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Dispersive estimates for solutions of multi-dimensional isotropic symmetric hyperbolic systems

Thomas C. Sideris

Abstract.

An approach to obtaining dispersive estimates for certain multi-dimensional linear hyperbolic systems will be described. The key result is a local energy decay estimate which provides the intermediate link between the generalized Sobolev inequality and the energy inequality.

§1. Introduction

Dispersive estimates play a central role in the perturbative existence theory of nonlinear hyperbolic systems. Traditionally, such estimates are proved by means of representation formulae. The generalized energy method offers an alternative approach in problems which possess Lorentz invariance [2] or, as has been more recently found, Galilean invariance [3, 4, 5, 6, 7]. The nonlinear analysis in these recent works relied heavily on key linear estimates which establish, in an *ad hoc* manner, local energy decay.

This note summarizes a unified point of view for obtaining these local energy decay estimates. The general framework is based on symmetric hyperbolic systems, in combination with a system of constraint equations. The constraints are essential because they rule out time-independent solutions for which decay cannot hold. The other key ingredient is an isotropic spectral assumption on the symbol associated to the problem, guaranteeing the existence of commuting vector fields. An additional dissipation term can be included at no extra cost.

The main result, appearing in section 4, shows that solutions decompose into individual wave families, corresponding to the eigenstates of

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the symbol. Thanks to the isotropy assumption, the characteristic cones are standard, and the components related to the positive eigenvalues concentrate along these cones. This is reminiscent of the one-dimensional picture where wave families propagate along characteristics, [1]. The remaining components, associated to the nonpositive eigenvalues, actually decay uniformly in L^2. In the anisotropic case, solutions still decay uniformly in L^2 in a region which is strictly interior to all characteristic cones, but detailed information along the cones is lost. A detailed proof of this result can be found in [8].

For completeness, an outline is also given for combining this result with energy and Sobolev inequalities to obtain pointwise dispersive estimates. The main point is that in the Galilean invariant case, one cannot pass directly from L^∞ to L^2 via the generalized Sobolev inequality. Local energy decay forms the intermediate link. This will be explained in the linear context, but the utility of these estimates is best seen in nonlinear applications coming from nonlinear elastodynamics, see [6], [7]. Specific examples of relevant linearized systems appear in [8].

§2. Framework

Let $\mathcal{V}$ and $\mathcal{W}$ be finite dimensional inner product spaces over $\mathbb{R}$. We will be concerned with $\mathcal{V}$-valued strong solutions $u : [0,T) \times \mathbb{R}^n \to \mathcal{V}$ of the linear system

$$\text{(PDE1)} \quad L(\partial)u - \nu\Delta u = f \quad \text{with} \quad L(\partial) = \partial_t + A(\nabla), \quad A(\nabla) = A_k\partial_k$$

together a system of constraints

$$\text{(PDE2)} \qquad B(\nabla)u = g \quad \text{with} \quad B(\nabla) = B_k\partial_k.$$

Here, we suppose that

$$A_k \in \mathcal{L}(\mathcal{V},\mathcal{V}), \quad B_k \in \mathcal{L}(\mathcal{V},\mathcal{W}), \quad k = 1,\dots,n$$

and

$$f : [0,T) \times \mathbb{R}^n \to \mathcal{V}, \quad g : [0,T) \times \mathbb{R}^n \to \mathcal{W}$$

are functions whose required regularity will become clear below.

The first assumption is the symmetry of the coefficients of (PDE1) as elements of $\mathcal{L}(\mathcal{V},\mathcal{V})$

$$\text{(A1)} \qquad A_k = A_k^*, \quad k = 1,\dots,n.$$

Associated to the differential operators $A(\nabla)$ and $B(\nabla)$, define the symbols

$$A(\xi) = A_k\xi^k \quad \text{and} \quad B(\xi) = B_k\xi^k, \quad \xi \in \mathbb{R}^n.$$

The second assumption is the nondegeneracy condition

$$\text{(A2)} \qquad \ker B(\xi) \cap \ker A(\xi) = \{0\}, \quad \text{for every} \quad 0 \neq \xi \in \mathbb{R}^n.$$

The third assumption is a sort of isospectral condition. Assume that there exist smooth maps taking the identity to the identity[1] such that

$$V : SO(\mathbb{R}^n) \to SO(\mathcal{V}) \quad \text{and} \quad W : SO(\mathbb{R}^n) \to \mathcal{L}(\mathcal{W}, \mathcal{W})$$

such that for every $\xi \in \mathbb{R}^n$ and $R \in SO(\mathbb{R}^n)$

$$\text{(A3)} \qquad A(R\xi) = V(R)A(\xi)V(R)^*$$

and

$$\text{(A4)} \qquad B(R\xi) = W(R)B(\xi)V(R)^*.$$

We will see momentarily that these assumptions imply that, in a certain sense, the system is isotropic and that there exists a useful collection of commuting vector fields.

§3. Basic consequences of the assumptions

Here we list several useful and important properties implied by the assumptions (A1)-(A4).

Isospectral property

Lemma 1. *The spectrum of $A(\omega)$ is real and independent of $\omega \in S^{n-1}$. The nonzero eigenvalues of $A(\omega)$ occur in plus/minus pairs.*

Invariance property

Lemma 2. *For any smooth function $u : \mathbb{R}^n \to \mathcal{V}$ and any $R \in SO(\mathbb{R}^n)$, the following hold*

$$\begin{aligned} A(\nabla)[V(R)u(R^*x)] &= V(R)[A(\nabla)u](R^*x), \\ B(\nabla)[V(R)u(R^*x)] &= W(R)[B(\nabla)u](R^*x), \\ \Delta[V(R)u(R^*x)] &= V(R)[\Delta u](R^*x). \end{aligned}$$

[1] In typical applications these maps are homomorphisms.

Vector fields

Let $\{e_i\}_{i=1}^n$ be the standard basis on $\mathbb{R}^n$, and define the anti-symmetric maps

$$S_{ij} = e_i \otimes e_j - e_j \otimes e_i \quad 1 \leq i < j \leq n.$$

Then $R_{ij}(\tau) = \exp(\tau S_{ij})$ is a smooth one-parameter family in $SO(\mathbb{R}^n)$. It is natural to consider the vector fields arising as the infinitesimal generators of the invariants

$$\frac{d}{d\tau} V(R_{ij}(\tau))u(R_{ij}(\tau)^* x)|_{\tau=0} = \Omega_{ij}u(x) + Z_{ij}u(x) \equiv \widetilde{\Omega}_{ij}u(x),$$

where $\Omega_{ij} = x^i\partial_j - x^j\partial_i$ are the standard angular momentum operators (note that we have used the fact that $V(I) = I$) and

$$Z_{ij} = \frac{d}{d\tau} V(R_{ij}(\tau))|_{\tau=0} \in \mathcal{L}(\mathcal{V}, \mathcal{V}).$$

Further, we define

$$Y_{ij} = \frac{d}{d\tau} W(R_{ij}(\tau))|_{\tau=0} \in \mathcal{L}(\mathcal{W}, \mathcal{W}).$$

We shall also make use of the scaling vector field

$$S = t\partial_t + r\partial_r.$$

Commutation properties

Lemma 3. *If u is a sufficiently regular solution to* (PDE1),(PDE2), *then*

$$[L(\partial) - \nu\Delta]\widetilde{\Omega}_{ij}u = \widetilde{\Omega}_{ij}f$$

and

$$B(\nabla)\widetilde{\Omega}_{ij}u = (\Omega_{ij} + Y_{ij})g.$$

In addition, for any positive integer p,

$$[L(\partial) - \nu\Delta]S^p u = (S+1)^p f - \sum_{j=0}^{p-1} (-1)^{p-j} \binom{p}{j} \nu\Delta S^j u,$$

and

$$B(\nabla)S^p u = (S+1)^p g.$$

This result combined with the energy method allows for the estimation of these derivatives in L^2, see Lemma 6.

Spectral projections

Let $\{\lambda_\beta\}$ denote the distinct eigenvalues of $A(\omega)$. For each $\omega \in S^{n-1}$, let $\mathcal{P}_\beta(\omega) \in \mathcal{L}(\mathcal{V}, \mathcal{V})$ be the orthogonal projection onto the eigenspace of $A(\omega)$ corresponding to the eigenvalue λ_β.

Lemma 4. *The orthogonal projections $\mathcal{P}_\beta(\omega)$ are smooth functions of $\omega = x/|x|$ on S^{n-1} which satisfy the commutation property $[\widetilde{\Omega}_{ij}, \mathcal{P}_\beta(\omega)] = 0$.*

Plane waves

Consider a plane wave solution of the operator $L(\partial)$:

$$u(t,x) = \phi(\langle \omega, x\rangle - \lambda_\beta t)\psi_\beta(\omega),$$

in which $\psi_\beta(\omega)$ is an eigenvector of $A(\omega)$ for λ_β. Our assumptions imply that the propagation speed λ_β is independent of the direction of propagation ω and that a rotation R of the propagation direction produces a corresponding rotation $V(R)$ of the eigenspace of the polarization vector $\psi_\beta(\omega)$. In this sense, the system is isotropic.

If the eigenvalue $\lambda_\beta = 0$, then the plane wave solution is stationary. This solution does not satisfy the homogeneous version of the constraint equation thanks to the nondegeneracy condition (A2).

§4. Dispersive estimates

From now on, we regard the projections $\{\mathcal{P}_\beta(\omega)\}$ onto the eigenspaces of $A(\omega)$ as homogeneous functions of degree zero on $\mathbb{R}^n$, by setting $\omega = x/|x|$.

Weighted L^2-inequality

Theorem 1. *Let $n \geq 2$ and $j = 1, \ldots, n$. Assume that conditions* (A1) *and* (A2) *hold. There are positive constants α and C, depending on the coefficients A_k and B_k, such that all sufficiently regular solutions of* (PDE1), (PDE2) *satisfy the estimate*

$$\begin{aligned}\alpha t\|\partial_j u\|_{L^2(\{r\leq \alpha t\},\mathcal{V})} &+ (\nu t)^{1/2}\|\nabla u\|_{L^2(\mathbb{R}^n,\mathcal{V})} + \nu t\|\Delta u\|_{L^2(\mathbb{R}^n,\mathcal{V})} \\ &\leq C\|u\|_{L^2(\mathbb{R}^n,\mathcal{V})} + \|Su\|_{L^2(\mathbb{R}^n,\mathcal{V})} + t\|f\|_{L^2(\mathbb{R}^n,\mathcal{V})} + t\|g\|_{L^2(\mathbb{R}^n,\mathcal{W})}.\end{aligned}$$

If, in addition, conditions (A3), (A4) *hold, then*

$$\begin{aligned}&\|(\lambda_\beta t - r)\mathcal{P}_\beta \partial_j u\|_{L^2(\{r\geq \alpha t\},\mathcal{V})} \\ &\qquad \leq C\left[\|\widetilde{\Omega} u\|_{L^2(\mathbb{R}^n,\mathcal{V})} + \|u\|_{L^2(\mathbb{R}^n,\mathcal{V})}\right] \\ &\qquad\qquad + \|Su\|_{L^2(\mathbb{R}^n,\mathcal{V})} + t\|f\|_{L^2(\mathbb{R}^n,\mathcal{V})},\end{aligned}$$

and

$$\|rB(\omega)\partial_j u\|_{L^2(\{r\geq\alpha t\},\mathcal{W})} \leq C\left[\|\widetilde{\Omega}u\|_{L^2(\mathbb{R}^n,\mathcal{V})} + \|u\|_{L^2(\mathbb{R}^n,\mathcal{V})}\right] + \|rg\|_{L^2(\mathbb{R}^n,\mathcal{W})}.$$

Although it is elementary, the proof will not be given. The interested reader can consult [7]. Nevertheless, here is the key step:

Lemma 5. *Let $n \geq 2$. Suppose that conditions* (A1) *and* (A2) *hold. All sufficiently regular solutions of* (PDE1) *satisfy the estimate*

$$\|(tA(\nabla) - r\partial_r)u\|^2_{L^2(\mathbb{R}^n,\mathcal{V})} + (n-2)\nu t\|\nabla u\|^2_{L^2(\mathbb{R}^n,\mathcal{V})} + (\nu t)^2\|\Delta u\|^2_{L^2(\mathbb{R}^n,\mathcal{V})} \leq \|Su - tf\|^2_{L^2(\mathbb{R}^n,\mathcal{V})}.$$

Proof of Lemma 5. Using the definition of S, we may rewrite (PDE1) as

$$tA(\nabla)u - r\partial_r u - \nu t\Delta u = -Su + tf.$$

Taking the L^2-norm of both sides, this immediately gives

$$\|tA(\nabla)u - r\partial_r u\|^2_{L^2(\mathbb{R}^n,\mathcal{V})} + 2\langle r\partial_r u - tA(\nabla)u, \nu t\Delta u\rangle_{L^2(\mathbb{R}^n,\mathcal{V})} + \|\nu t\Delta u\|^2_{L^2(\mathbb{R}^n,\mathcal{V})} \leq \|Su - tf\|^2_{L^2(\mathbb{R}^n,\mathcal{V})}.$$

Thanks to the symmetry of the coefficient matrices, we find using integration by parts that

$$\langle A(\nabla)u, \Delta u\rangle_{L^2(\mathbb{R}^n,\mathcal{V})} = \langle A_k\partial_k u, \Delta u\rangle_{L^2(\mathbb{R}^n,\mathcal{V})} = 0.$$

Again using integration by parts, we can rewrite the remaining cross term as follows:

$$\begin{aligned}
2\langle r\partial_r u,&\nu t\Delta u\rangle_{L^2(\mathbb{R}^n,\mathcal{V})}\\
&=2\nu t\langle x^j\partial_j u, \Delta u\rangle_{L^2(\mathbb{R}^n,\mathcal{V})}\\
&=-2\nu t\langle x^j\partial_j\partial_k u, \partial_k u\rangle_{L^2(\mathbb{R}^n,\mathcal{V})} - 2\nu t\langle\partial_k u, \partial_k u\rangle_{L^2(\mathbb{R}^n,\mathcal{V})}\\
&=n\nu t\langle\partial_k u, \partial_k u\rangle_{L^2(\mathbb{R}^n,\mathcal{V})} - 2\nu t\langle\partial_k u, \partial_k u\rangle_{L^2(\mathbb{R}^n,\mathcal{V})}\\
&=(n-2)\nu t\langle\partial_k u, \partial_k u\rangle_{L^2(\mathbb{R}^n,\mathcal{V})}\\
&=(n-2)\nu t\|\nabla u\|^2_{L^2(\mathbb{R}^n,\mathcal{V})}.
\end{aligned}$$

Q.E.D.

The proof of Theorem 1 proceeds by extracting the relevant information from the first quantity on the left in Lemma 5.

Higher-order estimates

Let $\Gamma = (\Gamma_1, \ldots, \Gamma_q)$ denote the collection of $q = n + 1 + n(n-1)/2$ vector fields ∂_j, S, $\widetilde{\Omega}_{ij}$. Let $a = (a_1, \ldots, a_m)$ be an m-tuple in $\{1, \ldots, q\}$. We denote by Γ^a the m^{th}-order operator $\Gamma_{a_1} \cdots \Gamma_{a_m}$.

We define the generalized energy norm

$$\|u(t)\|_s^2 = \sum_{|a| \le s} \|\Gamma^a u(t)\|^2_{L^2(\mathbb{R}^n, \mathcal{V})},$$

and we also define the weighted L^2 quantity

$$\mathcal{X}_s[u(t)] = \sum_{j=1}^n \sum_{|a| \le s-1} \Big\{ \|\langle t + r\rangle \mathcal{P}_0(\cdot) \partial_j \Gamma^a u(t, \cdot)\|_{L^2(\mathbb{R}^n, \mathcal{V})} + \sum_{\lambda_\beta \ne 0} \|\langle \lambda_\beta t - r\rangle \mathcal{P}_\beta(\cdot) \partial_j \Gamma^a u(t, \cdot)\|_{L^2(\mathbb{R}^n, \mathcal{V})} \Big\}.$$

We emphasize that the $\mathcal{V}$-inner product is used in the construction of these norms.

For $\mathcal{W}$-valued functions the s-norm is defined analogously using derivatives $\Gamma' = (\partial_j, S, \Omega_{ij} + Y_{ij})$.

Theorem 1 and Lemma 3 imply

Corollary 1. *Let $n \ge 2$. Assume that conditions* (A1)-(A4) *hold. Then all sufficiently regular solutions of* (PDE1), (PDE2) *satisfy*

$$\mathcal{X}_s[u(t)] \lesssim \|u(t)\|_s + t\|f(t)\|_s + t\|g(t)\|_s.$$

Energy estimate

Lemma 6. *Assume that conditions* (A1)-(A4) *hold. Then all sufficiently regular solutions of* (PDE1), (PDE2) *satisfy*

$$\|u(t)\|_s^2 + \int_0^t \nu \|\nabla u(\tau)\|_s^2 \, d\tau \lesssim \|u(0)\|_s^2 + \sum_{|a| \le s} \int_0^t |\langle \Gamma^a f(\tau), \Gamma^a u(\tau)\rangle_{L^2(\mathbb{R}^n, \mathcal{V})}| \, d\tau.$$

This follows by the standard energy method for symmetric systems, Lemma 3, and induction on the number of "S" derivatives.

From Lemma 6 and Corollary 1, we deduce the inequality

$$(\star) \qquad \mathcal{X}_s[u(t)] \lesssim \|u(0)\|_s + \int_0^t \|f(\tau)\|_s d\tau + t[\|f(t)\|_s + \|g(t)\|_s].$$

Sobolev inequality

Here we quote a result in three dimensions. The proof follows from the method in Lemmas 4.1 and 4.2 in [3] and Proposition 3.3 in [5].

Lemma 7. *For* $n = 3$, $|a| + 3 \leq s$, $u : [0, T) \times \mathbb{R}^3 \to \mathcal{V}$,

$$\langle r\rangle\langle t+r\rangle|\mathcal{P}_0(\omega)\partial_j\Gamma^a u(t,x)|_{\mathcal{V}} + \sum_{\lambda_\beta \neq 0} \langle r\rangle\langle \lambda_\beta t - r\rangle|\mathcal{P}_\beta(\omega)\partial_j\Gamma^a u(t,x)|_{\mathcal{V}} \lesssim \mathcal{X}_s[u(t)].$$

In combination with $(\star)$, this gives precise pointwise decay for solutions in the homogeneous case, for example. In nonlinear situations, more work is required, but the same basic strategy can be used.

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Department of Mathematics
University of California
Santa Barbara, CA 93106