U(\cdot)u_0 - iG(uv), \\ N_2(u) &= K(\cdot)v_1 + \dot{K}(\cdot)v_0 + H(\omega^2|u|^2), \\ (Gf)(t) &= \int_0^t U(t-t')f(t')dt', \\ (Hf)(t) &= \int_0^t K(t-t')f(t')dt'. \end{aligned}$$

We look for local solutions to (3) and (4) as fixed points of the mapping $N : (u, v) \mapsto N(u, v)$ on a closed ball in $(L^{8/n}(I; L^4) \cap L^\infty(I; L^2)) \oplus L^\infty(I; L^2)$. For that purpose we use the Strichartz estimates of the following form.

Proposition 1. [2,5,7] *Let $n, q_j, r_j, j = 0, 1, 2$, satisfy $0 \leq 2/q_j = n/2 - n/r_j \leq 1$ with the exception $(n, q_j, r_j) = (2, 2, \infty)$. Then the following estimates hold:*

$$\begin{aligned} \|U(\cdot)\phi; L^{q_0}(\mathbf{R}; L^{r_0})\| &\leq C\|\phi; L^2\|, \\ \|Gf; L^{q_1}(I; L^{r_1})\| &\leq C\|f; L^{q'_2}(I; L^{r'_2})\|, \end{aligned}$$

where C is independent of ϕ , f , $I = [0, T]$, and q' is the dual exponent to q defined by $1/q + 1/q' = 1$. Moreover, for any $\phi \in L^2$ and $f \in L^{q'_2}(I; L^{r'_2})$, $U(\cdot)\phi \in C(\mathbf{R}; L^2)$ and $Gf \in C(I; L^2)$.

For $R > 0$ we define the closed ball

$$\begin{aligned} B_T(R) &= \{(u, v) \in (L^{8/n}(I; L^4) \cap L^\infty(I; L^2)) \oplus L^\infty(I; L^2); \\ &\quad \| (u, v) \| \equiv \|u; L_t^{8/n}(L^4)\| \vee \|u; L_t^\infty(L^2)\| \vee \|v; L_t^\infty(L^2)\| \leq R\}, \end{aligned}$$

where $a \vee b = \max(a, b)$.

For any $(u, v) \in B_T(R)$ we estimate $N_1(u, v)$ in $L^{8/n}(I; L^4) \cap L^\infty(I; L^2)$ as

$$\begin{aligned} (5) \quad & \|N_1(u, v); L_t^{8/n}(L^4) \cap L_t^\infty(L^2)\| \\ & \equiv \|N_1(u, v); L_t^{8/n}(L^4)\| \vee \|N_1(u, v); L_t^\infty(L^2)\| \\ & \leq C\|u_0; L^2\| + C\|uv; L_t^{8/(8-n)}(L^{4/3})\| \\ & \leq C\|u_0; L^2\| + CT^{1-n/4}\|u; L_t^{8/n}(L^4)\| \|v; L_t^\infty(L^2)\| \\ & \leq C\|u_0; L^2\| + CT^{1-n/4}R^2. \end{aligned}$$

Similarly, for any $(u, v), (u', v') \in B_T(R)$,

$$\begin{aligned} (6) \quad & \|N_1(u, v) - N_1(u', v'); L_t^{8/n}(L^4) \cap L_t^\infty(L^2)\| \\ & = \|G((u - u')v + u'(v - v'))\|; L_t^{8/n}(L^4) \cap L_t^\infty(L^2)\| \\ & \leq CT^{1-n/4}(\|u - u'; L_t^{8/n}(L^4)\| \|v; L_t^\infty(L^2)\| \\ & \quad + \|u; L_t^{8/n}(L^4)\| \|v - v'; L_t^\infty(L^2)\|) \\ & \leq CT^{1-n/4}R \| (u, v) - (u', v') \|. \end{aligned}$$

We estimate $N_2(u)$ and $N_2(u) - N_2(u')$ as

$$\begin{aligned} (7) \quad & \|N_2(u); L_t^\infty(L^2)\| \\ & \leq T\|v_1; L^2\| + \|v_0; L^2\| + \|\omega|u|^2; L_t^1(L^2)\| \\ & \leq T\|\dot{v}_1; L^2\| + \|v_0; L^2\| + \| |u|^2; L_t^1(L^2) \| \\ & \leq T\|v_1; L^2\| + \|v_0; L^2\| + T^{1-n/4}\|u; L_t^{8/n}(L^4)\|^2 \\ & \leq T\|v_1; L^2\| + \|v_0; L^2\| + T^{1-n/4}R^2, \end{aligned}$$

$$\begin{aligned}
 (8) \quad & \|N_2(u) - N_2(u'); L_t^\infty(L^2)\| \\
 & \leq \|\omega(|u|^2 - |u'|^2); L_t^1(L^2)\| \\
 & \leq \| |u|^2 - |u'|^2; L_t^1(L^2)\| \\
 & \leq T^{1-n/4}(\|u; L_t^{8/n}(L^4)\| + \|u'; L_t^{8/n}(L^4)\|)\|u - u'; L_t^{8/n}(L^4)\| \\
 & \leq CT^{1-n/4}R\|(u, v) - (u', v')\|.
 \end{aligned}$$

Collecting (5)–(8), we obtain

$$\begin{aligned}
 (9) \quad & \| |N(u, v)| \| \leq C(\|u_0; L^2\| + \|v_0; L^2\| + T\|v_1; L^2\|) + CT^{1-n/4}R^2, \\
 (10) \quad & \| |N(u, v) - N(u', v')| \| \leq CT^{1-n/4}R\|(u, v) - (u', v')\|.
 \end{aligned}$$

It follows from (9) and (10) that N leaves $B_T(R)$ invariant and is a contraction provided that

$$\begin{aligned}
 T & \leq 1, \\
 C(\|u_0; L^2\| + \|v_0; L^2\| + \|v_1; L^2\|) & \leq R/2, \\
 CT^{1-n/4}R & \leq 1/2.
 \end{aligned}$$

This implies the existence of local solutions to (3) and (4) in $B_T(R)$ with $T > 0$ sufficiently small for $n \leq 3$ and with $T, R > 0$ sufficiently small for $n = 4$. Uniqueness of solutions in $(L^{8/n}(I; L^4) \cap L^\infty(I; L^2)) \oplus L^\infty(I; L^2)$ follows from similar estimates in the standard argument. Regularity in time of solutions follows from (3), (4), and Proposition 1. Continuous dependence of local solutions on the Cauchy data follows in the same way as above.

§3. Proof of Theorem 2

Local existence of solutions in $X^m([0, \tilde{T}])$ with \tilde{T} depending on $\|u_0; H^m\|, \|v_0; H^m\|, \|v_1; H^m\|$ follows in the same way as in the proof of Theorem 1 with the bilinear estimates

$$\begin{aligned}
 \|uv; H_{4/3}^m\| & \leq C\|u; H_4^m\|\|v; H^m\|, \\
 \|uv; H^m\| & \leq C\|u; H_4^m\|\|v; H_4^m\|.
 \end{aligned}$$

By Proposition 1, we have $(u, v, \partial_t v) \in Y^m([0, \tilde{T}])$. Let \tilde{T}_{\max} be the maximal existence time of solutions (u, v) with values in $H^m \oplus H^m$. In the same way as in the proof of Proposition 3.3 in [10], we see that $T < \tilde{T}_{\max}$.

§4. Proof of Theorem 3

We first prove the following

Proposition 2. *Let $n \leq 2$. Let $(u_0, v_0, v_1) \in H^2 \oplus H^2 \oplus H^2$ and let (u, v) be solutions of (3) and (4) with $(u, v, \partial_t v) \in Y^2(I)$ given by Theorems 1 and 2. Then:*

(1) *The following conservation laws hold for all $t \in I$:*

$$\|u(t); L^2\| = \|u_0; L^2\|, \quad E(t) = E(0),$$

where

$$\begin{aligned} & E(t) \\ &= \frac{1}{2} (\|\nabla u(t); L^2\|^2 + \|v(t); L^2\|^2 + \|\omega^{-1} \partial_t v(t); L^2\|^2) + (v(t), |u(t)|^2) \end{aligned}$$

and (\cdot, \cdot) denotes the scalar product on L^2 .

(2) *There exists a positive constant C depending only on $\|u_0; H^1\|$, $\|v_0; L^2\|$, $\|v_1; L^2 \cap \dot{H}^{-1}\|$ and T , such that*

$$\|u; L^\infty(I; H^1)\| \vee \|v; L^\infty(I; L^2)\| \vee \|\omega^{-1} \partial_t v; L^\infty(I; L^2)\| \leq C.$$

Proof of Proposition 2.

Since the solutions in question are H^2 -solutions, formal proofs of the standard conservation laws are justified as they are. Therefore we omit the details and proceed to the proof of Part (2). The main task here is to obtain a priori estimates of local solutions $(u, v, \partial_t v) \in H^1 \oplus L^2 \oplus (L^2 \cap \dot{H}^{-1})$ from two conserved quantities in Part (1).

By the Gagliardo–Nirenberg inequality and the conservation law of the L^2 -norm, we have

$$\begin{aligned} (11) \quad |(v, |u|^2)| &\leq \|v; L^2\| \|u; L^4\|^2 \\ &\leq \frac{1}{4} \|v; L^2\|^2 + \|u; L^4\|^4 \\ &\leq \frac{1}{4} \|v; L^2\|^2 + C \|u; L^2\|^{4-n} \|\nabla u; L^2\|^n \\ &= \frac{1}{4} \|v; L^2\|^2 + C \|u_0; L^2\|^{4-n} \|\nabla u; L^2\|^n. \end{aligned}$$

If $n = 1$, then the RHS of the last inequality in (11) is bounded by

$$\frac{1}{4} \|v; L^2\|^2 + \frac{1}{4} \|\nabla u; L^2\|^2 + C \|u_0; L^2\|^6$$

and therefore

$$\begin{aligned} & \frac{1}{4}(\|\nabla u; L^2\|^2 + \|v; L^2\|^2 + \|\omega^{-1}\partial_t v; L^2\|^2) \\ & \leq E(0) + C\|u_0; L^2\|^6 \\ & \leq C(\|\nabla u_0; L^2\|^2 + \|v_0; L^2\|^2 + \|\omega^{-1}v_1; L^2\|^2 + \|u_0; L^2\|^6), \end{aligned}$$

which provides the required a priori estimate.

If $n = 2$, then instead of (11) we estimate

$$\begin{aligned} |(v, |u|^2)| & \leq C\|v; L^2\|\|u; L^2\|\|\nabla u; L^2\| \\ & \leq \frac{1}{4}\|\nabla u; L^2\|^2 + C\|u_0; L^2\|^2\|v; L^2\|^2 \end{aligned}$$

and therefore

$$\begin{aligned} (12) \quad & \frac{1}{4}(\|\nabla u; L^2\|^2 + \|v; L^2\|^2 + \|\omega^{-1}\partial_t v; L^2\|^2) \\ & \leq E(0) + C\|u_0; L^2\|^2\|v; L^2\|^2. \end{aligned}$$

By the trivial estimate

$$\|v; L^2\| \leq \|v_0; L^2\| + \int_0^t \|\partial_s v; L^2\| ds$$

and the Schwarz inequality in time, we have

$$(13) \quad \|v; L^2\|^2 \leq C\|v_0; L^2\|^2 + CT \int_0^t \|\partial_s v; L^2\|^2 ds.$$

By (12) and (13), we have

$$\begin{aligned} (14) \quad \|\partial_t v; L^2\|^2 & \leq \|\partial_t v; L^2\|^2 + \|\partial_t v; \dot{H}^{-1}\|^2 \\ & = \|\omega^{-1}\partial_t v; L^2\|^2 \\ & \leq CE(0) + C\|u_0; L^2\|^2\|v_0; L^2\|^2 \\ & \quad + CT\|u_0; L^2\|^2 \int_0^t \|\partial_s v; L^2\|^2 ds. \end{aligned}$$

By the Gronwall lemma applied to (14), we have

$$(15) \quad \|\partial_t v; L^2\|^2 \leq C(E(0) + \|u_0; L^2\|^2\|v_0; L^2\|^2) \exp(C\|u_0; L^2\|^2 T^2).$$

Collecting (12), (13) and (15), we obtain

$$\begin{aligned} & \|\nabla u; L^2\|^2 + \|v; L^2\|^2 + \|\omega^{-1}\partial_t v; L^2\|^2 \\ \leq & CE(0) + C\|u_0; L^2\|^2 \\ & \cdot (\|v_0; L^2\|^2 + CT^2(E(0) + C\|u_0; L^2\|^2\|v_0; L^2\|^2) \exp(C\|u_0; L^2\|^2 T^2)) \\ \leq & C(\|\nabla u_0; L^2\|^2 + \|u_0; L^2\|^2\|v_0; L^2\|^2 + \|v_0; L^2\|^2 + \|v_1; L^2\| \bigcap \dot{H}^{-1}\|^2) \\ & \cdot (1 + \|u_0; L^2\|^2 T^2 \exp(C\|u_0; L^2\|^2 T^2)), \end{aligned}$$

which provides the required a priori estimate for $n = 2$.

Proof of Theorem 3.

Let $(u_0, v_0, v_1) \in H^1 \oplus L^2 \oplus (L^2 \cap \dot{H}^{-1})$ and let (u, v) be solutions such that $(u, v, \partial_t v) \in Y(I)$ given by Theorem 1. Let $\{(u_0^{(k)}, v_0^{(k)}, v_1^{(k)})\}$ be a sequence in $H^2 \oplus H^2 \oplus H^2$ such that $(u_0^{(k)}, v_0^{(k)}, v_1^{(k)}) \rightarrow (u_0, v_0, v_1)$ in $H^1 \oplus L^2 \oplus (L^2 \cap \dot{H}^{-1})$ as $k \rightarrow \infty$ and let $(u^{(k)}, v^{(k)})$ be the corresponding solutions such that $(u^{(k)}, v^{(k)}, \partial_t v^{(k)}) \in Y^2(I)$. By Proposition 2, we see that there exists a constant C depending only on $\|u_0; H^1\|, \|v_0; L^2\| \|v_1; L^2 \cap \dot{H}^{-1}\|$, and T , such that

$$\begin{aligned} \sup_{k \geq 1} (\|u^{(k)}; L^\infty(I; H^1)\| \vee \|v^{(k)}; L^\infty(I; L^2)\| \\ \vee \|\omega^{-1}\partial_t v^{(k)}; L^\infty(I; L^2)\|) \leq C. \end{aligned}$$

By a compactness argument and uniqueness, we conclude that $u \in L^\infty(I; H^1)$ and $\omega^{-1}\partial_t v \in L^\infty(I; L^2)$. Moreover, we have the conservation law of the L^2 norm $\|u(t); L^2\| = \|u_0; L^2\|$ for all $t \in I$ and energy inequality $E(t) \leq E(0)$ for almost all $t \in I$. By the continuity of u with values in L^2 , the boundedness of u with values in H^1 implies that u is weakly continuous with values in H^1 and that energy inequality holds for all $t \in I$. By the time-reversibility and uniqueness, the energy inequality turns out to be equality for all $t \in I$.

By the equation (2) we have

$$\omega^{-1}\partial_t v = \omega^{-1}v_1 - \omega \int_0^t (v + |u|^2)(t') dt'$$

so that the continuity of $\omega^{-1}\partial_t v$ in L^2 follows from

$$\begin{aligned} & \| \omega^{-1}\partial_t v(t) - \omega^{-1}\partial_t v(t_0); L^2 \| \\ &= \left\| \omega \int_{t_0}^t (v + |u|^2)(t') dt'; L^2 \right\| \\ &\leq \left| \int_{t_0}^t \|v(t'); L^2\| dt' \right| + \left| \int_{t_0}^t \|u(t'); L^4\|^2 dt' \right| \rightarrow 0 \end{aligned}$$

as $t \rightarrow t_0$ since $(u, v, \partial_t v) \in X(I)$. Then the energy conservation law implies the norm continuity of $\|\nabla u; L^2\|$ and therefore $\nabla u \in C(I; L^2)$.

Now by taking the limit a priori estimates from Proposition 2, we have a priori estimates for $(u, v, \partial_t v)$ in $H^1 \oplus L^2 \oplus (L^2 \cap \dot{H}^{-1})$. This ensures the global existence with values in the last space by the standard argument.

References

- [1] H. Added and S. Added, Existence globale de solutions fortes pour les équations de la turbulence de Langmuir en dimension 2, *C. R. Acad. Sci. Paris*, **299** (1984), 551–554.
- [2] T. Cazenave, *Semilinear Schrödinger Equations*, Courant Lect. Notes Math., **10**, Amer. Math. Soc., 2003.
- [3] C. Guowang, Cauchy problem for multidimensional couple system of nonlinear Schrödinger equation and generalized IMBq equation, *Comment. Math. Univ. Carolinae*, **39** (1998), 15–38.
- [4] M. Colin and T. Colin, On a quasilinear Zakharov system describing laser-plasma interactions, *Differential Integral Equations*, **17** (2004), 297–330.
- [5] J. Ginibre, Introduction aux équations de Schrödinger non linéaires, Cours de DEA 1994-1995, Paris onze édition L **161**, Université de Paris-Sud, Orsay, 1998.
- [6] J. Ginibre, Y. Tsutsumi and G. Velo, On the Cauchy problem for the Zakharov system, *J. Funct. Anal.*, **151** (1997), 384–463.
- [7] M. Keel and T. Tao, Endpoint Strichartz estimates, *Amer. J. Math.*, **120** (1998), 955–980.
- [8] V. G. Makhankov, Dynamics of classical solitons (in non-integrable systems), *Phys. Reports*, **35** (1978), 1–128.
- [9] F. Linares and A. Navas, On Schrödinger–Boussinesq equations, *Adv. Differential Equations*, **9** (2004), 159–176.
- [10] T. Ozawa and Y. Tsutsumi, Existence and smoothing effect of solutions for Zakharov equations, *Publ. RIMS, Kyoto Univ.*, **28** (1991), 329–361.
- [11] H. Pecher, Global well-posedness below energy space for the 1-dimensional Zakharov system, *Internat. Math. Res. Notices*, **19** (2001), 1027–1056.

- [12] H. Pecher, Global solutions with infinite energy for the one-dimensional Zakharov system, *Electron. J. Differential Equations*, **41** (2005), 18 pp. (electronic).
- [13] S. Wang and G. Chen, The Cauchy problem for the generalized IMBq equation in $W^{s,p}(\mathbf{R}^n)$, *J. Math. Anal. Appl.*, **266** (2002), 38–54.
- [14] S. Wang and G. Chen, Small amplitude solutions of the generalized IMBq equation, *J. Math. Anal. Appl.*, **274** (2002), 846–866.
- [15] S. Wang and G. Chen, Cauchy problem for the nonlinear Schrödinger–IMBq equations, preprint.
- [16] C. Sulem and P.-L. Sulem, *The Nonlinear Schrödinger Equation: Self-Focusing and Wave Collapse*, *Appl. Math. Sci.*, **139**, Springer, 1999.
- [17] M. I. Weinstein, Nonlinear Schrödinger equations and sharp interpolation estimates, *Comm. Math. Phys.*, **87** (1983), 567–576.

Tohru Ozawa
Department of Mathematics
Hokkaido University
Sapporo 060-0810
Japan

Kimitoshi Tsutaya
Department of Mathematics
Hokkaido University
Sapporo 060-0810
Japan