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Semi-classical analysis of the Hartree equation around and before the caustic

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Abstract.

We consider the asymptotic behavior of the solution to the semi-classical Hartree equation in the limit of short wave length, with an initial data which causes focusing at a point. It is known that there exists a critical index which indicates whether or not the asymptotic behavior reflects the effect of the nonlinearity in the neighborhood of the caustic. Investigating time range where the nonlinear effect appears, we improve previous convergence results. In particular, we show that the solution behaves as a free solution before the caustic in some super critical cases.

§1. Introduction

This paper is devoted to the study of the asymptotic behavior of the solution to the semi-classical Hartree equation

$$(\mathrm{HE}^\varepsilon) \qquad \begin{cases} i\varepsilon\partial_t u^\varepsilon + \frac{1}{2}\varepsilon^2\Delta u^\varepsilon = \lambda\varepsilon^\alpha(|x|^{-\gamma} * |u^\varepsilon|^2)u^\varepsilon, \\ u^\varepsilon_{|t=0}(x) = e^{-i\frac{x^2}{2\varepsilon}} f(x). \end{cases}$$

as positive parameter ε goes to zero. Here u^ε is a complex function defined in $n+1$ dimensional space-time, ∂_t denotes the time derivative, Δ denotes the Laplace operator in $\mathbb{R}^n$, and λ is a real constant. The initial datum f belongs to $\Sigma := H^1 \cap \mathcal{F}(H^1)$, where $\mathcal{F}$ denotes the Fourier transform. α and γ are two positive constants, which characterize the size of nonlinearity and the long distance behavior of interaction, respectively. In particular γ distinguishes between the short range case and the long range case.

In three dimensions, for the case where we convolute with the Newtonian potential $|x|^{-1}$, the Hartree equation is the Schrödinger–Poisson

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system:

$$\text{(1.1)} \qquad \begin{cases} i\varepsilon\partial_t u^\varepsilon + \frac{1}{2}\varepsilon^2\Delta u^\varepsilon = V u^\varepsilon, & (t,x) \in \mathbb{R}^{1+3} \\ \Delta V = |u^\varepsilon|^2, \\ u^\varepsilon_{|t=0} = u^\varepsilon_0. \end{cases}$$

This equation arises typically if we consider the quantum mechanical time evolution of electrons in the mean field approximation of the many body effect, modeled by the Poisson equation. The parameter ε corresponds to the Planck constant and the limit $\varepsilon \to 0$ is known as the semi-classical limit. It is relevant when coupling quantum models to classical models.

Up to a constant, the equation (1.1) is equivalent to the Hartree equation

$$\text{(1.2)} \qquad \begin{cases} i\varepsilon\partial_t u^\varepsilon + \frac{1}{2}\varepsilon^2\Delta u^\varepsilon = (|x|^{-1} * |u^\varepsilon|^2)u^\varepsilon, \\ u^\varepsilon_{|t=0} = u^\varepsilon_0. \end{cases}$$

We restrict our attention to the case that initial data have the form $u^\varepsilon_0 = \varepsilon^{\alpha/2}e^{-x^2/2\varepsilon}f$, where f is independent of ε and $\alpha > 1$. Note that "small data" is equivalent to "small nonlinearity", since, denoting $\varepsilon^{-\alpha/2}u^\varepsilon$ by again u^ε, we see that (1.2) becomes

$$\text{(1.3)} \qquad \begin{cases} i\varepsilon\partial_t u^\varepsilon + \frac{1}{2}\varepsilon^2\Delta u^\varepsilon = \varepsilon^\alpha(|x|^{-1} * |u^\varepsilon|^2)u^\varepsilon, \\ u^\varepsilon_{|t=0} = e^{-i\frac{x^2}{2\varepsilon}}f. \end{cases}$$

Now, we will consider the semi-classical Hartree equation (HE$^\varepsilon$) which has more general nonlinearity.

One seeks in general a solution of the form

$$\text{(1.4)} \qquad u^\varepsilon(t,x) \underset{\varepsilon\to 0}{\sim} U\left(t, x, \frac{\phi(t,x)}{\varepsilon}\right),$$

for some profile U independent of ε. The phase ϕ solves an eikonal equation. Since the initial datum of u^ε has quadratic oscillation, ϕ has a singular point, where the asymptotic expansion of geometrical optics ceases to be valid ([C1]). This phenomenon is well understood for linear equations. However, for a nonlinear equation, few results are available. In general, it is known that there exist two distinct notions of critical index depending on the equations, the nonlinearity, the amplitude of the initial datum and the geometry of propagation. One critical index tells us whether or not the nonlinear term is negligible outside the caustic, and another one tells the same argument near the caustic. In the case of Hartree equation, the first distinction is whether α is greater than 1 or not, and the second one is whether α is greater than γ or not. Such distinctions were proved by Joly, Métivier and Rauch ([JMR4]) for some nonlinear wave equations, and by Carles ([C1, C2]) for some nonlinear Schrödinger equations. In particular, Carles and Lannes present a general formulation, and apply it to Hartree equations and Klein-Gordon equations ([CL]).

The aim of this paper is to improve the results of [CL]. We treat only the case $\alpha > 1$ (" linear propagation ", see [C1, CL]). Roughly speaking, in [CL] they introduce the scaling

$$u^\varepsilon(t,x) = \varepsilon^{-n/2}\psi^\varepsilon\left(\frac{t-1}{\varepsilon}, \frac{x}{\varepsilon}\right), \tag{1.5}$$

and consider the equation for ψ^ε:

$$\begin{cases} i\partial_t\psi^\varepsilon + \dfrac{1}{2}\Delta\psi^\varepsilon = \lambda\varepsilon^{\alpha-\gamma}(|x|^{-\gamma} * |\psi^\varepsilon|^2)\psi^\varepsilon, \\ \psi^\varepsilon_{|t=-1/\varepsilon}(x) = \varepsilon^{n/2}e^{-i\varepsilon x^2/2}f(\varepsilon x). \end{cases} \tag{1.6}$$

Note that the equation of ψ^ε has no parameter in its linear part, and the size of the L^2 norm of its initial data is independent of ε. Therefore, it seems to be natural that if $\alpha > \gamma$, the nonlinear term is negligible, and that if $\alpha = \gamma$, it is not negligible, which follows from the results on (usual) Hartree equations. Moreover, since $(t-1)/\varepsilon$ goes to $-\infty$ (if $t < 1$) and ∞ (if $t > 1$) as $\varepsilon \to 0$, the nonlinear effect near the caustic (it causes the change of asymptotic behavior) is described by the scattering theory in the case $\alpha = \gamma$. However, this argument is not suitable for the case $\alpha < \gamma$ since the right hand side of the first line of the equation (1.6) diverges to ∞ as $\varepsilon \to 0$. Obviously, strong nonlinear effect should happen.

Our strategy is based on a converse idea, that is, we shall adapt the methods of usual Hartree equation to the original equation (HE^ε). It enables us to prove the convergence of the solution u^ε to a free solution in a stronger topology than that in [CL] (see Theorem 3.5 and Remark 3.6). Moreover, we can prove that the solution behaves as a free solution also in some supercritical case (the case $\alpha < \gamma$: Theorem 4.1). As above, in supercritical case we encounter very strong nonlinear effect. Therefore, the description of caustic crossing and the asymptotic behavior beyond the caustic are difficult (Remark 4.4).

This paper is organized as follows. In section 2, we collect a number of definitions and preliminary estimates. We shall then study the existence theorem and the asymptotic behavior of solutions in sections 3 and 4. We treat the case $\alpha > \gamma$ ("linear caustic") in section 3, and the case $\alpha \leqslant \gamma$ ("nonlinear caustic" and "supercritical caustic") in section 4 (for the terminology, see [C1, CL]).

§2. Preliminaries

2.1. Scaled Strichartz estimate and time decay estimate

In this section we summarize some elementary facts on semi-classical Schrödinger equation, which will be used in the following sections. Let us first consider the free semi-classical Schrödinger equation

$$(i\varepsilon\partial_t + (1/2)\varepsilon^2\Delta)u = 0. \tag{2.1}$$

This equation is solved by the use of unitary group $U^\varepsilon(t) = \exp(i(\varepsilon t/2)\Delta)$. That group can be written as

$$U^\varepsilon(t) = M^\varepsilon(t)D(\varepsilon t)\mathcal{F}M^\varepsilon(t), \tag{2.2}$$

where

$$M^\varepsilon(t) = \exp(ix^2/2\varepsilon t), \tag{2.3}$$

$$D(t)\phi(x) = (it)^{-n/2}\phi(x/t). \tag{2.4}$$

Obviously, it holds that

$$U^\varepsilon(t) = U_0(\varepsilon t), \tag{2.5}$$

where $U_0(t) = \exp(it\Delta/2)$ is a usual Schrödinger group which solves

$$(i\partial_t + (1/2)\Delta)u = 0.$$

Therefore, one can easily observe that U^ε inherits many properties of U_0. We make an essential use of the following scaled Strichartz inequalities. It is a scaled version of well known estimates on U_0. Before stating the details, we make several definitions. For real number r, we use the following notation:

$$\delta(r) := n\left(\frac{1}{2} - \frac{1}{r}\right),$$

where n denotes space dimensions. A pair of real numbers (q, r) is said to be *admissible* if $2 \leqslant r \leqslant 2n(n-2)$ ($2 \leqslant r \leqslant \infty$ if $n = 1$, $2 \leqslant r < \infty$ if $n = 2$) and

$$\frac{2}{q} = \delta(r).$$

Note that $(\infty, 2)$ is admissible for any n.

Proposition 2.1 (scaled Strichartz inequalities).

(1) *For any admissible pair (q,r), there exists C_r such that*

$$\varepsilon^{\frac{1}{q}}\|U^\varepsilon(t)u\|_{L^q(\mathbb{R};L^r)} \leqslant C_r\|u\|_{L^2}.$$

(2) *For any admissible pairs (q_1,r_1) and (q_2,r_2), and any interval I, there exists C_{r_1,r_2} such that*

$$\varepsilon^{\frac{1}{q_1}}\left\|\int_{I\cap\{s\leqslant t\}} U^\varepsilon(t-s)F(s)\,ds\right\|_{L^{q_1}(I;L^{r_1})} \leqslant C_{r_1,r_2}\varepsilon^{-\frac{1}{q_2}}\|F\|_{L^{q_2'}(I;L^{r_2'})}.$$

The constants above are independent of ε and I.

From (2.2), the following formula are obtained by elementary computations

$$\begin{aligned} U^\varepsilon(t-1)\frac{x}{\varepsilon}U^\varepsilon(-t+1) &= \frac{x}{\varepsilon}+i(t-1)\nabla \\ &= M^\varepsilon(t-1)i(t-1)\nabla M^\varepsilon(-t+1). \end{aligned} \tag{2.6}$$

We denote this by $J^\varepsilon(t)$. Namely,

$$J^\varepsilon(t) := \frac{x}{\varepsilon}+i(t-1)\nabla. \tag{2.7}$$

$J^\varepsilon(t)$ is the Galilean operator which is adapted to our scaled problem, and has the following properties:

- commutation property

$$\left[J^\varepsilon(t), i\varepsilon\partial_t+\frac{1}{2}\varepsilon^2\Delta\right]=0.$$

- the modified Sobolev inequality

$$\|u\|_{L^r} \leqslant C|t-1|^{-\delta(r)}\|u\|_{L^2}^{1-\delta(r)}\|J^\varepsilon u\|_{L^2}^{\delta(r)}$$

  for r which satisfies $0\leqslant\delta(r)<1$.

Combining the modified Sobolev inequality and the Gagliardo-Nirenberg inequality, we obtain the following time decay estimate.

Proposition 2.2 (time decay estimate). *Let r satisfy $0\leqslant\delta(r)<1$. Then*

$$\|u\|_{L^r} \leqslant C(\varepsilon+|t-1|)^{-\delta(r)}\|u\|_{L^2}^{1-\delta(r)}\left(\|\varepsilon\nabla u\|_{L^2}+\|J^\varepsilon u\|_{L^2}\right)^{\delta(r)}$$

for all $t\in\mathbb{R}$.

2.2. Estimates on the Hartree type nonlinearity

The following estimates are useful for the Hartree type nonlinearity.

Proposition 2.3. *Let r, s_i satisfy $0 \leqslant \delta(r), \delta(s_i) < 1$ $(i = 1,2,3)$, $0 < \gamma < \min(4, n)$. Then*

$$\|(|x|^{-\gamma} * (v_1 v_2)) v_3\|_{L^{r'}} \leqslant \|v_1\|_{L^{s_1}} \|v_2\|_{L^{s_2}} \|v_3\|_{L^{s_3}},$$

where

$$0 < \delta(r) + \delta(s_3) = \gamma - \delta(s_1) - \delta(s_2) < \gamma.$$

Proof. By the Hölder and the Hardy-Littlewood-Sobolev inequalities, we obtain

$$\begin{aligned}\|(|x|^{-\gamma} * (v_1 v_2)) v_3\|_{L^{r'}} &\leqslant \||x|^{-\gamma} * (v_1 v_2)\|_{L^m} \|v_3\|_{L^{s_3}} \\ &\leqslant \|v_1\|_{L^{s_1}} \|v_2\|_{L^{s_2}} \|v_3\|_{L^{s_3}},\end{aligned}$$

provided

$$\begin{aligned}\delta(r) + \delta(s_3) \equiv n\left(\frac{1}{r'} - \frac{1}{s_3}\right) &= \frac{n}{m} \\ &= n\left(\frac{\gamma}{n} + \left(\frac{1}{s_1} + \frac{1}{s_2}\right) - 1\right) \equiv \gamma - \delta(s_1) - \delta(s_2)\end{aligned}$$

and $0 < 1/m < \gamma/n$. Q.E.D.

If $\delta(r) + \delta(s_3) = 0$, the above proposition fails. Therefore, we use the following one in place of it.

Proposition 2.4. *Let $0 < \gamma < 2$ and $\eta > 0$ be sufficiently small such that $l_\pm$ satisfy $0 < \delta(l_\pm) = (\gamma \pm \eta)/2 < 1$. Then we have*

$$\|(|x|^{-\gamma} * (v_1 v_2))\|_{L^\infty} \leqslant C\sqrt{\|v_1\|_{L^{l_+}} \|v_1\|_{L^{l_-}} \|v_2\|_{L^{l_+}} \|v_2\|_{L^{l_-}}}.$$

Proof. We estimate by the Hölder inequality

$$\begin{aligned}\left||x|^{-\gamma} * (v_1 v_2)\right| &\leqslant \left|\{|x|^{-\gamma}\chi(|x| \leqslant a)\} * (v_1 v_2)\right| \\ &\quad + \left|\{|x|^{-\gamma}\chi(|x| \geqslant a)\} * (v_1 v_2)\right| \\ &\leqslant C(a^{\eta}\|v_1\|_{L^{l_+}} \|v_2\|_{L^{l_+}} + a^{-\eta}\|v_1\|_{L^{l_-}} \|v_2\|_{L^{l_-}}).\end{aligned}$$

By minimizing the right-hand side with respect to a, we have

$$\cdots \leqslant C\sqrt{\|v_1\|_{L^{l_+}} \|v_1\|_{L^{l_-}} \|v_2\|_{L^{l_+}} \|v_2\|_{L^{l_-}}}.$$

Q.E.D.

2.3. Conservation laws and a priori estimates

Next we see basic conservation results. Multiplying (HE^ε) by $\bar{u}^\varepsilon$, integrating over $\mathbb{R}^n$, and taking imaginary part, we obtain conservation of charge

$$\partial_t \|u^\varepsilon(t)\|_{L^2}^2 = 0. \tag{2.8}$$

Next, multiplying the equation by $\overline{\partial_t u^\varepsilon}$, integrating over $\mathbb{R}^n$, and taking real part, we obtain conservation of energy

$$\partial_t E(u^\varepsilon(t)) = 0, \tag{2.9}$$

where the energy E is given by

$$E(u^\varepsilon) = \|\varepsilon \nabla u^\varepsilon\|_{L^2}^2 + G(u^\varepsilon) \tag{2.10}$$

with

$$G(u^\varepsilon) = \lambda \varepsilon^\alpha \int_{\mathbb{R}^n} (|x|^{-\gamma} * |u^\varepsilon|^2)|u^\varepsilon|^2 dx. \tag{2.11}$$

From above conservation laws, we deduce a priori estimates. From (2.8),

$$\sup_{t\in\mathbb{R}} \|u^\varepsilon(t)\|_{L^2} = \|u^\varepsilon(0)\| = \|f\|_{L^2} \tag{2.12}$$

Note that it is independent of ε. Secondly, we obtain

$$\begin{aligned} E(u^\varepsilon(t)) = E(u^\varepsilon(0)) &= \|\varepsilon \nabla e^{-i\frac{x^2}{2\varepsilon}} f\|^2 + \lambda \varepsilon^\alpha \int_{\mathbb{R}^n} (|x|^{-\gamma} * |f|^2)|f|^2 dx \\ &\leqslant C\|xf\|_{L^2}^2 + C\varepsilon^2 \|\nabla f\|_{L^2}^2 + C\lambda\varepsilon^\alpha \|f\|_{L^r}^4 \end{aligned}$$

by (2.9) and Proposition 2.3 with $\delta(r) = \gamma/4$. If $\lambda > 0$, we deduce that

$$\sup_{\varepsilon<1, t\in\mathbb{R}} \|\varepsilon \nabla u^\varepsilon(t)\|_{L^2} \leqslant C(\lambda, \|f\|_\Sigma). \tag{2.13}$$

2.4. Function spaces

At the end of this section, we introduce several function spaces. For an admissible pair (q, r) and an interval $I \subset \mathbb{R}$, we define

$$\Sigma = \{\phi \in H^1 : \|\phi\|_\Sigma \equiv \|\phi\|_{H^1} + \|x\phi\|_{L^2} < \infty\},$$

$$\begin{aligned} \Sigma^{\varepsilon,r}(I) = \{\phi \in L^q(I, L^r) : \|\phi\|_{\Sigma^{\varepsilon,r}(I)} &\equiv \|\phi\|_{L^q(I,L^r)} \\ &+ \|\varepsilon\nabla\phi\|_{L^q(I,L^r)} + \|J^\varepsilon \phi\|_{L^q(I,L^r)} < \infty\}. \end{aligned}$$

And, for an interval $I \subset \mathbb{R}$ and $\rho > 0$ we define the spaces $X^{\varepsilon}(I)$ and $X^{\varepsilon}_{\rho}(I)$ as follows.

$$X^{\varepsilon}(I) = \left\{\phi \in C(I, L^2) : \|\phi\|_{X^{\varepsilon}(I)} < \infty\right\},$$
$$X^{\varepsilon}_{\rho}(I) = \left\{\phi \in C(I, L^2) : \|\phi\|_{X^{\varepsilon}(I)} < \rho\right\},$$

where

$$\|\cdot\|_{X^{\varepsilon}(I)} = \sup_{(q,r):\text{admissible}} \varepsilon^{\frac{1}{q}} \|\cdot\|_{\Sigma^{\varepsilon,r}(I)}.$$

We multiply $\varepsilon^{1/q}$ to $\|\cdot\|_{\Sigma^{\varepsilon,q}(I)}$ for the reason of adapting them for the use of scaled Strichartz inequalities.

Remark 2.5. If $n = 2$ then the pair $(2, \infty)$ is not admissible. Therefore, we understand that the above supremum is took over all admissible pairs (q, r) which satisfy $2 \leqslant r \leqslant r_0$ for a fixed sufficiently large r_0. We also note that the endpoint Strichartz estimates hold for $n \geqslant 3$ ([KT]).

§3. Linear Caustic

3.1. Existence Theorem

Theorem 3.1. *Let $n \geqslant 2$, $\alpha > \gamma$, and $\alpha > 1$. Assume that if $\gamma = 1$, ν is any positive constant and if $\gamma < 1$, $0 < \nu < (\alpha - 1)/(1 - \gamma)$. We put*

$$I = \begin{cases} \mathbb{R} & \text{if } \gamma > 1, \\ [1 - C_0\varepsilon^{-\nu}, 1 + C_0\varepsilon^{-\nu}] & \text{if } \gamma \leqslant 1. \end{cases}$$

Then for any $f \in \Sigma$ there exists $\varepsilon^* = \varepsilon^*(\|f; \Sigma\|, \alpha, \gamma, \nu, C_0)$ such that (HE$^{\varepsilon}$) *has a unique solution in $X^{\varepsilon}(I)$ for all ε satisfying $0 < \varepsilon < \varepsilon^*$. Here, C_0 is a positive constant.*

Its proof is done by contraction argument. We first rewrite (HE$^{\varepsilon}$) to an integral equation, that is

$$(\mathrm{IHE}^{\varepsilon}) \quad u^{\varepsilon}(t) = U^{\varepsilon}(t) u^{\varepsilon}_{|t=0} - i\lambda\varepsilon^{\alpha-1} \int_0^t U^{\varepsilon}(t-s)(|x|^{-\gamma} * |u^{\varepsilon}|^2) u^{\varepsilon}(s) ds.$$

Now, let us denote the right hand side of (IHE$^{\varepsilon}$) by $F(u^{\varepsilon})$. We shall show that F is a contraction map from $X^{\varepsilon}(I)$ to itself with suitable choice of I.

Lemma 3.2. *Under the assumption of Theorem 3.1, there exist two constants ρ and ε^* such that $F : X^{\varepsilon}_{\rho}(I) \to X^{\varepsilon}_{\rho}(I)$ for $0 < \varepsilon < \varepsilon^*$.*

Proof. What we have to show is that the norm $\varepsilon^{\frac{1}{q}}\|F(u^\varepsilon)\|_{\Sigma^{\varepsilon,r}}$ is bounded by ρ for all admissible pair (q,r).

Step 1. Let us start with estimate for the norm of $F(u^\varepsilon)$. Let (q_1,r_1), (q_2,r_2) be admissible pairs. From scaled Strichartz estimate, we have

$$(3.1)\quad \varepsilon^{\frac{1}{q_1}}\|F(u^\varepsilon)\|_{L^{q_1}(I;L^{r_1})} \leqslant C\|f\|_{L^2} + C\varepsilon^{\alpha-1-\frac{1}{q_2}}\|(|x|^{-\gamma}*|u^\varepsilon|^2)u^\varepsilon\|_{L^{q_2'}(I;L^{r_2'})}.$$

From technical reason, we can not treat entire range of γ at once. So, we first consider the case $\gamma>4/3$. We let $I=\mathbb{R}$ and choose r_2 so that $\delta(r_2)=\gamma/4$, then $q_2=2/\delta(r_2)=8/\gamma$. Now, we estimate $L^{r_2'}$ norm in (3.1) by Proposition 2.3.

$$\|(|x|^{-\gamma}*|u^\varepsilon|^2)u^\varepsilon\|_{L^{r_2'}} \leqslant C\|u^\varepsilon\|_{L^k}^2\times\|u^\varepsilon\|_{L^k},$$

where

$$\delta(r_2)+\delta(k)=\gamma-2\delta(k),$$

that is,

$$\delta(k)=\delta(r_2)=\frac{\gamma}{4}.$$

Applying time decay estimate to $\|u^\varepsilon\|_{L^k}^2$, we can derive time decaying term:

$$(3.2)\quad \|(|x|^{-\gamma}*|u^\varepsilon|^2)u^\varepsilon\|_{L^{r_2'}} \leqslant C(\varepsilon+|t-1|)^{-\frac{\gamma}{2}}\|u^\varepsilon\|_{L^2}^{2-\frac{\gamma}{2}} \times(\|\varepsilon\nabla u^\varepsilon\|_{L^2}+\|J^\varepsilon(t)u^\varepsilon\|_{L^2})^{\frac{\gamma}{2}}\|u^\varepsilon\|_{L^k}.$$

Taking $L^{q_2'}(\mathbb{R})$ norm in time, we have

(3.3)

$$\|(|x|^{-\gamma}*|u^\varepsilon|^2)u^\varepsilon\|_{L_t^{q_2'}L_x^{r_2'}} \leqslant C\left(\int_{\mathbb{R}}(\varepsilon+|t-1|)^{-\frac{\gamma y}{2}}dt\right)^{\frac{1}{y}}\|u^\varepsilon\|_{L_t^\infty L_x^2}^{2-\frac{\gamma}{2}} \times(\|\varepsilon\nabla u^\varepsilon\|_{L_t^\infty L_x^2}+\|J^\varepsilon u^\varepsilon\|_{L_t^\infty L_x^2})^{\frac{\gamma}{2}}\|u^\varepsilon\|_{L_t^{\frac{8}{\gamma}}L_x^k},$$

where $1/q_2'=1/y+\gamma/8$, that is, $y=4/(4-\gamma)$. We denote $L^q(I:L^r)$ by $L_t^qL_x^r$, for short. If $\gamma y/2>1$, the integral in right hand side of (3.3) is convergent. This condition is equivalent to $\gamma>4/3$, which we impose.

Therefore,

$$(3.4)\quad \left(\int_{\mathbb{R}} (\varepsilon + |t-1|)^{-\frac{\gamma y}{2}} dt\right)^{\frac{1}{y}} = \left(2\int_0^\infty (\varepsilon + T)^{-\frac{2\gamma}{4-\gamma}} dT\right)^{\frac{4-\gamma}{4}} \leqslant C\varepsilon^{(-\frac{2\gamma}{4-\gamma}+1)(\frac{4-\gamma}{4})} = C\varepsilon^{1-\frac{3}{4}\gamma}.$$

From definition of function space X^ε, we have

$$\|u^\varepsilon\|_{L_t^{\frac{8}{\gamma}} L_x^k} \leqslant \varepsilon^{-\frac{\gamma}{8}} \left(\varepsilon^{\frac{\gamma}{8}} \|u^\varepsilon\|_{L_t^{\frac{8}{\gamma}} L_x^k}\right) \leqslant \varepsilon^{-\frac{\gamma}{8}} \|u\|_{X^\varepsilon}.$$

Then, from (3.1) and (3.3) we obtain

$$(3.5)\quad \varepsilon^{\frac{1}{q_1}} \|F(u^\varepsilon)\|_{L^{q_1}(\mathbb{R};L^{r_1})} \leqslant C\|f\|_{L^2} + C\varepsilon^{\alpha-1-\frac{\gamma}{8}} \times \varepsilon^{1-\frac{7}{8}\gamma}\|u\|^3_{X^\varepsilon(\mathbb{R})} \leqslant C\|f\|_{L^2} + C\varepsilon^{\alpha-\gamma}\|u\|^3_{X^\varepsilon(\mathbb{R})}.$$

Next, let us consider the case $1 < \gamma \leqslant 4/3$. We assume $I = \mathbb{R}$, and choose $r_2 = 2$, then $q_2 = \infty$. We estimate the L^2 norm by Proposition 2.4.

$$\|(|x|^{-\gamma} * |u^\varepsilon|^2)u^\varepsilon\|_{L^2} \leqslant \|(|x|^{-\gamma} * |u^\varepsilon|^2)\|_{L^\infty}\|u^\varepsilon\|_{L^2} \leqslant C\|u\|_{L^{l_+}}\|u\|_{L^{l_-}}\|u\|_{L^2},$$

where $l_\pm$ satisfy $\delta(l_\pm) = (\gamma \pm \eta)/2$ with $\eta > 0$. η is so small that $0 < \delta(l_\pm) < 1$. Note that $\delta(l_+) + \delta(l_-) = \gamma$. Then, we can derive time decay effect from $\|u^\varepsilon\|_{L^{l_\pm}}$ as previous case:

$$\begin{aligned}\|u\|_{L^{l_+}}\|u\|_{L^{l_-}} &\leqslant C(\varepsilon + |t-1|)^{-(\delta(l_+)+\delta(l_-))}\|u^\varepsilon\|_{L^2}^{2-(\delta(l_+)+\delta(l_-))} \\ &\quad \times (\|\varepsilon\nabla u^\varepsilon\|_{L^2} + \|J^\varepsilon(t)u^\varepsilon\|_{L^2})^{\delta(l_+)+\delta(l_-)} \\ &= C(\varepsilon + |t-1|)^{-\gamma}\|u^\varepsilon\|_{L^2}^{2-\gamma}(\|\varepsilon\nabla u^\varepsilon\|_{L^2} + \|J^\varepsilon(t)u^\varepsilon\|_{L^2})^\gamma.\end{aligned}$$

Therefore,

$$(3.6)\quad \|(|x|^{-\gamma} * |u^\varepsilon|^2)u^\varepsilon\|_{L^2} \leqslant C(\varepsilon + |t-1|)^{-\gamma}\|u^\varepsilon\|_{L^2}^{3-\gamma} (\|\varepsilon\nabla u^\varepsilon\|_{L^2} + \|J^\varepsilon(t)u^\varepsilon\|_{L^2})^\gamma.$$

Taking $L^1(\mathbb{R})$ norm in time,

$$(3.7)\quad \|(|x|^{-\gamma} * |u^\varepsilon|^2)u^\varepsilon\|_{L_t^1 L_x^2} \leqslant C\left(\int_{\mathbb{R}} (\varepsilon + |t-1|)^{-\gamma} dt\right)\|u^\varepsilon\|_{L_t^\infty L_x^2}^{3-\gamma} \times (\|\varepsilon\nabla u^\varepsilon\|_{L_t^\infty L_x^2} + \|J^\varepsilon(t)u^\varepsilon\|_{L_t^\infty L_x^2})^\gamma.$$

Since we assume $\gamma > 1$, the integral is convergent and bounded by $C\varepsilon^{1-\gamma}$. So, from (3.1) and (3.7), we conclude

$$\varepsilon^{\frac{1}{q_1}} \|F(u^\varepsilon)\|_{L^{q_1}(\mathbb{R};L^{r_1})} \leqslant C\|f\|_{L^2} + C\varepsilon^{\alpha-\gamma}\|u\|^3_{X^\varepsilon(\mathbb{R})}. \tag{3.8}$$

Let us proceed to the case $0 < \gamma \leqslant 1$. The choice of r_2 is the same as previous case $1 < \gamma \leqslant 4/3$, that is $(q_2, r_2) = (\infty, 2)$. We can obtain estimate (3.6) in the same way, however, the time integral in (3.7) does not converge with $I = \mathbb{R}$. Therefore we set

$$I = [1 - C_0\varepsilon^{-\nu}, 1 + C_0\varepsilon^{-\nu}], \tag{3.9}$$

where C_0 and ν are positive constants. Then, there exists some constant C such that

$$\int_I (\varepsilon + |t-1|)^{-\gamma} dt < \begin{cases} C(\varepsilon + C_0\varepsilon^{-\nu})^{1-\gamma} & \text{if } 0 < \gamma < 1 \\ C|\log(\varepsilon + C_0\varepsilon^{-\nu})| & \text{if } \gamma = 1. \end{cases} \tag{3.10}$$

Note that we can suppose $\varepsilon < C_0\varepsilon^{-\nu}$ and $C_0\varepsilon^{-\nu} \gg 1$ for small ε. Combining (3.1), (3.6) and (3.10), we obtain

$$\varepsilon^{\frac{1}{q_1}} \|F(u^\varepsilon)\|_{L^{q_1}(I;L^{r_1})} \leqslant C\|f\|_{L^2} + C\varepsilon^{\alpha-1-\nu(1-\gamma)}\|u\|^3_{X^\varepsilon(I)} \tag{3.11}$$

for $0 < \gamma < 1$, and

$$\varepsilon^{\frac{1}{q_1}} \|F(u^\varepsilon)\|_{L^{q_1}(I;L^{r_1})} \leqslant C\|f\|_{L^2} + C\varepsilon^{\alpha-1}|\log(2C_0\varepsilon^{-\nu})| \cdot \|u^\varepsilon\|^3_{X^\varepsilon(I)} \tag{3.12}$$

for $\gamma = 1$.

Step2. We estimate $\varepsilon\nabla F(u^\varepsilon)$. Since the operator $\varepsilon\nabla$ commutes $U^\varepsilon(t)$, it is obtained from (IHE$^\varepsilon$) that

$$\varepsilon\nabla F(u^\varepsilon) = U^\varepsilon(t)(\varepsilon\nabla u^\varepsilon_{|t=0}) - i\lambda\varepsilon^{\alpha-1}\int_0^t U^\varepsilon(t-s)\big(\varepsilon\nabla(|x|^{-\gamma} * |u^\varepsilon|^2)u^\varepsilon(s)\big)ds.$$

Therefore, for admissible pairs (q_1, r_1) and (q_2, r_2), we deduce that

$$\begin{aligned} \varepsilon^{\frac{1}{q_1}} \|\varepsilon\nabla F(u^\varepsilon)\|_{L^{q_1}(I;L^{r_1})} &\leqslant C\|\varepsilon\nabla e^{-ix^2/2\varepsilon} f\|_{L^2} \\ &\quad + C\varepsilon^{\alpha-1-\frac{1}{q_2}} \|\varepsilon\nabla(|x|^{-\gamma} * |u^\varepsilon|^2)u^\varepsilon\|_{L^{q_2'}(I;L^{r_2'})} \end{aligned} \tag{3.13}$$

by scaled Strichartz estimate. One easily verifies by elementary computations that

$$\|\varepsilon\nabla e^{-ix^2/2\varepsilon} f\|_{L^2} = \|(-ix+\varepsilon\nabla)f\|_{L^2} \leqslant \|xf\|_{L^2} + \varepsilon\|\nabla f\|_{L^2}, \tag{3.14}$$

and

$$|\varepsilon\nabla(|x|^{-\gamma} * |u^\varepsilon|^2)u^\varepsilon| \leqslant |(|x|^{-\gamma} * 2\mathrm{Re}(\overline{u^\varepsilon}\varepsilon\nabla u^\varepsilon))u^\varepsilon| + |(|x|^{-\gamma} * |u^\varepsilon|^2)\varepsilon\nabla u^\varepsilon|. \tag{3.15}$$

We first consider the case $\gamma > 4/3$. We choose r_2 so that $\delta(r_2) = \gamma/4$ and estimate the L^{r_2} norm in (3.13) by Proposition 2.3:

$$\begin{aligned} \|\varepsilon\nabla(|x|^{-\gamma} * |u^\varepsilon|^2)u^\varepsilon\|_{L^{r_2'}} &\leqslant \|(|x|^{-\gamma} * 2\mathrm{Re}(\overline{u^\varepsilon}\varepsilon\nabla u^\varepsilon))u^\varepsilon\|_{L^{r_2'}} \\ &\quad + \|(|x|^{-\gamma} * |u^\varepsilon|^2)\varepsilon\nabla u^\varepsilon\|_{L^{r_2'}} \\ &\leqslant 3\|u^\varepsilon\|_{L^k}^2 \|\varepsilon\nabla u^\varepsilon\|_{L^k}, \end{aligned} \tag{3.16}$$

where k satisfies $\delta(k) = \gamma/4$. Note that $\|u\|_{L^k}^2$ appears in (3.16). This is the only term which produces time decay effect in Step 1, and now we have it. Therefore we can obtain

$$\|\varepsilon\nabla(|x|^{-\gamma} * |u^\varepsilon|^2)u^\varepsilon\|_{L_t^{(8/\gamma)'} L_x^{r_2'}} \leqslant C\varepsilon^{1-\frac{7}{8}\gamma} \|u\|_{X^\varepsilon}^2 \left(\varepsilon^{\frac{\gamma}{8}} \|\varepsilon\nabla u^\varepsilon\|_{L_t^{8/\gamma} L_x^k}\right). \tag{3.17}$$

in the exactly same way. Now, since we have

$$\left(\varepsilon^{\frac{\gamma}{8}} \|\varepsilon\nabla u^\varepsilon\|_{L^{8/\gamma}(\mathbb{R};L^k)}\right) \leqslant \|u^\varepsilon\|_{X^\varepsilon(\mathbb{R})}$$

from the definition of X^ε, we conclude from (3.13), (3.14), and (3.17) that

$$\varepsilon^{\frac{1}{q_1}} \|\varepsilon\nabla F(u^\varepsilon)\|_{L^{q_1}(\mathbb{R};L^{r_1})} \leqslant C\|xf\|_{L^2} + C\varepsilon\|\nabla f\|_{L^2} + C\varepsilon^{\alpha-\gamma}\|u\|_{X^\varepsilon(\mathbb{R})}^3. \tag{3.18}$$

The case $1 < \gamma \leqslant 4/3$. We choose $(q_2, r_2) = (\infty, 2)$. From (3.15), we have

$$\begin{aligned} \|\varepsilon\nabla(|x|^{-\gamma} * |u^\varepsilon|^2)u^\varepsilon\|_{L^2} &\leqslant \|(|x|^{-\gamma} * 2\mathrm{Re}(\overline{u^\varepsilon}\varepsilon\nabla u^\varepsilon))u^\varepsilon\|_{L^2} \\ &\quad + \|(|x|^{-\gamma} * |u^\varepsilon|^2)\varepsilon\nabla u^\varepsilon\|_2. \\ &\leqslant \|(|x|^{-\gamma} * 2\mathrm{Re}(\overline{u^\varepsilon}\varepsilon\nabla u^\varepsilon))u^\varepsilon\|_{L^2} \\ &\quad + \||x|^{-\gamma} * |u^\varepsilon|^2\|_{L^\infty} \|\varepsilon\nabla u^\varepsilon\|_2. \end{aligned}$$

Then, estimating the first term by Proposition 2.3, and the last term by Proposition 2.4, we obtain

$$\begin{aligned}(3.19)\quad \|\varepsilon\nabla(|x|^{-\gamma} * |u^\varepsilon|^2)u^\varepsilon\|_{L^2} \leqslant 2\|u^\varepsilon\|_{L^l}^2\|\varepsilon\nabla u^\varepsilon\|_{L^2} \\ + \|u^\varepsilon\|_{L^{l_+}}\|u^\varepsilon\|_{L^{l_-}}\|\varepsilon\nabla u^\varepsilon\|_{L^2},\end{aligned}$$

where $\delta(l) = \gamma/2$ and $\delta(l_\pm) = (\gamma \pm \eta)/2$ with small $\eta > 0$. In this case, the term $\|u^\varepsilon\|_{L^{l_+}}\|u^\varepsilon\|_{L^{l_-}}$ provides time decay effect. Note that the term $\|u^\varepsilon\|_{L^l}^2$ can play the same role, since $2\delta(l) = \gamma = \delta(l_+)+\delta(l_-)$ (see (3.6)). Therefore, we obtain

$$\begin{aligned}(3.20)\quad \varepsilon^{\frac{1}{q_1}}\|\varepsilon\nabla F(u^\varepsilon)\|_{L^{q_1}(\mathbb{R};L^{r_1})} \leqslant C\|xf\|_{L^2} \\ + C\varepsilon\|\nabla f\|_{L^2} + C\varepsilon^{\alpha-\gamma}\|u\|_{X^\varepsilon(\mathbb{R})}^3\end{aligned}$$

as Step 1.

From above argument, we can derive an estimate similar to that for $F(u^\varepsilon)$ also in the case $0 < \gamma \leqslant 1$. Namely, if $0 < \gamma < 1$ we have

$$\begin{aligned}(3.21)\quad \varepsilon^{\frac{1}{q_1}}\|\varepsilon\nabla F(u^\varepsilon)\|_{L^{q_1}(I;L^{r_1})} \leqslant C\|xf\|_{L^2} + C\varepsilon\|\nabla f\|_{L^2} + \\ C\varepsilon^{\alpha-1-\nu(1-\gamma)}\|u\|_{X^\varepsilon(I)}^3,\end{aligned}$$

and if $\gamma = 1$ we have

$$\begin{aligned}(3.22)\quad \varepsilon^{\frac{1}{q_1}}\|\varepsilon\nabla F(u^\varepsilon)\|_{L^{q_1}(I;L^{r_1})} \leqslant C\|xf\|_{L^2} + C\varepsilon\|\nabla f\|_{L^2} \\ + C\varepsilon^{\alpha-1}|\log(2C_0\varepsilon^{-\nu})| \cdot \|u^\varepsilon\|_{X^\varepsilon(I)}^3,\end{aligned}$$

where I satisfies (3.9).

The estimate for $J^\varepsilon(t)F(u^\varepsilon)$ is essentially equal to that for $\varepsilon\nabla F(u^\varepsilon)$, because $J^\varepsilon(t)U_0(\varepsilon(t-s)) = U_0(\varepsilon(t-s))J^\varepsilon(s)$ and $J^\varepsilon(t)$ operates like derivative on Gauge invariant function. The only difference is estimate for initial value. Namely, we make use of

$$(3.23)\quad \|J^\varepsilon(t)U(\varepsilon t)e^{-i\frac{x^2}{2\varepsilon}}f\|_{L^2} = \|U(\varepsilon t)J^\varepsilon(0)e^{-i\frac{x^2}{2\varepsilon}}f\|_{L^2} = \|\nabla f\|_{L^2}$$

instead of (3.14). So we omit the detail.

Step 3. We collect the results in Steps 1 and 2, and complete the proof by combining them. From (3.5), (3.8), (3.18), (3.20) and (3.23),

$$\begin{aligned}(3.24)\quad \varepsilon^{\frac{1}{q}}\|F(u^\varepsilon)\|_{\Sigma^{\varepsilon,r}(\mathbb{R})} \leqslant C\|f\|_{L^2} + C\|xf\|_{L^2} \\ + C(\varepsilon+1)\|\nabla f\|_{L^2} + C\varepsilon^{\alpha-\gamma}\|u\|_{X^\varepsilon(\mathbb{R})}^3.\end{aligned}$$

for $\gamma > 1$, where (q,r) is admissible. Taking supremum all over (q,r), finally we obtain

$$\|F(u^\varepsilon)\|_{X^\varepsilon(\mathbb{R})} \leqslant C\|f\|_\Sigma + C\varepsilon^{\alpha-\gamma}\|u^\varepsilon\|^3_{X^\varepsilon(\mathbb{R})}. \tag{3.25}$$

Now, we put $\rho = 2C\|f;\Sigma\|$. Then, if $\alpha > \gamma$ there exist a constant ε^* depending on n, α, γ, and $\|f;\Sigma\|$ such that

$$\|F(u^\varepsilon)\|_{X^\varepsilon(\mathbb{R})} \leqslant \rho$$

for all $0 < \varepsilon < \varepsilon^*$ and all $u^\varepsilon \in X^\varepsilon_\rho(\mathbb{R})$. It goes without saying that it means $F : X^\varepsilon_\rho(\mathbb{R}) \to X^\varepsilon_\rho(\mathbb{R})$.

For $0 < \gamma < 1$, we use (3.11), (3.21), and (3.23). These inequalities gives us

$$\|F(u^\varepsilon)\|_{X^\varepsilon(I)} \leqslant C\|f\|_\Sigma + C\varepsilon^{\alpha-1-\nu(1-\gamma)}\|u\|^3_{X^\varepsilon(I)}, \tag{3.26}$$

where $I = [1 - C_0\varepsilon^{-\nu}, 1 + C_0\varepsilon^{-\nu}]$.

Again, we put $\rho = 2C\|f;\Sigma\|$ and assume that $u^\varepsilon \in X^\varepsilon_\rho(\mathbb{R})$. In this case, we need the condition that ν satisfies $\alpha - 1 - \nu(1-\gamma) > 0$, that is $\nu < (\alpha-1)/(1-\gamma)$ in addition to $\alpha > 1$. Then there exists a constant ε^* depending on n, α, γ, ν, C_0, and $\|f;\Sigma\|$ such that

$$\|F(u^\varepsilon)\|_{X^\varepsilon(\mathbb{R})} \leqslant \rho$$

for $0 < \varepsilon < \varepsilon^*$. Lemma has been proven for $0 < \gamma < 1$.

In the case of $\gamma = 1$, from the inequalities (3.12), (3.22), and (3.23) we deduce that

$$\|F(u^\varepsilon)\|_{X^\varepsilon(I)} \leqslant C\|f\|_\Sigma + C\varepsilon^{\alpha-1}|\log(2C_0\varepsilon^{-\nu})| \cdot \|u^\varepsilon\|^3_{X^\varepsilon(I)}, \tag{3.27}$$

where I satisfies (3.9). Therefore, if $\alpha > 1$ there exists ε^* depending on n, α, ν, C_0, and $\|f;\Sigma\|$ such that

$$\|F(u^\varepsilon)\|_{X^\varepsilon(\mathbb{R})} \leqslant \rho$$

with some constant ρ which is proportional to $\|f\|_\Sigma$. Q.E.D.

Lemma 3.3. *For any $\rho > 0$, there exists an $\varepsilon^{**} > 0$ such that $F : X^\varepsilon_\rho(I) \to X^\varepsilon_\rho(I)$ is a contraction map for $0 < \varepsilon < \varepsilon^{**}$.*

Proof. Let $u_1, u_2 \in X^\varepsilon_\rho(I)$ be a solution to (IHE$^\varepsilon$). One easily verifies the following identities:

$$(3.28)\quad F(u_1) - F(u_2) = -i\lambda\varepsilon^{\alpha-1}\int_0^t U(\varepsilon(t-s))\Big\{(|x|^{-\gamma} * |u_1|^2)u_1(s) - (|x|^{-\gamma} * |u_2|^2)u_2(s)\Big\}ds.$$

(3.29)

$$\begin{aligned}(|x|^{-\gamma} * |u_1|^2)u_1 - (|x|^{-\gamma} * |u_2|^2)u_2 =&(|x|^{-\gamma} * |u_1|^2)(u_1 - u_2)\\ &+ (|x|^{-\gamma} * \{u_1(\overline{u_1 - u_2})\})u_2\\ &+ (|x|^{-\gamma} * \{(u_1 - u_2)\overline{u_2}\})u_2.\end{aligned}$$

In exactly the same way as in the proof of Lemma 3.2, we obtain for $\gamma > 1$

$$(3.30)\qquad \|F(u_1) - F(u_2)\|_{X^\varepsilon(\mathbb{R})} \leqslant C\varepsilon^{\alpha-\gamma}\rho^2\|u_1 - u_2\|_{X^\varepsilon(\mathbb{R})},$$

for $0 < \gamma < 1$

$$(3.31)\qquad \|F(u_1) - F(u_2)\|_{X^\varepsilon(I)} \leqslant C\varepsilon^{\alpha-1-\nu(1-\gamma)}\rho^2\|u_1 - u_2\|_{X^\varepsilon(I)},$$

and for $\gamma = 1$

$$(3.32)\quad \|F(u_1) - F(u_2)\|_{X^\varepsilon(I)} \leqslant C\varepsilon^{\alpha-1}|\log(2C_0\varepsilon^{-\nu})|\rho^2\|u_1 - u_2\|_{X^\varepsilon(I)}$$

with $I = [1 - C_0\varepsilon^{-\nu}, 1 + C_0\varepsilon^{-\nu}]$. Therefore, we deduce that F is a contraction map for small ε, provided α and ν satisfies the assumption of Theorem 3.1. Q.E.D.

Now, Theorem 3.1 immediately follows from Lemmas 3.2 and 3.3.

Remark 3.4. Equation (HE$^\varepsilon$) can have unique solution also in the outside of I for any fixed ε. However, the norm X^ε may be divergent as $\varepsilon \to 0$ on the subset of $\mathbb{R} \setminus I$. Therefore I is the interval where the norm $X^\varepsilon(I)$ is bounded uniformly in ε.

3.2. Asymptotic behavior

We next consider the asymptotic behavior of u^ε. It is an immediate consequence of Theorem 3.1. Define $w^\varepsilon(t,x)$ to be a solution to

$$(3.33)\qquad \begin{cases} i\varepsilon\partial_t w^\varepsilon + \frac{1}{2}\varepsilon^2\Delta w^\varepsilon = 0,\\ w^\varepsilon_{|t=0} = u^\varepsilon_{|t=0}.\end{cases}$$

Then, we have

$$w^\varepsilon(t) = U^\varepsilon(t) u^\varepsilon_{|t=0}. \tag{3.34}$$

Therefore, we obtain the following theorem.

Theorem 3.5. *Let (q,r) be an admissible pair and I be an interval which satisfies the condition in Theorem 3.1.*

- *If $\alpha > \gamma > 1$ and $1/q < \alpha - \gamma$, then*

$$\|u^\varepsilon - w^\varepsilon\|_{\Sigma^{\varepsilon,r}(\mathbb{R})} \to 0 \qquad (\varepsilon \to 0).$$

- *If $\alpha > 1 \geqslant \gamma$ and $1/q < \alpha - 1 - \nu(1-\gamma)$, then*

$$\|u^\varepsilon - w^\varepsilon\|_{\Sigma^{\varepsilon,r}(I)} \to 0 \qquad (\varepsilon \to 0).$$

Proof. From (3.34), we deduce that

$$\|u^\varepsilon - w^\varepsilon\|_{X^\varepsilon(I)} < \begin{cases} C\varepsilon^{\alpha-\gamma} & \text{if } \gamma > 1, \\ C\varepsilon^{\alpha-1}\log(2C_0\varepsilon^{-\nu}) & \text{if } \gamma = 1, \\ C\varepsilon^{\alpha-1-\nu(1-\gamma)} & \text{if } \gamma < 1. \end{cases} \tag{3.35}$$

Therefore, the theorem follows from the definition of X^ε. Q.E.D.

Remark 3.6. In [CL], this asymptotics is shown for $\gamma > 1$ and $(q,r) = (\infty, 2)$. The norm X^ε gives us more information about admissible pairs. Therefore we obtain more precise result. This theorem says that larger α provides better convergence. It seems to be natural because the constant α denotes the size of nonlinearity.

Remark 3.7. The case $0 < \gamma \leqslant 1$ is called the long range case. If $t \notin I$, the solution u^ε does not behaves as a free solution. In this sense, the nonlinear effect appears near $t = \pm\infty$, that is, in the interval $(-\infty, 1 - C_0\varepsilon^{-\nu}]$ and $[1 + C_0\varepsilon^{-\nu}, \infty)$. If $\gamma = 1$ then ν is arbitrary, and if $\gamma < 1$ then $\nu < (\alpha-1)/(1-\gamma)$. Moreover, in the case $\gamma < 1$, it seems that the convergence becomes worse for larger ν (namely for larger $|t|$).

Remark 3.8. In fact, we can choose I so that $|I| = O(\varepsilon^{-(C_0\varepsilon^{-\nu})})$ with $\nu < \alpha - 1$ in the case $\gamma = 1$.

§4. Nonlinear and Supercritical Caustic

4.1. Existence result and asymptotic behavior before the caustic

Theorem 4.1. *Let $n \geqslant 2$. Assume $\gamma \geqslant \alpha > \max(1, \gamma/2)$, $\gamma > 1$ and $I = (-\infty, 1 - C_0\varepsilon^\mu]$ with*

$$\mu < \mu_{\alpha,\gamma} := \frac{\alpha - \max(1,\gamma/2)}{\gamma - \max(1,\gamma/2)}. \tag{4.1}$$

Then for any $f \in \Sigma$ there exists $\varepsilon^* = \varepsilon^*(\|f;\Sigma\|, \alpha, \gamma, I)$ such that (HE$^\varepsilon$) *has unique solution in $X^\varepsilon(I)$ for all ε satisfying $0 < \varepsilon < \varepsilon^*$ and it holds that*

$$\|u^\varepsilon - w^\varepsilon\|_{X^\varepsilon(I)} \to 0 \qquad (\varepsilon \to 0). \tag{4.2}$$

Note that $\mu_{\alpha,\gamma} \leqslant 1$ under the above condition.

Proof. Again, we use contraction method. We denote the right hand side of (IHE$^\varepsilon$) by $F(u^\varepsilon)$. We shall restrict our attention to the estimate for $F(u^\varepsilon)$, because that for $\varepsilon\nabla F(u^\varepsilon)$ and $J^\varepsilon(t)F(u^\varepsilon)$ is obtained in the same way.

Let us begin with the case $1 < \gamma < 3$. Letting $(q_2, r_2) = (\infty, 2)$ and applying Proposition 2.3, we deduce from (3.1) that

$$\|(|x|^{-\gamma} * |u^\varepsilon|^2)u^\varepsilon\|_{L^2} \leqslant \|u^\varepsilon\|_{L^{s_1}} \|u^\varepsilon\|_{L^{s_2}} \|u^\varepsilon\|_{L^{s_3}}, \tag{4.3}$$

with $\delta(s_1) = \delta(s_2) = (\gamma - \delta(s_3))/2$. Applying time decay estimate to $\|u^\varepsilon\|_{L^{s_1}}$ and $\|u^\varepsilon\|_{L^{s_2}}$, and taking $L^1(I)$ norm in time with $I = (-\infty, 1 - C_0\varepsilon^\mu]$, we obtain

$$\varepsilon^{\frac{1}{q_1}} \|F(u^\varepsilon)\|_{L^{q_1}(I;L^{r_1})} \leqslant C\|f\|_{L^2} + C\varepsilon^{\beta_1} \|u^\varepsilon\|^3_{X^\varepsilon(I)} \tag{4.4}$$

with

$$\beta_1 = \alpha - 1 - \frac{\delta(s_3)}{2} - \mu\left(\gamma - 1 - \frac{\delta(s_3)}{2}\right). \tag{4.5}$$

Now, we shall decide $\delta(s_3)$. Since β_1 should be positive, we would like to choose $\delta(s_3)$ as small as possible in order to expand the range of α. The conditions on $\delta(s_3)$ are $0 \leqslant \delta(s_3) < 1$, $0 < \delta(s_1), \delta(s_2) < 1 \Leftrightarrow \gamma - 2 < \delta(s_3) < \gamma$ and $\gamma - 1 - \delta(s_2)/2 > 0$ (It comes from integrability). Therefore we set

$$\delta(s_3) = \begin{cases} 0 & \text{if } 1 < \gamma < 2 \\ \gamma - 2 + \zeta & \text{if } 2 \leqslant \gamma < 3 \end{cases} \tag{4.6}$$

with small ζ. Note that in the case $\delta(s_3) = 0$ we make use of Proposition 2.4 instead of Proposition 2.3. With this choice of s_3, if α and μ satisfies

$$\alpha > \max\left(1, \frac{\gamma}{2}\right), \tag{4.7}$$

$$\mu < \frac{\alpha - \max(1, \gamma/2)}{\gamma - \max(1, \gamma/2)} \tag{4.8}$$

then $\beta_1 > 0$ for sufficiently small ζ.

Let us proceed to the case $3 \leqslant \gamma < \min(4, n)$. Evidently, space dimension n is larger than or equal to 4. We let (q_2, r_2) to be chosen later. By Proposition 2.3, we have

$$\|(|x|^{-\gamma} * |u^\varepsilon|^2)u^\varepsilon\|_{L^{r_2'}} \leqslant \|u^\varepsilon\|_{L^s}^2 \times \|u^\varepsilon\|_{L^s},$$

with $\delta(s) = (\gamma - \delta(r_2))/3$. We assume $I = (-\infty, 1 - C_0\varepsilon^\mu]$. Applying time decay estimate to $\|u^\varepsilon\|_{L^s}^2$, and taking $L^{q_2'}(I)$ norm in time, we deduce that

$$\varepsilon^{\frac{1}{q_1}} \|F(u^\varepsilon)\|_{L^{q_1}(I;L^{r_1})} \leqslant C\|f\|_{L^2} + C\varepsilon^{\beta_2}\|u^\varepsilon\|_{X^\varepsilon(I)}^3. \tag{4.9}$$

Here,

$$\beta_2 = \alpha - \frac{1}{6}\gamma - 1 - \frac{1}{3}\delta(r_2) - \mu\left(\frac{5}{6}\gamma - 1 - \frac{1}{3}\delta(r_2)\right). \tag{4.10}$$

It should be positive, and we would like to take $\delta(r_2)$ as small as possible at the same time. Since r_2 satisfies $0 \leqslant \delta(r_2) < 1$, $\gamma - 3 < \delta(r_2) < \gamma (\Leftrightarrow 0 < \delta(s) < 1)$ and $5\gamma/6 - 1 - \delta(r_2)/3 > 0$, we choose $\delta(r_2) = \gamma - 3 + \zeta$ with small ζ. Therefore, we conclude that if

$$\alpha > \frac{\gamma}{2},$$
$$\mu < \frac{\alpha - \gamma/2}{\gamma/2},$$

then $\beta_2 > 0$ for small ζ.

In the same way as the linear caustic case, the result follows from the estimates (4.4) and (4.9). Q.E.D.

The condition that α is not greater than γ restricts the distance between the caustic ($t = 1$) and the range where the solution u^ε asymptoticly behaves as a free solution. The nonlinear effect appears near the caustic in this sense.

Through the scaling $u^\varepsilon(t,x) = \varepsilon^{-(\alpha-\gamma)/2-n/2}\psi^\varepsilon((t-1)/\varepsilon, x/\varepsilon)$, we rewrite ($\mathrm{HE}^\varepsilon$) as follows:

$$(\mathrm{SHE}^\varepsilon) \qquad \begin{cases} i\partial_t\psi^\varepsilon + \dfrac{1}{2}\Delta\psi^\varepsilon = \lambda(|x|^{-\gamma} * |\psi^\varepsilon|^2)\psi^\varepsilon, \\ \psi^\varepsilon_{|t=-1/\varepsilon}(x) = \varepsilon^{(\alpha-\gamma)/2+n/2}e^{-i\varepsilon x^2/2}f(\varepsilon x). \end{cases}$$

Let us define ϕ^ε by $\phi^\varepsilon(t,x) = U_0(t)\psi^\varepsilon_{|t=-1/\varepsilon}(x)$. Then, Theorem 4.1 implies that under those assumption, (SHE^ε) has unique solution and ψ^ε behaves as free solution ϕ^ε on I', where

$$I' = (-\infty, -C_0\varepsilon^{\mu-1}].$$

Note that if $0 < \mu < 1$, $I \to \mathbb{R}\setminus\{0\}$ and $I' \to \emptyset$ as $\varepsilon \to 0$. Thus, from the view of usual Hartree equation this theorem seems to be almost meaningless. Our approach makes some divergent quantity stay finite, and hence makes the nonlinear effect clear.

In [CL], the following $v^\varepsilon(t,x)$ is used as asymptotic behavior:

$$v^\varepsilon(t,x) = \frac{1}{(1-t)^{n/2}} f\left(\frac{x}{t-1}\right) e^{i\frac{x^2}{2\varepsilon(t-1)}}. \tag{4.11}$$

This is exactly of the form (1.4). It holds that there exists a continuous function $h : \mathbb{R} \to \mathbb{R}$ which satisfies $h(0) = 0$ such that

$$\|w^\varepsilon - v^\varepsilon\|_{\Sigma^{\varepsilon,2}(I)} \leqslant C(\|f\|_\Sigma) \sup_{t\in I} h\left(\frac{\varepsilon}{t-1}\right) \tag{4.12}$$

(by an argument similar to [C2], Lemma 1). Therefore the following asymptotics holds. It is an extension of the result in [CL].

Corollary 4.2. *Under the assumptions of Theorem 4.1, it holds for all $f \in \Sigma$ that*

$$\|u^\varepsilon - v^\varepsilon\|_{\Sigma^{\varepsilon,2}(I)} \to 0 \qquad (\varepsilon \to 0). \tag{4.13}$$

Remark 4.3. In [CL], it is shown that nonlinear caustic crossing causes the change of asymptotic profile, and that it is described by the scattering operator associated to equation

$$i\partial_t\psi + \frac{1}{2}\Delta\psi = \lambda(|x|^{-\gamma} * |\psi|^2)\psi. \tag{4.14}$$

Remark 4.4. If $\lambda > 0$ and α satisfies

$$(\gamma \geqslant)\alpha > \begin{cases} \dfrac{4\gamma-3}{2\gamma-1} & \text{if } \dfrac{3}{2} < \gamma < 2, \\ \dfrac{\gamma}{2} + \dfrac{2\gamma}{\gamma+4} & \text{if } \gamma \geqslant 2, \end{cases}$$

then we can describe the asymptotic behavior of the solution also beyond caustic. The proof will appear elsewhere.

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