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Plane curve singularities whose Milnor and Tjurina numbers differ by three

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Abstract.

Bayer and Hefez described irreducible plane curve singularities whose Milnor and Tjurina numbers differ by one or two, modulo analytic equivalence. After their work, we classify the case in which their difference is three.

§ Introduction

We first define a *plane curve singularity*, which is the main subject in the present paper. Let f be an irreducible element of $\mathbb{C}[[X, Y]]$ such that its partial derivatives f_X and f_Y belong to the maximal ideal (X, Y). Set

$$C := \{u \cdot f \mid u \ \ \textit{is a unit of}\ \mathbb{C}[[X, Y]]\}.$$

If f is a convergent power series, $f = 0$ defines a singular germ of a plane curve at the origin. So it is natural that we call C an irreducible plane curve singularity. The Milnor and Tjurina numbers of C at the origin are defined by,

$$\mu := \dim_{\mathbb{C}} \mathbb{C}[[X, Y]]/(f_X, f_Y) \text{ and } \tau := \dim_{\mathbb{C}} \mathbb{C}[[X, Y]]/(f, f_X, f_Y).$$

It follows from these definitions that $\mu \geq \tau$. We set $r := \mu - \tau$. Let n be the multiplicity of C at the origin. Then there exists a positive integer m with $m > n$ and $n \nmid m$ such that C has the following parametrization at the origin:

$$(1) \qquad x = t^n, \quad y = t^m + a_{m+1}t^{m+1} + \cdots ,$$

where $x \equiv X \bmod (f)$ and $y \equiv Y \bmod (f)$. The local ring of C is defined by $\mathcal{O}_C := \mathbb{C}[[X, Y]]/(f)$. Using the parametrization (1), we have the

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following isomorphism: $\mathcal{O}_C \cong \mathbb{C}[[x, y]] = \mathbb{C}[[t^n, t^m + a_{m+1}t^{m+1} + \cdots]]$. Let D be an irreducible plane curve singularity. Then C and D are said to be *analytically equivalent*, if there exists a $\mathbb{C}$-algebra isomorphism $\mathcal{O}_C \cong \mathcal{O}_D$.

Zariski ([7]) showed that $r = 0$ if and only if C is analytically equivalent to the singularity $Y^n - X^m = 0$ with $\gcd(n, m) = 1$. When $r \neq 0$, he introduced an important invariant λ. Recently, Bayer and Hefez ([3]) classified irreducible plane curve singularities with $r = 1$ and 2. Their work was reviewed by Azevedo in [2]. The aim of this paper is to classify irreducible plane curve singularities with $r = 3$.

Theorem. *Let C be an irreducible plane curve singularity whose parametrization is of the form* (1). *Then we have $r = 3$ if and only if $\gcd(n, m) = 1$ and the parametrization takes one of the following three types. We write $m = pn + q$ with $0 < q < n$.*

Type (i): $\lambda = (n-1)m - 4n$.

(A) $x = t^n$, $y = t^m + t^\lambda$, *where* $n \geq 3$, $p \geq 2$.

(B) $x = t^n$, $y = t^m + t^\lambda + at^{(n-2)m-2n}$, *where* $n \geq 5$, $p = 1$ *and* $a \in \mathbb{C}$.

Type (ii): $\lambda = (n-2)m - 2n$.

(C) $x = t^n$, $y = t^m + t^\lambda + at^{(n-1)m-4n} + bt^{(n-1)m-3n}$, *where* $n \geq 5$, $p \geq 2$ *and* $a\ (\neq 0)$, $b \in \mathbb{C}$.

(D) $x = t^4$, $y = t^m + t^\lambda + at^{3m-16} + bt^{3m-12}$, *where* $p \geq 2$ *and* $a\ (\neq (3m-8)/2m)$, $b \in \mathbb{C}$.

Type (iii) $\lambda = (n-3)m - 2n$

(E) $x = t^n$, $y = t^m + t^\lambda + \sum_{i=1}^{p}(a_i t^{m_i} + b_i t^{n_i}) + \sum_{i=p+1}^{2p} b_i t^{n_i}$, *where* $n > 2q$, $n \geq 5$, $m > 2n/(n-4)$, $a_i, b_i \in \mathbb{C}$, $m_i = (n-2)m - (p+3-i)n$ *and* $n_i = (n-1)m - (2p+3-i)n$.

(F) $x = t^n$, $y = t^m + t^\lambda + \sum_{i=1}^{p}(a_i t^{m_i} + b_i t^{n_i}) + \sum_{i=p+1}^{2p+1} a_i t^{m_i}$, *where* $n < 2q$, $n \geq 5$, $m > 2n/(n-4)$, $a_i, b_i \in \mathbb{C}$, $m_i = (n-1)m - (2p+4-i)n$ *and* $n_i = (n-2)m - (p+4-i)n$.

Furthermore, the coefficients in the parametrizations (E) *and* (F) *must satisfy the relations given in* Tables 1 *and* 2 *in* Section 4, *respectively.*

The present paper is organized as follows: In Section 1, we recall some results on the parametrization of plane curve singularities. The notion "genus" g of an irreducible plane curve singularity plays an important role. We infer from a result of Bayer and Hefez that if $r = 3$, then $g = 1$ or $g = 2$. In Section 2, we study the properties of plane curve singularities of genus one. In particular, we consider the certain types of λ which are needed in the proof of Theorem. In Section 3, we prove the following fact.

Proposition 1. *If $r = 3$, then we have $g = 1$.*

In Section 4, we develop the method in [3] and prove Theorem by using it.

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§1. Semigroups and differentials

Let $\widetilde{C}$ be the nonsingular model of C and we denote by $\mathcal{O}_{\widetilde{C}}$ its local ring. Since $\mathcal{O}_{\widetilde{C}} \cong \mathbb{C}[[t]]$, the order function ν on $\mathbb{C}((t))$ gives a discrete normalized valuation of $\mathcal{O}_{\widetilde{C}}$. We define the *semigroup* of C to be $S := \{\nu(A) \mid A \in \mathcal{O}_C\}$. The *conductor* c of S is characterized by the following properties:

$$c - 1 \notin S \text{ and } c + n \in S \quad \text{for any } n \in \mathbb{N}. \tag{2}$$

It is well known that $\mu = c$ (See [6], Theorem 1). An element of $G := \mathbb{N} \cup \{0\} \setminus S$ is called a *gap* of S. The properties (2) implies that $c - 1$ is the biggest gap of S. We define two sequences (e_i) and (β_i) associated to the parametrization (1) as follows:

$$\begin{aligned} &e_0 = \beta_0 = n, \quad \beta_j = \min\{i \mid i \not\equiv 0 \bmod e_{j-1} \text{ and } a_i \neq 0\}, \\ &e_j = \gcd(e_{j-1}, \beta_j). \end{aligned}$$

It follows that $\beta_1 = m$. Since the relevant exponents in a parametrization of C are coprime, there exists an integer g such that $e_{g-1} \neq 1$ and $e_g = 1$. We call this integer g and the set $\{\beta_0, \ldots, \beta_g\}$ the *genus* of C and the *characteristic* of C respectively. The characteristic of C is denoted by $\mathrm{Ch}(C)$. Define the integers n_i by

$$n_0 = 1 \text{ and } e_{i-1} = n_i e_i, \quad (i = 1, \ldots, g).$$

It follows that $n = n_1 \cdots n_g$. The semigroup S of C is minimally generated by the set of integers $\{v_0, v_1, \ldots, v_g\}$, defined by

$$v_0 = n \text{ and } v_i = n_{i-1} v_{i-1} + \beta_i - \beta_{i-1}, \quad (i = 1, \ldots, g).$$

(See [8], Theorem 3.9) We easily see that $v_1 = m$ and $v_0 < v_1 < \cdots < v_g$.

We denote by Ω^1_C and $\Omega^1_{\widetilde{C}}$ the module of differentials of $\mathcal{O}_C$ and that of $\mathcal{O}_{\widetilde{C}}$, respectively. Note that Ω^1_C is the $\mathcal{O}_C$-module generated by

dx and dy, modulo $f_x dx + f_y dy = 0$. Similarly, $\Omega^1_{\widetilde{C}}$ is the $\mathcal{O}_{\widetilde{C}}$-module generated by dt. Consider the map π^* from Ω^1_C to $\Omega^1_{\widetilde{C}}$ defined by

$$\pi^*\big(A(x, y)dx + B(x, y)dy\big) = A\left(t^n, \varphi(t)\right) dt^n + B(t^n, \varphi(t))d(\varphi(t)),$$

where $(t^n, \varphi(t))$ is the parametrization of C. We naturally extend the valuation ν of $\mathcal{O}_{\widetilde{C}}$ to $\Omega^1_{\widetilde{C}}$ through π^*. Namely, for $\zeta = H(t)dt \in \Omega^1_{\widetilde{C}}$, we define $\nu(\zeta)$ to be $\nu(H(t))$. Let ξ be an element of Ω^1_C. Since Ω^1_C can be regarded as a submodule of $\Omega^1_{\widetilde{C}}$ through its image of π^*, we define $\nu(\xi)$ to be $\nu\big(\pi^*(\xi)\big)$. A differential ξ is said to be *exact*, if there exists an element $A \in \mathcal{O}_C$ such that $\xi = dA$. We denote by $d\mathcal{O}_C$ the set of all exact differentials. Set $V := \nu\left(\Omega^1_C\right) \setminus \nu(d\mathcal{O}_C)$. Since $\nu(d\mathcal{O}_C) = \{l - 1 \in \mathbb{N} \mid l \in S\}$, we have $V = \left\{l - 1 \in \nu\left(\Omega^1_C\right) \mid l \in G\right\}$. Zariski ([7]) showed that

$$r = \dim_{\mathbb{C}}\left(\Omega^1_C / d\mathcal{O}_C\right) = \sharp(V). \tag{3}$$

For the case where $r > 0$, he also showed that $\lambda = \min\{V\} - n + 1$ is an analytic invariant with the following property:

$$\lambda,\ \lambda + n \notin S \text{ and } m < \lambda \leq \beta_2 = v_2 - (n_1 - 1)v_1. \tag{4}$$

We call λ the *Zariski invariant* of C. The differential $\omega := mydx - nxdy$ gives the minimal order $\lambda + n - 1$ in V (See [7]). Furthermore, C is analytically equivalent to the plane curve singularity given by

$$x = t^n, \quad y = t^m + t^\lambda + \cdots. \tag{5}$$

By the way, any integer t can be written in a unique way as

$$t = t_g v_g + \cdots + t_1 v_1 - t_0 v_0, \tag{6}$$

where $t_0, \ldots, t_g$ are integers such that $0 \leq t_i \leq n_i - 1$ for $i = 1, \ldots, g$ (See [1], Lemma I.2.4). It follows from (6) that t belongs to S if and only if $t_0 \leq 0$. The biggest gap of S, $c - 1$, is expressed as $(n_g - 1)v_g + \cdots + (n_1 - 1)v_1 - v_0$. The Zariski invariant is also written as $\lambda = \lambda_g v_g + \cdots + \lambda_1 v_1 - \lambda_0 v_0$ where $0 \leq \lambda_i \leq n_i - 1$ for $i = 1, \ldots, g$. By the properties (4), we easily see that $\lambda_0 \geq 2$.

The $\mathbb{C}$-vector space Ω^1_C is expressed as the following form ([3], Proposition 2):

$$\Omega^1_C = \mathcal{O}_C\omega + d\mathcal{O}_C. \tag{7}$$

So any element of Ω^1_C can be written as $A\omega + dB$ for some $A, B \in \mathcal{O}_C$.

Definition 2. We define the subsets V_0, V_1 of V by

$$V_0 := \nu(\mathcal{O}_C\omega) \setminus \nu(d\mathcal{O}_C), \quad V_1 := V \setminus V_0.$$

Furthermore, define the sets V^+, V_0^+ and V_1^+ by

$$V^+ := \{\alpha + 1 \mid \alpha \in V\}, \quad V_0^+ := \{\alpha + 1 \mid \alpha \in V_0\},$$
$$V_1^+ := \{\alpha + 1 \mid \alpha \in V_1\}.$$

Note that we have $V^+ = V_0^+ \cup V_1^+ \subset G$ where $V_0^+ \cap V_1^+ = \emptyset$. A positive integer α is contained in V if and only if $\alpha + 1$ is contained in V^+. So we have $\sharp(V) = \sharp(V^+)$. It is also clear that the relations $\sharp(V^+) = \sharp(V) = \sharp(V_0) + \sharp(V_1)$, $\sharp(V_0^+) = \sharp(V_0)$ and $\sharp(V_1^+) = \sharp(V_1)$ hold. The formula (3) can be rewritten as

$$r = \sharp(V_0^+) + \sharp(V_1^+). \tag{8}$$

Lemma 3. *Let the genus of C be 1. Then we have*

$$\sharp(V_0^+) = (\lambda_0 - 1)(n - \lambda_1).$$

Proof. Recall that $v_0 = n$, $v_1 = m$ and $\nu(\omega) = \lambda_1 m - (\lambda_0 - 1)n - 1$. If $\gamma \in V_0^+$, then we have $\gamma = \nu(A\omega) + 1$ for some $A \in \mathcal{O}_C$. The gap γ is expressed as $(l_1 + \lambda_1)m - (\lambda_0 - l_0 - 1)n$ where $\nu(A) = l_1 m + l_0 n$. So we have

$$V_0^+ = \left\{ (l_1 + \lambda_1)m - (\lambda_0 - l_0 - 1)n \in V^+ \;\middle|\; \begin{array}{l} 0 \le l_0 \le \lambda_0 - 2, \\ l_1 \ge 0 \end{array} \right\}.$$

Define a subset U_0^+ of V_0^+ by

$$U_0^+ = \left\{ (l_1 + \lambda_1)m - (\lambda_0 - l_0 - 1)n \;\middle|\; \begin{array}{l} 0 \le l_0 \le \lambda_0 - 2, \\ 0 \le l_1 + \lambda_1 \le n - 1 \end{array} \right\}.$$

We prove that $V_0^+ = U_0^+$, which gives the desired result. It is enough to show that $V_0^+ \subset U_0^+$. Take an element $\gamma = (l_1 + \lambda_1)m - (\lambda_0 - l_0 - 1)n$ from V_0^+. If $l_1 + \lambda_1 \le n - 1$, then there is nothing to prove. So we assume that $l_1 + \lambda_1 > n - 1$. Then there exists a positive integer k such that $0 \le l_1 + \lambda_1 - kn \le n - 1$. By using this k, we rewrite γ as the form of (6). That is,

$$\gamma = (l_1 + \lambda_1 - kn)m - (\lambda_0 - km - l_0 - 1)n.$$

Since $\gamma \in V^+$, the inequality $\lambda_0 - km - l_0 - 1 > 0$ holds. Then γ is given by $\nu(x^{km+l_0}\omega) + 1$, so $\gamma \in U_0^+$. Q.E.D.

Remark 4. We infer from Lemma 3 that $\sharp(V_0^+)$ is determined by λ for the case where $g = 1$.

For any genus, the following relations hold (See [3], Proposition 1, Corollary 5):

$$r \geq (\lambda_0 - 1)(n_1 - \lambda_1) \cdots (n_g - \lambda_g), \tag{9}$$

$$r \geq 2^{g-1}. \tag{10}$$

Let $\zeta = \big(at^{\nu(\zeta)} + \text{terms of higher degree}\big)dt$ be an element of $\Omega^1_{\widetilde{C}}$. We denote by $\mathrm{LT}(\zeta)$ the leading term $at^{\nu(\zeta)}$. Let $\mathrm{LC}(\zeta)$ denote the leading coefficient a. For $\xi \in \Omega^1_C$, we simply write $\mathrm{LT}(\xi) = \mathrm{LT}(\pi^*(\xi))$ and $\mathrm{LC}(\xi) = \mathrm{LC}(\pi^*(\xi))$.

Lemma 5. *Let C be an irreducible plane curve singularity of genus g. If $\xi = A\omega + dB$ is an element of Ω^1_C with $\nu(\xi) + 1 \in V_1^+$, then ξ satisfies the following conditions*:

$$\mathrm{LT}(A\omega) + \mathrm{LT}(dB) = 0, \tag{11}$$

$$\nu(\xi) + 1 < \sum_{i=1}^{g} (n_i - 1)v_i - v_0. \tag{12}$$

Proof. If $\mathrm{LT}(A\omega) + \mathrm{LT}(dB) \neq 0$, then $\nu(\xi)$ belongs to V_0 or to $\nu(d\mathcal{O}_C)$. Hence the condition (11) must occur. Let $\lambda = \sum_{i=1}^{g} \lambda_i v_i - \lambda_0 v_0$ be the Zariski invariant of C. Then $\nu(\omega) + 1$ is expressed as $\sum_{i=1}^{g} \lambda_i v_i - (\lambda_0 - 1)v_0$. We know that $\max\{V^+\} \leq \sum_{i=1}^{g}(n_i - 1)v_i - v_0$. Let z_i be an element of $\mathcal{O}_C$ with $\nu(z_i) = v_i$ for $i = 0, \ldots, g$. Then the differential $z_0^{\lambda_0 - 2}\Pi_{i=1}^{g} z_i^{(n_i - \lambda_i - 1)}\omega$ gives the order $\sum_{i=1}^{g}(n_i - 1)v_i - v_0 - 1$. Hence $\sum_{i=1}^{g}(n_i - 1)v_i - v_0 \in V_0^+$. We have the desired consequence. Q.E.D.

There are some criteria for simplifying the parametrization of C modulo analytic equivalence (See [4] and [8], Ch.III, Proposition 1.2; Ch.IV, Lemma 2.6 and Proposition 3.1).

Lemma 6. *Let $a_s t^s$ be a term of y in the parametrization* (5) *where $s > \lambda$ and $a_s \neq 0$. If either*

(EC 1): *s belongs to S, or*
(EC 2): *$s + n = lm$ for some $l \in \mathbb{N}$, or*
(EC 3): *$s - \lambda$ belongs to the subset of S generated by n and m,*

then C is analytically equivalent to an irreducible plane curve singularity given by a parametrization of the same form, but with $a_s = 0$ and a_i unchanged for $i < s$.

Applying Lemma 6 to the parametrization (5), we have the following parametrization:

$$x = t^n, \quad y = t^m + t^\lambda + \sum_{i \in G} a_i t^i.$$

We always consider such parametrizations of C in this paper.

§2. Singularities of genus one

In this section, we consider irreducible plane curve singularities of genus 1. Note that $g = 1$ if and only if $\gcd(n, m) = 1$. In this case, we have $v_0 = n = n_1$ and $v_1 = m$. We write $m = pn + q$ where $0 < q < n$. We first prove the following proposition:

Proposition 7. *If C is given by*

$$x = t^n, \quad y = t^m + t^{(n-1)m-(R+1)n}, \tag{13}$$

where $1 \leq R \leq p+1$, then we have $r = R$.

Proof. Note that $\lambda = (n-1)m - (R+1)n$ in (13). If $1 \leq R \leq p$, then we have $(n-2)m - n < (n-1)m - (p+1)n \leq (n-1)m - (R+1)n$. So the gaps which are greater than λ are

$$(n-1)m - Rn, \ldots, (n-1)m - n. \tag{14}$$

If $R = p+1$, then we have $(n-1)m - (R+1)n < (n-2)m - n < (n-1)m - Rn$. The gaps which are greater than λ are

$$(n-2)m - n, (n-1)m - Rn, \ldots, (n-1)m - n.$$

For both cases, clearly, we have $V_0 = \{\nu(\omega), \nu(x\omega), \ldots, \nu(x^{R-1}\omega)\}$. Note that $\nu(x^i\omega) + 1 = (n-1)m - (R-i)n$ for $i = 0, \ldots, R-1$. Since $\min\{V^+\} = \nu(\omega) + 1$, we conclude that $V_1^+ = \emptyset$. Hence we conclude that $r = R$ by (8). Q.E.D.

Remark 8. Since $\lambda = (n-1)m - (R+1)n$, we infer from Proposition 7 that r is determined by λ for the plane curve singularity given by the parametrization (13).

Remark 9. The cases where $R = 1$ and $R = 2$ in Proposition 7 correspond to Theorems 7 and 17 in [3], respectively.

Corollary 10. *Fix a positive integer n (≥ 3). For any positive integer R, there exists an irreducible plane curve singularity of $g = 1$ with multiplicity n and $r = R$.*

Proof. Put $m = (R+1)n+1$ and $\lambda = (n-1)m-(R+1)n$. Then we have $\lambda > m$ and $(n-1)m-(R+1)n > (n-2)m-n$. Hence the gaps which are greater than λ are same as (14). Therefore the parametrization

$$x = t^n, \quad y = t^m + t^{(n-1)m-(R+1)n}$$

gives the desired singularity. Q.E.D.

In what follows, we consider three types of the values of λ: (i) $\lambda = (n-1)m-4n$, (ii) $\lambda = (n-2)m-2n$, (iii) $\lambda = (n-3)m-2n$, which will be used in the proof of Theorem.

2.1. Type (i): $\lambda = (n-1)m-4n$

Since $\lambda > m$, we must have $(n-2)m > 4n$, hence $n \geq 3$. We first consider the case in which $p \geq 2$. Furthermore, if $p \geq 3$, then the gaps which are greater than λ are

$$(n-1)m-3n, \ (n-1)m-2n, \ (n-1)m-n.$$

On the other hand, if $p = 2$, then we have the following gaps:

$$(n-2)m-n, \ (n-1)m-3n, \ (n-1)m-2n, \ (n-1)m-n.$$

For both cases, by Lemma 6, the parametrization of C can be taken as

$$x = t^n, \quad y = t^m + t^\lambda. \tag{15}$$

Next we consider the case in which $p = 1$. This case occurs only when $n \geq 5$. The gaps which are greater than λ are

$$(n-2)m-2n, \ (n-1)m-3n, \ (n-2)m-n,$$
$$(n-1)m-2n, \ (n-1)m-n.$$

By Lemma 6, the parametrization of C can be taken as

$$x = t^n, \quad y = t^m + t^\lambda + at^{(n-2)m-2n}, \quad (a \in \mathbb{C}). \tag{16}$$

Remark 11. According to the conditions: (1) $\gcd(n, m) = 1$, (2) $m > 4n/(n-2)$, we have some restrictions on p, q. First of all, we must have $q \geq 1$. We also infer that $\gcd(n, q) = 1$.

2.2. Type (ii): $\lambda = (n-2)m - 2n$

It follows from $\lambda > m$ that $n \geq 4$ and $m > 2n/(n-3)$. Since $\nu(\omega) + 1 = (n-2)m - n$, we find that $V_0 = \{\nu(\omega), \nu(y\omega)\}$. The gaps which are greater than λ are

$$(n-1)m - (p+2)n,\ (n-2)m - n,\ (n-1)m - (p+1)n,$$
$$(n-1)m - pn,\ \ldots,\ (n-1)m - n.$$

By Lemma 6, the parametrization of C can be taken as

$$x = t^n, \quad y = t^m + t^\lambda + \sum_{i=1}^{p} a_i t^{m_i}, \tag{17}$$

where $m_i = (n-1)m - (p+3-i)n$ and $a_i \in \mathbb{C}$.

Definition 12. Define the differentials η_k for $k \geq 1$ by

$$\eta_k := x^k\omega + d\left(u_k x^{k-1} y^{n-2}\right) \text{ where } u_k = \frac{-n(m-\lambda)}{(k-1)n + (n-2)m}.$$

Then we have

$$\eta_k = x^k\omega + u_k\left\{(k-1)x^{k-2}y^{n-2}dx + (n-2)x^{k-1}y^{n-3}dy\right\}.$$

Furthermore, we see that

$$\begin{aligned}\pi^*(\eta_k) = n\Bigg[&(m-\lambda)t^{(n-2)m+(k-1)n-1} \\ &+ \sum_{i=1}^{p} a_i(m-m_i)t^{(n-1)m-(p+2-i-k)n-1}\Bigg]dt \\ &+ u_k(k-1)n\Big[t^{(n-2)m+(k-1)-1} \\ &\qquad + (n-2)t^{(2n-5)m-(3-k)n-1} + \cdots\Big]dt \\ &+ u_k(n-2)\Big[mt^{(n-2)m+(k-1)-1} \\ &\qquad + \{m(n-3)+\lambda\}t^{(2n-5)m-(3-k)n-1} + \cdots\Big]dt.\end{aligned} \tag{18}$$

So we have $\mathrm{LT}(x^k\omega) + \mathrm{LT}\left(d\left(u_k x^{k-1}y^{n-2}\right)\right) = 0$ for any k. Comparing $(n-1)m - (3-k)n - 1$ with $(2n-5)m - (3-k)n - 1$ in (18), we have the following relations according to n.

$$(n-1)m - (3-k)n - 1 < (2n-5)m - (3-k)n - 1 \text{ for } n \geq 5,$$
$$(n-1)m - (3-k)n - 1 = (2n-5)m - (3-k)n - 1 \text{ for } n = 4.$$

So we consider the cases (C) $n \geq 5$ and (D) $n = 4$ separately.

Lemma 13. *Let ξ be an element of Ω^1_C with $\nu(\xi)+1 \in V_1^+$. Then we have $\nu(\xi) \geq \nu(\eta_k)$ for some k.*

Proof. Put $\xi = A\omega + dB$ where A, $B \in \mathcal{O}_C$. There exists only one term $c_1x^{k_1}y^{l_1}$ in A such that $\nu(A) = \nu(c_1x^{k_1}y^{l_1})$. Then we must have $l_1 = 0$. Indeed, if not, we have

$$\nu(A\omega) + 1 = (l_1 + n - 2)m + (k_1 - 1)n \geq (n-1)m - n.$$

By (12), we see that $\nu(\xi) + 1 \notin V^+$, which is a contradiction.

Since the cancellation (11) occurs, we have $\nu(dB) = (n-2)m + (k_1 - 1)n - 1$. So the function B contains only one term $h_1x^{k_1-1}y^{n-2}$ such that $\nu(dB) = \nu(h_1x^{k_1-1}y^{n-2})$. Since $\mathrm{LT}\left(c_1x^{k_1}\omega\right) + \mathrm{LT}\left(d\left(h_1x^{k_1-1}y^{n-2}\right)\right) = 0$, we easily see that $h_1 = c_1u_{k_1}$. Hence ξ can be written as $c_1\eta_{k_1} + \xi_1$ where $\xi_1 = \left(A - c_1x^{k_1}\right)\omega + d\left(B - h_1x^{k_1-1}y^{n-2}\right)$. If $\nu(\xi_1) + 1 \in V_1^+$, then we can apply the same argument to ξ_1. Namely, there exists η_{k_2} such that $\xi = c_1\eta_{k_1} + c_2\eta_{k_2} + \xi_2$ where $\xi_2 = \left(A - c_1x^{k_1} - c_2x^{k_2}\right)\omega + d\left(B - h_1x^{k_1-1}y^{n-2} - h_2x^{k_2-1}y^{n-2}\right)$. Note that $\nu(A - c_1x^{k_1} - c_2x^{k_2}) > \nu(A - c_1x^{k_1})$. We can continue this procedure successively. After the j-th step, we have

$$\xi = \sum_{i=1}^{j} c_i\eta_{k_i} + \xi_j,$$

$$\text{where } \xi_j = \left(A - \sum_{i=1}^{j} c_ix^{k_i}\right)\omega + d\left(B - \sum_{i=1}^{j} h_ix^{k_i-1}y^{n-2}\right).$$

Since we have $\nu\left(A - \sum_{i=1}^{j+1} c_ix^{k_i}\right) > \nu\left(A - \sum_{i=1}^{j} c_ix^{k_i}\right)$, there exists a positive integer j such that $\nu(\xi_j) \geq \nu\left(\left(A - \sum_{i=1}^{j} c_ix^{k_i}\right)\omega\right) > (n-1)m - n - 1$. It follows from (12) that $\nu(\xi_j) + 1 \notin V_1^+$. So we have $\xi = \sum c_i\eta_{k_i} + \xi_j$ where $\nu(\sum c_i\eta_{k_i}) < \nu(\xi_j)$. Thus we obtain $\nu(\xi) \geq \min\{\nu(\eta_{k_i})\}$. Q.E.D.

Lemma 14. *Let C be an irreducible plane curve singularity given by (17). If $n \geq 5$ and $p = 1$, then we have $V_1^+ = \emptyset$.*

Proof. Assume that $V_1 \neq \emptyset$. Let ξ be a differential with $\nu(\xi) \in V_1$. Then we have $\nu(\xi) \geq \nu(\eta_k)$ for some k by Lemma 13. However we have

$$\nu(\eta_k) \geq \begin{cases} (n-1)m - n - 1 & \text{for } k = 1, \\ (n-1)m + (k-2)n - 1 & \text{for } k \geq 2. \end{cases}$$

Since $\nu(\eta_k)+1 \geq (n-1)m-n$ for any k, we have $\nu(\xi)+1 \notin V_1^+$ by (12). This is a contradiction. Q.E.D.

Lemma 15. *If $V_1^+ \neq \emptyset$, then we have $\nu(\eta_1)+1 = \min\{V_1^+\}$.*

Proof. We here prove this lemma for Case (C). We can similarly deal with Case (D). We have

$$\pi^*(\eta_1) = \left[n \sum_{i=1}^{p-1} a_i(m-m_i)t^{(n-1)m-(p+1-i)n-1} + \cdots \right] dt, \tag{19}$$

where we abbreviate the terms whose degree is greater than $(n-1)m-2n-1$. Assume that $V_1^+ \neq \emptyset$. We must have $p \geq 2$ by Lemma 14. We first show that $\nu(\eta_1) \in V_1$. If $\nu(\eta_1) \notin V_1$, then $\nu(\eta_1) \in V_0$ or $\nu(\eta_1) \in \nu(d\mathcal{O}_C)$. Now we have $V_0 = \{\nu(\omega), \nu(y\omega)\}$. If $\nu(\eta_1) \in V_0$, then we have $\nu(\eta_1) = \nu(y\omega)$ by the definition of η_1. At least the coefficients in (19) must satisfy

$$a_i = 0 \text{ for } i = 1, \ldots, p-1. \tag{20}$$

Let ξ be a differential with $\nu(\xi) \in V_1$. By Lemma 13, we have $\nu(\xi) \geq \nu(\eta_k)$ for some k. Under the conditions (20), if $k \geq 2$, then we have

$$\pi^*(\eta_k) = \left[a_p n(m-m_p)t^{(n-1)m+(k-2)n-1} + \cdots \right] dt.$$

Since $\nu(\eta_k)+1 \geq (n-1)m-n$ for all k, we have $\nu(\xi)+1 \notin V_1^+$ by (12), which is a contradiction. On the other hand, if $\nu(\eta_1) \in \nu(d\mathcal{O}_C)$, then (20) must hold again. Since same contradiction occurs, we have $\nu(\eta_1) \in V_1$.

Next we show that $\min\{V_1^+\} = \nu(\eta_1)+1$. It suffices to consider the case where $\sharp(V_1^+) \geq 2$. Let ξ be an element of Ω_C^1 with $\nu(\xi) \in V_1$ and $\nu(\xi) \neq \nu(\eta_1)$. By Lemma 13, we have $\nu(\xi) \geq \nu(\eta_k)$ for some k (≥ 2). We have

$$\pi^*(\eta_k) = \left[n \sum_{i=1}^{p} a_i(m-m_i)t^{(n-1)m-(p+2-i-k)n-1} + \cdots \right] dt. \tag{21}$$

Set $N := \min\{i \mid a_i \neq 0\}$. Then we have $\nu(\eta_k) = (n-1)m-(p+2-N-k)n-1$. It follows from (12) that $(n-1)m-(p+2-N-k)n-1 < (n-1)m-n-1$. It yields the inequality

$$N < p+1-k. \tag{22}$$

On the other hand, it follows from (19) that $\nu(\eta_1) = (n-1)m-(p+1-N)n-1$. We see that $\nu(\eta_1) < \nu(\eta_k)$ by (22), which gives the desired consequence. Q.E.D.

2.3. Type (iii): $\lambda = (n-3)m - 2n$

It follows from $\lambda > m$ that $n \geq 5$ and $m > 2n/(n-4)$. Since $\nu(\omega) = (n-3)m - n - 1$, we find that $V_0 = \{\nu(\omega), \nu(y\omega), \nu(y^2\omega)\}$. We divide Type (iii) into two cases: (E) $n > 2q$, (F) $n < 2q$.

(E): $n > 2q$. The following gaps are greater than λ:

$$\begin{gathered}(n-2)m-(p+2)n,\ (n-1)m-(2p+2)n,\ (n-3)m-n,\\ (n-2)m-(p+1)n,\ (n-1)m-(2p+1)n,\ (n-2)m-pn,\\ (n-1)m-2pn,\ (n-2)m-(p-1)n,\ \dots,\ (n-2)m-n,\\ (n-1)m-(p+1)n,\ \dots,\ (n-1)m-n.\end{gathered}$$

By Lemma 6 with the above gaps, we see that C has the parametrization

$$x = t^n, \quad y = t^m + t^\lambda + \sum_{i=1}^{p}(a_i t^{m_i} + b_i t^{n_i}) + \sum_{i=p+1}^{2p} b_i t^{n_i}, \tag{23}$$

where

$$m_i = (n-2)m - (p+3-i)n,\ n_i = (n-1)m - (2p+3-i)n$$

and $a_i,\ b_i \in \mathbb{C}$.

(F): $n < 2q$. The gaps which are greater than λ are

$$\begin{gathered}(n-1)m-(2p+3)n,\ (n-2)m-(p+2)n,\ (n-3)m-n,\\ (n-1)m-(2p+2)n,\ (n-2)m-(p+1)n,\\ (n-1)m-(2p+1)n,\ (n-2)m-pn,\ (n-1)m-2pn,\ \dots\\ (n-2)m-n,\ (n-1)m-(p+1)n,\ \dots,\ (n-1)m-n.\end{gathered}$$

Then C has the following parametrization:

$$x = t^n, \quad y = t^m + t^\lambda + \sum_{i=1}^{p}(a_i t^{m_i} + b_i t^{n_i}) + \sum_{i=p+1}^{2p+1} a_i t^{m_i}, \tag{24}$$

where

$$m_i = (n-1)m - (2p+4-i)n,\ n_i = (n-2)m - (p+3-i)n$$

and $a_i, b_i \in \mathbb{C}$.

Definition 16. Define the differentials ζ_{kl} ($k \geq 1$, $l \geq 0$) by

$$\zeta_{kl} := x^k y^l \omega + d\left(s_{kl} x^{k-1} y^{n+l-3}\right),$$

$$\text{where } s_{kl} = \frac{-n(m-\lambda)}{(k-1)n + (n+l-3)m}.$$

We rewrite the differentials ζ_{kl} as follows:

$$\zeta_{1l} = y^l \phi_{1l} \text{ and } \zeta_{kl} = x^{k-2} y^l \phi_{kl},$$

$$\text{where } \begin{cases} \phi_{1l} = x\omega + s_{kl}(n+l-3)y^{n-4}dy, \\ \phi_{kl} = x^2\omega + s_{kl}\{(k-1)y^{n-3}dx \\ \qquad\qquad + (n+l-3)xy^{n-4}dy\}(k \geq 2). \end{cases}$$

We can easily check that $\mathrm{LT}\left(x^k y^l \omega\right) + \mathrm{LT}\left(d\left(s_{kl} x^{k-1} y^{n+l-3}\right)\right) = 0$. Note that $\phi_{10} = \zeta_{10}$ and $\phi_{20} = \zeta_{20}$. The following lemma is an analogue of Lemma 13.

Lemma 17. *If ξ is an element of Ω_C^1 with $\nu(\xi) + 1 \in V_1^+$, then ξ has the form $a\zeta_{kl} + \xi'$ for some ζ_{kl} where $\nu(\zeta_{kl}) \leq \nu(\xi')$ and $a \in \mathbb{C}$.*

Proof. This proof is similar to that of Lemma 13. So we omit it. Q.E.D.

§3. Singularities of genus two

We consider irreducible plane curve singularities of genus 2 in this section. The aim of this section is to prove Proposition 1. We first prove some technical auxiliary results needed in the proof of Proposition 1. Recall that if $g = 2$, then we have $S = \langle v_0, v_1, v_2 \rangle$ where $v_0 < v_1 < v_2$, $v_0 = n = n_1 n_2$ with $n_i \geq 2$ $(i = 1, 2)$, $v_1 = m = e_1 m_1$ for some positive integer m_1 and $e_1 = n_2$. Set $\lambda = \lambda_2 v_2 + \lambda_1 v_1 - \lambda_0 v_0$. In case $r = 3$, by (9), we have

$$3 \geq (\lambda_0 - 1)(n_1 - \lambda_1)(n_2 - \lambda_2) > 0. \tag{25}$$

Lemma 18 (Nishiyama). *If C is an irreducible plane curve singularity of genus 2 with $r = 3$, then we have $\lambda = (n_1 - 1)m - 2n$ and $\mathrm{Ch}(C) = \{3n_1, 3m_1, \beta_2\}$ where n_1 and m_1 are coprime, $n_1 < m_1$ and β_2 is not divisible by 3.*

Proof. We first show that $n_2 \neq 2$ (cf. Lemma 10 in [3]). If $n_2 = 2$, then we have $S = \langle 2p, 2q, v_2 \rangle$ where $p < q$, $\gcd(p, q) = 1$, $p = n_1$ and $v_2 > n_1 v_1 = 2pq$. Furthermore, we can rewrite S with some positive and odd integer d as

$$S = \langle 2p, 2q, 2pq + d \rangle.$$

Luengo and Pfister ([5]) showed that the irreducible plane curve singularity C with such semigroup has $\tau = \mu - (p-1)(q-1)$. That is,

$r = (p-1)(q-1)$. So if we set $r = (p-1)(q-1) = 3$, then we have $p = 2$ and $q = 4$. This implies that $m = 2n$, which is a contradiction.

Let $\lambda = \lambda_2 v_2 + \lambda_1 v_1 - \lambda_0 v_0$ be the Zariski invariant of C. We first consider the case where $\lambda_2 \neq 0$. Recall that $\lambda \leq \beta_2$ (See (4)). Assume that $\lambda < \beta_2$. Since $v_2 = n_1 v_1 + \beta_2 - \beta_1$ and β_2 can not be divisible by e_1, λ is also not divisible by e_1. This contradicts the definition of β_2. Hence we have $\lambda = \beta_2 = v_2 + v_1 - m_1 v_0$. It follows that $\lambda_2 = 1$, $\lambda_1 = 1$, $\lambda_0 = m_1$. Since $m_1 > n_1 \geq 2$, we easily see that $n_2 = 2$ by (25). So the case in which $\lambda_2 \neq 0$ does not occur by the above argument.

On the other hand, if $\lambda_2 = 0$, then we must have $n_2 = 3$ by the above argument and (25). It follows that $S = \langle 3n_1, 3m_1, v_2 \rangle$. We also obtain $\lambda_1 = n_1 - 1$ and $\lambda_0 = 2$ by (25). The corresponding characteristic is $\mathrm{Ch}(C) = \{3n_1, 3m_1, \beta_2\}$ where $\beta_2 = v_2 - (n_1 - 1)m$. We have completed the proof of Lemma 18. Q.E.D.

By Lemma 18, we have only to consider the case where

$$\lambda = (n_1 - 1)m - 2n \text{ and } \mathrm{Ch}(C) = \{3n_1, 3m_1, \beta_2\}.$$

In this case, We have $S = \langle v_0, v_1, v_2 \rangle$ where $v_0 = n = 3n_1$, $v_1 = m = 3m_1$ and $v_2 = 2m + \beta_2$. We also have $n_1 \geq 3$ by $\lambda > m$ and the following conditions are satisfied:

$$m_1 \geq \begin{cases} n_1 + 1 & \text{for } n_1 \geq 4. \\ 7 & \text{for } n_1 = 3. \end{cases}$$

Lemma 19. *Let C be an irreducible plane curve singularity with $\mathrm{Ch}(C) = \{3n_1, 3m_1, \beta_2\}$ and $\lambda = (n_1 - 1)m - 2n$. Then the parametrization of C can be taken as*

$$x = t^n, \quad y = t^m + t^\lambda + at^{\beta_2} + \cdots, \quad (a \neq 0). \tag{26}$$

Proof. Let h_1 be the biggest positive integer satisfying $m + h_1 e_1 < \beta_2$. Note that $e_1 = 3$ and $\lambda = m + 3\{(n_1 - 2)m_1 - 2n_1\}$. So we can take the parametrization of C as

$$x = t^n, \quad y = t^m + t^\lambda + \sum_{(n_1-2)m_1 - n_1 \leq i \leq h_1} a_i t^{m+3i} + at^{\beta_2} + \cdots, \tag{27}$$

where $m + 3i \in G$ for any i. Since each $m + 3i$ is a gap, it is written in a unique way as

$$m + 3i = t_2 v_2 + t_1 v_1 - t_0 v_0, \tag{28}$$

where $0 \leq t_2 \leq 2$, $0 \leq t_1 \leq n_1 - 1$ and $t_0 > 0$ (See (6)). Since the left hand side of (28) is divisible by 3 and v_2 is not divisible by 3, we must have $t_2 = 0$. Since $\lambda = (n_1 - 1)m - 2n$, no integer satisfies this condition other than $(n_1 - 1)m - n$. If $\beta_2 < (n_1 - 1)m - n$, then we obtain (26). On the other hand, if $(n_1 - 1)m - n < \beta_2$, then (27) becomes

$$x = t^n, \quad y = t^m + t^\lambda + a_{(n_1-2)m_1-n_1} t^{(n_1-1)m-n} + at^{\beta_2} + \cdots .$$

By using (EC 2) in Lemma 6, we can rewrite this as (26). Q.E.D.

Lemma 20. *If a positive integer $k = am + bn$ $(a, b \in \mathbb{Z})$ is greater than $(n_1 - 1)m - n$, then we have $k \in \langle n, m\rangle \subset S$.*

Proof. By (6), we can rewrite k as $l_2v_2 + l_1v_1 - l_0v_0$ where $0 \leq l_2 \leq 2$ and $0 \leq l_1 \leq n_1 - 1$. Now we have $l_2 = 0$. Indeed, if not, then we have

$$l_2v_2 = 3\{(a - l_1)m_1 + (b + l_0)m_1\}.$$

Since l_2 is equal to 1 or 2, the integer v_2 must be divisible by 3, which is a contradiction. Thus we have $k = l_1m - l_0n$. Since the biggest gap of such form is $(n_1 - 1)m - n$, the positive integer k is contained in S. Q.E.D.

Proof of Proposition 1. It follows from (10) that if $r = 3$, then $g = 1$ or 2. We shall show that if $g = 2$, then $r \neq 3$. It is enough to consider the plane curve singularity C with $\lambda = (n_1 - 1)m - 2n$ and $\mathrm{Ch}(C) = \{3n_1, 3m_1, \beta_2\}$ by Lemma 18. By Lemma 19, we may assume that C is given by (26). Since $(\lambda_0 - 1)(n_1 - \lambda_1)(n_2 - \lambda_2) = 3$, there exist three distinct elements of V_0. They are given by $\nu(\omega)$, $\nu(z\omega)$ and $\nu(z^2\omega)$ where $z \in \mathcal{O}_C$ with $\nu(z) = v_2$. We shall inductively construct a differential ξ such that

$$\pi^*(\xi) = \Big\{am(m - \beta_2)t^{\beta_2+2n-1} + \cdots\Big\}dt. \tag{29}$$

Since $\beta_2 + 2n - 1 = v_2 + m - (m_1 - 2)n - 1$ is different from $\nu(\omega)$, $\nu(z\omega)$ and $\nu(z^2\omega)$, we would have $r \geq 4$ by (8). We first set $\xi_0 = (m/n)x\omega$. Then we have

$$\pi^*(\xi_0) = \Big\{m(m - \lambda)t^{(n_1-1)m-1} + am(m - \beta_2)t^{\beta_2+2n-1} + \cdots\Big\}dt.$$

Next we set $\xi_1 = \xi_0 - (m - \lambda)y^{n_1-2}dy$ as the first step. We have

$$
(30)\quad
\begin{aligned}
y^{n_1-2} = t^{(n_1-2)m}\Bigg[1 &+ \binom{n_1-2}{1}\{t^{\lambda-m} + at^{\beta_2-m} + \cdots\} \\
&+ \binom{n_1-2}{2}\left\{t^{2(\lambda-m)} + 2at^{\beta_2+\lambda-2m} + \cdots\right\} \\
&+ \binom{n_1-2}{3}\left\{t^{3(\lambda-m)} + \cdots\right\} + \cdots \\
&+ \binom{n_1-2}{n_1-2}\left\{t^{(n_1-2)(\lambda-m)} + \cdots\right\}\Bigg].
\end{aligned}
$$

We consider the cases where $\beta_2 - m < 2(\lambda - m)$ and where $2(\lambda - m) < \beta_2 - m$ separately.

If $\beta_2 - m < 2(\lambda - m)$, then we have

$$
\begin{aligned}
\pi^*\left(y^{n_1-2}dy\right) =& \Big[mt^{(n_1-1)m-1} + \{m(n_1-2)+\lambda\}t^{(2n_1-3)m-2n-1} \\
&+ a\{m(n_1-2)+\beta_2\}t^{\beta_2+(n_1-2)m-1} + \cdots\Big]dt.
\end{aligned}
$$

Since $(2n_1-3)m - 2n > \beta_2 + 2n$, we have

$$
\pi^*(\xi_1) = \left[am(m-\beta_2)t^{\beta_2+2n-1} + \cdots\right]dt,
$$

which is the desired differential. In particular, if $n_1 = 3$, then this case always occurs.

Next we consider the case where $2(\lambda - m) < \beta_2 - m$. This case occurs only when $n_1 \geq 4$. Set $N_1 := \max\{i \mid i(\lambda-m) < \beta_2 - m \text{ and } 2 \leq i \leq n_1 - 2\}$. Then (30) becomes

$$
\begin{aligned}
y^{n_1-2} =& t^{(n_1-2)m} + \sum_{i=1}^{N_1}\binom{n_1-2}{i}t^{(i+1)(n_1-2)m-2in} \\
&+ a(n_1-2)t^{\beta_2+(n_1-3)m} + \cdots .
\end{aligned}
$$

So we have

$$
\begin{aligned}
\pi^*\left(y^{n_1-2}dy\right) = &\Bigg[mt^{(n_1-1)m-1} \\
&+ \sum_{i=1}^{N_1}\left\{m\binom{n_1-2}{i} + \lambda\binom{n_1-2}{i-1}\right\}t^{n_i-1} + \cdots\Bigg]dt,
\end{aligned}
$$

where $n_i = \{(i+1)n_1 - 2i - 1\}m - 2in$. In a similar manner as in the previous case, we set $\xi_1 = \xi_0 - (m-\lambda)y^{n_1-2}dy$. Since $n_{N_1-1} < \beta_2 + 2n <$

n_{N_1} holds, we have

$$\pi^*(\xi_1) = \left[-(m-\lambda) \sum_{i=1}^{N_1-1} \left\{ m\binom{n_1-2}{i} + \lambda\binom{n_1-2}{i-1} \right\} t^{n_i-1} \right.$$
$$\left. + am(m-\beta_2)t^{\beta_2+2n-1} + \cdots \right] dt.$$

Note that $n_i \in \langle n, m\rangle$ by Lemma 20. Starting with ξ_1, we inductively define a differential ξ_k. Assume that ξ_k $(k \geq 1)$ satisfies the following condition:

$$(31)\quad \pi^*(\xi_k) = \left\{ \sum_{\text{finite sum}} c_{k,\alpha} t^{m_\alpha - 1} + am(m-\beta_2)t^{\beta_2+2n-1} + \cdots \right\} dt,$$

where $m_\alpha \in \langle n, m\rangle$. Putting $\nu(\xi_k) = a_k m + b_k n - 1$, we set

$$\xi_{k+1} := \begin{cases} \xi_k - \big(\mathrm{LT}(\xi_k)/n\big) x^{b_k} dx, & \text{if } a_k = 0 \text{ and } b_k \neq 0. \\ \xi_k - \big(\mathrm{LT}(\xi_k)/n\big) x^{b_k-1} y^{a_k} dx, & \text{if } a_k \neq 0 \text{ and } b_k \neq 0. \\ \xi_k - \big(\mathrm{LT}(\xi_k)/m\big) y^{a_k-1} dy, & \text{if } a_k \neq 0 \text{ and } b_k = 0. \end{cases}$$

It follows from this definition that $\nu(\xi_{k+1}) > \nu(\xi_k)$. We prove that ξ_{k+1} above satisfies the condition (31).

Case 1) $a_k = 0$ and $b_k \neq 0$. We have

$$\pi^*(\xi_{k+1}) = \left\{ \sum c_{k+1,\alpha} t^{m_\alpha - 1} + am(m-\beta_2)t^{\beta_2+2n-1} + \cdots \right\} dt,$$

where $m_\alpha \in \langle n, m\rangle$. Note that the number of m_α is finite. The differential ξ_{k+1} satisfies the condition (31).

Case 2) $a_k \neq 0$ and $b_k \neq 0$. Consider the differential $x^{b_k-1}y^{a_k}dx$. Writing

$$y^{a_k} = t^{a_k m} \left[1 + \binom{a_k}{1} \left\{ t^{\lambda-m} + at^{\beta_2-m} + \cdots \right\} \right.$$
$$+ \binom{a_k}{2} \left\{ t^{2(\lambda-m)} + 2at^{\beta_2+\lambda-2m} + \cdots \right\}$$
$$+ \binom{a_k}{3} \left\{ t^{3(\lambda-m)} + \cdots \right\} + \cdots$$
$$\left. + \binom{a_k}{a_k} \left\{ t^{a_k(\lambda-m)} + \cdots \right\} \right],$$

we set $N_{k+1} := \max\{i \mid i(\lambda - m) < \beta_2 - m \text{ and } 2 \leq i \leq a_k\}$. Then we have

$$\pi^*\left(x^{b_k-1}y^{a_k}dx\right) = n\Bigg[t^{a_k m + b_k n - 1} + \sum_{i=1}^{N_{k+1}} c_i t^{\{(n_1-2)i+a_k\}m+(b_k-2i)n-1} + a a_k t^{\beta_2+(a_k-1)m+b_k n-1} + \cdots\Bigg]dt.$$

By Lemma 20, the integers $\{(n_1 - 2)i + a_k\}m + (b_k - 2i)n$ belong to $\langle n, m\rangle$. It is easy to see that $\beta_2 + (a_k - 1)m + b_k n - 1 > \beta_2 + 2n - 1$. So ξ_{k+1} satisfies the condition (31).

Case 3) $a_k \neq 0$ and $b_k = 0$. We consider the differential $y^{a_k-1}dy$. By the same argument as Case 2, we define $N_{k+1} := \max\{i \mid i(\lambda - m) < \beta_2 - m \text{ and } 2 \leq i \leq a_k - 1\}$ for y^{a_k-1}. We have

$$\pi^*\left(y^{a_k-1}dy\right) = \Bigg[m t^{a_k m - 1} + \sum_{\text{finite sum}} c_\alpha t^{m_\alpha - 1} + a(m + \beta_2)t^{\beta_2+(a_k-1)m-1} + \cdots\Bigg]dt,$$

where $m_\alpha \in \langle n, m\rangle$. So we see that ξ_{k+1} satisfies the condition (31).

We can therefore inductively construct ξ_{k+1} from ξ_k. Since there exist finitely many elements of $\langle n, m\rangle$ which are smaller than $\beta_2 + 2n - 1$ and $\nu(\xi_0) < \nu(\xi_1) < \cdots < \nu(\xi_k) < \cdots$ holds, we obtain ξ with $\pi^*(\xi) = \left[am(m - \beta_2)t^{\beta_2+2n-1} + \cdots\right]dt$ after finitely many steps. Q.E.D.

§4. Proof of Theorem

By Proposition 1, it is enough to consider the case where $g = 1$. Substituting $r = 3$ and $g = 1$ to (9), we obtain $3 = r \geq (\lambda_0 - 1)(n - \lambda_1)$. This inequality yields the following possible five types of λ:

(i) $\lambda = (n - 1)m - 4n$, (ii) $\lambda = (n - 2)m - 2n$, (iii) $\lambda = (n - 3)m - 2n$,
(iv) $\lambda = (n - 1)m - 2n$, (v) $\lambda = (n - 1)m - 3n$.

Lemma 21. *If λ is either of type* (iv) *or of type* (v), *then* $r \neq 3$.

Proof. For type (iv) (resp. type (v)), letting $R = 1$ (resp. $R = 2$) in Proposition 7, we conclude that $r = 1$ (resp. $r = 2$). Q.E.D.

We consider the remaining three types separately. We freely use the notations and the results in Section 2.

Type (i): $\lambda = (n-1)m - 4n$. We show that $r = 3$. We first consider the case in which $p \geq 2$. We may assume that C is given by (15). By Proposition 7, we have $r = 3$.

Next we consider the case in which $p = 1$. The parametrization of C has the form (16). We have

$$\pi^*(\omega) = n\left\{(m-\lambda)t^{(n-1)m-3n-1} + a(m-m_1)t^{(n-2)m-n-1}\right\}dt.$$

It follows from $\nu(\omega) = (n-1)m-3n-1$ that $V_0 = \{\nu(\omega),\, \nu(x\omega),\, \nu(x^2\omega)\}$. By (8), we have $r = 3$ if and only if $V_1^+ = \emptyset$. Assume that $V_1^+ \neq \emptyset$. Let $\xi = A\omega + dB$ be an element of Ω_C^1 with $\nu(\xi) \in V_1$. If we set $(A\omega) = \left(ux^ky^l + \cdots\right)\omega$ where $u \in \mathbb{C}$, then we have

$$\nu(A\omega) = (n+l-1)m + (k-3)n - 1. \tag{32}$$

Since $\nu(A\omega) \in d\mathcal{O}_C$ by (11), we have $k \geq 3$ or $l \geq 1$. Suppose $k \geq 3$. Then we see from (32) that $\nu(A\omega) > (n-1)m - n - 1$. So the order $\nu(\xi)$ can not belong to V_1 by (12). Thus we must have $l \geq 1$. If $l \geq 2$, then we have $\nu(A\omega) > (n-1)m - n - 1$ again. Hence $l = 1$. Then (12) yields $(k-1)n + q < 0$. We infer from this that $k = 0$. Thus we have $A\omega = (uy + \text{terms of higher degree})\omega$. We have

$$\pi^*(uy\omega) = un\left[(m-\lambda)t^{(m-3)n-1} - abt^{(n-1)m-n-1}\right]dt.$$

where $b := (n-3)m - 2n$. Since (11) holds, the differential dB has the form $d\left(-u(m-\lambda)x^{m-3}/(m-3) + \cdots\right)$. So ξ can be rewritten as $uy\omega + d\left(-u(m-\lambda)x^{m-3}/(m-3)\right) + \xi'$. If we write $\xi' = (u'x^ky^l + \cdots)\omega + dB'$, then $(k,\, l) \neq (0,\, 1)$ and hence $\nu(\xi') \notin V_1$. This fact implies that $\nu(\xi) = \nu(y\omega) + d\left(-u(m-\lambda)x^{m-3}/(m-3)\right)$ If $n = 5,\, 6$, we find that $\nu(A\omega) > (n-2)m - n - 1$. Since $(n-1)m - n$ is the only gap greater than $(n-2)m - n$, there exists no element of V_1. That is, $V_1 = \emptyset$. For $7 \geq n$, we have

$$\pi^*\left(uy\omega + d\left(\frac{-u(m-\lambda)}{(m-3)}x^{m-3}\right)\right) = -abunt^{(n-1)m-n-1}dt.$$

By (12), $\nu(\xi)$ can not be in V_1. We conclude that $V_1 = \emptyset$ for $7 \geq n$.

Type (ii): $\lambda = (n-2)m - 2n$. By Lemma 3, $\sharp(V_0^+) = 2$ holds. So we have $r = 3$ if and only if $\sharp(V_1^+) = 1$ by (8). Furthermore, by Lemma 15, we obtain $\sharp(V_1^+) = 1$ if and only if $V_1^+ = \{\nu(\eta_1) + 1\}$.

(C): $n \geq 5$. It follows from the inequality $\lambda > m$ that $p \geq 1$ for $n \geq 5$. However we must have $p \geq 2$ by Lemma 14. If $\sharp(V_1^+) = 1$, then the coefficients in (19) must satisfy

$$a_i = 0 \text{ for } i = 1, \ldots, p-2 \text{ and } a_{p-1} \neq 0. \tag{33}$$

Conversely, assume that the coefficients in the parametrization (17) satisfy (33). Then we have $\nu(\eta_1) = (n-1)m - 2n - 1$. Since $(n-1)m - n$ is the only one gap of S which is greater than $(n-1)m - 2n$, by Lemma 15, we have $V_1^+ = \{\nu(\eta_1) + 1\}$.

(D): $n = 4$. It follows from the inequality $\lambda > m$ that $p \geq 2$. We have

$$\pi^*(\eta_1) = \left[4\sum_{i=1}^{p-2} a_i(m - m_i)t^{3m-4(p+1-i)-1}\right.$$
$$\left. + 4\left\{a_{p-1}(m - m_{p-1}) - \frac{m^2 - \lambda^2}{m}\right\}t^{3m-8-1} + \cdots\right]dt.$$

If $\sharp(V_1^+) = 1$, then we must have the following condition:

$$a_i = 0 \text{ for } i = 1, \ldots, p-2 \text{ and } a_{p-1} \neq \frac{3m-8}{2m}. \tag{34}$$

Conversely, if C is given by the parametrization (17) with (34), then we find that $\sharp(V_1^+) = 1$ by the same argument as in (C).

Type (iii): $\lambda = (n-3)m - 2n$. Since $\sharp(V_0^+) = 3$, we have $r = 3$ if and only if $V_1^+ = \emptyset$ (See (8)). We here prove Case (E). We can similarly deal with Case (F). Now we have

$$\pi^*(\zeta_{10}) = n\left[(m - \lambda)t^{(n-3)m-1}\right.$$
$$+ \sum_{i=1}^{p}\{a_i(m - m_i)t^{(n-2)m-(p+1-i)n-1}\}$$
$$+ \sum_{i=1}^{p}\{b_i(m - n_i)t^{(n-1)m-(2p+1-i)n-1}\}$$
$$\left. + \sum_{i=p+1}^{2p} b_i(m - n_i)t^{(n-1)m-(2p+1-i)n-1}\right]dt$$
$$- s_{10}\left[(n-3)mt^{(n-3)m-1}\right.$$
$$\left. + \{\lambda + m(n-4)\}t^{(2n-7)m-2n-1} + \cdots\right]dt.$$

Comparing the exponent $(2n-7)m-2n-1$ with $(n-1)m-n-1$, the following three subcases occur:

(E1): $(2n-7)m-2n-1 > (n-1)m-n-1$ for $n \geq 7$.
(E2): $(2n-7)m-2n-1 = (n-1)m-2n-1$ for $n = 6$.
(E3): $(2n-7)m-2n-1 < (n-1)m-2n-1$ for $n = 5$.

It follows from $\lambda > m$ and $n > 2q$ that the conditions (i) $p \geq 1$ for $n \geq 7$, (ii) $p \geq 1$ and $q = 1$ for $n = 6$, (iii) $p \geq 2$ and $q = 1, 2$ for $n = 5$.
(E1): $n \geq 7$. The differential $\pi^*(\zeta_{10})$ becomes

$$\pi^*(\zeta_{10}) = \left[n\sum_{i=1}^{p} a_i(m-m_i)t^{(n-2)m-(p+1-i)n-1} \right.$$
$$(35) \qquad + n\sum_{i=1}^{p} b_i(m-n_i)t^{(n-1)m-(2p+1-i)n-1}$$
$$\left. + n\sum_{i=p+1}^{2p} b_i(m-m_i)t^{(n-1)m-(2p+1-i)n-1} + \cdots \right] dt.$$

If $V_1^+ = \emptyset$, then the order $\nu(\zeta_{10})$ must belong to $\nu(d\mathcal{O}_C)$ or V_0. Furthermore, if $\nu(\zeta_{10}) \in V_0$, then $\nu(\zeta_{10})$ equals $\nu(y\omega)$ or $\nu(y^2\omega)$.
E1.1: $\nu(\zeta_{10}) \neq \nu(y\omega)$. If $V_1^+ = \emptyset$, then the coefficients in (35) must satisfy the following conditions.

$$(36) \qquad \begin{aligned} &a_i = 0 \text{ for } i = 1, \ldots, p \\ &b_i = 0 \text{ for } i = 1, \ldots, 2p-1 \text{ and } \forall b_{2p}. \end{aligned}$$

Conversely, assume that (23) has (36). If $V_1^+ \neq \emptyset$, then there exists a differential ξ with $\nu(\xi)+1 \in V_1^+$. By Lemma 17, ξ has the form $\xi = a\zeta_{kl} + \xi'$. Recall that there exists the following relation between $\nu(\zeta_{kl})$ and $\nu(\phi_{kl})$:

$$(37) \qquad \nu(\zeta_{kl}) = \begin{cases} \nu(\phi_{1l}) + lm, & \text{if } k = 1. \\ \nu(\phi_{kl}) + lm + (k-2)m, & \text{if } k \geq 2. \end{cases}$$

(See Subsection 2.3). If $k = 1$, then we have

$$\pi^*(\phi_{1l}) = \left[b_{2p}n(m-n_{2p})t^{(n-1)m-n-1} + \text{higher degree terms} \right] dt.$$

So we have $\nu(\xi) \geq \nu(\phi_{1l}) \geq (n-1)m-n-1$. By Lemma 12, $\nu(\xi)+1$ can not be in V_1^+.

On the other hand, if $k \geq 2$, then we have

$$\pi^*(\phi_{kl}) = \left[b_{2p}(m-n_p)t^{(n-1)m-1} + \text{higher degree terms} \right] dt.$$

Since $\nu(\xi) \geq \nu(\phi_{kl}) > (n-1)m - n - 1$, we have $\nu(\xi) + 1 \notin V_1^+$ again. Thus, we conclude that $V_1^+ = \emptyset$.

E1.2: $\nu(\zeta_{10}) = \nu(y\omega)$. If $V_1^+ = \emptyset$, then the parametrization (35) must have the following coefficients:

$$a_i = b_i = 0 \text{ for } i = 1, \ldots, p-1 \text{ and } a_p \neq 0. \tag{38}$$

For the parametrization (35) with the condition (38), we consider the differential $\zeta_{10} - (\mathrm{LT}(\zeta_{10})/\,\mathrm{LT}(y\omega))y\omega$.

$$\begin{aligned}\pi^*\left(\zeta_{10} - \frac{\mathrm{LT}(\zeta_{10})}{\mathrm{LT}(y\omega)}y\omega\right) = &\left[\sum_{i=p}^{2p-2} b_i n(m-n_i)t^{(n-1)m-(2p+1-i)n-1}\right. \\ &+ \left\{b_{2p-1}(m-n_{2p-1}) - \frac{a_p^2(m-m_p)^2}{(m-\lambda)}\right\} nt^{(n-1)m-2n-1} \\ &\left. - \frac{a_p b_p mn(m-m_p)(m-n_p)}{(m-\lambda)}t^{(m-p-2)n-1} + \cdots\right] dt.\end{aligned} \tag{39}$$

In (39), we must put

$$\begin{aligned} b_i &= 0 \text{ for } i = p, \ldots, 2p-2 \\ b_{2p-1} &= \frac{a_p^2(m-m_p)^2}{(m-n_{2p-1})(m-\lambda)}. \end{aligned} \tag{40}$$

Conversely, assume that the parametrization (23) has (38) and (40). If $V_1^+ \neq \emptyset$, then we take a differential ξ with $\nu(\xi)+1 \in V_1^+$. By Lemma 17, ξ has the form $c_1\zeta_{k_1l_1} + \xi_1$ where $\nu(\zeta_{k_1l_1}) \leq \nu(\xi_1)$. Note that ξ_1 does not contain $\zeta_{k_1l_1}$. We first consider the case where $k_1 = 1$. Then we have

$$\pi^*(\phi_{1l_1}) = \left[a_p n(m-m_p)t^{(n-2)m-n-1} + \cdots\right] dt.$$

If $l_1 \geq 1$, then we have $\nu(\zeta_{11}) \geq (n-1)m - n - 1$ by (37). Since $\nu(\xi)+1$ can not be an element of V_1^+ by (12), it contradicts assumption. So we must have $l_1 = 0$. Then we have $\nu(\zeta_{10}) = \nu(y\omega) = (n-2)m - n - 1$. So the relation $\mathrm{LT}(\zeta_{10}) + \mathrm{LT}(\xi_1) = 0$ must need for $\nu(\xi)+1 \in V_1^+$. We show that ξ_1 has the form $\{-c_1\,\mathrm{LT}(\zeta_{10})/\,\mathrm{LT}(y\omega)\}y\omega + \xi_1'$. Set $\xi_1 = A_1\omega + dB_1$. If $\nu(A_1\omega) > \nu(dB_1)$, then $\nu(\xi_1) \in d\mathcal{O}_C$. This case does not occur. If $\nu(A_1\omega) = \nu(dB_1)$, then $\mathrm{LT}(A_1\omega) + \mathrm{LT}(dB_1) = 0$ holds. By the same argument as in the proof of Lemma 13, we have the expression $\xi_1 = a\zeta_{kl} + \xi_1'$ for some ζ_{kl}. It is clear that $k \neq 1$. For $k \geq 2$, we have

$$\pi^*(\phi_{kl}) = \left[a_p(m-m_p)t^{(n-2)m-1} + \cdots\right] dt. \tag{41}$$

It follows from (37) and (41) that $\nu(\zeta_{kl}) > \nu(y\omega)$. Thus, only the case where $\nu(A_1\omega) < \nu(dB_1)$ occurs. By the same argument as in the proof of Lemma 13, there exists only one term ax^ky^l in A_1 such that $\nu(ax^ky^l) = \nu(A_1\omega)$. It follows from $\nu(\xi_1) = \nu(A_1\omega) = \nu(y\omega)$ that $k = 0$ and $l = 1$. Since $\mathrm{LT}(\zeta_{10})+\mathrm{LT}(\xi_1) = 0$, we must set $a = -c_1\,\mathrm{LT}(\zeta_{10})/\,\mathrm{LT}(y\omega)$. Putting $\xi_1' = \xi_1 + \{c_1\,\mathrm{LT}(\zeta_{10})/\,\mathrm{LT}(y\omega)\}y\omega$, we obtain the desired expression. Now we have $\xi = c_1\zeta_{10} - \{c_1\,\mathrm{LT}(\zeta_{10})/\,\mathrm{LT}(y\omega)\}y\omega + \xi_1'$. Since

$$\pi^*\left(c_1\zeta_{10} - \frac{c_1\,\mathrm{LT}(\zeta_{10})}{\mathrm{LT}(y\omega)}y\omega\right) = \\ c_1\left[b_{2p}n(m-n_{2p})t^{(n-1)m-n-1} + \cdots\right]dt$$

holds, the order $\nu(\xi_1')$ must equal $\nu(\xi)$.

$(*)$ By Lemma 17, there exists $\zeta_{k_2l_2}$ such that $\xi_1' = c_2\zeta_{k_2l_2} + \xi_2$ where ξ_2 does not contain $\zeta_{k_2l_2}$. Now $\zeta_{k_2l_2}$ is different from ζ_{10} and ζ_{11}. So we must have $k_2 \geq 2$. If $l_2 \geq 1$, then $\nu(\zeta_{kl}) > (n-1)m-n-1$ by (37). We must have $l_2 = 0$. Note that we have

$$\begin{aligned}\pi^*(\zeta_{k0}) = \Big[&a_pn(m-m_p)t^{(n-2)m+(k-2)n-1} \\ &+ b_{2p-1}n(m-n_{2p-1})t^{(n-1)m+(k-3)n-1} + \cdots\Big]dt,\end{aligned} \tag{42}$$

for $k \geq 2$. Since $\nu(\zeta_{k_20}) \in \nu(d\mathcal{O}_C)$, the equality $\mathrm{LT}(c_2\zeta_{k_20}) + \mathrm{LT}(\xi_2) = 0$ must hold for $\nu(\xi) \in V_1$. Write $\xi_2 = A_2\omega + dB_2$. By the same argument as in the proof of Lemma 13, there exists only one term $e_2x^ky^l$ in A_2 such that $\nu(e_2x^ky^l) = \nu(A_2\omega)$. Similarly, B_2 contains only one term $h_2x^ky^l$ such that $\nu(h_2x^ky^l) = \nu(dB_2)$. It is easily checked that ξ_2 has the form

$$\xi_2 = \left(e_2x^{k_2-1}y + \cdots\right)\omega + d\left(h_2x^{k_2-2}y^{n-2} + \cdots\right),$$

where $\mathrm{LT}(c_2\zeta_{k_20}) + \mathrm{LT}\left(e_2x^{k_2-1}y\omega + d\left(h_2x^{k_2-2}y^{n-2}\right)\right) = 0$. Furthermore, if we set $\xi_2' = \xi_2 - \left\{e_2x^{k_2-1}y\omega + d\left(h_2x^{k_2-2}y^{n-2}\right)\right\}$, then we have $\nu(\xi_2') > \nu\left(e_2x^{k_2-1}y\omega + d\left(h_2x^{k_2-2}y^{n-2}\right)\right) = \nu(\zeta_{k_20})$ holds. The differential ξ_1' is expressed as

$$\xi_1' = c_2\zeta_{k_2l_2} + e_2x^{k_2-1}y\omega + d\left(h_2x^{k_2-2}y^{n-2}\right) + \xi_2'.$$

Since ξ_2 dose not contain ζ_{k_20}, so does not ξ_2'. We easily see that $\nu(\xi_1' - \xi_2') \geq (n-1)m-n-1$. Hence we must have $\nu(\xi) = \nu(\xi_2')$. The argument started from $(*)$ is applicable to ξ_2'. So we obtain

$$\xi_2' = c_3\zeta_{k_30} + e_3x^{k_3-1}y\omega + d\left(h_3x^{k_3-2}y^{n-2}\right) + \xi_3',$$

where $\nu(\xi_3') > \nu\left(c_3\zeta_{k_30} + e_3x^{k_3-1}y\omega + d\left(h_3x^{k_3-2}y^{n-2}\right)\right) \in \nu(d\mathcal{O}_C)$. Note that $\nu(\zeta_{k0}) < \nu(\zeta_{k'0})$ if and only if $k < k'$ by (42). Since $\nu(\zeta_{k_20}) < \nu(\xi_3')$, we obtain $k_2 < k_3$. We continue this process successively and after j-th step we have

$$\xi_{j-1}' = c_j\zeta_{k_j0} + e_jx^{k_j-1}y\omega + d\left(h_jx^{k_j-2}y^{n-2}\right) + \xi_j'.$$

where $\nu(\xi_j') > \nu\left(c_j\zeta_{k_j0} + e_jx^{k_j-1}y\omega + d\left(h_jx^{k_j-2}y^{n-2}\right)\right) \in \nu(d\mathcal{O}_C)$. Then ξ is rewritten as

$$\xi = c_1\zeta_{10} - \frac{c_1\,\mathrm{LT}(\zeta_{10})}{\mathrm{LT}(y\omega)}y\omega + \sum_{i=2}^{j}\left\{c_i\zeta_{k_i0} + e_ix^{k_i+1}y\omega + d\left(h_ix^{k_i}y^{n-2}\right)\right\} + \xi_j'.$$

where $k_2 < k_3 < \ldots < k_j$ and $\nu(\xi_j') > \nu(\zeta_{k_j0})$. Since $\nu\left(\xi - \xi_j'\right) \notin V_1$, we must have $\nu(\xi) = \nu(\xi_j')$. However, the inequalities $\nu(\xi_j') > \nu(\zeta_{k_j0}) > (n-1)m - n - 1$ occur after finitely many steps. It contradicts the assumption $\nu(\xi) + 1 \in V_1^+$. Hence we have $V_1^+ = \emptyset$.

For the case where $k_1 \geq 2$, we can apply the argument started from $(*)$ to ξ by replacing ξ_1' by ξ. Then we find $V_1^+ = \emptyset$, so $r = 3$.

The proofs of (E2), (E3) and Case (F) are essentially same. So we omit them. Q.E.D.

We summarized the consequences for Case (E) and Case (F). If $r = 3$, then the parametrizations (23) and (24) have the coefficients in Table 1 and in Table 2 respectively.

Table 1

No.	Conditions	Coefficients
E1.1	$n \geq 7$	$a_i = 0\ (i = 1, \ldots, p)$, $b_i = 0\ (i = 1, \ldots, 2p-1)$, $\forall b_{2p}$.
E1.2	$n \geq 7$	$a_i = 0\ (i = 1, \ldots, p-1)$, $a_p \neq 0$, $b_i = 0\ (i = p, \ldots, 2p-2)$, $b_{2p-1} = a_p^2(m - m_p)^2/(m - n_{2p-1})(m - \lambda)$, $\forall b_{2p}$.
E2.1	$n = 6$, $m = 6p+1$	$a_i = 0\ (i = 1, \ldots, p)$, $b_i = 0\ (i = 1, \ldots, 2p-2)$, $b_{2p-1} = (5m - 12)/2m$, $\forall b_{2p}$.
E2.2	$n = 6$, $m = 6p+1$	$a_i = 0\ (i = 1, \ldots, p-1)$, $a_p \neq 0$, $b_i = 0\ (i = 1, \ldots, 2p-2)$, $b_{2p-1} = \left(9a_p^2m + 20m - 48\right)/8m$, $\forall b_{2p}$.

E3.1	$n=5$ $p\geq 3$	$a_i=0\ (i=1,\dots,p-2,\ p)$, $a_{p-1}=(3m-10)/2m$, $b_i=0\ (i=1,\dots,2p-4,\ 2p-2,\ 2p-1)$, $b_{2p-3}=4(m-5)(2m-5)/3m^2,\ \forall b_{2p}$.
E3.2	$n=5$ $p\geq 3$	$a_i=0\ (i=1,\dots,p-2),\ a_{p-1}=(3m-10)/2m$, $a_p\neq 0,\ b_i=0\ (i=1,\dots,2p-4)$, $b_{2p-3}=4(m-5)(2m-5)/3m^2$, $b_{2p-2}=3a_p(4m^2-45m+100)/m(3m-25)$, $b_{2p-1}=a_p^2(2m-15)^2/(3m-20)(m-10),\ \forall b_{2p}$.
E3.3	$n=5$ $m=11$	$a_1=23/22,\ \forall a_2,\ b_1=136/121$, $b_2=440215/56689952+267a_2/88,\ \forall b_4$, $b_3=-103195941517/43159874536064$ $-440813a_2/66997216+49a_2^2/13$.
E3.4	$n=5$ $m=12$	$a_1=13/12,\ a_2=0$, $b_1=133/108,\ b_2=0,\ b_3=5225/559872,\ \forall b_4$.
E3.5	$n=5$ $m=12$	$a_1=13/12,\ a_2\neq 0,\ b_1=133/108,\ b_2=34a_2/11$, $b_3=81a_2^2/32+5225/559872,\ \forall b_4$.

Table 2

No.	Conditions	Coefficients
F1.1	$n\geq 7$	$a_i=0\ (i=1,\dots,2p),\ \forall a_{2p+1}$, $b_i=0\ (i=1,\dots,p)$.
F1.2	$n\geq 7$	$a_i=0\ (i=1,\dots,2p-1)$, $b_i=0\ (i=1,\dots,p-1),\ b_p\neq 0$, $a_{2p}=b_p^2(m-n_p)^2/(m-m_{2p})(m-\lambda),\ \forall a_{2p+1}$.
F2.1	$n=6$ $m=6p+5$	$a_i=0\ (i=1,\dots,2p-1),\ a_{2p}=(5m-12)/2m$, $\forall a_{2p+1},\ b_i=0\ (i=1,\dots,p)$.
F2.2	$n=6$ $m=6p+5$ $m=6p+5$	$a_i=0\ (i=1,\dots,2p-1)$ $a_{2p}=\left(9b_p^2m+20m-48\right)/8m,\ \forall a_{2p+1}$, $b_i=0\ (i=1,\dots,p-1),\ b_p\neq 0$.
F3.1	$n=5$ $p\geq 2$	$a_i=0\ (i=1,\dots,2p-3,\ 2p-1,\ 2p)$, $a_{2p-2}=4(m-5)(2m-5)/3m^2$, $\forall a_{2p+1},\ b_i=0\ (i=1,\dots,p-2,\ p)$, $b_{p-1}=(3m-10)/2m$.
F3.2	$n=5$ $p\geq 3$	$a_i=0\ (i=1,\dots,2p-3)$, $a_{2p-2}=4(m-5)(2m-5)/3m^2$, $a_{2p-1}=3b_p(4m^2-45m+100)/m(3m-25)$, $a_{2p}=b_p^2(2m-15)^2/(3m-20)(m-10)$, $\forall a_{2p+1},\ b_i=0\ (i=1,\cdots,p-2)$, $b_{p-1}=(3m-10)/2m,\ b_p\neq 0$.

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