Advanced Studies in Pure Mathematics 47-1, 2007
Asymptotic Analysis and Singularities
pp. 253–272

Wave maps in gravitational theory

Makoto Narita

Abstract.

We survey results on wave maps in gravitational theory. In the first part, we review global properties of cosmological spacetimes with two spacelike commutable Killing vector fields in the Einstein-Maxwell-dilaton-axion system, which is arising in the low energy effective superstring theory. It is shown that the dynamical evolution parts of this Einstein-matter equations become a system of wave map equations and global existence and inextendibility for the system are discussed. Asymptotic behavior of constant mean curvature foliations is also analyze. In the second part, we show a global existence theorem of a wave map on black hole background by using Choquet-Bruhat's arguments. Moreover, asymptotic decay property is shown by conformal transformation.

§1. Introduction

General relativity is the theory for describing the dynamics of spacetime geometry and the fundamental equations of the theory are the Einstein(-matter) equations. The Cauchy problem provides a way of achieving to understand what kind of solutions these equations possess. It deals with the most general solutions to the equations being considered and characterized them by their initial data on some appropriate initial hypersurfaces. Fortunately, the local Cauchy problem for the Einstein equations is well understood as far as smooth solutions are concerned. Once the local problem has been solved, one can proceed to consider about the global one. However, the global Cauchy problem is still open. The question whether the Cauchy problem for the Einstein equations can be solved globally is closely related to the important physical question of the existence and nature of spacetime singularities. In particular, the *strong cosmic censorship* (SCC) *conjecture* [37] is most naturally expressed in terms of the global Cauchy problem. This conjecture is one

Received October 31, 2005.
Revised March 21, 2006.

of the most important and unsolved questions in classical gravitational theory. In addition, it is important to study the conjecture in scope of superstring/M-theory, because it is natural to expect that global structure of spacetimes must be ultimately found on the theory which is the most promising candidate for the unified theory.

Roughly speaking, the SCC states that there is no naked singularity. If not so, local observers may see singularities. This means violation of predictability, because we cannot put appropriate boundary conditions on the singularities. Since predictability is a fundamental requirement of classical physics, it seems reasonable to require it to be valid in the whole spacetime. This implies that physical spacetimes should be globally hyperbolic in Leray's sense. Motivated by these considerations, Penrose proposed the SCC conjecture in physical words and Klainerman formulated it mathematically as follows:

Conjecture 1 (Klainerman [26]). *Generic Cauchy data sets have maximal Cauchy developments which are locally inextendible as Lorentzian manifolds.*

We need two steps to prove the validity of the conjecture: (1) show global existence theorems for solutions to the Einstein(-matter) equations in an appropriate time coordinate, (2) analyze asymptotic behavior of the solutions and show inextendiblity of spacetime manifold. Thus, an important aspect of this SCC conjecture is relation to the global Cauchy problem for the Einstein(-matter) equations.

Now, wave maps play important roles to tackle global problems in general relativity. For example, dynamical evolution equations of the (four dimensional) vacuum Einstein equations for spacelike $U(1)$-symmetric cosmological spacetimes can be written as a wave map [9]. Also, it is known that nonlinearity of wave maps is similar with one of the Einstein equations, that is "null form" [11, 25]. This nonlinearity is a key to prove global existence theorems for the Einstein equations and wave maps in small initial data in four dimensions. Thus, we expects that fundamental parts of the Einstein equations are wave maps or have the same structure of wave maps.

In this article, we will survey some results on wave maps relating to gravitational theory. In Section 2, we show global properties of Gowdy symmetric spacetimes in low energy effective superstring theory. Unfortunately, the problem of proving global existence theorems for the full Einstein-matter equations is beyond the reach of the mathematics presently available [39]. To make some progress, it is necessary to concentrate on simplified models. The typical simplification is to look at solutions with various types of symmetry. Gowdy symmetric spacetimes are the most simplest inhomogeneous cosmological models because the spacetimes have two commutable spacelike Killing vector and then dynamical evolution equations become $1+1$-dimensional nonlinear wave equations which are equivalent to a wave map.

From the viewpoint of the SCC, it is important to study whether black hole spacetimes are stationary limit after dynamical evolution (i.e. naked singularities never appear by generic gravitational collapse) and/or the spacetimes are stable or not. Christodoulou has shown that generic spherically symmetric gravitational collapse leads to the Schwarzschild spacetime as the final state [12]. For $D \geq 3$, it has been shown that $D+1$-dimensional static and spherical symmetric black hole spacetimes are stable against linear perturbation [3, 21]. These results support the validity of the SCC conjecture, that is, the stability of the black holes. Next, nonlinear perturbation should be considered. However, we have no mathematical tool to analyze full nonlinear perturbation for curved spacetimes at the present time[1] and the only one result of stability against nonlinear perturbation is for the Minkowski spacetime [13, 5]. Then, we will consider nonlinear scalar fields as *test fields* on curved spacetimes. Recently, some global results for nonlinear wave equations on the (four dimensional) Schwarzschild background are proven with power nonlinearity [2, 16]. Our choice for the test fields are wave maps, which play an important role in gravitational theory as already mentioned. In Section 3, we will show a global existence theorem of a wave map on black hole spacetimes in any dimensions by using Choquet-Bruhat's arguments, which has been used to prove the theorem in some Sobolev classes in four-dimensions [4, 6, 7, 8]. Furthermore, asymptotic decay property is shown by conformal transformation [11, 10]. These results suggest nonlinear stability of black hole spacetimes and support the validity of the SCC.

In the remaining of this section, let us summarize wave maps. Let (V,g) and (M,h) be $D+1$-dimensional Lorentzian and d-dimensional Riemannian manifolds. A mapping $u:(V,g)\to(M,h)$ is called a *wave map* if it satisfies the following PDE in local coordinates on V and M:

$$g^{\alpha\beta}\nabla_\alpha\partial_\beta u^a = g^{\alpha\beta}\left(\partial^2_{\alpha\beta}u^a - \Gamma^\lambda_{\alpha\beta}\partial_\lambda u^a + \Gamma^a_{bc}\partial_\alpha u^b\partial_\beta u^c\right) = 0,$$

where $\Gamma^\lambda_{\alpha\beta}$ and Γ^a_{bc} are the Christoffel symbols of the base V and target M, respectively. Here, Greek indices runs from 0 to D and Roman

[1]Recently, nonlinear orbital stability of five-dimensional static and spherically symmetric black hole (Schwarzschild-Tangherlini) spacetimes has been shown [14].

indices run from 1 to d. This wave map system is a critical point of the following action:

$$S_{\mathrm{WM}} = \int d^{D+1}x\sqrt{-g}g^{\alpha\beta}h_{ab}\partial_\alpha u^a \partial_\beta u^b.$$

Then, the wave map equations read as a semilinear quasidiagonal system of second order partial differential equations for d scalar functions u^a. The principal part is the wave operator for the Lorentzian metric g and the nonlinear terms are quadratic forms in the derivative of u with coefficient $\Gamma(u)$, which satisfy the null condition [11, 25].

Let us consider the Cauchy problem, that is, given values on a given spacelike hypersurface S_0 in V, $u(x)|_{S_0} = \phi(x) \in M$ and $\partial_t u(x)|_{S_0} = \psi(x) \in T_{\phi(x)}M$, the construction of a wave map. Here, $x \in S_0$ and ϕ is a map from the initial hypersurface S_0 into the target M.

Now we put an assumption for target manifolds.

Definition 1. *The* (M, h) *is said to be* regularly embedded *in the Euclidean manifold* (R^N, Q) *if it is defined by P smooth scalar equations* $F^{(P)} = 0$ *on* R^N, *of rank P on M, and h is the pullback of the metric Q under this embedding. We denote by $\nu^{(P)}$ the normal to M in* R^N *defined by the gradient of $F^{(P)}$. We set $\nu_{(I)} = m_{IJ}\nu^{(J)}$, where m_{IJ} is the inverse matrix with elements $m^{IJ} = Q^{AB}\nu_A^{(I)}\nu_B^{(J)}$.*

Under this assumption, the equations for U read as a semilinear quasidiagonal system of second order partial differential equations for N scalar functions on V (see (1)).

To consider the Cauchy problem for the wave map on generic curved manifolds, we need to suppose conditions on the base manifold.

Definition 2. *The* (V, g) *is said to be* regularly hyperbolic *if:* (a) *It is globally hyperbolic. Then V is of the type $S \times \mathrm{R}$, with S a D-dimensional oriented smooth spacelike manifold, the past of any compact subset of V intersects any $S_t \equiv S \times \{t\}$ along a compact set.* (b) *The metric g, assumed here for simplicity to be smooth, can be written*

$$g = -N^2dt^2 + g_{ij}\theta^i\theta^j, \qquad \theta^i \equiv dx^i + \beta^i dt,$$

with $0 < B_1 \leq N \leq B_2$ on V, where B_1 and B_2 are positive and continuous functions of t. (c) *The metrics $g_t = g_{ij}dx^i dx^j$, induced by g on S_t, are uniformly equivalent to a given smooth Riemannian metric e on S, that is, there exist positive and continuous functions of t, A_1 and A_2 which, for any vector field ξ on S and t, satisfy $A_1 e(\xi, \xi) \leq g_t(\xi, \xi) \leq A_2 e(\xi, \xi)$ on S.*

We have the standard local existence and uniqueness theorem for quasi-linear wave equations (see theorem 6.4.11 of [20] or theorem 4.1 of [43]). Then, the following theorem is obtained:

Theorem 1. [4] *Let (V, g) be a smooth regularly hyperbolic manifold. Let (M, h) be a smooth complete Riemannian manifold regularly embedded by i in a Euclidean space (R^N, Q). Then $u : (V, g) \to (M, h)$ is a wave map if and only if the mapping $U = i \circ u : V \to \mathrm{R}^N$ satisfies the system of d scalar semilinear equations on (V, g), such that*

$$g^{\alpha\beta}\left(\nabla_\beta \partial_\alpha U^A + \partial_\alpha U^B \partial_\beta U^C Q^{AD} \nu_{(I)D} \partial_B \nu_C^{(I)}\right) = 0, \tag{1}$$

and the initial data for U are that $U^A|_{S_0} =: \Phi^A$ take their values in M and $\partial_t U^A|_{S_0} =: \Psi^A$ take their values in the tangent space to M.

Let $\phi : S \to M$ and $\psi : S \to TM$ be Cauchy data for u. Suppose that the $2N$ functions $\Phi = i(\phi)$ and $\Psi = i'(\psi)$ are respectively in $W^{s,2}$ and $W^{s-1,2}$, with $s > \frac{D}{2} + 1$, then there exists a number $T > 0$ and a wave map $u : (S \times [0, T]) \to (M, h)$, $u \in E_s(T)$ taking the given data. The interval T of existence for any given s is equal to the interval corresponding to s_0, smallest integer greater than $\frac{D}{2} + 1$. The solution is unique and depends continuously on the data. In particular, $u \in C^\infty(S \times [0, T])$ if $\phi, \psi \in C_0^\infty$. □

§2. Wave maps in cosmological spacetimes

In this section, we would like to consider global properties of Gowdy symmetric spacetimes in the Einstein-Maxwell-Dilaton-Axion (EMDA) system, which is arising in heterotic superstring theory.

2.1. Einstein-Maxwell-Dilaton-Axion system

The action of the EMDA theory is

$$\begin{aligned} S &= \int d^4x \sqrt{|g|}\left[R - e^{-2\phi}F^2 - 2(\nabla\phi)^2 - \frac{1}{3}e^{-4\phi}H^2\right] \\ &= \int d^4x \sqrt{|g|}\left[R - e^{-2\phi}F^2 - 2(\nabla\phi)^2 - \frac{1}{2}e^{4\phi}(\nabla\kappa)^2 - \kappa F\tilde{F}\right], \end{aligned} \tag{2}$$

where ϕ is the dilaton field, $F_{\mu\nu} := 2\nabla_{[\mu}A_{\nu]}$, $\tilde{F}_{\mu\nu} := \frac{1}{2}\sqrt{-g}\epsilon_{\mu\nu\delta\rho}F^{\delta\rho}$, $H_{\alpha\beta\gamma} := \nabla_\alpha B_{\beta\gamma} - A_\alpha F_{\beta\gamma} + \text{cyclic} = \frac{1}{2}\epsilon_{\alpha\beta\gamma\delta}e^{4a\phi}\nabla^\delta\kappa$ is the three-index antisymmetric tensor field dual to the axion field κ [44]. Greek indices run from 0 to 3. Varying the action (2) with respect to the functions, we have the following field equations.

$$\nabla_\mu(e^{-2\phi}F^{\mu\nu} + \kappa\tilde{F}^{\mu\nu}) = 0, \tag{3}$$

$$\nabla_\mu(\tilde{F}^{\mu\nu}) = 0, \tag{4}$$

$$\nabla_\mu \nabla^\mu \phi - \frac{1}{2} e^{4\phi} \nabla_\mu \kappa \nabla^\mu \kappa + \frac{1}{2} e^{-2\phi} F^2 = 0, \tag{5}$$

$$\nabla_\mu \nabla^\mu \kappa + 4 \nabla_\mu \phi \nabla^\mu \kappa - e^{-4\phi} F_{\mu\nu} \tilde{F}^{\mu\nu} = 0, \tag{6}$$

$$\begin{aligned} R_{\mu\nu} &= 2\nabla_\mu \phi \nabla_\nu \phi + \frac{1}{2} e^{4\phi} \nabla_\mu \kappa \nabla_\nu \kappa \\ &\quad + e^{-2\phi}(2F_{\mu\rho} F^\rho_\nu - \frac{1}{2} g_{\mu\nu} F^2). \end{aligned} \tag{7}$$

2.2. G_1-symmetric spacetimes

As mentioned in Introduction, wave maps appear in gravitational theory by spacetime symmetric reduction. The method in [23] is used to get the reduced equations[2]. Suppose that the metric is

$$ds^2 = e^{-2P} \gamma_{mn} dx^m dx^n + e^{2P} (\xi_m dx^m + dx^3)^2,$$

where Latin indices run from 0 to 2. ∂_3 is a spacelike Killing vector, then P, ξ_m and γ_{mn} depend only on x^m. The same symmetry is assumed for matter fields. In three dimensions, vector fields can be parametrized by scalar potentials, Φ and Ψ, via

$$F_{n3} = \partial_n A_3 =: \frac{1}{\sqrt{2}} \partial_n \Phi, \qquad e^{-2\phi} F^{mn} + \kappa \tilde{F}^{mn} =: \frac{e^{2P}}{\sqrt{2}\sqrt{-\gamma}} \epsilon^{mnp} \partial_p \Psi.$$

The three dimensional Kaluza-Klein field ξ_m is dualized by

$$E^p := \frac{e^{4P}}{\sqrt{-\gamma}} \epsilon^{pmn} \partial_m \xi_n.$$

One can solve the constraint equation which is $(m3)$-component of Einstein equations (7) by introducing the twist potential Q,

$$\partial_n Q := E_n + \Phi \partial_n \Psi - \Psi \partial_n \Phi,$$

so we obtain the effective action for new variables as follows:

$$S_{\mathrm{WM}} = \int d^3x \sqrt{-\gamma} \gamma^{mn} (-\mathcal{R}_{mn} + h_{AB} \nabla_m U^A \nabla_n U^B),$$

[2]The similar reduction was obtained in the case of stationary spacetimes [17]

where

$$
\begin{aligned}
(8) \qquad h_{AB}dU^A dU^B &= 2dP^2 + \frac{1}{2}e^{-4P}(dQ + \Psi d\Phi - \Phi d\Psi)^2 \\
&\quad +2d\phi^2 + \frac{1}{2}e^{4\phi}d\kappa^2 \\
&\quad +e^{-2P}\left[e^{-2\phi}d\Phi^2 + e^{2\phi}(d\Psi - \kappa d\Phi)^2\right],
\end{aligned}
$$

where $\mathcal{R}_{mn}$ is the Ricci tensor for γ. Thus, equations for G_1-symmetric spacetimes in the EMDA system are given by $2+1$-dimensional Einstein equations coupled with a wave map. Note that this target space of the wave map is an Einstein space. Indeed, one can calculate the Ricci and the Einstein tensors as follows:

$$G_{AB} = 6h_{AB}, \qquad R_{AB} = -3h_{AB},$$

where the scalar curvature $R = -18$. In particular, the target space becomes two dimensional hyperbolic space in vacuum case.

2.3. Gowdy symmetric spacetimes

The Gowdy symmetric spacetimes are spatially compact, globally hyperbolic with commutable spacelike Killing vector fields on the Cauchy surfaces, which twist constant vanish. Here, we will assume that spatial topology is T^3. If one consider Gowdy symmetric spacetimes, the metric is as follows:

$$ds^2 = e^{-2P}\gamma_{mn}dx^m dx^n + e^{2P}(Ady + dx)^2,$$

and

$$\gamma_{mn}dx^m dx^n = e^{2\eta}(-dt^2 + d\theta^2) + R^2 dy^2,$$

where $x^0 = t$, $x^1 = \theta$, $x^2 = y$ and $x^3 = x$. Furthermore, $\partial_2 = \partial_y$ is another spacelike Killing vector. Thus, all functions depend on $t \in (0, \infty)$ and $\theta \in [0, 2\pi)$ only. From three dimensional Einstein equations for γ, we have

$$\mathcal{G}_{tt} - \mathcal{G}_{\theta\theta} = \partial_t^2 R - \partial_\theta^2 R = 0,$$

where $\mathcal{G}_{mn}$ is the Einstein tensor for γ. Then, we can take the areal time coordinate $R = t$, where R means the area density 2-tori spanned by two Killing vector fields. In this coordinate, equations for the function η are given as follows.

$$(9) \qquad \mathcal{G}_{tt} = \frac{\partial_t \eta}{t} = \frac{1}{2}h_{AB}\left(\partial_t U^A \partial_t U^B + \partial_\theta U^A \partial_\theta U^B\right) = \mathcal{G}_{\theta\theta},$$

$$\mathcal{G}_{t\theta} = \frac{\partial_\theta \eta'}{t} = h_{AB}\partial_t U^A \partial_\theta U^B, \tag{10}$$

and

$$\frac{e^{2\eta}}{t^2}\mathcal{G}_{yy} = (\partial_t^2 - \partial_\theta^2)\eta = \frac{1}{2}h_{AB}\left(\partial_t U^A \partial_t U^B - \partial_\theta U^A \partial_\theta U^B\right). \tag{11}$$

Note that the function η is decoupled with other functions. Thus, an action for evolution equations of the EMDA system with Gowdy symmetry is obtained:

$$S_{\text{Gowdy}} = \int d^3x\ {}^0g^{mn} h_{AB}\partial_m U^A \partial_n U^B, \tag{12}$$

where ${}^0g = -dt^2 + d\theta^2 + t^2 d\psi^2$, $0 \leq t$, $0 \leq \theta, \psi \leq 2\pi$ and U is independent on ψ. This is a wave map $U : (R^{2+1}, {}^0g) \to (N^6, h)$, where h is given by (8).

Now, we give explicit form of the wave map equations. The evolution equations for gravitational sector (P, Q) are given by

$$\begin{aligned} \partial_t(t\partial_t P) - \partial_\theta(t\partial_\theta P) &= -\frac{t}{2}e^{-4P}E_n E^n \\ &\quad -\frac{t}{2}e^{-2P}\left(e^{-2\phi}(\nabla\Phi)^2 + e^{2\phi}(\nabla\Psi - \kappa\nabla\Phi)^2\right), \end{aligned} \tag{13}$$

and

$$\partial_t\left(te^{-4P}E_t\right) - \partial_\theta\left(te^{-4P}E_\theta\right) = 0. \tag{14}$$

The evolution equations for the Maxwell fields (Φ, Ψ) become

$$\begin{aligned} &\partial_t[te^{-2(P+\phi)}\partial_t\Phi] - \partial_\theta[te^{-2(P+\phi)}\partial_\theta\Phi] \\ &= te^{-4P}H_m(E^m - e^{2(P+\phi)}\partial^m\kappa). \end{aligned} \tag{15}$$

and

$$\partial_t\left(te^{-2(P-\phi)}H_t\right) - \partial_\theta\left(te^{-2(P-\phi)}H_\theta\right) = -te^{-4P}\partial_m\Phi E^m. \tag{16}$$

The wave equations for dilaton and axion fields (ϕ, κ) are

$$\begin{aligned} \partial_t(t\partial_t\phi) - \partial_\theta(t\partial_\theta\phi) &= \frac{t}{2}e^{4\phi}(\nabla\kappa)^2 \\ &\quad -\frac{t}{2}e^{-2P}(e^{-2\phi}(\nabla\Phi)^2 - e^{2\phi}(\nabla\Psi - \kappa\nabla\Phi)^2), \end{aligned} \tag{17}$$

and

$$\partial_t(te^{4\phi}\partial_t\kappa) - \partial_\theta(te^{4\phi}\partial_\theta\kappa) = -2te^{-2(P-\phi)}\partial^m\Phi H_m. \tag{18}$$

Here $H_m := \partial_m \Psi - \kappa \partial_m \Phi$. For the above system, we have the following global existence theorem:

Theorem 2 ([31]). *Consider the wave map equations (13)-(18). Given smooth initial data at $t = t_0 \in (0, \infty)$, there is a unique solution for any $t \in (0, \infty)$.* $\square$

The main method of the proof of Theorem 2 is *light cone estimate* which has been used in vacuum Gowdy case [30]. Presently, this method is applied to the cases which are the Einstein-Vlasov [1], the Einstein-stringy matter [32, 34], higher-dimensional [33] systems. Note that the above results are restricted answers for an open problem proposed in [45], that is Open Problem 6.(b).

From the viewpoint of the SCC, we should analyze inextendibility of spacetimes obtained in Theorem 2. Fortunately, a very useful tool-kit for this was given in [15], which states that geometric quantity R, which comes from spacetime symmetries, can extend continuously to spacetime boundaries if the spacetimes admit symmetries. In our case, since $R = t$ goes to infinity into the future direction, then future extension of the original spacetimes is impossible. Thus,

Theorem 3. *The globally hyperbolic maximal development of data for the Gowdy symmetric EMDA system cannot be extended in a C^2-category into the expanding direction.* $\square$

The same conclusion holds for vacuum, Einstein-Vlasov, Einstein-stringy matter and higher-dimensional cases. From theorems 2 and 3, the validity of the SCC is expected in our spacetimes.

To analyze global properties of Gowdy symmetric spacetimes, the areal time coordinate is useful because the evolution equations (wave maps) decoupled with the constraint equations ((9)-(11) in our case). In relativity, since *time* is a just parameter, there are many choice for it. Another good time coordinate is constant mean curvature time [38]. In this coordinate, time is measured by mean curvature of Cauchy surfaces S, where the values of the mean curvature on S is constant. Now, we have the following theorem:

Theorem 4. *Let (M, g) be a globally hyperbolic maximal development of data for the vacuum Gowdy system. Then there exists a unique constant mean curvature foliation of (M, g) which covers (M, g). Moreover, the mean curvature* $\mathrm{tr}k$ *of Cauchy surfaces approaches* (a) $-\infty$ *as* $t \to 0$, *and* (b) 0 *as* $t \to \infty$.

Proof: Results for global foliation and (a) have been shown in [22] and the same ones are also true in the cases of non-vacuum (Vlasov and stringy matter) and higher-dimension. In that paper, it was conjectured that problem (b) would be true. Then, we concentrate problem (b).

The energy decay estimates given in [41] will be used[3]. To adjust my notation with that paper, we put metric function as follows:

$$2\eta \Rightarrow P + \frac{\lambda}{2} + \frac{1}{2}\ln t, \qquad 2P \Rightarrow P + \ln t, \qquad A \Rightarrow Q.$$

Under this transformation, the mean curvature of the hypersurfaces of constant areal time is

$$\mathrm{tr}k = -\frac{1}{4}e^{-\lambda/4}t^{1/4}\left[\partial_t\lambda + \frac{3}{t}\right].$$

From Theorem 1.7 in [41], we have

$$|\partial_t\lambda| \leq K,$$

and also, for large enough t,

$$\lambda \simeq c_\lambda t \pm K\ln t,$$

where K and c_λ are positive constants. Then,

$$0 > \lim_{t\to\infty}\mathrm{tr}k \geq \lim_{t\to\infty}\left(-\frac{1}{4}e^{-c_\lambda t}t^{(1\pm K)/4}\left[K + \frac{3}{t}\right]\right) = 0.$$

Thus, the conjecture in [22] (that is, problem (b)) has been shown. □

We did not mention asymptotic behavior of Gowdy symmetric spacetimes near singularity ($t = 0$). Some results concerning this are given in references [24, 35, 33, 34]. Complete arguments for inextendibility beyond the singularity has been shown only in the vacuum case [40].

§3. Wave maps on black holes

Let us consider $D+1$-dimensional static and spherically symmetric black hole spacetimes $V \sim \mathrm{R}_t \times (0,\infty)_r \times S^n$ (thus, $n = D-1$), as the base manifold for wave maps, endowed with the Lorentzian metric

$$g = -f_n(r)dt^2 + \frac{dr^2}{f_n(r)} + r^2 d\sigma_n^2, \tag{19}$$

[3]To get the estimates, wave map structure of Gowdy equations was analyzed fully. In particular, symmetry of the target is important.

where $f_n \in C^\infty((0,\infty))$ and $d\sigma_n^2$ is the n-dimensional standard spherical symmetric metric. We assume the existence of a value r_g of r, $0 < r_g < \infty$, the only possible zeros of f, such that

$$f_n(r_g) = 0, \qquad \partial_r f_n(r_g) = 2\kappa_g \neq 0,$$
$$f_n(r) > 0 \ \text{ for } \ r \in (r_g, \infty), \qquad f_n(r) < 0 \ \text{ for } \ r \in (0, r_g).$$

One example satisfying the above conditions is a higher-dimensional black hole spacetime which is a solution of the vacuum Einstein equations, $G_{\alpha\beta} = 0$, where $G_{\alpha\beta}$ is the Einstein tensor:

$$f_n(r) = 1 - \frac{C}{r^{n-1}}, \tag{20}$$

where C is a integration constant.

We will use the Regge-Wheeler coordinate:

$$\frac{dr}{d\rho} = f_n(r).$$

Then we have the following metric

$$g = f_n(r)\left(-dt^2 + d\rho^2\right) + r^2 d\sigma_n^2, \tag{21}$$

with $-\infty < \rho < +\infty$, $r = r(\rho)$ is a C^∞ function of ρ, increasing from r_g to $+\infty$.

Suppose that a spherically symmetric wave map is one that depends only on t and r, i.e., $u = u(t, r)$. If we assume static spherically symmetric spacetimes, (21) with (20), as the base manifold for the wave map and , the wave map equation (1) becomes as follows:

$$\begin{aligned} - & \quad \partial_t^2 U^A + \partial_\rho^2 U^A + l_\rho \partial_\rho U^A \\ + & \quad Q^{AD} \nu_{(P)D} \partial_B \nu_C^{(P)} \left(-\partial_t U^B \partial_t U^C + \partial_\rho U^B \partial_\rho U^C\right) = 0, \end{aligned} \tag{22}$$

where $l_\rho := n r^{-1} f_n(r)$ is a C^∞ function of ρ. Thus, our problem becomes to analyze this $1+1$-dimensional semilinear wave equations with smooth coefficients.

3.1. Global existence theorem of wave maps on black holes in higher dimension

Now, we will show a global existence theorem for (22):

Theorem 5. *Let (M, h) be a Riemannian manifold C^∞ regularly embedded into a Euclidean space. If $\Phi \in W^{2,2}$ and $\Psi \in W^{1,2}$, then there exists a global spherically symmetric wave map $u \in C^k([0,\infty), W^{2-k,2})$, 0*

$\leq k \leq 2$, from the exterior of a higher dimensional spherically symmetric black hole into (M, h) taking these Cauchy data. The solution is unique. The solution in a compact set depends only on the data in the intersection of the initial line with the past of this set. In particular, if the Cauchy data Φ and Ψ belong to $C_0^\infty(\mathrm{R}_\rho)$, then there exists a unique global solution which belongs to $C^\infty(\mathrm{R}_t \times \mathrm{R}_\rho)$.

As arguments for the first and second energies (Definition 4) are the same as Choquet-Bruhat's [4, 6, 7, 8], we omit them. In her papers, global existence in some Sobolev classes has been already shown. To show existence of global solutions to the wave map (22) in smooth category, we need to estimate higher order energy.

Definition 3. *We define the $k+1$-th energy tensor $T^{(k)}_{\alpha\beta}$ of $U : (R^2, \eta) \to (R^d, Q)$ and the $k+1$-th energy-momentum vector $\mathcal{P}^{(k)\alpha}$ on R^2 given by*

$$T^{(k)}_{\alpha\beta} := \delta_{AB}\left(\partial_\alpha U^{(k),A}\partial_\beta U^{(k),B} - \frac{1}{2}\eta_{\alpha\beta}\partial_\lambda U^{(k),A}\partial^\lambda U^{(k),B}\right),$$

and

$$\mathcal{P}^{(k)\alpha} := T^{(k)\alpha\beta}X_\beta,$$

where $U^{(k),A} := \partial_\rho^k U^A$.

Definition 4. *Suppose that X is the timelike Killing vector field on (R^2, η) with components: $X_0 = 1$ and $X_1 = 0$. We define the $k+1$-th energy density $\mathcal{P}^{(k)0} := \mathcal{P}^{(k)\alpha}X_\alpha$ and the $k+1$-th energy $E^{(k)}_t(U)$ of U at time t given by*

$$\mathcal{P}^{(k)0} := \frac{1}{2}\delta_{AB}\left(\partial_t U^{(k),A}\partial_t U^{(k),B} + \partial_\rho U^{(k),A}\partial_\rho U^{(k),B}\right),$$

and

$$E^{(k)}_t(U) := \frac{1}{2}\int_{-\infty}^{+\infty} d\rho\,\mathcal{P}^{(k)0},$$

respectively.

By deriving the wave map equation (22) with respect to ρ, we have the $k+1$-th order evolution equation as follows:

$$\begin{aligned}\partial_t^2 U^{(k),A} - \partial_\rho^2 U^{(k),A} &= \left(l_\rho \partial_\rho U^A\right)^{(k)} \\ &+\delta^{AD}\nu_{(P)D}\partial_B \nu_C^{(P)}\left(-\partial_t U^B \partial_t U^C + \partial_\rho U^B \partial_\rho U^C\right)^{(k)} \\ &+\delta^{AD}\sum_{i=1}^{k} {}_k\mathrm{C}_i \left(\nu_{(P)D}\partial_B \nu_C^{(P)}\right)^{(i)} \\ &\times \left(-\partial_t U^B \partial_t U^C + \partial_\rho U^B \partial_\rho U^C\right)^{(k-i)},\end{aligned} \tag{23}$$

where $F(t,\rho)^{(k)} := \partial_\rho^k F(t,\rho)$ for $k \geq 0$. Note that the first order equations are the original ones.

Now, we will calculate the $k+1$-th energy estimate. The divergence of the energy momentum vector is given by

$$\partial_\alpha \mathcal{P}^{(k)\alpha} = \delta_{AE}\partial_t U^{(k),E}\eta^{\alpha\beta}\partial_\alpha\partial_\beta U^{(k),A} = -\delta_{AE}\partial_t U^{(k),E}\tilde{f}^A,$$

where

$$\begin{aligned}\tilde{f}^A &= \left(l_\rho \partial_\rho U^A\right)^{(k)} \\ &+\delta^{AD}\nu_{(P)D}\partial_B \nu_C^{(P)}\left(-\partial_t U^B \partial_t U^C + \partial_\rho U^B \partial_\rho U^C\right)^{(k)} \\ &+\delta^{AD}\sum_{i=1}^{k} {}_k\mathrm{C}_i \left(\nu_{(P)D}\partial_B \nu_C^{(P)}\right)^{(i)} \times \\ &\times \left(-\partial_t U^B \partial_t U^C + \partial_\rho U^B \partial_\rho U^C\right)^{(k-i)}.\end{aligned}$$

From the definition of regular embedded manifolds, we have

$$\partial_\rho^k \partial_t F(U) = \partial_t U^{(k),A}\nu_{(P)A} + \sum_{i=1}^{k} {}_k\mathrm{C}_i \partial_t U^{(k-i),A}\nu_{(P)A}^{(i)} = 0 \tag{24}$$

Combining (23)-(24), we have

$$\partial_\alpha \mathcal{P}^{(k)\alpha} = \mathrm{I} + \mathrm{II} + \mathrm{III}, \tag{25}$$

where

$$\mathrm{I} = -\delta_{AE}\partial_t U^{(k),E}\left(l_\rho \partial_\rho U^A\right)^{(k)}, \tag{26}$$

$$\begin{aligned}\mathrm{II} &= -\delta_{AE}\partial_t U^{(k),E}\delta^{AD}\sum_{i=1}^{k} {}_k\mathrm{C}_i \left(\nu_{(P)D}\partial_B \nu_C^{(P)}\right)^{(i)} \times \\ &\times \left(-\partial_t U^B \partial_t U^C + \partial_\rho U^B \partial_\rho U^C\right)^{(k-i)},\end{aligned} \tag{27}$$

and

$$\begin{aligned} \mathrm{III} &= \left\{ \sum_{i=1}^{k} {}_k\mathrm{C}_i \partial_t U^{(k-i),A} \nu^{(i)}_{(P)A} \right\} \times \\ &\quad \times \partial_B \nu_C^{(P)} \left(-\partial_t U^B \partial_t U^C + \partial_\rho U^B \partial_\rho U^C \right)^{(k)}. \end{aligned} \tag{28}$$

Integrating (25), we have

$$\begin{aligned} \int_0^t d\tau \int_{-\infty}^{\infty} d\rho \partial_\alpha \mathcal{P}^{(k)\alpha} &= E_t^{(k)}(U) - E_0^{(k)}(U) \\ &= \int_0^t d\tau \int_{-\infty}^{\infty} d\rho \mathrm{I} + \mathrm{II} + \mathrm{III}, \end{aligned}$$

From the term I (26), we have

$$\begin{aligned} \int_0^t d\tau \int_{-\infty}^{\infty} d\rho \mathrm{I} &\leq C \int_0^t d\tau E_\tau^{(k)}(U) \\ &\quad + C \int_0^t d\tau E_\tau^{(k)}(U)^{1/2} E_\tau^{(k-1)}(U)^{1/2} \\ &\quad + \cdots + C \int_0^t d\tau E_\tau^{(k)}(U)^{1/2} E_\tau^{(0)}(U)^{1/2}. \end{aligned}$$

When one estimates $k+1$-th energy, terms of $(k-1)$-order and less than this order are not important, because these terms can be controlled by i-th energy for $i = k, \cdots, 0$ by using the generalized Hölder and the Sobolev inequalities. For example, define $\mathrm{IV} := (\partial U)^{k+2} \partial U^{(k)}$, then

$$\begin{aligned} \int_{-\infty}^{\infty} d\rho \mathrm{IV} &\leq \|(\partial U)^{k+2}\|_{L^2(\mathrm{R})} \|\partial U^{(k)}\|_{L^2(\mathrm{R})} \\ &\leq E^{(k)1/2} \|(\partial U)^{2(k+2)}\|_{L^1(\mathrm{R})}^{1/2} \\ &\leq E^{(k)1/2} \left(\|(\partial U)^{2(k+1)}\|_{L^1(\mathrm{R})} \|(\partial U)^2\|_{C_b^0(\mathrm{R})} \right)^{1/2} \\ &\leq E^{(k)1/2} \left(\|(\partial U)^2\|_{L^1(\mathrm{R})} \|(\partial U)^2\|_{C_b^0(\mathrm{R})}^{k+1} \right)^{1/2} \\ &\leq E^{(k)1/2} \times \\ &\quad \times \left[E \left\{ \|(\partial U)^2\|_{L^1(\mathrm{R})} + \|\partial \left((\partial U)^2 \right) \|_{L^1(\mathrm{R})} \right\}^{k+1} \right]^{1/2} \\ &\leq E^{(k)1/2} E^{1/2} \left[E + E^{1/2} E^{(1)1/2} \right]^{(k+1)/2} \\ &\leq E^{(k)1/2} E^{(k+3)/4} \left[E^{1/2} + E^{(1)1/2} \right]^{(k+1)/2}. \end{aligned}$$

At the fourth inequality, the Sobolev inequality was used k times. Therefore, we will estimate only terms which include $(k-1)$-order.

In the cases of $i = 1$ and $i = k$ in the terms II (27) and III (28), $k+1$-th and k-th terms appear. They can be estimated by the same ways:

$$\int_{-\infty}^{\infty} d\rho \mathrm{V} \leq E^{(k)1/2} E^{(k-1)1/2} E^{1/2} \left(E^{1/2} + E^{(1)1/2} \right),$$

where $\mathrm{V} := \partial U^{(k)} (\partial U)^2 \partial U^{(k-1)}$, and

$$\int_{-\infty}^{\infty} d\rho \mathrm{VI} \leq E^{(k-1)} \left(E^{1/2} E^{(1)1/2} + E^{(1)} + E^{1/2} E^{(2)1/2} \right),$$

where $\mathrm{VI} := (\partial U^{(k-1)})^2 \partial U \partial U^{(1)}$.

In the case of $k = 2$, we need to estimate the L^6 norm of the first terms. as follows;

$$\begin{aligned} \int_{-\infty}^{\infty} d\rho \mathrm{VII} &\leq \|\partial U\|_{L^2(\mathrm{R})} \|(\partial U^{(1)})^3\|_{L^2(\mathrm{R})} \\ &\leq E(U) \|(\partial U^{(1)})^2\|_{L^1(\mathrm{R})}^{1/2} \|(\partial U^{(1)})^2\|_{C_b^0(\mathrm{R})} \\ &\leq E(U) E^{(1)}(U) \left(E^{(1)}(U)^{1/2} + E^{(2)}(U)^{1/2} \right), \end{aligned}$$

where $\mathrm{VII} := \partial U (\partial U^{(1)})^3$.

Thus, the resulting integral inequality is linear with respect to $E^{(k)}(U)$ and then, $k+1$-th energy bounds for any $t \in [0, \infty)$ if energies $E^{(i)}(U)$ for $i = 0, \cdots, k$ bound. Since the boundedness of the first and the second energies has been shown [4, 6, 7, 8], for any k, we have the boundedness of the k-th energy. By the standard Sobolev embedding theorem, the conclusion of Theorem 5 is obtained. $\square$

3.2. Asymptotic behavior of solutions

3.2.1. *Conformal transformation* The Christodoulou's idea [11] is that a global problem is transformed into a local problem by conformal transformations. Originally, this method was introduced by Penrose to analyze global structure of spacetimes [36]. Since we have already global smooth solutions in the original spacetime, by the Kelvin transformation [28], we have a local solution, which corresponds to the original global solution, in the conformal spacetime as follows.

Let $\kappa : \mathrm{R}^{1+1} \ni (T, R) \to (t, \rho) \in \mathrm{R}^{1+1}$ and $\tilde{U} : \mathrm{R}^{1+1} \to \mathrm{R}^p$ be defined by

$$(29)\, x^i = \kappa^i(T, R) = \frac{4y^i}{T^2 - R^2} = \frac{4y^i}{\Omega}, \qquad \tilde{U} := \Omega^{-(D-1)/2} U \circ \kappa,$$

where $x^0 = t$, $x^1 = \rho$, $y^0 = T$ and $y^1 = R$. If the metric in the (t, ρ)-coordinates is the $D+1$-dimensional Schwarzschild one, $g = f_n(r)(-dt^2 + d\rho^2) + r^2 d\sigma_n^2$, then the pull-back metric $\overline{g}$ in the (T, R)-coordinates is given by

$$\overline{g} = \Omega^2 g = f_n(r)(-dT + dR) + \left(\frac{Rr}{\rho}\right)^2 d\sigma_n^2.$$

According to [4, 10], we can see U satisfies the original equations

$$\left(\Box_g - \frac{D-1}{4D} R_g\right) U^A + \lambda_I \left[\left(Q^{AB}\nu_B^{(I)}\right) \circ U\right] = 0,$$

with

$$\lambda_I = m_{IJ} g^{\alpha\beta} \partial_\alpha U^A \partial_\beta U^B \left[\left(\partial_A \nu_B^{(J)}\right) \circ U\right],$$

and

$$F^{(p)}(U) = 0.$$

if and only if

$$\begin{aligned}(30)\left(\Box_{\overline{g}} - \frac{D-1}{4D} R_{\overline{g}}\right) \tilde{U}^A &= -\Omega^{-(3+D)/2} \times \\ &\times \bar{\lambda}_I \left[\left(Q^{AB}\nu_B^{(I)}\right) \circ \left(\Omega^{(D-1)/2}\tilde{U}\right)\right],\end{aligned}$$

with

$$\begin{aligned}(31)\quad \bar{\lambda}_I &= m_{IJ}\Omega^2 \overline{g}^{\alpha\beta} \partial_\alpha \left(\Omega^{(D-1)/2}\tilde{U}^A\right) \partial_\beta \left(\Omega^{(D-1)/2}\tilde{U}^B\right) \times \\ &\times \left[\left(\partial_A \nu_B^{(J)}\right) \circ \left(\Omega^{(D-1)/2}\tilde{U}\right)\right],\end{aligned}$$

and

$$F^{(p)}\left(\Omega^{(D-1)/2}\tilde{U}\right) = 0,$$

where $\Box_{\overline{g}}$ and $\Box_g$ are the d'Alembertian in the metrics $\overline{g}$ and g, respectively.

Now, we should confirm smoothness of coefficients in equation (30). From (31), we find the following,

$$\begin{aligned}(32)\qquad \partial_\alpha \left(\Omega^{(D-1)/2}\tilde{U}^A\right) &= \Omega^{(D-1)/2}\partial_\alpha \tilde{U} \\ &+ \frac{D-1}{2}\Omega^{(D-3)/2}\partial_\alpha \Omega \tilde{U}^A,\end{aligned}$$

where,

$$\bar{g}^{\alpha\beta}\partial_\alpha\Omega\partial_\beta\Omega = -f_n^{-1}\Omega. \tag{33}$$

Combining (31), (32) and (33), we can see that coefficients in the third term of (30) are smooth if $D \geq 3$ is odd. Also, for $-\infty < t, \rho < \infty$, the Ricci scalar $R_{\bar{g}}$ for $\bar{g}$ is smooth and bounded, which is given by

$$R_{\bar{g}} = \frac{n(n+1)}{4}\left(-\frac{\rho}{r} + \frac{1}{f_n}\right)(t^2 - \rho^2).$$

Therefore, the conformal equations (30) becomes a semi-linear, semi-diagonal second order hyperbolic system with smooth coefficients for the map $\tilde{U}$ if $D \geq 3$ is odd.

3.2.2. *Reduction to a local problem* Under the Kelvin transformation (29), the correspondence between global smooth solutions in the original spacetime and local smooth solutions in the conformal spacetime can be shown by the argument in [27].

We have already a global solution in the original spacetime for any $t \in [t_I, +\infty)$. This means that there exists the corresponding local solution in the conformal spacetime for any $T \in (0, T_I]$, where there are correspondence $t_I \Leftrightarrow T_I$ and $+\infty \Leftrightarrow 0$. Therefore, we can apply the standard local arguments for (30), which means that there is a constant $T_0 \in (0, T_I]$ such that $\sup_{T_0 \leq T < T_I} |\tilde{U}(T)| \leq C_0\epsilon$, where a constant $\epsilon > 0$ depends only on size of the initial data and C_0 is a arbitrary constant but fixed [42]. Note that C_0 is independent of T_0.

Proposition 1. *If a solution $\tilde{U} \in C_B^\infty((0, T_I] \times I)$ of the equation (30) exists, then there exist positive constants $\epsilon_0 \geq \epsilon > 0$ and C such that*

$$\sup_{0 < T \leq T_I} |\tilde{U}(T)| \leq C\epsilon,$$

where $|f(s)| := \sup_{y \in I} |f(s, y)|$.

Proof. The argument is similar with [18]. Let C_1 be that $|\tilde{U}(0)| \leq C_1\epsilon$. For $\gamma > C_1$, put $\mathcal{J} = \{T_0 \in (0, T_I)$: equation (30) has a solution $\tilde{U} \in C_B^\infty((T_0, T_I] \times I)$ such that $\sup_{T_0 < T \leq T_I} |\tilde{U}(T)| \leq \gamma\epsilon\}$. It is clear that $\mathcal{J} \neq \emptyset$ and that $\mathcal{J}$ is closed set in (T_0, T_I). As we have already seen, if $s \in (0, T_I)$, we have a unique local solution, which is corresponding to the global solution, such that $\sup_{s \leq s' \leq T_I} |\tilde{U}(s)| \leq C_2\epsilon$, where C_2 is a constant independent of γ. Now, if $C_2 < \gamma$, a standard continuity argument (see Theorem 2.2. in [29], Theorem 4.2. in Chapter 1 in [43] or Theorem 6.4.11. in [20]) implies that $\mathcal{J}$ is open set in $(0, T_I)$. Then, if $\gamma > \max\{C_1, C_2\}$ and $\epsilon_0 \geq \epsilon > 0$, we have $\mathcal{J} = (0, T_I)$, which shows the conclusion of this proposition. $\square$

Now, we know that, if the function $\tilde{U}$ satisfies the equation (30), the function U, given by

$$U = \Omega^{(D-1)/2}\tilde{U}$$

satisfies the original equation (22), and conversely. Thus, we have the following theorem:

Theorem 6. *Given Cauchy data $\Phi, \Psi \in C_0^\infty(\mathrm{R}_\rho)$ which are bounded by $\mathcal{E} > 0$, for $t - |\rho| > 0$, the solution has the following decay property*

$$|u(t,\rho)| \leq C\mathcal{E}(t+\rho)^{-(D-1)/2}(t-|\rho|)^{-(D-1)/2},$$

where $\mathcal{E}$ depends only on initial data and $D \geq 3$ is odd. $\square$

References

[1] H. Andréasson, Global foliations of matter spacetimes with Gowdy symmetry, Comm. Math. Phys. **206** (1999), 337–365.

[2] D. Catania and V. Georgiev, Blow up for the semilinear wave equation in Schwarzschild metric, math-ph/0407001.

[3] S. Chandrasekhar, The mathematical theory of black holes, International Series of Monographs on Physics, **69**, Oxford Univ. Press, New York, 1983.

[4] Y. Choquet-Bruhat, Global existence theorems for hyperbolic harmonic maps, Ann. Inst. Henri Poincaré, **46** (1987), 97–111.

[5] Y. Choquet-Bruhat, Global existence theorems for Einstein equations in high dimensions, In: Gravity, Particles and Space-Time, (eds. P. Pronin and G. Sardanashvily), World Sci. Publishing, River Edge, NJ, 1996, pp. 19–28.

[6] Y. Choquet-Bruhat, Wave maps in general relativity, In: On Einstein path, (ed. A. Harvey), Springer-Verlag, New York, 1999, pp. 147–169.

[7] Y. Choquet-Bruhat, Global wave maps on curved space times, In: Mathematical and quantum aspects of relativity and cosmology, (eds. S. Cotsakis and G. W. Gibbons), Lecture Notes in Phys., **537**, Springer-Verlag, Berlin, 2000, pp. 1–29.

[8] Y. Choquet-Bruhat, Global wave maps on black holes, In: Proceedings of Jean Leray '99 Conference, (ed. M. de Gosson), Kluwer Academic Publishers, Netherlands, 2003, pp. 469–482.

[9] Y. Choquet-Bruhat, Future complete S^1 symmetric Einsteinian spacetimes, the unpolarized case, C. R. Acad. Sci. Paris Ser. I, **337** (2003), 129–136.

[10] Y. Choquet-Bruhat and C. DeWitt-Morette, Analysis, manifolds and physics. Part II. 92 applications, North-Holland Publishing, Amsterdam, 1989.

[11] D. Christodoulou, Global solutions of nonlinear hyperbolic equations for small initial data, Commun. Pure Appl. Math., **39** (1986), 267–282.

[12] D. Christodoulou, On the global initial value problem and the issue of singularities, Class. Quantum Grav., **16** (1999), A23–A35.

[13] D. Christodoulou and S. Klainerman, The global nonlinear stability of the Minkowski space, Princeton Mathematical Series, **41**, Princeton University Press, Princeton, 1993.

[14] M. Dafermos and G. Holzegel, On the nonlinear stability of higher-dimensional triaxial Bianchi IX black holes, gr-qc/0510051.

[15] M. Dafermos and A. D. Rendall, Inextendibility of expanding cosmological models with symmetry, Class. Quantum Grav., **22** (2005), L143–L147.

[16] M. Dafermos and I. Rodnianski, Small-amplitude nonlinear waves on a black hole background, preprint, math.AP/0503024.

[17] D. V. Gal'tsov, Integrable systems in stringy gravity, Phys. Rev. Lett., **74** (1995), 2863–2866.

[18] P. Godin, Global existence of solutions to some exterior radial quasilinear Cauchy–Dirichlet problems, Amer. J. Math., **117** (1995), 1475–1505.

[19] S. W. Hawking and G. F. R. Ellis, The large scale structure of space-time, Cambridge Monographs on Mathematical Physics, Cambridge Univ. Press, London, 1973.

[20] L. Hörmander, Lectures on nonlinear hyperbolic differential equations, Mathématiques & Applications, **26**, Springer-Verlag, Berlin, 1997.

[21] A. Ishibashi and H. Kodama, Stability of higher-dimensional Schwarzschild black holes, Progr. Theoret. Phys., **110** (2003), 901–919.

[22] J. Isenberg and V. Moncrief, The existence of constant mean curvature foliations of Gowdy 3-torus spacetimes, Comm. Math. Phys., **86** (1982), 485–493.

[23] W. Israel and G. A. Wilson, A class of stationary electromagnetic vacuum fields, J. Math. Phys., **13** (1972), 865–867.

[24] S. Kichenassamy and A. D. Rendall, Analytic description of singularities in Gowdy spacetimes, Class. Quantum Grav., **15** (1998), 1339–1355.

[25] S. Klainerman, The null condition and global existence to nonlinear wave equations, Lectures in Appl. Math., **23** (1986), 293–326.

[26] S. Klainerman, Mathematical challenges of general relativity, In: Proceedings of the Ninth Marcel Grossmann Meeting, (eds. V. G. Gurzadyan, R. T. Jantzen and R. Ruffini), World Scientific, 2002, pp. 28–43.

[27] H. Lindblad, A remark on global existence for small initial data of the minimal surface equation in Minkowskian space time, Proc. Amer. Math. Soc., **132** (2004), 1095–1102.

[28] P. D. Lax and R. S. Phillips, Scattering Theory, Revised Edition, Pure Appl. Math., **26**, Academic Press, New York, 1989.

[29] A. Majda, Compressible fluid flow and systems of conservation laws in several space variables, Appl. Math. Sci., **53**, Springer-Verlag, New York, 1984.

[30] V. Moncrief, Global properties of Gowdy spacetimes with $T^3 \times \mathbb{R}$ topology, Ann. Physics, **132** (1981), 87–107.

[31] M. Narita, On the existence of global solutions for T^3-Gowdy spacetimes with stringy matter, Class. Quantum Grav., **19** (2002), 6279–6288.

[32] M. Narita, Global existence problem in T^3-Gowdy symmetric IIB superstring cosmology, Class. Quantum Grav., **20** (2003), 4983–4994.

[33] M. Narita, Global properties of higher-dimensional inhomogeneous spacetimes, Class. Quantum Grav., **21** (2004), 2071–2087.

[34] M. Narita, On initial conditions and global existence for accelerating cosmologies from string theory, Ann. Henri Poincaré, **6** (2005), 821–847.

[35] M. Narita, T. Torii and K. Maeda, Asymptotic singular behaviour of Gowdy spacetimes in string theory, Class. Quantum Grav., **17** (2000), 4597–4613.

[36] R. Penrose, Confomal treatment of infinity, In: Relativity, Groups and Topology, Les Houches, 1963, (eds. C. DeWitt and B. DeWitt), Gordon and Breach, New York, 1964, pp. 563–584.

[37] R. Penrose, Singularities and time asymmetry, In: General Relativity, an Einstein centenarysurvey, (eds. S. W. Hawking and W. Israel), Cambridge Univ. Press, Cambridge, 1979.

[38] A. D. Rendall, Constant mean curvature foliations in cosmological spacetimes, Helv. Phys. Acta, **69** (1996), 490–500.

[39] A. D. Rendall, Theorems on Existence and Global Dynamics for the Einstein Equations, Living Rev. Relativ., **8** (2005), 68pp.

[40] H. Ringström, Strong cosmic censorship in T^3-Gowdy spacetimes, to appear in Ann. of Math.

[41] H. Ringström, On a wave map equation arising in general relativity, Commun. Pure Appl. Math., **57** (2004), 657–703.

[42] R. Racke, Lectures on nonlinear evolution equations: Initial value problems, Aspects Math., **E19**, Friedr. Vieweg & Sohn, Braunschweig, 1992.

[43] C. D. Sogge, Lectures on nonlinear wave equations, Monogr. Anal., **II**, International Press, Boston, 1995.

[44] A. Shapere, S. Trivedi and F. Wilczek, Dual dilaton dyons, Mod. Phys. Lett., **A6** (1991), 2677–2686.

[45] D. Tataru, The wave maps equation, Bull. Amer. Math. Soc. (N.S.), **41** (2004), 185–204.

Center for Relativity and Geometric Physics Studies
Department of Physics
National Central University
Jhongli 320
Taiwan

Present address
Department of Mathematics
National Taiwan University
1, Sec. 4, Roosevelt Rd.
Taipei 106
Taiwan