Steady motions of the Navier-Stokes fluid around a rotating body

Toshiaki Hishida

Abstract.

Consider the 3-dimensional Navier-Stokes fluid filling an infinite space exterior to a rotating body with constant angular velocity. By using the coordinate system attached to the body, the problem is reduced to an equivalent one in the fixed exterior domain. The reduced equation then involves an important drift operator, which is not subordinate to the usual Stokes operator and causes some difficulties. Based on [13], [26] and [12], this survey article addresses steady solutions to the reduced problem.

§1. Introduction and summary

1.1. *Navier-Stokes fluid around a rotating body*

Let us consider the physical situation that a compact rigid body moves in a viscous incompressible fluid in a prescribed way. We would like to know the motion of the fluid, which is governed by the Navier-Stokes equation in a time-dependent exterior domain. In his series of papers, Finn considered the problem with translating bodies and started its mathematical analysis, see [14], [9], [15], [31] and the references therein. It is certainly interesting to take also the rotation of the body into account. In the last decade a lot of efforts have been made by several mathematicians on some related problems with rotating bodies, however, our mathematical understanding is still far from complete.

In this survey article, based on the works [13], [26] and [12] jointly with R. Farwig and D. Müller, we provide some recent results on steady motions. We discuss the purely rotating case and thus the translation of the body is absent. Let D be an exterior domain in $\mathbb{R}^3$ with smooth

boundary ∂D. The rigid body $\mathbb{R}^3 \setminus D$ is rotating about y_3-axis (without loss of generality) with the constant angular velocity $\omega = (0, 0, |\omega|)^T$. Here and hereafter, superscript-T denotes the transpose and all vectors are column ones. Unless the body is axisymmetric, the domain $D(t) = O(|\omega|t)D = \{y = O(|\omega|t)x; x \in D\}$ occupied by the fluid actually varies with time t, where

$$O(t) = \begin{pmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The problem we are going to consider is the Navier-Stokes equation

$$\partial_t v + v \cdot \nabla_y v = \Delta_y v - \nabla_y \pi + g, \qquad \text{div}_y v = 0,$$

for $y \in D(t), t > 0$ subject to the non-slip boundary condition on the surface of the body

$$v|_{\partial D(t)} = \omega \wedge y$$

and the rest condition at space infinity

$$v \to 0 \quad \text{as } |y| \to \infty$$

together with initial condition

$$v(y, 0) = a(y), \qquad y \in D.$$

Here, $v(y, t) = (v_1, v_2, v_3)^T$ and $\pi(y, t)$ respectively denote unknown velocity and pressure of the fluid, while $g(y, t) = (g_1, g_2, g_3)^T$ and $a(y) = (a_1, a_2, a_3)^T$ are given external force and initial velocity. The symbol $\wedge$ stands for the usual exterior product of 3-dimensional vectors; thus,

$$\omega \wedge y = |\omega|(-y_2, y_1, 0)^T = \frac{d}{dt}O(|\omega|t)x$$

which is the rotating velocity of the rigid body. It is reasonable to reduce the problem to an equivalent one in the exterior domain D by using the reference frame attached to the rotating body (although there is another possibility, see [8]). Namely, the following change of unknowns (v, π) and the force g is made:

$$u(x, t) = O(|\omega|t)^T v\left(O(|\omega|t)x, t\right), \qquad p(x, t) = \pi\left(O(|\omega|t)x, t\right),$$

$$f(x, t) = O(|\omega|t)^T g\left(O(|\omega|t)x, t\right).$$

Our problem is then reduced to

$$\partial_t u + u \cdot \nabla u = \Delta u + (\omega \wedge x) \cdot \nabla u - \omega \wedge u - \nabla p + f,$$
$$\text{div } u = 0,$$

in $D \times (0, \infty)$ subject to

$$u|_{\partial D} = \omega \wedge x, \qquad u \to 0 \text{ as } |x| \to \infty, \qquad u(x,0) = a(x).$$

The most interesting and difficult feature is that the drift term $(\omega \wedge x) \cdot \nabla u$ is not subordinate to the viscous term Δu and thus cannot be treated as a simple perturbation even if $|\omega|$ is small. In fact, unlike the Laplace operator, the fundamental solution $\Gamma(x, y)$ of the linear operator

$$(1.1) \qquad\qquad L = -\Delta - (\omega \wedge x) \cdot \nabla + \omega \wedge$$

cannot be estimated from above by $C/|x - y|$; to be precise, its final component $\Gamma_{33}(x, y)$ satisfies

$$(1.2) \qquad\qquad \Gamma_{33}(x_\rho, y_\rho) \geq \frac{C \log \rho}{\rho}$$

for $\rho > 1$ when one takes, for example, $x_\rho = (\rho, 0, 0)^T$ and $y_\rho = (0, \rho, 0)^T$, see [13], [26]. Furthermore, unlike the heat semigroup $e^{t\Delta}$, the generated semigroup

$$\left(e^{-tL} f\right)(x) = O(|\omega|t)^T \left(e^{t\Delta} f\right)(O(|\omega|t)x)$$

on $L_2(\mathbb{R}^3)$ is not analytic, although it possesses some smoothing properties. The related semigroup [22] for the exterior problem enjoys such properties as well, see [23], [24], [25] and also the recent work [21].

There are some studies on the Navier-Stokes initial value problem above in exterior domains within the framework of L_2 space; weak solution [3], local unique solution [23], local and global strong solutions [18]. Recently, a local unique solution has been constructed by [21] within the framework of L_q space. We also mention the related topic on the steady falling motion of a body in an infinite fluid. The body must undergo a translation and a rotation which are to be determined from equilibrium conditions on the boundary, see [37] and [16].

1.2. *Steady problem*

The present article is devoted to the study of the steady problem in exterior domains:

$$(1.3) \qquad Lu + \nabla p + u \cdot \nabla u = f, \qquad \text{div } u = 0 \qquad \text{in } D,$$

see (1.1), subject to

(1.4) $u|_{\partial D} = \omega \wedge x, \qquad u \to 0$ as $|x| \to \infty.$

Note that the steady motion in the frame attached to the body corresponds to the time-periodic one in the original frame. It is possible to construct solutions of class $\nabla u \in L_2$ to (1.3)-(1.4) by means of the Galerkin method in L_2 framework for arbitrary ω and $f = \text{div } F$ with $F \in L_2$, see [3], [16], [30] and [33]. When ω is small enough and $f = \text{div } F$ satisfies $|x|^2|F(x)| + |x|^3|f(x)| + |x|^4|\text{div } f(x)| \leq c_0$ with some small $c_0 > 0$, Galdi [17] derived remarkable pointwise estimates

(1.5) $|x||u(x)| + |x|^2(|\nabla u(x)| + |p(x)|) + |x|^3|\nabla p(x)| \leq C$

of a unique solution. These decay properties are very interesting and important in some studies of stability ([3], [18]), but, at the first glance, rather surprising as we know (1.2). See also [19], in which the translation of the body is also taken into account. We may expect some anisotropic decay structures of solutions (that the Oseen case reminds us, see [14], [9], [15], [31]), but as far as simple isotropic decay estimates are concerned, the result of [17] shows that the rate of the decay of the Navier-Stokes flow at infinity is the same as the usual case $\omega = 0$ in spite of slightly worse behavior (1.2) of the fundamental solution.

    In Theorem 2.5 of this article we provide another outlook on the pointwise estimates (1.5) in a different framework by use of function spaces; to be precise, we show the existence of a unique solution to (1.3)-(1.4) with the force $f \in \dot{W}^{-1}_{3/2,\infty}$ in the class $(\nabla u, p) \in L_{3/2,\infty}$ when both f and ω are small enough, where $L_{3/2,\infty}$ is the weak-$L_{3/2}$ space, one of the Lorentz spaces. We here note that $f \in \dot{W}^{-1}_{3/2,\infty}$ if and only if $f = \text{div } F$ with $F \in L_{3/2,\infty}$. Our class of solutions is consistent with (1.5), and our class of external forces is larger than [17]. The complete proof of our result will be given in [12]. For the case $\omega = 0$, the same result as ours has been already proved by Kozono and Yamazaki [29].

    1.3. *Linearized problem in L_q*

    The first step toward the result above is the L_q-analysis of the linear whole space problem initiated by [13], in which the fundamental L_q estimate

(1.6) $\|\nabla^2 u\|_{L_q(\mathbb{R}^3)} \leq C\|Lu\|_{L_q(\mathbb{R}^3)}$

was proved, where L is the linear operator given by (1.1). For the proof of (1.6), it is sufficient to show $\|\Delta u\|_{L_q(\mathbb{R}^3)} \leq C\|Lu\|_{L_q(\mathbb{R}^3)}$, which is never trivial since the integral kernel $\Delta_x \Gamma(x,y)$ of the operator $f \mapsto \Delta u$ does not seem to be of Calderón-Zygmund type. L_2 estimate is, however, easy to show. So, the well-known standard argument due to Calderón-Zygmund is to establish the so-called weak $(1,1)$ inequality (L_1-$L_{1,\infty}$ estimate), which implies the L_q estimate on account of the Marcinkiewicz interpolation theorem. Although the weak $(1,1)$ inequality is of own interest, the proof given by [13] provides another route. In [13], roughly speaking, we split Δu into $\{(\Delta u)_j\}_{j\in\mathbb{Z}}$ by using the homogeneous Littlewood-Paley dyadic decomposition

$$(1.7) \qquad \sum_{j\in\mathbb{Z}} \alpha_j(\xi) = 1 \qquad \left(\xi \in \mathbb{R}^3 \setminus \{0\}\right).$$

In order to prove not only that the series $\Delta u = \sum_{j\in\mathbb{Z}}(\Delta u)_j$ makes sense in L_q but also the desired L_q estimate, we make use of the square function of Littlewood-Paley type

$$(1.8) \qquad (Sf)(x) = \left(\int_0^\infty |(\phi_s * f)(x)|^2 \frac{ds}{s}\right)^{1/2}$$

to find the estimate

$$(1.9) \qquad \|S(\Delta u)_j\|_{L_q(\mathbb{R}^3)} \leq C2^{-2|j|}\|SLu\|_{L_q(\mathbb{R}^3)}$$

for each $j \in \mathbb{Z}$ provided $2 < q < \infty$. An important fact is that $\|S(\cdot)\|_{L_q(\mathbb{R}^n)}$ is equivalent to $\|\cdot\|_{L_q(\mathbb{R}^n)}$ under some conditions on the family $\{\phi_s\}_{s>0}$ of rapidly decreasing functions. Therefore, the method of the square function enables us to reduce the study of L_q-norms to that of quadratic expressions. The L_q boundedness of the Hardy-Littlewood maximal function is also employed in the proof of (1.9). The other case $1 < q < 2$ follows from a duality argument. Concerning the tools above from harmonic analysis, we refer to [35], [36]. The result of [13] has been generalized by [10] (see also [11]) for the case where the translation of the body is also taken into account.

In [26] the harmonic-analytic approach explained above has been developed to prove the L_q estimate of $(\nabla u, p)$ for the linear whole space problem

$$Lu + \nabla p = f, \qquad \text{div } u = 0 \qquad \text{in } \mathbb{R}^3,$$

when the external force f is taken from the space $\dot{W}_q^{-1}(\mathbb{R}^3)$. Since the kernel $\Gamma(x,y)$ is not symmetric, the second derivative $\nabla_x\nabla_y\Gamma(x,y)$ is more complicated; nevertheless, the essential part of the argument of [13] works well. By means of a localization procedure, [26] has studied the existence, uniqueness and L_q estimate

$$(1.10) \qquad \|\nabla u\|_{L_q(D)} + \|p\|_{L_q(D)} \leq C\|f\|_{\dot{W}_q^{-1}(D)}$$

of weak solutions to the linear exterior problem

$$(1.11) \qquad Lu + \nabla p = f, \qquad \text{div } u = 0 \qquad \text{in } D; \qquad u|_{\partial D} = 0,$$

provided $n/(n-1) = 3/2 < q < 3 = n$. This result is regarded as a generalization of [4], [20], [27] and [28] for the usual Stokes problem (the case $\omega = 0$), since the restriction on the exponent q is the same. It is worth while emphasizing that the restriction above is optimal; that is, $q > n/(n-1)$ is necessary for the solvability in the class $(u,p) \in \dot{W}_q^1 \times L_q$ for all $f \in \dot{W}_q^{-1}$, and so is $q < n$ for the uniqueness in that class. Because of lack of the case $q = 3/2 = n/2$, which comes from the nonlinearity $u \cdot \nabla u$, the L_q-theory does not help us to solve the steady Navier-Stokes problem (1.3)-(1.4); in fact, $\|u \cdot \nabla u\|_{\dot{W}_q^{-1}(D)} \leq C\|\nabla u\|_{L_q(D)}^2$ holds if and only if $q = n/2$. Therefore, we have to replace $L_{3/2}$ by a larger space. To do so, in the case $\omega = 0$, Kozono and Yamazaki [29] first introduced the Lorentz space. Our result tells us that, in the problem (1.3)-(1.4) as well, a right class to find the solution is $(\nabla u, p) \in L_{3/2,\infty}$.

   1.4. *Linearized problem in $L_{3/2,\infty}$*
   In [12], instead of (1.10), the $L_{3/2,\infty}$ estimate

$$(1.12) \qquad \|\nabla u\|_{L_{3/2,\infty}(D)} + \|p\|_{L_{3/2,\infty}(D)} \leq C\|f\|_{\dot{W}_{3/2,\infty}^{-1}(D)}$$

for the problem (1.11) is derived. Once this is established, a fixed point argument gives us a unique solution of (1.3)-(1.4) in the class $(\nabla u, p) \in L_{3/2,\infty}$. In the proof of the solvability of (1.11) for all $f \in \dot{W}_{3/2,\infty}^{-1}$, a duality argument due to [29] does not seem to be applied to our problem because of lack of homogeneity of the equation unlike the usual case $\omega = 0$. We thus follow, in principle, the argument of Shibata and Yamazaki [32], in which the solution is constructed without any duality argument for the Oseen problem to study the uniformity of solutions with respect to constant flow at infinity. Note that one cannot use any continuity argument since C_0^∞ is not dense in $L_{q,\infty}$. So, as in [32], given $f \in$

$\dot{W}_{3/2,\infty}^{-1}$, we try to construct directly the solution to (1.11). Though cut-off procedures were carried out twice in [32], we use such a procedure only once to obtain the solution as below; in this point, the proof of [32] is simplified. Let (v, π) be a parametrix (an approximation of the solution), which is constructed by use of solutions in the whole space and in a bounded domain combined with the Bogovskii operator [2]. Then (v, π) satisfies $Lv + \nabla\pi = f + Rf$ and div $v = 0$ in D with $v|_{\partial D} = 0$, where Rf is a remainder term with compact support. It is possible to show that the operator $1 + R$ has a bounded inverse in $\dot{W}_{3/2,\infty}^{-1}$.

In the next section we present the main theorems: Theorems 2.1 and 2.2 for the linear whole space problem (2.7) below, Theorems 2.3 and 2.4 for the linear exterior problem (1.11) and Theorem 2.5 for the Navier-Stokes problem (1.3)-(1.4). We sketch the proof of only Theorems 2.2 and 2.4, which are the central parts of the proof of Theorem 2.5, in the final section.

§2. Results

2.1. *Function spaces*

To begin with, we introduce notation. Let Ω be a smooth domain in $\mathbb{R}^3$; especially, we need function spaces on $\Omega = D, \mathbb{R}^3$, or a bounded domain. By $C_0^\infty(\Omega)$ we denote the class of smooth functions with compact supports in Ω. For $1 \le q \le \infty$, the usual Lebesgue spaces are denoted by $L_q(\Omega)$ with norm $\|\cdot\|_{q,\Omega}$. We need the Lorentz spaces $L_{q,r}(\Omega)$, with norm $\|\cdot\|_{q,r,\Omega}$, that are defined by use of average functions; for details, see [1]. For $1 < q < \infty$ and $1 \le r \le \infty$, the Lorentz spaces can be constructed via real interpolation

$$L_{q,r}(\Omega) = (L_1(\Omega), L_\infty(\Omega))_{1-1/q, r}.$$

For

(2.1) $1 < q < \infty, \quad 1 < r \le \infty, \quad \dfrac{1}{q'} + \dfrac{1}{q} = 1, \quad \dfrac{1}{r'} + \dfrac{1}{r} = 1,$

we have the duality relation

$$L_{q,r}(\Omega) = L_{q',r'}(\Omega)^*.$$

In particular, $L_{q,\infty}(\Omega) = L_{q',1}(\Omega)^*$ is well known as the weak-L_q space, in which $C_0^\infty(\Omega)$ is not dense, and f is in $L_{q,\infty}(\Omega)$ if and only if

$$\sup_{\sigma>0} \sigma \, |\{x \in \Omega; |f(x)| > \sigma\}|^{1/q} < \infty,$$

where $|\cdot|$ stands for the Lebesgue measure. In what follows, we adopt the same symbols for denoting the vector and scalar function spaces as long as there is no confusion, and we use abbreviations $\|\cdot\|_q = \|\cdot\|_{q,D}$ and $\|\cdot\|_{q,r} = \|\cdot\|_{q,r,D}$ for the exterior domain D.

We need the homogeneous Sobolev spaces. For $1 < q < \infty$, let $\dot{W}_q^1(\Omega)$ be the completion of $C_0^\infty(\Omega)$ with respect to the norm $\|\nabla(\cdot)\|_{q,\Omega}$, and $\dot{W}_q^{-1}(\Omega)$ the dual space of $\dot{W}_{q'}^1(\Omega)$ where $1/q' + 1/q = 1$. Let

$$(2.2) \quad 1 < q_0 < q < q_1 < \infty, \quad 1/q = (1-\theta)/q_0 + \theta/q_1, \quad 1 \le r \le \infty.$$

We then define

$$\dot{W}_{q,r}^1(\Omega) = \left(\dot{W}_{q_0}^1(\Omega), \dot{W}_{q_1}^1(\Omega)\right)_{\theta,r},$$

which is independent of the choice of (q_0, q_1), with norm $\|\nabla(\cdot)\|_{q,r,\Omega}$. When $r = \infty$, we note that $C_0^\infty(\Omega)$ is not dense in $\dot{W}_{q,\infty}^1(\Omega)$.

For (q,r) satisfying (2.1), the space $\dot{W}_{q,r}^{-1}(\Omega)$ is defined as the dual space of $\dot{W}_{q',r'}^1(\Omega)$; by duality theorem for interpolation spaces (see [1, 3.7.1]), we see that

$$\dot{W}_{q,r}^{-1}(\Omega) = \left(\dot{W}_{q_0}^{-1}(\Omega), \dot{W}_{q_1}^{-1}(\Omega)\right)_{\theta,r}$$

for q, q_0, q_1, r satisfying (2.2) but $r \ne 1$. Let $1 < q < \infty$ and $1 < r \le \infty$. Then, due to Kozono and Yamazaki [29, Lemma 2.2], for every $f \in \dot{W}_{q,r}^{-1}(\Omega)$, there is a vector function $F \in L_{q,r}(\Omega)$ such that

$$\text{div } F = f, \qquad \|F\|_{q,r,\Omega} \le C\|f\|_{\dot{W}_{q,r}^{-1}(\Omega)}$$

with some $C > 0$.

For the exterior domain D and $1 \le r \le \infty$, if in particular $1 < q < 3 = n$, we then have the characterization

$$(2.3) \qquad \dot{W}_{q,r}^1(D) = \{u \in L_{q_*,r}(D); \nabla u \in L_{q,r}(D), u|_{\partial D} = 0\}$$

together with

$$(2.4) \qquad \|u\|_{q_*,r} \le C\|\nabla u\|_{q,r},$$

where $1/q_* = 1/q - 1/3$. When $q = 3 = n$, we have also $\dot{W}_{3,1}^1(D) \hookrightarrow L_\infty(D) \cap C(D)$ with

$$(2.5) \qquad \|u\|_\infty \leq \frac{1}{3}\|\nabla u\|_{3,1},$$

which will play an important role. Concerning the embedding inequalities (2.4) and (2.5), see [29, Lemma 2.1].

For a bounded domain Ω, $1 < q < \infty$ and $1 \leq r \leq \infty$, we have

$$\dot{W}^1_{q,r}(\Omega) = \{u \in L_{q,r}(\Omega); \nabla u \in L_{q,r}(\Omega), u|_{\partial\Omega} = 0\},$$

with the Poincaré inequality $\|u\|_{q,r,\Omega} \leq C\|\nabla u\|_{q,r,\Omega}$ for $u \in \dot{W}^1_{q,r}(\Omega)$ by real interpolation.

2.2. *Main theorems for the linearized problems*

Let $1 < q < \infty$ and $1 < r \leq \infty$. Let us consider the linear exterior problem (1.11). Given $f \in \dot{W}^{-1}_{q,r}(D)$, the pair of functions (u,p) is called (q,r)-weak solution (q-weak solution when $q = r$) of (1.11) if

(1) $(u,p) \in \dot{W}^1_{q,r}(D) \times L_{q,r}(D)$;
(2) div $u = 0$ in $L_{q,r}(D)$;
(3) $(\omega \wedge x) \cdot \nabla u - \omega \wedge u \in \dot{W}^{-1}_{q,r}(D)$;
(4) (u,p) satisfies $Lu + \nabla p = f$ in the sense of distributions, that is,

$$(2.6) \qquad \langle \nabla u, \nabla \varphi \rangle - \langle (\omega \wedge x) \cdot \nabla u - \omega \wedge u, \varphi \rangle - \langle p, \text{div } \varphi \rangle = \langle f, \varphi \rangle$$

holds for all $\varphi \in C_0^\infty(D)$, where $\langle \cdot, \cdot \rangle$ stands for various duality pairings; by continuity (note $r > 1$), (u,p) satisfies (2.6) for all $\varphi \in \dot{W}^1_{q',r'}(D)$.

When $1 < q < 3$, we have $u \in L_{q_*,r}(D)$ by (2.3), so that $u \to 0$ at infinity in this weak sense.

Since we make use of a cut-off technique, we must consider the whole space problem as well with the inhomogeneous divergence condition

$$(2.7) \qquad Lu + \nabla p = f, \qquad \text{div } u = g \qquad \text{in } \mathbb{R}^3,$$

a weak solution of which is defined in the same way as above.

For (2.7) we study a strong solution too, which is defined as follows. Let $1 < q < \infty$. Given $f \in L_q(\mathbb{R}^3)$ and $g \in \dot{W}^1_q(\mathbb{R}^3)$, the pair of functions (u,p) is called q-strong solution of (2.7) if

(1) $(u,p) \in \dot{W}^2_q(\mathbb{R}^3) \times \dot{W}^1_q(\mathbb{R}^3)$;
(2) div $u = g$ in $\dot{W}^1_q(\mathbb{R}^3)$;

(3) $(\omega \wedge x) \cdot \nabla u - \omega \wedge u \in L_q(\mathbb{R}^3)$;

(4) (u, p) satisfies $Lu + \nabla p = f$ in $L_q(\mathbb{R}^3)$.

Here, the space $\dot{W}_q^1(\mathbb{R}^3)$ has been already introduced, but we here give its characterization

$$\dot{W}_q^1(\mathbb{R}^3) = \{g \in L_{q,loc}(\mathbb{R}^3); \nabla g \in L_q(\mathbb{R}^3)\}/\{\text{constants}\};$$

and also,

$$\dot{W}_q^2(\mathbb{R}^3) = \{v \in L_{q,loc}(\mathbb{R}^3); \nabla^2 v \in L_q(\mathbb{R}^3)\}/\{\text{polynomials of degree} \leq 1\}.$$

The results on the existence, uniqueness and L_q estimates of solutions to (2.7) and to (1.11) are as follows.

Theorem 2.1. ([13]) *Let* $1 < q < \infty$ *and suppose that*

$$f \in L_q(\mathbb{R}^3), \quad g \in \dot{W}_q^1(\mathbb{R}^3), \quad (\omega \wedge x)g \in L_q(\mathbb{R}^3).$$

Then the problem (2.7) possesses a q-strong solution $(u, p) \in \dot{W}_q^2(\mathbb{R}^3) \times \dot{W}_q^1(\mathbb{R}^3)$ *subject to the estimate*

$$
\begin{aligned}
(2.8) \quad & \|\nabla^2 u\|_{q,\mathbb{R}^3} + \|\nabla p\|_{q,\mathbb{R}^3} + \|(\omega \wedge x) \cdot \nabla u - \omega \wedge u\|_{q,\mathbb{R}^3} \\
& \leq C \left(\|f\|_{q,\mathbb{R}^3} + \|\nabla g\|_{q,\mathbb{R}^3} + \|(\omega \wedge x)g\|_{q,\mathbb{R}^3} \right),
\end{aligned}
$$

with some $C > 0$ *independent of* $|\omega|$. *The solution is unique in the class above up to a linear combination of* ω, $\omega \wedge x$ *and* $(x_1, x_2, -2x_3)^T$ *for* u, *and up to a constant for* p.

Theorem 2.2. ([26]) *Let* $1 < q < \infty$ *and suppose that*

$$f \in \dot{W}_q^{-1}(\mathbb{R}^3), \quad g \in L_q(\mathbb{R}^3), \quad (\omega \wedge x)g \in \dot{W}_q^{-1}(\mathbb{R}^3).$$

Then the problem (2.7) possesses a q-weak solution $(u, p) \in \dot{W}_q^1(\mathbb{R}^3) \times L_q(\mathbb{R}^3)$ *subject to the estimate*

$$
\begin{aligned}
(2.9) \quad & \|\nabla u\|_{q,\mathbb{R}^3} + \|p\|_{q,\mathbb{R}^3} + \|(\omega \wedge x) \cdot \nabla u - \omega \wedge u\|_{\dot{W}_q^{-1}(\mathbb{R}^3)} \\
& \leq C \left(\|f\|_{\dot{W}_q^{-1}(\mathbb{R}^3)} + \|g\|_{q,\mathbb{R}^3} + \|(\omega \wedge x)g\|_{\dot{W}_q^{-1}(\mathbb{R}^3)} \right),
\end{aligned}
$$

with some $C > 0$ *independent of* $|\omega|$. *The solution is unique in the class above up to a constant multiple of* ω *for* u.

Theorem 2.3. ([26]) *Let* $3/2 < q < 3$. *For every* $f \in \dot{W}_q^{-1}(D)$, *there exists a unique q-weak solution* $(u,p) \in \dot{W}_q^1(D) \times L_q(D)$ *of the problem (1.11) subject to the estimate*

$$(2.10) \quad \|\nabla u\|_q + \|p\|_q + \|(\omega \wedge x) \cdot \nabla u - \omega \wedge u\|_{\dot{W}_q^{-1}(D)} \le C\|f\|_{\dot{W}_q^{-1}(D)},$$

with some $C > 0$ independent of $|\omega| \in [0, \delta]$, *where $\delta > 0$ is arbitrary.*

The next theorem provides the existence, uniqueness and $L_{3/2,\infty}$ estimate of solutions to (1.11).

Theorem 2.4. ([12]) *Let* $f \in \dot{W}_{3/2,\infty}^{-1}(D)$. *Then the problem (1.11) possesses a unique $(3/2, \infty)$-weak solution* $(u,p) \in \dot{W}_{3/2,\infty}^1(D) \times L_{3/2,\infty}(D)$ *subject to the estimate*

$$(2.11) \quad \begin{aligned} &\|\nabla u\|_{3/2,\infty} + \|p\|_{3/2,\infty} \\ &\quad + \|(\omega \wedge x) \cdot \nabla u - \omega \wedge u\|_{\dot{W}_{3/2,\infty}^{-1}(D)} \le C\|f\|_{\dot{W}_{3/2,\infty}^{-1}(D)}, \end{aligned}$$

with some $C > 0$ independent of $|\omega| \in [0, \delta]$, *where $\delta > 0$ is arbitrary.*

2.3. *Main theorem for the Navier-Stokes problem*

We take a cut-off function $\zeta \in C_0^\infty(\mathbb{R}^3; [0,1])$ satisfying $\zeta = 1$ near the boundary ∂D, and set

$$b(x) = -\frac{1}{2} \operatorname{rot} \left(\zeta(x)|x|^2 \omega \right).$$

Then we see that div $b = 0$ and $b|_{\partial D} = \omega \wedge x$. We thus intend to find the solution to (1.3)-(1.4) as the form $u = v + b$, so that (v,p) should obey

$$(2.12) \quad \begin{cases} Lv + \nabla p = f - \Phi(v,b), & \operatorname{div} v = 0 \quad \text{in } D, \\ v|_{\partial D} = 0, & v \to 0 \quad \text{as } |x| \to \infty, \end{cases}$$

with

$$\begin{aligned} \Phi(v,b) &= (v+b) \cdot \nabla(v+b) + Lb \\ &= \operatorname{div}\left[(v+b) \otimes (v+b) - \nabla b - (\omega \wedge x) \otimes b + b \otimes (\omega \wedge x)\right], \end{aligned}$$

where $w \otimes \widetilde{w} = (w_j \widetilde{w}_k)$; here, note that

$$(\omega \wedge x) \cdot \nabla b = \operatorname{div}\left[(\omega \wedge x) \otimes b\right], \qquad \omega \wedge b = \operatorname{div}\left[b \otimes (\omega \wedge x)\right].$$

Let $f \in \dot{W}_{3/2,\infty}^{-1}(D)$. Since $v \in \dot{W}_{3/2,\infty}^{1}(D)$ implies $\Phi(v,b) \in \dot{W}_{3/2,\infty}^{-1}(D)$, one can define weak solution $(v,p) \in \dot{W}_{3/2,\infty}^{1}(D) \times L_{3/2,\infty}(D)$ of (2.12) by $(3/2,\infty)$-weak solution of (1.11) with f replaced by $f - \Phi(v,b)$.

Theorem 2.5. ([12]) *There is a constant $\eta = \eta(D) > 0$ such that if $f \in \dot{W}_{3/2,\infty}^{-1}(D)$ with*

$$|\omega| + \|f\|_{\dot{W}_{3/2,\infty}^{-1}(D)} \leq \eta,$$

then the problem (2.12) possesses a unique weak solution

$$(v,p) \in \dot{W}_{3/2,\infty}^{1}(D) \times L_{3/2,\infty}(D)$$

subject to the estimate

$$(2.13) \quad \|\nabla v\|_{3/2,\infty} + \|v\|_{3,\infty} + \|p\|_{3/2,\infty} \leq C \left(|\omega| + \|f\|_{\dot{W}_{3/2,\infty}^{-1}(D)} \right),$$

with some $C > 0$ independent of $|\omega|$ and f.

§3. Outline of the proof

3.1. *On the proof of Theorem 2.2* ([26])
For the proof of Theorem 2.2, it suffices to consider

$$(3.1) \qquad\qquad\qquad Lu = f \qquad \text{in } \mathbb{R}^3.$$

For rapidly decreasing forces f, the equation (3.1) admits a solution of the form

$$(3.2) \quad u(x) = \int_{\mathbb{R}^3} \Gamma(x,y)f(y)dy = \int_0^\infty O(|\omega|t)^T \left(e^{t\Delta} f \right) (O(|\omega|t)x)dt$$

with the kernel

$$(3.3) \qquad \Gamma(x,y) = \int_0^\infty O(|\omega|t)^T E_t(O(|\omega|t)x - y)dt,$$

where $E(x) = (4\pi)^{-3/2} e^{-|x|^2/4}$ and $E_t(x) = t^{-3/2} E(x/\sqrt{t})$.

By [27] the class $\{\text{div } F; F \in C_0^\infty(\mathbb{R}^3)\}$ is dense in $\dot{W}_q^{-1}(\mathbb{R}^3)$. Therefore, the essential step is to show

(3.4) $$\|\nabla u\|_{q,\mathbb{R}^3} \le C \|F\|_{q,\mathbb{R}^3},$$

for the force term of the form $f = \operatorname{div} F$ with $F \in C_0^\infty(\mathbb{R}^3)$; hereafter, we will concentrate ourselves on (3.4). Set

(3.5) $$(TF)(x) = \nabla u(x) = -\int_{\mathbb{R}^3} \nabla_x \nabla_y \Gamma(x,y) : F(y)dy,$$

which we rewrite as the form $TF = (T_{\ell m}F)_{1 \le \ell,m \le 3}$ for $F = (F_{\mu\nu})_{1 \le \mu,\nu \le 3}$ with

(3.6) $$\begin{aligned} (T_{\ell m}F)(x) &= \partial_{x_m} u_\ell(x) \\ &= \sum_{\mu,\nu,k} \int_0^\infty O(|\omega|t)_{\ell\mu}^T O(|\omega|t)_{km} (H_{k\nu,t} * F_{\mu\nu})(O(|\omega|t)x) \frac{dt}{t}, \end{aligned}$$

where $H = (H_{k\nu})_{1 \le k,\nu \le 3}$ is the Hessian matrix of E, that is,

$$H_{k\nu}(x) = \partial_{x_\nu} \partial_{x_k} E(x), \qquad H_{k\nu,t}(x) = t^{-3/2} H_{k\nu}(x/\sqrt{t}).$$

By use of the Littlewood-Paley decomposition (1.7), we decompose the function H as

$$H_{k\nu} = \sum_{j \in \mathbb{Z}} H_{k\nu}^{(j)}, \qquad \widehat{H_{k\nu}^{(j)}} = \alpha_j \widehat{H_{k\nu}}.$$

In (3.6) we replace H by $H^{(j)} = \left(H_{k\nu}^{(j)}\right)_{1 \le k,\nu \le 3}$ to define the decomposed operators $T^{(j)} = \left(T_{\ell m}^{(j)}\right)_{1 \le \ell,m \le 3}$ with

$$\left(T_{\ell m}^{(j)}F\right)(x) = \sum_{\mu,\nu,k} \int_0^\infty O(|\omega|t)_{\ell\mu}^T O(|\omega|t)_{km} (H_{k\nu,t}^{(j)} * F_{\mu\nu})(O(|\omega|t)x) \frac{dt}{t},$$

where $H_{k\nu,t}^{(j)}(x) = t^{-3/2} H_{k\nu}^{(j)}(x/\sqrt{t})$.

In order to estimate $T_{\ell m}^{(j)}F$, we make use of the square function (1.8), where $\{\phi_s\}_{s>0}$ is a fixed family of rapidly decreasing and radially symmetric functions satisfying

$$\phi_s(x) = s^{-3/2}\phi_1(x/\sqrt{s}) \qquad \left(\widehat{\phi_s}(\xi) = \widehat{\phi_1}(\sqrt{s}\xi)\right)$$

and

$$(3.7) \qquad \int_{\mathbb{R}^3} \phi_s(x)dx = 0; \qquad \int_0^\infty \widehat{\phi_s}(\xi)^2 \frac{ds}{s} = 1 \qquad (\xi \in \mathbb{R}^3 \setminus \{0\}),$$

together with supp $\widehat{\phi_s} \subset \left\{\xi; \frac{1}{2\sqrt{s}} < |\xi| < \frac{2}{\sqrt{s}}\right\}$. Since $\|S(\cdot)\|_{q,\mathbb{R}^3}$ is equivalent to $\|\cdot\|_{q,\mathbb{R}^3}$ ([36, Chapter I, 8.23]), we have

$$(3.8) \qquad \|T_{\ell m}^{(j)}F\|_{q,\mathbb{R}^3}^2 \le C\|ST_{\ell m}^{(j)}F\|_{q,\mathbb{R}^3}^2 = C\|(ST_{\ell m}^{(j)}F)^2\|_{q/2,\mathbb{R}^3}.$$

Assume now that $1 < q/2 < \infty$. Then one can estimate

$$(3.9) \qquad \langle (ST_{\ell m}^{(j)}F)^2, w \rangle \equiv \int_{\mathbb{R}^3} w(x) \int_0^\infty |(\phi_s * T_{\ell m}^{(j)}F)(x)|^2 \frac{ds}{s} dx$$

for $w \in L_{q/(q-2)}(\mathbb{R}^3)$ to obtain

$$(3.10)$$
$$|\langle (ST_{\ell m}^{(j)}F)^2, w \rangle| \le C \sum_{\mu,\nu,k} \|H_{k\nu}^{(j)}\|_{1,\mathbb{R}^3} \int_{\mathbb{R}^3} \left(M_{k\nu}^{(j)}w\right)(x)\,(SF_{\mu\nu})(x)^2 dx$$

with

$$(3.11) \qquad \left(M_{k\nu}^{(j)}w\right)(x) = \sup_{r>0} \int_{2^{-4}r}^{2^4 r} \left(|\widetilde{H_{k\nu,t}^{(j)}}| * |w\left(O(|\omega|t)^T(\cdot)\right)|\right)(x)\frac{dt}{t},$$

where $\widetilde{H_{k\nu,t}^{(j)}}(x) = H_{k\nu,t}^{(j)}(-x)$. From the pointwise estimate

$$|H_{k\nu}^{(j)}(x)| \le C2^{-2|j|}\psi_{2^{-2j}}(x),$$

where $\psi(x) = (1 + |x|^2)^{-2}$ and $\psi_t(x) = t^{-3/2}\psi(x/\sqrt{t})$, we obtain

Proposition 3.1. *Let* $1 < p < \infty$. *Then the sublinear operator defined by (3.11) enjoys*

$$\|M_{k\nu}^{(j)}w\|_{p,\mathbb{R}^3} \le C2^{-2|j|}\|w\|_{p,\mathbb{R}^3},$$

with some $C = C(p) > 0$ *independent of* $w \in L_p(\mathbb{R}^3), j \in \mathbb{Z}, 1 \le k, \nu \le 3$ *and* $|\omega|$.

In view of (3.10), we use Proposition 3.1 as well as $\|H_{k\nu}^{(j)}\|_{1,\mathbb{R}^3} \le C2^{-2|j|}$ to see that

$$|\langle (ST_{\ell m}^{(j)}F)^2, w\rangle| \le C\left(2^{-2|j|}\right)^2 \|w\|_{q/(q-2),\mathbb{R}^3} \sum_{\mu,\nu} \|F_{\mu\nu}\|_{q,\mathbb{R}^3}^2,$$

for all $w \in L_{q/(q-2)}(\mathbb{R}^3)$. By duality and by (3.8) we arrive at

$$\|T_{\ell m}^{(j)}F\|_{q,\mathbb{R}^3} \le C2^{-2|j|}\|F\|_{q,\mathbb{R}^3},$$

with some $C > 0$ independent of $F \in C_0^\infty(\mathbb{R}^3), j \in \mathbb{Z}, 1 \le \ell, m \le 3$ and $|\omega|$. Hence, as long as $2 < q < \infty$,

$$T = (T_{\ell m})_{1\le \ell,m\le 3} \quad \text{with} \quad T_{\ell m} = \sum_{j\in\mathbb{Z}} T_{\ell m}^{(j)}$$

is well-defined as a bounded operator on $L_q(\mathbb{R}^3)$. For $1 < q < 2$, we use the adjoint operator T^*. The same argument as above implies that T^* is also a bounded operator on $L_{q/(q-1)}(\mathbb{R}^3)$; so, T is L_q-bounded for $1 < q < 2$ as well. We have thus proved (3.4) for $1 < q < \infty$.

3.2. *On the proof of Theorem 2.4* ([12])

By real interpolation Theorem 2.2 implies

Proposition 3.2. *Let $1 < q < \infty$ and suppose that*

$$f \in \dot{W}_{q,\infty}^{-1}(\mathbb{R}^3), \quad g \in L_{q,\infty}(\mathbb{R}^3), \quad (\omega \wedge x)g \in \dot{W}_{q,\infty}^{-1}(\mathbb{R}^3).$$

Then the problem (2.7) possesses a (q, ∞)-weak solution

$$(u, p) \in \dot{W}_{q,\infty}^1(\mathbb{R}^3) \times L_{q,\infty}(\mathbb{R}^3)$$

subject to the estimate

(3.12)
$$\|\nabla u\|_{q,\infty,\mathbb{R}^3} + \|p\|_{q,\infty,\mathbb{R}^3} + \|(\omega \wedge x) \cdot \nabla u - \omega \wedge u\|_{\dot{W}_{q,\infty}^{-1}(\mathbb{R}^3)}$$
$$\le C\left(\|f\|_{\dot{W}_{q,\infty}^{-1}(\mathbb{R}^3)} + \|g\|_{q,\infty,\mathbb{R}^3} + \|(\omega \wedge x)g\|_{\dot{W}_{q,\infty}^{-1}(\mathbb{R}^3)}\right),$$

where $C > 0$ is independent of $|\omega|$. The solution is unique in the class above up to a constant multiple of ω for u.

Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with smooth boundary $\partial\Omega$, and let us consider

(3.13) $\qquad Lu + \nabla p = f, \qquad \text{div } u = 0 \qquad \text{in } \Omega; \qquad u|_{\partial\Omega} = 0.$

For the usual Stokes problem (the case $\omega = 0$), L_q- and $L_{q,\infty}$-theories are known, see [6], [34], [27] and [29]. In bounded domains, the operator L can be treated as a perturbation to the Laplace operator; thus, we have

Proposition 3.3. *Let Ω be as above and let $1 < q < \infty$. Suppose that $f \in \dot{W}_{q,\infty}^{-1}(\Omega)$. Then the problem (3.13) possesses a unique (up to an additive constant for p) (q, ∞)-weak solution $(u, p) \in \dot{W}_{q,\infty}^{1}(\Omega) \times L_{q,\infty}(\Omega)$ subject to the estimate*

$$(3.14) \qquad \|\nabla u\|_{q,\infty,\Omega} + \|u\|_{q,\infty,\Omega} + \|p - \overline{p}\|_{q,\infty,\Omega} \leq C\|f\|_{\dot{W}_{q,\infty}^{-1}(\Omega)},$$

with some $C > 0$ independent of $|\omega| \in [0, \delta]$, where $\delta > 0$ is arbitrary and $\overline{p} = \dfrac{1}{|\Omega|} \displaystyle\int_{\Omega} p(x)dx$.

We first see that the uniqueness part of Theorem 2.4 follows from the L_q-theory (Theorem 2.3) by using the similar cut-off procedure to [26, Lemma 5.2]. Given $f \in \dot{W}_{3/2,\infty}^{-1}(D)$, we next intend to construct the solution of (1.11) with use of the solutions in the whole space and in a bounded domain. We fix $\rho > 0$ so large that $\mathbb{R}^3 \setminus D \subset B_{\rho-5}$ (the open ball centered at the origin with radius $\rho - 5$), and take the cut-off functions $\phi_j \in C^\infty(\mathbb{R}^3; [0, 1]), j = 0, 1, 2$, satisfying

$$\phi_1(x) = \begin{cases} 0, & |x| \leq \rho - 5, \\ 1, & |x| \geq \rho - 4, \end{cases}$$

$$\phi_j(x) = \begin{cases} 1, & |x| \leq \rho - 3 + j, \\ 0, & |x| \geq \rho - 2 + j, \end{cases} \qquad (j = 0, 2).$$

We set

$$D_\rho = D \cap B_\rho, \qquad A = \{x \in \mathbb{R}^3; \rho - 4 < |x| < \rho - 1\}.$$

Consider (2.7) with f replaced by $\phi_1 f$ and $g = 0$ in the whole space $\mathbb{R}^3$. We see that $\phi_1 f \in \dot{W}_{3/2,\infty}^{-1}(\mathbb{R}^3)$ with $\|\phi_1 f\|_{\dot{W}_{3/2,\infty}^{-1}(\mathbb{R}^3)} \leq C\|f\|_{\dot{W}_{3/2,\infty}^{-1}(D)}$. Let (u_∞, p_∞) be the solution obtained in Proposition 3.2 for the external force $\phi_1 f$, and by

$$(Q_\infty, \Pi_\infty) : \dot{W}_{3/2,\infty}^{-1}(D) \ni f \mapsto (u_\infty, p_\infty) \in \dot{W}_{3/2,\infty}^{1}(\mathbb{R}^3) \times L_{3/2,\infty}(\mathbb{R}^3)$$

we denote the solution operator. Here, u_∞ is uniquely chosen in such a way that $u_\infty \in L_{3,\infty}(\mathbb{R}^3)$. We also consider (3.13) with f replaced by $\phi_2 f$ in the bounded domain $\Omega = D_\rho$. We easily see that $\phi_2 f \in$

$\dot{W}_{3/2,\infty}^{-1}(D_\rho)$ with $\|\phi_2 f\|_{\dot{W}_{3/2,\infty}^{-1}(D_\rho)} \leq C\|f\|_{\dot{W}_{3/2,\infty}^{-1}(D)}$. Let (u_0, p_0) be the solution obtained in Proposition 3.3 for the external force $\phi_2 f$ and by

$$(Q_0, \Pi_0) : \dot{W}_{3/2,\infty}^{-1}(D) \ni f \mapsto (u_0, p_0) \in \dot{W}_{3/2,\infty}^1(D_\rho) \times L_{3/2,\infty}(D_\rho)$$

we denote the solution operator. Here, p_0 is uniquely chosen in such a way that $\int_{D_\rho} p_0(x)dx = 0$. As a parametrix (an approximation of the solution) for the exterior problem, we take

$$(3.15) \quad \begin{cases} Qf = (1 - \phi_0)Q_\infty f + \phi_0 Q_0 f + B[(Q_\infty f - Q_0 f) \cdot \nabla \phi_0], \\ \Pi f = (1 - \phi_0)\Pi_\infty f + \phi_0 \Pi_0 f, \end{cases}$$

where B is the Bogovskii operator, which makes the correction of divergence, in the bounded domain A, see [2], [5], [15]. Concerning the class of $(Qf, \Pi f)$, we have

Proposition 3.4. *Let* $f \in \dot{W}_{3/2,\infty}^{-1}(D)$. *Then* $(Qf, \Pi f) \in \dot{W}_{3/2,\infty}^1(D) \times L_{3/2,\infty}(D)$ *with*

$$\|\nabla Qf\|_{3/2,\infty} + \|\Pi f\|_{3/2,\infty} \leq C\|f\|_{\dot{W}_{3/2,\infty}^{-1}(D)},$$

for some $C > 0$ *independent of* $|\omega| \in [0, \delta]$, *where* $\delta > 0$ *is arbitrary.*

We see that $(v, \pi) = (Qf, \Pi f)$ is a distribution solution to

$$(3.16) \quad Lv + \nabla \pi = f + Rf, \quad \text{div } v = 0 \quad \text{in } D; \quad v|_{\partial D} = 0,$$

where $R : f \mapsto$ (remainder); that is,

$$Rf = - 2\nabla \phi_0 \cdot \nabla (Q_\infty f - Q_0 f) - [\Delta \phi_0 + (\omega \wedge x) \cdot \nabla \phi_0](Q_\infty f - Q_0 f)$$
$$- LB[(Q_\infty f - Q_0 f) \cdot \nabla \phi_0] + (\nabla \phi_0)(\Pi_\infty f - \Pi_0 f),$$

for which we have the following lemma.

Lemma 3.1. *Let* $f \in \dot{W}_{3/2,\infty}^{-1}(D)$. *Then* $Rf \in \dot{W}_{3/2,\infty}^{-1}(D)$ *with*

$$\|Rf\|_{\dot{W}_{3/2,\infty}^{-1}(D)} \leq C\|f\|_{\dot{W}_{3/2,\infty}^{-1}(D)}.$$

In the proof, the embedding relation (2.5) plays a fundamental role. By Lemma 3.1 and Proposition 3.4, we find that $(v, \pi) = (Qf, \Pi f)$ is a $(3/2, \infty)$-weak solution of (3.16).

Proposition 3.5. *The operator R is compact from $\dot{W}_{3/2,\infty}^{-1}(D)$ into itself. And further, $1 + R$ has a bounded inverse in $\dot{W}_{3/2,\infty}^{-1}(D)$.*

By Propositions 3.4 and 3.5 the pair of

$$u = Q(1 + R)^{-1}f, \qquad p = \Pi(1 + R)^{-1}f,$$

provides a $(3/2,\infty)$-weak solution of (1.11) with $f \in \dot{W}_{3/2,\infty}^{-1}(D)$ and the estimate (2.11) holds. This shows the existence part. Finally, one can show that the constant $C > 0$ in (2.11) is independent of $|\omega| \in [0,\delta]$ by means of a contradiction argument.

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Department of Applied Mathematics
Faculty of Engineering
Niigata University
Niigata 950-2181 Japan