Advanced Studies in Pure Mathematics 45, 2006
Moduli Spaces and Arithmetic Geometry (Kyoto, 2004)
pp. 387–432

Moduli of stable parabolic connections, Riemann-Hilbert correspondence and geometry of Painlevé equation of type VI, part II

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Dedicated to Professor Masaki Maruyama on his 60th birthday

Abstract.

In this paper, we show that the family of moduli spaces of $\boldsymbol{\alpha}'$-stable $(\mathbf{t},\boldsymbol{\lambda})$-parabolic ϕ-connections of rank 2 over $\mathbf{P}^1$ with 4-regular singular points and the fixed determinant bundle of degree -1 is isomorphic to the family of Okamoto–Painlevé pairs introduced by Okamoto [O1] and [STT]. We also discuss about the generalization of our theory to the case where the rank of the connections and genus of the base curve are arbitrary. Defining isomonodromic flows on the family of moduli space of stable parabolic connections via the Riemann-Hilbert correspondences, we will show that a property of the Riemann-Hilbert correspondences implies the Painlevé property of isomonodromic flows.

§1. Introduction

In part I [IIS1], we established a complete geometric background for Painlevé equations of type VI or more generally for Garnier systems from view points of moduli spaces of rank 2 stable parabolic connections, moduli spaces of SL_2-representations of $\pi_1(\mathbf{P}^1 \setminus D(\mathbf{t}))$ and the Riemann-Hilbert correspondences between them.

In this formulation, Painlevé equations of type VI or Garnier systems are vector fields or systems of vector fields on each corresponding family of moduli spaces of stable parabolic connections arising from

Received December 5, 2005.
[1]Partly supported by Grant-in Aid for Scientific Research (Wakate-B-15740018).
[2]Partly supported by Grant-in Aid for Scientific Research (B-12440043).
[3]Partly supported by Grant-in Aid for Scientific Research (B-16340009), (Houga-16654004), and JSPS-NWO exchange program.

isomonodromic deformations of linear connections. Most notably, we can give a complete geometric proof of the Painlevé property of Painlevé equations of type VI and Garnier systems by proving that the Riemann-Hilbert correspondences are *bimeromorphic proper surjective holomorphic maps.* Moreover, one can prove that the Riemann-Hilbert correspondences give analytic resolutions of singularities of moduli spaces of the SL_2-representations. Then on the inverse image of each singular point, which is a family of compact subvarieties in the family of moduli spaces of connections, the vector fields admit classical solutions such as Riccati solutions in Painlevé VI case. See [Iw1], [Iw2], [SU], [IIS0], [STe] and [IIS3], for further applications of our approach to explicit dynamics of the Painlevé VI equations such as the classification of Riccati solutions and rational solutions, nonlinear monodromy, and Bäklund transformations as well as the relation with the former results [Miwa], [Mal] on the Painlevé property.

In this paper, with the notation in §3, we study in detail the moduli space $\overline{M_4^{\boldsymbol{\alpha}'}}(\mathbf{t}, \boldsymbol{\lambda}, -1)$ of $\boldsymbol{\alpha}'$-stable $(\mathbf{t}, \boldsymbol{\lambda})$-parabolic ϕ-connections of rank 2 over $\mathbf{P}^1$ with the fixed determinant bundle of degree -1 as well as the moduli space $M_4^{\boldsymbol{\alpha}}(\mathbf{t}, \boldsymbol{\lambda}, -1)$ of corresponding $\boldsymbol{\alpha}$-stable $(\mathbf{t}, \boldsymbol{\lambda})$-parabolic connections of rank 2 over $\mathbf{P}^1$. From a general result ([Theorem 1.1, [IIS1]] or [Theorem 5.1, §3]) which is also valid for $n \geq 5$, we can show that

- $\overline{M_4^{\boldsymbol{\alpha}'}}(\mathbf{t}, \boldsymbol{\lambda}, -1)$ is a projective surface,
- $M_4^{\boldsymbol{\alpha}}(\mathbf{t}, \boldsymbol{\lambda}, -1)$ is a smooth irreducible algebraic surface with a holomorphic symplectic structure and
- there exists a natural embedding $M_4^{\boldsymbol{\alpha}}(\mathbf{t}, \boldsymbol{\lambda}, -1) \hookrightarrow \overline{M_4^{\boldsymbol{\alpha}'}}(\mathbf{t}, \boldsymbol{\lambda}, -1)$.

In Theorem 4.1, which is the main theorem in this paper, we will show that the moduli space $\overline{M_4^{\boldsymbol{\alpha}'}}(\mathbf{t}, \boldsymbol{\lambda}, -1)$ is isomorphic to a *smooth* projective rational surface $\overline{\mathcal{S}}_{\mathbf{t},\boldsymbol{\lambda}}$. Moreover we can show that there exists a unique effective anti-canonical divisor $\mathcal{Y}_{\mathbf{t},\boldsymbol{\lambda}} \in | - K_{\overline{\mathcal{S}}_{\mathbf{t},\boldsymbol{\lambda}}}|$ of $\overline{\mathcal{S}}_{\mathbf{t},\boldsymbol{\lambda}}$ such that $\overline{\mathcal{S}}_{\mathbf{t},\boldsymbol{\lambda}} \setminus \mathcal{Y}_{\mathbf{t},\boldsymbol{\lambda},red} \simeq M_4^{\boldsymbol{\alpha}}(\mathbf{t}, \boldsymbol{\lambda}, -1)$. Moreover $(\overline{\mathcal{S}}_{\mathbf{t},\boldsymbol{\lambda}}, Y_{\mathbf{t},\boldsymbol{\lambda}})$ is a non-fibered rational Okamoto–Painlevé pairs of type $D_4^{(1)}$ which is defined in [STT] (cf. [Sakai]). Note that $\overline{\mathcal{S}}_{\mathbf{t},\boldsymbol{\lambda}} \setminus \mathcal{Y}_{\mathbf{t},\boldsymbol{\lambda},red}$ is isomorphic to the space of initial conditions for Painlevé equations of type VI constructed by Okamoto [O1].

We should mention here that an algebraic moduli space of parabolic connections without stability conditions was essentially considered by D. Arinlin and S. Lysenco in [AL1], [AL2] and [A] and they constructed a nice moduli space for generic $\boldsymbol{\lambda}$. However for special $\boldsymbol{\lambda}$, we should consider certain stability condition to construct a nice moduli space.

There are also different approaches [N], [Ni] for constructions of moduli spaces of logarithmic connections with or without parabolic structures.

The rough plan of this paper is as follows. In §2, we will explain about motivation of this paper and the theory of Okamoto–Painlevé pairs in [STa] and [STT]. In §3, we review results in part I [IIS1]. In §4, we will state Theorem 4.1 and the rest of the section will be devoted to show this theorem. In §5, we give a formulation of moduli theory of stable parabolic connection with regular singularities of any rank over any smooth curve. We also define the moduli space of representations of the fundamental group of n-punctured curve of genus g. Then we state the existence theorem of moduli space due to Inaba [Ina] without proof. In §6, we define the Riemann-Hilbert correspondence and state, also without proof, Theorem 6.1 which says that the Riemann–Hilbert correspondence is a proper surjective bimeromorphic analytic morphism. In §7, we will define isomonodromic flows on the family of the moduli spaces of $\boldsymbol{\alpha}$-stable parabolic connections. Assuming that Theorem 6.1 is true, we will show that isomonodromic flows satisfy the Painlevé property. (Note that, if rank $r = 2$ and over $\mathbf{P}^1$, a proof of Theorem 6.1 is found in [IIS1]).

Throughout in this paper, we will work over the field $\mathbf{C}$ of complex numbers.

§2. Motivation–Painlevé equations of type VI and Okamoto–Painlevé pairs

Let us recall the theory of space of initial conditions of Painlevé equation of type VI. Fix $\boldsymbol{\lambda} = (\lambda_1, \cdots, \lambda_4) \in \Lambda_4 = \mathbf{C}^4$ and consider the following ordinary differential equation of Painlevé VI type $P_{VI}(\boldsymbol{\lambda})$ parameterized by $\boldsymbol{\lambda}$:

(1)

$$
\begin{aligned}
P_{VI}(\boldsymbol{\lambda}):&\\
\frac{d^2x}{dt^2} \;=\;& \frac{1}{2}\left(\frac{1}{x}+\frac{1}{x-1}+\frac{1}{x-t}\right)\left(\frac{dx}{dt}\right)^2 - \\
&\left(\frac{1}{t}+\frac{1}{t-1}+\frac{1}{x-t}\right)\left(\frac{dx}{dt}\right) + \frac{x(x-1)(x-t)}{t^2(t-1)^2}\times \\
&\left[2(\lambda_4-\frac{1}{2})^2 - 2\lambda_1^2\frac{t}{x^2} + 2\lambda_2^2\frac{t-1}{(x-1)^2} + \left(\frac{1}{2}-2\lambda_3^2\right)\frac{t(t-1)}{(x-t)^2}\right].
\end{aligned}
$$

It is known that this algebraic differential equation $P_{VI}(\boldsymbol{\lambda})$ is equivalent to the following nonautonomous Hamiltonian system:

$$
(2) \qquad (H_{VI}(\boldsymbol{\lambda})): \begin{cases} \dfrac{dx}{dt} &= \dfrac{\partial H_{VI}}{\partial y}, \\ \dfrac{dy}{dt} &= -\dfrac{\partial H_{VI}}{\partial x}, \end{cases}
$$

where the Hamiltonian is given as follows.

$$
\begin{aligned} H_{VI}(x,y,t) &= \frac{1}{t(t-1)}\left[x(x-1)(x-t)y^2 - \{2\lambda_1(x-1)(x-t)\right. \\ &\quad \left. +2\lambda_2 x(x-t) + (2\lambda_3 - 1)x(x-1)\} y + \lambda(x-t)\right] \end{aligned}
$$

$$
\left(\lambda := \left\{(\lambda_1+\lambda_2+\lambda_3-1/2)^2 - (\lambda_4 - \frac{1}{2})^2\right\}\right).
$$

Let us set $T = \mathbf{C} \setminus \{0,1\}$ and consider the following algebraic vector fields on $\mathcal{S}^{(0)} = \mathbf{C}^2 \times T \times \Lambda_4 \ni (x,y,t,\boldsymbol{\lambda})$

$$
(3) \qquad v = \frac{\partial}{\partial t} + \frac{\partial H_{VI}}{\partial y}\frac{\partial}{\partial x} - \frac{\partial H_{VI}}{\partial x}\frac{\partial}{\partial y}
$$

Taking a relative compactification $\overline{\mathcal{S}}^{(0)} = \Sigma_0 \times T \times \Lambda_4$ of $\mathcal{S}^{(0)}$ where $\Sigma_0 = \mathbf{P}^1 \times \mathbf{P}^1$ and setting $\mathcal{D}^{(0)} = \overline{\mathcal{S}}^{(0)} \setminus \mathcal{S}^{(0)}$, we obtain the commutative diagram:

$$
(4) \qquad \begin{array}{ccccc} \mathcal{S}^{(0)} & \hookrightarrow & \overline{\mathcal{S}}^{(0)} & \hookleftarrow & \mathcal{D}^{(0)} \\ & \searrow_{\pi} & \downarrow \overline{\pi}^{(0)} & \swarrow & \\ & & T \times \Lambda_4. & & \end{array}
$$

We can extend the vector field v in (3) on $\mathcal{S}^{(0)}$ to a rational vector field

$$
(5) \qquad \tilde{v} \in H^0(\overline{\mathcal{S}}^{(0)}, \Theta_{\overline{\mathcal{S}}^{(0)}}(*\mathcal{D}^{(0)})).
$$

In general, the rational vector field $\tilde{v}$ has accessible singularities at the boundary divisor $\mathcal{D}^{(0)}$. In [O1], Okamoto gave explicit resolutions of accessible singularities by successive blowings-up at points on the boundary divisor. Then finally, we obtain a smooth family of smooth projective rational surfaces

$$
(6) \qquad \begin{array}{ccccc} \mathcal{S} & \hookrightarrow & \overline{\mathcal{S}} & \hookleftarrow & \mathcal{D} \\ & \searrow_{\pi} & \downarrow \overline{\pi} & \swarrow & \\ & & T \times \Lambda_4. & & \end{array}
$$

such that $\mathcal{D} := \overline{\mathcal{S}} \setminus \mathcal{S}$ is a reduced normal crossing divisor and $\mathcal{S}$ contains $\mathcal{S}^{(0)}$ as a Zariski open set. Moreover one can show that

$$\tilde{v} \in H^0(\overline{\mathcal{S}}, \Theta_{\overline{\mathcal{S}}}(-\log \mathcal{D})(\mathcal{D})), \tag{7}$$

where $\Theta_{\overline{\mathcal{S}}}(-\log \mathcal{D})$ denotes the sheaf of germs of regular vector fields with logarithmic zero along $\mathcal{D}$ (cf. [STT]). The extended rational vector field $\tilde{v}$ on $\overline{\mathcal{S}}$ has poles of order 1 along $\mathcal{D}$ and is regular on $\mathcal{S} = \overline{\mathcal{S}} \setminus \mathcal{D}$.

For each fixed $(t, \boldsymbol{\lambda}) \in T \times \Lambda_4$, the fiber $\overline{\pi}^{-1}((t, \boldsymbol{\lambda})) = \overline{\mathcal{S}}_{t,\boldsymbol{\lambda}}$ has a unique effective anti-canonical divisor $\mathcal{Y}_{t,\boldsymbol{\lambda}} \in |-K_{\overline{\mathcal{S}}_{t,\boldsymbol{\lambda}}}|$ with the irreducible decomposition

$$\mathcal{Y}_{t,\boldsymbol{\lambda}} = 2D_0 + D_1 + D_2 + D_3 + D_4$$

such that $\mathcal{Y}_{t,\boldsymbol{\lambda},red} = \sum_{i=0}^{4} D_i = \mathcal{D}_{t,\boldsymbol{\lambda}}$. Moreover it satisfies the following numerical conditions

$$\boxed{\mathcal{Y}_{t,\boldsymbol{\lambda}} \cdot D_i = \deg(-K_{\overline{\mathcal{S}}_{t,\boldsymbol{\lambda}}|D_i}) = 0 \text{ for } i = 0, \ldots, 4.} \tag{8}$$

In [STT], we give the following

Definition 2.1. (Cf. [STT], [STa], [Sakai]). A pair (S, Y) of a smooth projective rational surface with an anti-canonical divisor $Y \in |-K_S|$ with the irreducible decomposition $Y = \sum_i m_i Y_i$ is called a rational *Okamoto–Painlevé pair* if it satisfies the condition

$$\boxed{Y \cdot Y_i = \deg(-K_{\overline{\mathcal{S}}_{t,\boldsymbol{\lambda}}|Y_i}) = 0 \text{ for all } i.} \tag{9}$$

A rational Okamoto–Painlevé pair (S, Y) is called *of fibered-type* if there exists an elliptic fibration $f : S \longrightarrow \mathbf{P}^1$ such that $f^*(\infty) = nY$ for some $n \geq 1$.

It is easy to see that for a rational Okamoto–Painlevé pair the configuration of Y is in the list of degenerate fibers of elliptic surfaces due to Kodaira, which was classified by affine Dynkin diagrams. Therefore, we have a classification of rational Okamoto–Painlevé pairs (S, Y) by the Dynkin diagram of Y. For the case of Painlevé VI, we can say that the pair $(\overline{\mathcal{S}}_{t,\boldsymbol{\lambda}}, \mathcal{Y}_{t,\boldsymbol{\lambda}})$ appeared in a fiber of the family (6) is a rational Okamoto–Painlevé pair of type $D_4^{(1)}$. The family of the complement of the divisor $\mathcal{D}$ in (6) $\mathcal{S} \longrightarrow T \times \Lambda_4$, where the rational vector field $\tilde{v}$ is regular, should be the family of the space of initial conditions of Painlevé equations of type VI or the phase space of the vector field $\tilde{v}$. Note that $\mathcal{S} \longrightarrow T \times \Lambda_4$ contains the original family $\mathcal{S}^{(0)} \longrightarrow T \times \Lambda_4$ as a *proper* Zariski open subset, that is, $\mathcal{S}^{(0)} \subsetneq \mathcal{S}$. Here we recall the following technical lemma proved in [Proposition 1.3, [STT]].

Lemma 2.1. *Let (S, Y) be a rational Okamoto–Painlevé pair. Then the following conditions are equivalent to each other.*

(1) *(S, Y) is non-fibered type.*

(2) *A regular algebraic functions on the complement $S \setminus Y_{red}$ must be a constant function.*

In particular, for a non-fibered rational Okamoto–Painlevé pair (S, Y), the complement $S \setminus Y_{red}$ is never an affine variety.

Since one can show that an Okamoto–Painlevé pair $(\overline{\mathcal{S}}_{t,\boldsymbol{\lambda}}, \mathcal{Y}_{t,\boldsymbol{\lambda}})$ which appeared in a fiber of $\overline{\pi}$ in (6) is non-fibered type, we obtain the following

Corollary 2.1. *As for the family (6) for Painlevé equations of type VI constructed by Okamoto* [O1], *each fiber $\mathcal{S}_{t,\boldsymbol{\lambda}} = \overline{\mathcal{S}}_{t,\boldsymbol{\lambda}} \setminus \mathcal{D}_{t,\boldsymbol{\lambda}}$ is not an affine variety.*

In Theorem 4.1, we will show that the family (6) $\overline{\mathcal{S}} \longrightarrow T \times \Lambda_4$ constructed by Okamoto in [O1] is isomorphic to the family of moduli spaces

$$\overline{M_4^{\boldsymbol{\alpha}'}}(-1) \longrightarrow T_4 \times \Lambda_4$$

of $\boldsymbol{\alpha}'$-stable parabolic ϕ-connections of rank 2 over $\mathbf{P}^1$ with 4 regular singular points. (In order to identify, we need to normalize 4 points (t_1, t_2, t_3, t_4) to $(0, 1, t, \infty)$).

In [IIS1], for $\mathbf{a} = (a_1, \cdots, a_4) \in \mathcal{A}_4 \simeq \mathbf{C}^4$, we can also consider the moduli space $\mathcal{R}(\mathcal{P}_{4,t})_{\mathbf{a}}$ of $SL_2(\mathbf{C})$-representations ρ of $\pi_1(\mathbf{P}^1 \setminus D(\mathbf{t}))$ with the conditions $\mathrm{Tr}[\rho(\gamma_i)] = a_i$. Then we can define the Riemann-Hilbert correspondence

$$\mathbf{RH}_{t,\boldsymbol{\lambda}} : S_{t,\boldsymbol{\lambda}} \simeq M_4^{\boldsymbol{\alpha}}(t, \boldsymbol{\lambda}, -1) \longrightarrow \mathcal{R}(\mathcal{P}_{4,t})_{\mathbf{a}} \tag{10}$$

where $a_i = 2\cos 2\pi\lambda_i$.

Note that the Riemann–Hilbert correspondence is a highly transcendental analytic morphism, which is never an algebraic morphism. From results in [IIS1], we can show the following Theorem, which shows highly transcendental nature of the Riemann–Hilbert correspondence $\mathbf{RH}_{t,\boldsymbol{\lambda}}$.

Proposition 2.1. (Cf. [Theorem 1.4, Theorem 1.3, [IIS1]])

(1) *For all $(t, \boldsymbol{\lambda}) \in T \times \Lambda_4$, the Riemann–Hilbert correspondence $\mathbf{RH}_{t,\boldsymbol{\lambda}}$ is a bimeromorphic proper surjective analytic morphism. If $\boldsymbol{\lambda} \in \Lambda_4$ is generic, $\mathbf{RH}_{t,\boldsymbol{\lambda}}$ is an analytic isomorphism.*

(2) *For all $\mathbf{a} \in \mathcal{A}_4$, $\mathcal{R}(\mathcal{P}_{4,t})_{\mathbf{a}}$ is an affine variety, while $S_{t,\boldsymbol{\lambda}} \simeq M_4^{\boldsymbol{\alpha}}(t, \boldsymbol{\lambda}, -1)$ is not an affine variety. Hence if $\lambda \in \Lambda_4$ is generic, $\mathbf{RH}_{t,\boldsymbol{\lambda}}$ gives an analytic isomorphism between a non-affine variety $S_{t,\boldsymbol{\lambda}} \simeq M_4^{\boldsymbol{\alpha}}(t, \boldsymbol{\lambda}, -1)$ and an affine variety $\mathcal{R}(\mathcal{P}_{4,t})_{\mathbf{a}}$.*

(3) *For a generic* $\boldsymbol{\lambda} \in \Lambda_4$, $S_{t,\boldsymbol{\lambda}} \simeq M_4^{\boldsymbol{\alpha}}(t, \boldsymbol{\lambda}, -1)$ *is a Stein manifold, but not an affine variety.*

In §4, in order to obtain Okamoto-Painlevé pairs $(\overline{\mathcal{S}}_{\mathbf{t},\boldsymbol{\lambda}}, \mathcal{Y}_{\mathbf{t},\boldsymbol{\lambda}})$, we use a process of blowings-up which is a little bit different from Okamoto's in [O1]. The process can be explained as follows. Take $\Sigma_2 = \mathbf{P}(\mathcal{O}_{\mathbf{P}^1}(2) \oplus \mathcal{O}_{\mathbf{P}^1}) \longrightarrow \mathbf{P}^1$, which is the Hirzebruch surface of degree 2. Let D_0 denote the unique infinite section with $D_0^2 = -2$ and take the fibers F_i over t_i for $i = 1, \ldots, 4$. From the data λ_i, we can determine two points b_i^+ and b_i^- on F_i. (See §4 for precise definition of $b_i^{\pm}$). By blowing-up of Σ_2 at 8-points $\{b_i^{\pm}\}_{i=1}^4$, we obtain the rational surface $\overline{\mathcal{S}}_{\mathbf{t},\boldsymbol{\lambda}}$ and the unique effective anti-canonical divisor $\mathcal{Y}_{\mathbf{t},\boldsymbol{\lambda}}$ can be given by $\mathcal{Y}_{\mathbf{t},\boldsymbol{\lambda}} = 2D_0 + D_1 + D_2 + D_3 + D_4$ where D_i denotes the proper transform of F_i, (see Fig. 1).

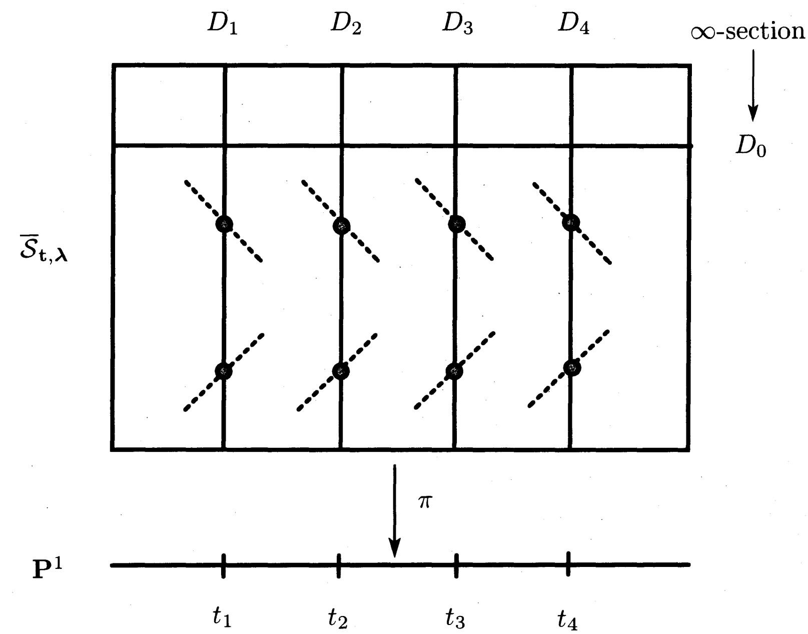

Fig. 1. Okamoto–Painlevé pair of type $D_4^{(1)}$

§3. Moduli spaces of rank 2 stable parabolic connections on $\mathbf{P}^1$ and their compactifications. A review of Part I.

In this section, we reproduce basic notation and definition in part I [IIS1] for reader's convenience.

3.1. Parabolic connections on $\mathbf{P}^1$.

Let $n \geq 3$ and set

$$T_n = \{(t_1, \dots, t_n) \in (\mathbf{P}^1)^n \quad | \quad t_i \neq t_j, (i \neq j)\}, \tag{11}$$

$$\Lambda_n = \{\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_n) \in \mathbf{C}^n\}. \tag{12}$$

Fixing a data $(\mathbf{t}, \boldsymbol{\lambda}) = (t_1, \dots, t_n, \lambda_1, \dots, \lambda_n) \in T_n \times \Lambda_n$, we define a reduced divisor on $\mathbf{P}^1$ as

$$D(\mathbf{t}) = t_1 + \cdots + t_n. \tag{13}$$

Moreover we fix a line bundle L on $\mathbf{P}^1$ with a logarithmic connection $\nabla_L : L \longrightarrow L \otimes \Omega^1_{\mathbf{P}^1}(D(\mathbf{t}))$.

Definition 3.1. A (rank 2) $(\mathbf{t}, \boldsymbol{\lambda})$-parabolic connection on $\mathbf{P}^1$ with the determinant (L, ∇_L) is a quadruplet $(E, \nabla, \varphi, \{l_i\}_{1 \leq i \leq n})$ which consists of

(1) a rank 2 vector bundle E on $\mathbf{P}^1$,
(2) a logarithmic connection $\nabla : E \longrightarrow E \otimes \Omega^1_{\mathbf{P}^1}(D(\mathbf{t}))$
(3) a bundle isomorphism $\varphi : \wedge^2 E \xrightarrow{\simeq} L$
(4) one dimensional subspace l_i of the fiber E_{t_i} of E at t_i, $l_i \subset E_{t_i}$, $i = 1, \dots, n$, such that
  (a) for any local sections s_1, s_2 of E,

$$\varphi \otimes id(\nabla s_1 \wedge s_2 + s_1 \wedge \nabla s_2) = \nabla_L(\varphi(s_1 \wedge s_2)),$$

  (b) $l_i \subset \mathrm{Ker}(\mathrm{res}_{t_i}(\nabla) - \lambda_i)$, that is, λ_i is an eigenvalue of the residue $\mathrm{res}_{t_i}(\nabla)$ of ∇ at t_i and l_i is a one-dimensional eigensubspace of $\mathrm{res}_{t_i}(\nabla)$.

Definition 3.2. Two $(\mathbf{t}, \boldsymbol{\lambda})$-parabolic connections

$$(E_1, \nabla_1, \varphi, \{l_i\}_{1 \leq i \leq n}), \quad (E_2, \nabla_2, \varphi', \{l'_i\}_{1 \leq i \leq n})$$

on $\mathbf{P}^1$ with the determinant (L, ∇_L) are isomorphic to each other if there is an isomorphism $\sigma : E_1 \xrightarrow{\sim} E_2$ and $c \in \mathbf{C}^\times$ such that the diagrams

$$\begin{array}{ccc} E_1 & \xrightarrow{\nabla_1} & E_1 \otimes \Omega^1_{\mathbf{P}^1}(D(\mathbf{t})) \\ \sigma \downarrow \cong & & \cong \downarrow \sigma \otimes \mathrm{id} \\ E_2 & \xrightarrow{\nabla_2} & E_2 \otimes \Omega^1_{\mathbf{P}^1}(D(\mathbf{t})) \end{array} \qquad \begin{array}{ccc} \bigwedge^2 E_1 & \xrightarrow[\cong]{\varphi} & L \\ \wedge^2\sigma \downarrow \cong & & c \downarrow \cong \\ \bigwedge^2 E_2 & \xrightarrow[\cong]{\varphi'} & L \end{array} \tag{14}$$

commute and $(\sigma)_{t_i}(l_i) = l'_i$ for $i = 1, \dots, n$.

3.2. The set of local exponents $\boldsymbol{\lambda} \in \Lambda_n$

Note that a data $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_n) \in \Lambda_n \simeq \mathbf{C}^n$ specifies the set of eigenvalues of the residue matrix of a connection ∇ at $\mathbf{t} = (t_1, \ldots, t_n)$, which will be called a set of *local exponents* of ∇.

Definition 3.3. A set of local exponents $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_n) \in \Lambda_n$ is called *special* if

(1) $\boldsymbol{\lambda}$ is *resonant*, that is, for some $1 \leq i \leq n$,

$$2\lambda_i \in \mathbf{Z}, \tag{15}$$

(2) or $\boldsymbol{\lambda}$ is *reducible*, that is, for some $(\epsilon_1, \ldots, \epsilon_n) \in \{\pm 1\}^n$

$$\sum_{i=1}^{n} \epsilon_i \lambda_i \in \mathbf{Z}. \tag{16}$$

If $\boldsymbol{\lambda} \in \Lambda_n$ is not special, $\boldsymbol{\lambda}$ is said to be *generic*.

3.3. Parabolic degrees and $\boldsymbol{\alpha}$-stability

Let us fix a series of positive rational numbers $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_{2n})$, which is called *a weight*, such that

$$0 \leq \alpha_1 < \alpha_2 < \cdots < \alpha_i < \cdots < \alpha_{2n} < \alpha_{2n+1} = 1. \tag{17}$$

For a $(\mathbf{t}, \boldsymbol{\lambda})$-parabolic connection on $\mathbf{P}^1$ with the determinant (L, ∇_L), we can define the parabolic degree of $E = (E, \nabla, \varphi, l)$ with respect to the weight $\boldsymbol{\alpha}$ by

$$\begin{aligned} \text{pardeg}_{\boldsymbol{\alpha}} E &= \deg E + \sum_{i=1}^{n} (\alpha_{2i-1} \dim E_{t_i}/l_i + \alpha_{2i} \dim l_i) \\ &= \deg L + \sum_{i=1}^{n} (\alpha_{2i-1} + \alpha_{2i}). \end{aligned} \tag{18}$$

Let $F \subset E$ be a rank 1 subbundle of E such that $\nabla F \subset F \otimes \Omega^1_{\mathbf{P}^1}(D(\mathbf{t}))$. We define the parabolic degree of $(F, \nabla_{|F})$ by

(19)

$$\text{pardeg}_{\boldsymbol{\alpha}} F = \deg F + \sum_{i=1}^{n} (\alpha_{2i-1} \dim F_{t_i}/l_i \cap F_{t_i} + \alpha_{2i} \dim l_i \cap F_{t_i}).$$

Definition 3.4. Fix a weight $\boldsymbol{\alpha}$. A $(\mathbf{t}, \boldsymbol{\lambda})$-parabolic connection (E, ∇, φ, l) on $\mathbf{P}^1$ with the determinant (L, ∇_L) is said to be $\boldsymbol{\alpha}$*-stable* (resp. *$\boldsymbol{\alpha}$-semistable*) if for every rank-1 subbundle F with $\nabla(F) \subset F \otimes \Omega^1_{\mathbf{P}^1}(D(\mathbf{t}))$

$$\text{pardeg}_{\boldsymbol{\alpha}} F < \frac{\text{pardeg}_{\boldsymbol{\alpha}} E}{2}, \quad (\text{resp. pardeg}_{\boldsymbol{\alpha}} F \leq \frac{\text{pardeg}_{\boldsymbol{\alpha}} E}{2}). \tag{20}$$

(For simplicity, "$\boldsymbol{\alpha}$-stable" will be abbreviated to "stable").

We define the coarse moduli space by

(21)

$$M^{\boldsymbol{\alpha}}_n(\mathbf{t}, \boldsymbol{\lambda}, L) = \left\{ (E, \nabla, \varphi, l); \begin{array}{l} \text{an } \boldsymbol{\alpha}\text{-stable } (\mathbf{t}, \boldsymbol{\lambda})\text{-parabolic} \\ \text{connection with} \\ \text{the determinant } (L, \nabla_L) \end{array} \right\} /\text{isom.}$$

3.4. Stable parabolic ϕ-connections

If $n \geq 4$, the moduli space $M^{\boldsymbol{\alpha}}_n(\mathbf{t}, \boldsymbol{\lambda}, L)$ never becomes projective nor complete. In order to obtain a compactification of the moduli space $M^{\boldsymbol{\alpha}}_n(\mathbf{t}, \boldsymbol{\lambda}, L)$, we will introduce the notion of a stable parabolic ϕ-connection, or equivalently, a stable parabolic Λ-triple. Again, let us fix $(\mathbf{t}, \boldsymbol{\lambda}) \in T_n \times \Lambda_n$ and a line bundle L on $\mathbf{P}^1$ with a connection $\nabla_L : L \to L \otimes \Omega^1_{\mathbf{P}^1}(D(\mathbf{t}))$.

Definition 3.5. The data $(E_1, E_2, \phi, \nabla, \varphi, \{l_i\}_{i=1}^n)$ is said to be a $(\mathbf{t}, \boldsymbol{\lambda})$-parabolic ϕ-connection of rank 2 with the determinant (L, ∇_L) if E_1, E_2 are rank 2 vector bundles on $\mathbf{P}^1$ with $\deg E_1 = \deg L$, $\phi : E_1 \to E_2$, $\nabla : E_1 \to E_2 \otimes \Omega^1_{\mathbf{P}^1}(D(\mathbf{t}))$ are morphisms of sheaves, $\varphi : \bigwedge^2 E_2 \xrightarrow{\sim} L$ is an isomorphism and $l_i \subset (E_1)_{t_i}$ are one dimensional subspaces for $i = 1, \ldots, n$ such that

(1) $\phi(fa) = f\phi(a)$ and $\nabla(fa) = \phi(a) \otimes df + f\nabla(a)$ for $f \in \mathcal{O}_{\mathbf{P}^1}$, $a \in E_1$,
(2) $(\varphi \otimes \text{id})(\nabla(s_1) \wedge \phi(s_2) + \phi(s_1) \wedge \nabla(s_2)) = \nabla_L(\varphi(\phi(s_1) \wedge \phi(s_2)))$ for $s_1, s_2 \in E_1$ and
(3) $(\text{res}_{t_i}(\nabla) - \lambda_i \phi_{t_i})|_{l_i} = 0$ for $i = 1, \ldots, n$.

Definition 3.6.
Two $(\mathbf{t}, \boldsymbol{\lambda})$ parabolic ϕ-connections

$$(E_1, E_2, \phi, \nabla, \varphi, \{l_i\}), \quad (E'_1, E'_2, \phi', \nabla', \varphi', \{l'_i\})$$

are said to be isomorphic to each other if there are isomorphisms $\sigma_1 : E_1 \xrightarrow{\sim} E_1'$, $\sigma_2 : E_2 \xrightarrow{\sim} E_2'$ and $c \in \mathbf{C} \setminus \{0\}$ such that the diagrams

$$\begin{array}{ccc} E_1 & \xrightarrow{\phi} & E_2 \\ {\scriptstyle \sigma_1}\downarrow{\scriptstyle \cong} & & {\scriptstyle \cong}\downarrow{\scriptstyle \sigma_2} \\ E_1' & \xrightarrow{\phi'} & E_2' \end{array} \qquad \begin{array}{ccc} E_1 & \xrightarrow{\nabla} & E_2 \otimes \Omega^1_{\mathbf{P}^1}(D(\mathbf{t})) \\ {\scriptstyle \sigma_1}\downarrow{\scriptstyle \cong} & & {\scriptstyle \cong}\downarrow{\scriptstyle \sigma_2 \otimes \mathrm{id}} \\ E_1' & \xrightarrow{\nabla'} & E_2' \otimes \Omega^1_{\mathbf{P}^1}(D(\mathbf{t})) \end{array}$$

$$\begin{array}{ccc} \bigwedge^2 E_2 & \xrightarrow[\cong]{\varphi} & L \\ {\scriptstyle \wedge^2 \sigma_2}\downarrow{\scriptstyle \cong} & & {\scriptstyle c}\downarrow{\scriptstyle \cong} \\ \bigwedge^2 E_2' & \xrightarrow[\cong]{\varphi'} & L \end{array}$$

commute and $(\sigma_1)_{t_i}(l_i) = l_i'$ for $i = 1, \ldots, n$.

Remark 3.1. Assume that two vector bundles E_1, E_2 and morphisms $\phi : E_1 \to E_2$, $\nabla : E_1 \to E_2 \otimes \Omega^1_{\mathbf{P}^1}(D(\mathbf{t}))$ satisfying $\phi(fa) = f\phi(a)$, $\nabla(fa) = \phi(a) \otimes df + f\nabla(a)$ for $f \in \mathcal{O}_{\mathbf{P}^1}$, $a \in E_1$ are given. If ϕ is an isomorphism, then $(\phi \otimes \mathrm{id})^{-1} \circ \nabla : E_1 \to E_1 \otimes \Omega^1_{\mathbf{P}^1}(D(\mathbf{t}))$ becomes a connection on E_1.

Fix rational numbers $\alpha_1', \alpha_2', \ldots, \alpha_{2n}', \alpha_{2n+1}'$ satisfying

$$0 \leq \alpha_1' < \alpha_2' < \cdots < \alpha_{2n}' < \alpha_{2n+1}' = 1$$

and positive integers β_1, β_2. Setting $\boldsymbol{\alpha}' = (\alpha_1', \ldots, \alpha_{2n}'), \boldsymbol{\beta} = (\beta_1, \beta_2)$, we obtain a *weight* $(\boldsymbol{\alpha}', \boldsymbol{\beta})$ for parabolic ϕ-connections.

Definition 3.7. Fix a sufficiently large integer γ. Let

$$(E_1, E_2, \phi, \nabla, \varphi, \{l_i\}_{i=1}^n)$$

be a parabolic ϕ-connection. For any subbundles $F_1 \subset E_1$, $F_2 \subset E_2$ satisfying $\phi(F_1) \subset F_2$, $\nabla(F_1) \subset F_2 \otimes \Omega^1_{\mathbf{P}^1}(D(\mathbf{t}))$, we define

$$\mu((F_1, F_2))_{\boldsymbol{\alpha}'\boldsymbol{\beta}} = \frac{1}{\beta_1 \operatorname{rank}(F_1) + \beta_2 \operatorname{rank}(F_2)} (\beta_1(\deg F_1(-D(\mathbf{t})))$$
$$+ \beta_2(\deg F_2 - \gamma \operatorname{rank}(F_2)) + \sum_{i=1}^n \beta_1(\alpha_{2i-1}' d_{2i-1}(F_1) + \alpha_{2i}' d_{2i}(F_1))$$

where $d_{2i-1}(F) = \dim((F_1)_{t_i}/l_i \cap (F_1)_{t_i})$, $d_{2i}(F_1) = \dim((F_1)_{t_i} \cap l_i)$.

A parabolic ϕ-connection $(E_1, E_2, \phi, \nabla, \varphi, \{l_i\}_{i=1}^n)$ is said to be $(\boldsymbol{\alpha}', \boldsymbol{\beta})$-stable (resp. $(\boldsymbol{\alpha}', \boldsymbol{\beta})$-semistable) if for any subbundles $F_1 \subset E_1$, $F_2 \subset$

E_2 satisfying $\phi(F_1) \subset F_2$, $\nabla(F_1) \subset F_2 \otimes \Omega^1_{\mathbf{P}^1}(D(\mathbf{t}))$ and $(F_1, F_2) \neq (E_1, E_2), (0,0)$, the inequality

$$\mu((F_1, F_2))_{\boldsymbol{\alpha}'\boldsymbol{\beta}} < \mu((E_1, E_2))_{\boldsymbol{\alpha}'\boldsymbol{\beta}}, \quad \text{(resp. } \mu((F_1, F_2))_{\boldsymbol{\alpha}'\boldsymbol{\beta}} \leq \mu((E_1, E_2))_{\boldsymbol{\alpha}'\boldsymbol{\beta}}.) \tag{22}$$

We define the coarse moduli space of $(\boldsymbol{\alpha}', \boldsymbol{\beta})$-stable $(\mathbf{t}, \boldsymbol{\lambda})$-parabolic ϕ-connections with the determinant (L, ∇_L) by

$$\overline{M_n^{\boldsymbol{\alpha}'\boldsymbol{\beta}}}(\mathbf{t}, \boldsymbol{\lambda}, L) := \{(E_1, E_2, \phi, \nabla, \varphi, \{l_i\})\} / \text{isom}. \tag{23}$$

For a given weight $(\boldsymbol{\alpha}', \boldsymbol{\beta})$ and $1 \leq i \leq 2n$, define a rational number α_i by

$$\alpha_i = \frac{\beta_1}{\beta_1 + \beta_2} \alpha_i'. \tag{24}$$

Then $\boldsymbol{\alpha} = (\alpha_i)$ satisfies the condition

$$0 \leq \alpha_1 < \alpha_2 < \cdots < \alpha_{2n} < \frac{\beta_1}{(\beta_1 + \beta_2)} < 1, \tag{25}$$

hence $\boldsymbol{\alpha}$ defines a weight for parabolic connections. It is easy to see that if we take γ sufficiently large $(E, \nabla, \varphi, \{l_i\})$ is $\boldsymbol{\alpha}$-stable if and only if the associated parabolic ϕ-connection $(E, E, \mathrm{id}_E, \nabla, \varphi, \{l_i\})$ is stable with respect to $(\boldsymbol{\alpha}', \boldsymbol{\beta})$. Therefore we see that the natural map

$$(E, \nabla, \varphi, \{l_i\}) \mapsto (E, E, \mathrm{id}_E, \nabla, \varphi, \{l_i\}) \tag{26}$$

induces an injection

$$M_n^{\boldsymbol{\alpha}}(\mathbf{t}, \boldsymbol{\lambda}, L) \hookrightarrow \overline{M_n^{\boldsymbol{\alpha}'\boldsymbol{\beta}}}(\mathbf{t}, \boldsymbol{\lambda}, L). \tag{27}$$

Conversely, assuming that $\boldsymbol{\beta} = (\beta_1, \beta_2)$ are given, for a weight $\boldsymbol{\alpha} = (\alpha_i)$ satisfying the condition (25), we can define $\alpha_i' = \alpha_i \frac{\beta_1 + \beta_2}{\beta_1}$ for $1 \leq i \leq 2n$. Since $0 \leq \alpha_1' < \alpha_2' < \cdots < \alpha_{2n}' = \alpha_{2n} \frac{\beta_1 + \beta_2}{\beta_1} < 1$, $(\boldsymbol{\alpha}', \boldsymbol{\beta})$ give a weight for parabolic ϕ-connections.

Moreover, considering the relative setting over $T_n \times \Lambda_n$, we can define two families of the moduli spaces

$$\overline{\pi}_n : \overline{M_n^{\boldsymbol{\alpha}'\boldsymbol{\beta}}}(L) \longrightarrow T_n \times \Lambda_n, \quad \pi_n : M_n^{\boldsymbol{\alpha}}(L) \longrightarrow T_n \times \Lambda_n \tag{28}$$

such that the following diagram commutes;

$$\begin{array}{ccc} M_n^{\boldsymbol{\alpha}}(L) & \stackrel{\iota}{\hookrightarrow} & \overline{M_n^{\boldsymbol{\alpha}'\boldsymbol{\beta}}}(L) \\ {\scriptstyle \pi_n}\downarrow & & \downarrow{\scriptstyle \overline{\pi}_n} \\ T_n \times \Lambda_n & = & T_n \times \Lambda_n. \end{array} \tag{29}$$

Here the fibers of π_n and $\overline{\pi_n}$ over $(\mathbf{t}, \boldsymbol{\lambda}) \in T_n \times \Lambda_n$ are

$$\pi_n^{-1}(\mathbf{t}, \boldsymbol{\lambda}) = M^{\boldsymbol{\alpha}}(\mathbf{t}, \boldsymbol{\lambda}, L), \quad \overline{\pi_n}^{-1}(\mathbf{t}, \boldsymbol{\lambda}) = \overline{M^{\boldsymbol{\alpha}'\boldsymbol{\beta}}}(\mathbf{t}, \boldsymbol{\lambda}, L). \tag{30}$$

3.5. The existence of moduli spaces and their properties

The following theorem was proved in [IIS1].

Theorem 3.1. ([Theorem 2.1, [IIS1]]).

(1) *Fix a weight* $\boldsymbol{\beta} = (\beta_1, \beta_2)$. *For a generic weight* $\boldsymbol{\alpha}'$,

$$\overline{\pi_n} : \overline{M_n^{\boldsymbol{\alpha}'\boldsymbol{\beta}}}(L) \longrightarrow T_n \times \Lambda_n$$

is a projective *morphism. In particular, the moduli space* $\overline{M^{\boldsymbol{\alpha}'\boldsymbol{\beta}}}(\mathbf{t}, \boldsymbol{\lambda}, L)$ *is a* projective *algebraic scheme for all* $(\mathbf{t}, \boldsymbol{\lambda}) \in T_n \times \Lambda_n$.

(2) *For a generic weight* $\boldsymbol{\alpha}$, $\pi_n : M_n^{\boldsymbol{\alpha}}(L) \longrightarrow T_n \times \Lambda_n$ *is a* smooth morphism *of relative dimension* $2n - 6$ *with irreducible closed fibers. Therefore, the moduli space* $M_n^{\boldsymbol{\alpha}}(\mathbf{t}, \boldsymbol{\lambda}, L)$ *is a* smooth, irreducible *algebraic variety of dimension* $2n-6$ *for all* $(\mathbf{t}, \boldsymbol{\lambda}) \in T_n \times \Lambda_n$.

Remark 3.2. (1) The structures of moduli spaces $M_n^{\boldsymbol{\alpha}}(L)$ and $\overline{M_n^{\boldsymbol{\alpha}'\boldsymbol{\beta}}}(L)$ may depend on the weights $\boldsymbol{\alpha}$, $(\boldsymbol{\alpha}', \boldsymbol{\beta})$ and $\deg L$.

(2) The moduli spaces $M_n^{\boldsymbol{\alpha}}(L)$ is a fine moduli space. In fact, we have the universal families over these moduli spaces.

(3) The moduli space $M_n^{\boldsymbol{\alpha}}(\mathbf{t}, \boldsymbol{\lambda}, L)$ admits a natural holomorphic symplectic structure. (See [Proposition 6.2, [IIS1]). This fact is a part of the reason why Painlevé VI and Garnier systems can be written in nonautonomous Hamiltonian systems.

(4) In case of $n = 4$, we can show that $\overline{M_4^{\boldsymbol{\alpha}'\boldsymbol{\beta}}}(\mathbf{t}, \boldsymbol{\lambda}, L)$ is smooth (cf. Proposition 4.3). However we do not know whether $\overline{M_n^{\boldsymbol{\alpha}'\boldsymbol{\beta}}}(\mathbf{t}, \boldsymbol{\lambda}, L)$ is smooth or not for $n \geq 5$.

When we describe the explicit algebraic or geometric structure of the moduli spaces $M_n^{\boldsymbol{\alpha}}(L)$ and $\overline{M_n^{\boldsymbol{\alpha}'\boldsymbol{\beta}}}(L)$, it is convenient to fix a determinant line bundle (L, ∇_L). As a typical example of the determinant bundle is

$$(L, \nabla_L) = (\mathcal{O}_{\mathbf{P}^1}(-t_n), d) \tag{31}$$

where the connection is given by

$$\nabla_L(z - t_n) = d(z - t_n) = (z - t_n) \otimes \frac{dz}{z - t_n}. \tag{32}$$

Here z is an inhomogeneous coordinate of $\mathbf{P}^1 = \operatorname{Spec} \mathbf{C}[z] \cup \{\infty\}$. For this $(L, \nabla_L) = (\mathcal{O}_{\mathbf{P}^1}(-t_n), d)$, we set

$$M_n^{\boldsymbol{\alpha}}(\mathbf{t}, \boldsymbol{\lambda}, -1) = M_n^{\boldsymbol{\alpha}}(\mathbf{t}, \boldsymbol{\lambda}, L), \quad (\text{resp. } \overline{M_n^{\boldsymbol{\alpha}'\boldsymbol{\beta}}}(\mathbf{t}, \boldsymbol{\lambda}, -1) = \overline{M_n^{\boldsymbol{\alpha}'\boldsymbol{\beta}}}(\mathbf{t}, \boldsymbol{\lambda}, L) \quad).$$

§4. Explicit construction of moduli spaces for the case of $n = 4$ (Painlevé VI case).

In this section, we will deal with the case of $n = 4$ in detail. Let us fix a sufficiently large integer γ and take a weight $(\boldsymbol{\alpha}', \boldsymbol{\beta})$ for parabolic ϕ-connections where $\boldsymbol{\alpha}' = (\alpha_1', \ldots, \alpha_8')$, $\boldsymbol{\beta} = (\beta_1, \beta_2), \gamma$ and fix $(\mathbf{t}, \boldsymbol{\lambda}) = (t_1, \ldots, t_4, \lambda_1, \ldots, \lambda_4) \in T_4 \times \Lambda_4$.

Then the corresponding weight $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_8)$ for parabolic connections can be given by

$$\alpha_i = \alpha_i' \frac{\beta_1}{\beta_1 + \beta_2} \quad 1 \leq i \leq 8.$$

For simplicity, we will assume that $\beta_1 = \beta_2 = 1$, hence $\boldsymbol{\alpha} = \boldsymbol{\alpha}'/2$. We also assume $(L, \nabla_l) = (\mathcal{O}_{\mathbf{P}^1}(-t_n), d)$ and set

$$\overline{M_4^{\boldsymbol{\alpha}'}}(\mathbf{t}, \boldsymbol{\lambda}, -1) = \overline{M_4^{\boldsymbol{\alpha}'\boldsymbol{\beta}}}(\mathbf{t}, \boldsymbol{\lambda}, L), \quad \overline{M_4^{\boldsymbol{\alpha}'}}(-1) = \overline{M_4^{\boldsymbol{\alpha}'\boldsymbol{\beta}}}(L).$$

From Theorem 3.1, we can obtain the commutative diagram:

$$\begin{array}{ccc} M_4^{\boldsymbol{\alpha}}(-1) & \overset{\iota}{\hookrightarrow} & \overline{M_4^{\boldsymbol{\alpha}'}}(-1) \\ {\scriptstyle \pi_4}\downarrow & & \downarrow{\scriptstyle \overline{\pi}_4} \\ T_4 \times \Lambda_4 & = & T_4 \times \Lambda_4, \end{array} \tag{33}$$

such that $\pi_4^{-1}((\mathbf{t}, \boldsymbol{\lambda})) \simeq M_4^{\boldsymbol{\alpha}}(\mathbf{t}, \boldsymbol{\lambda}, -1)$ and $\overline{\pi}_4^{-1}(\mathbf{t}, \boldsymbol{\lambda}) \simeq \overline{M_4^{\boldsymbol{\alpha}'}}(\mathbf{t}, \boldsymbol{\lambda}, -1)$. (Note that $\boldsymbol{\alpha} = \boldsymbol{\alpha}'/2$). From Theorem 3.1, we see that for a generic weight $\boldsymbol{\alpha}'$, $\overline{\pi}_4$ is a projective morphism and π_4 is a smooth morphism of relative dimension 2.

4.1. Main Theorem (Explicit description for $n = 4$ case).

Putting $\beta_1 = \beta_2 = 1$, we further assume that $|\alpha_j'| \ll 1$ for $i = 1, \ldots, 8$. Let $\tilde{t}_1, \ldots, \tilde{t}_4 \subset \mathbf{P}^1 \times \Lambda_4 \times T_4$ be the pull-back of the universal sections on $\mathbf{P}^1 \times T_4$ over T_4. Put $D(\tilde{\mathbf{t}}) := \tilde{t}_1 + \cdots + \tilde{t}_4$ and consider the projective bundle

$$\pi : \mathbf{P}\left(\Omega^1_{\mathbf{P}^1 \times T_4 \times \Lambda_4 / T_4 \times \Lambda_4}(D(\tilde{\mathbf{t}})) \oplus \mathcal{O}_{\mathbf{P}^1 \times T_4 \times \Lambda_4}\right) \longrightarrow \mathbf{P}^1 \times T_4 \times \Lambda_4.$$

Note that since $\Omega^1_{\mathbf{P}^1}(D(\mathbf{t})) \simeq \mathcal{O}_{\mathbf{P}^1}(2)$ the fiber of $p_{23} \circ \pi$ over $(\mathbf{t}, \boldsymbol{\lambda}) \in T_4 \times \Lambda_4$ is isomorphic to

$$\mathbf{P}(\mathcal{O}_{\mathbf{P}^1}(2) \oplus \mathcal{O}_{\mathbf{P}^1}) \simeq \Sigma_2$$

where Σ_2 is the Hirzebruch surface of degree 2.

Let $\tilde{D}_i \subset \mathbf{P}\left(\Omega^1_{\mathbf{P}^1 \times T_4 \times \Lambda_4 / T_4 \times \Lambda_4}(D(\tilde{\mathbf{t}})) \oplus \mathcal{O}_{\mathbf{P}^1 \times T_4 \times \Lambda_4}\right)$ be the inverse image of $\tilde{t}_i$. Since the residue map induces an isomorphism

$$\Omega^1_{\mathbf{P}^1 \times T_4 \times \Lambda_4 / T_4 \times \Lambda_4}(D(\tilde{\mathbf{t}}))|_{\tilde{t}_i} \xrightarrow{\sim} \mathcal{O}_{\tilde{t}_i},$$

we have a canonical isomorphism $\tilde{D}_i \xrightarrow{\sim} \mathbf{P}^1 \times T_4 \times \Lambda_4$. Let $\tilde{b}_i^+ \subset \tilde{D}_i$ (resp. $\tilde{b}_i^- \subset \tilde{D}_i$) be the inverse image of $[\lambda_i^+ : 1] \subset \mathbf{P}^1 \times T_4 \times \Lambda_4$ (resp. $[\lambda_i^- : 1] \subset \mathbf{P}^1 \times T_4 \times \Lambda_4$). We denote by B^+ (resp. B^-) the reduced induced structure on $\tilde{b}_1^+ \cup \cdots \cup \tilde{b}_4^+$ (resp. $\tilde{b}_1^- \cup \cdots \cup \tilde{b}_4^-$) and we consider the reduced induced structure on $B = B^+ \cup B^-$. Let

$$g : Z \to \mathbf{P}\left(\Omega^1_{\mathbf{P}^1 \times T_4 \times \Lambda_4 / T_4 \times \Lambda_4}(D(\tilde{\mathbf{t}})) \oplus \mathcal{O}_{\mathbf{P}^1 \times T_4 \times \Lambda_4}\right)$$

be the blow-up along B^+ and $\overline{S}$ be the blow-up of Z along the closure of $g^{-1}(B^- \setminus (B^+ \cap B^-))$. (It is easy to see that $\overline{S} \longrightarrow T_4 \times \Lambda_4$ is isomorphic to the family constructed by Okamoto [O1]). Note that Z is isomorphic to the blow-up of Z along $g^{-1}(B)$.

The main purpose of this section is to prove the following theorem:

Theorem 4.1. *Take $\boldsymbol{\alpha}' = (\alpha'_i)_{1 \le i \le 2n}$, $\boldsymbol{\beta} = (\beta_1, \beta_2)$ and γ such that $\beta_1 = \beta_2 = 1$, $\gamma \gg 0$, $|\alpha'_i| \ll 1$ for $1 \le i \le 2n$, $\alpha'_{2i} - \alpha'_{2i-1} < \sum_{j \ne i}(\alpha'_{2j} - \alpha'_{2j-1})$ for $1 \le i \le n$ and that any $(\boldsymbol{\alpha}', \boldsymbol{\beta})$-semistable parabolic ϕ-connection is $(\boldsymbol{\alpha}', \boldsymbol{\beta})$-stable.*

(1) *There exists an isomorphism*

$$\overline{M_4^{\boldsymbol{\alpha}'}}(\mathcal{O}_{\mathbf{P}^1}(-\tilde{t}_4)) \xrightarrow{\sim} \overline{S} \tag{34}$$

over $T_4 \times \Lambda_4$.

(2) *Let $\mathcal{Y}$ be the closed subscheme of $\overline{M_4^{\boldsymbol{\alpha}'}}(\mathcal{O}_{\mathbf{P}^1}(-\tilde{t}_4))$ defined by the condition $\wedge^2 \phi = 0$. Then*

$$M_4^{\boldsymbol{\alpha}'/2}(\mathcal{O}_{\mathbf{P}^1}(-\tilde{t}_4)) = \overline{M_4^{\boldsymbol{\alpha}'}}(\mathcal{O}_{\mathbf{P}^1}(-\tilde{t}_4)) \setminus \mathcal{Y}. \tag{35}$$

(3) *For each $(\mathbf{t}, \boldsymbol{\lambda}) \in T_4 \times \Lambda_4$, the fiber $\mathcal{Y}_{(\mathbf{t},\boldsymbol{\lambda})}$ is the anti-canonical divisor of $\overline{M_4^{\boldsymbol{\alpha}'}}(\mathbf{t}, \boldsymbol{\lambda}, \mathcal{O}_{\mathbf{P}^1}(-\tilde{t}_4))$ and the pair*

$$(\overline{M_4^{\boldsymbol{\alpha}'}}(\mathbf{t}, \boldsymbol{\lambda}, \mathcal{O}_{\mathbf{P}^1}(-\tilde{t}_4)), \mathcal{Y}_{(\mathbf{t},\boldsymbol{\lambda})}) \tag{36}$$

is an Okamoto-Painlevé pair of type $D_4^{(1)}$.

4.2. Construction of the morphism $\overline{M_4^{\boldsymbol{\alpha}'}}(\mathbf{t},\boldsymbol{\lambda},-1)\to\Sigma_2$

We assume that (α_i) satisfies the condition of Lemma 4.2 below.

Take any point $(E_1,E_2,\phi,\nabla,\varphi,\{l_i\})\in\overline{M_4^{\boldsymbol{\alpha}'}}(\mathbf{t},\boldsymbol{\lambda},-1)$. There are unique trivial subbundles $L_1^{(0)}\subset E_1$, $L_2^{(0)}\subset E_2$, whose existence is confirmed by Proposition 4.1 bellow. Since the composite

$$\mathcal{O}_{\mathbf{P}^1}\cong L_1^{(0)}\hookrightarrow E_1\xrightarrow{\phi}E_2\to E_2/L_2^{(0)}\cong\mathcal{O}_{\mathbf{P}^1}(-1)$$

is zero, the composite

(37)

$$u:L_1^{(0)}\hookrightarrow E_1\xrightarrow{\nabla}E_2\otimes\Omega^1_{\mathbf{P}^1}(D(\mathbf{t}))\to E_2/L_2^{(0)}\otimes\Omega^1_{\mathbf{P}^1}(D(\mathbf{t}))\cong\mathcal{O}_{\mathbf{P}^1}(1)$$

becomes a homomorphism. By Proposition 4.1 bellow, there is a unique point $q\in\mathbf{P}^1$ satisfying $u(q)=0$. Put $L_1^{(-1)}:=E_1/L_1^{(0)}$, $L_2^{(-1)}:=E_2/L_2^{(0)}$ and let $p_j:E_j\to L_j^{(-1)}$ be the projection for $j=1,2$. We define a homomorphism $B:E_1\to L_2^{(-1)}\otimes\Omega^1_{\mathbf{P}^1}(D(\mathbf{t}))$ by $B(a):=(p_2\otimes\mathrm{id})\nabla(a)-d(p_2\phi(a))$ for $a\in E_1$, where d is the canonical connection on $L_2^{(-1)}\cong\mathcal{O}_{\mathbf{P}^1}(-t_4)$. Since $u_q=0$, B_q induces a homomorphism $h_1:(L_1^{(-1)})_q\to\left(L_2^{(-1)}\otimes\Omega^1_{\mathbf{P}^1}(D(\mathbf{t}))\right)_q$ which makes the diagram

$$\begin{array}{ccccccc}0\to&(L_1^{(0)})_q&\longrightarrow&(E_1)_q&\longrightarrow&(L_1^{(-1)})_q&\to 0\\&&\searrow_{u_q=0}&{\scriptstyle B_q}\downarrow&\exists h_1\swarrow&&\\&&&\left(L_2^{(-1)}\otimes\Omega^1_{\mathbf{P}^1}(D(\mathbf{t}))\right)_q&&&\end{array}$$

commute. On the other hand, ϕ induces the following commutative diagram

$$\begin{array}{ccccccccc}0&\longrightarrow&L_1^{(0)}&\longrightarrow&E_1&\longrightarrow&L_1^{(-1)}&\longrightarrow&0\\&&{\scriptstyle\phi_1}\downarrow&&{\scriptstyle\phi}\downarrow&&{\scriptstyle\phi_2}\downarrow&&\\0&\longrightarrow&L_2^{(0)}&\longrightarrow&E_2&\longrightarrow&L_2^{(-1)}&\longrightarrow&0.\end{array}$$

We put $h_2:=\phi_2(q)$. Then h_1,h_2 determine a homomorphism

(38)

$$\iota:(L_1^{(-1)})_q\longrightarrow\left(L_2^{(-1)}\otimes\Omega^1_{\mathbf{P}^1}(D(\mathbf{t}))\oplus L_2^{(-1)}\right)_q;\quad a\mapsto(-h_1(a),h_2(a)).$$

By Proposition 4.2, ι is injective and the inclusion

$$\iota:(L_1^{(-1)})_q\hookrightarrow(L_2^{(-1)})_q\otimes(\Omega^1_{\mathbf{P}^1}(D(\mathbf{t}))\oplus\mathcal{O}_{\mathbf{P}^1})_q$$

determines a point $p(E_1, E_2, \phi, \nabla, \varphi, \{l_i\})$ of $\mathbf{P}_*(\Omega^1_{\mathbf{P}^1}(D(\mathbf{t})) \oplus \mathcal{O}_{\mathbf{P}^1})$, where $\mathbf{P}_*(\Omega^1_{\mathbf{P}^1}(D(\mathbf{t})) \oplus \mathcal{O}_{\mathbf{P}^1})$ means $\operatorname{Proj} S((\Omega^1_{\mathbf{P}^1}(D(\mathbf{t})) \oplus \mathcal{O}_{\mathbf{P}^1})^\vee)$. So we can define a morphism

$$(39) \qquad \begin{array}{cccc} p: & \overline{M_4^{\boldsymbol{\alpha}'}}(\mathbf{t}, \boldsymbol{\lambda}, -1) & \longrightarrow & \mathbf{P}_*(\Omega^1_{\mathbf{P}^1}(D(\mathbf{t})) \oplus \mathcal{O}_{\mathbf{P}^1}); \\ & (E_1, E_2, \phi, \nabla, \varphi, \{l_i\}) & \mapsto & p(E_1, E_2, \phi, \nabla, \varphi, \{l_i\}). \end{array}$$

Proposition 4.1. *For any member*

$$(E_1, E_2, \phi, \nabla, \varphi, \{l_i\}) \in \overline{M_4^{\boldsymbol{\alpha}'}}(\mathbf{t}, \boldsymbol{\lambda}, -1),$$

we have

$$E_1 \cong E_2 \cong \mathcal{O}_{\mathbf{P}^1} \oplus \mathcal{O}_{\mathbf{P}^1}(-1).$$

Proof. Take decompositions

$$E_1 = \mathcal{O}_{\mathbf{P}^1}(d_1) \oplus \mathcal{O}_{\mathbf{P}^1}(-d_1 - 1) \quad (d_1 \geq 0)$$
$$E_2 = \mathcal{O}_{\mathbf{P}^1}(d_2) \oplus \mathcal{O}_{\mathbf{P}^1}(-d_2 - 1) \quad (d_2 \geq 0).$$

Assume that $d_1 + d_2 > 1$. Then we have $\phi(\mathcal{O}_{\mathbf{P}^1}(d_1)) \subset \mathcal{O}_{\mathbf{P}^1}(d_2)$. The composite

$$\mathcal{O}_{\mathbf{P}^1}(d_1) \to E_1 \xrightarrow{\nabla} E_2 \otimes \Omega^1_{\mathbf{P}^1}(D(\mathbf{t})) \to \mathcal{O}_{\mathbf{P}^1}(-d_2-1) \otimes \Omega^1_{\mathbf{P}^1}(D(\mathbf{t})) \cong \mathcal{O}_{\mathbf{P}^1}(1-d_2)$$

becomes a homomorphism and must be zero since $H^0(\mathcal{O}_{\mathbf{P}^1}(1 - (d_1 + d_2))) = 0$. So we have $\nabla(\mathcal{O}_{\mathbf{P}^1}(d_1)) \subset \mathcal{O}_{\mathbf{P}^1}(d_2) \otimes \Omega^1(D(\mathbf{t}))$. Then the subbundles $(\mathcal{O}_{\mathbf{P}^1}(d_1), \mathcal{O}_{\mathbf{P}^1}(d_2))$ breaks the stability of $(E_1, E_2, \phi, \nabla, \varphi, \{l_i\})$.

If $d_1 = 1$ and $d_2 = 0$, then $\phi(\mathcal{O}_{\mathbf{P}^1}(1)) = 0$ and the composite

$$f : \mathcal{O}_{\mathbf{P}^1}(1) \hookrightarrow E_1 \xrightarrow{\nabla} E_2 \otimes \Omega^1_{\mathbf{P}^1}(D(\mathbf{t}))$$

becomes a homomorphism.

Put $L := (\operatorname{Im} f) \otimes \Omega^1(D(\mathbf{t}))^\vee$. Then L is a vector bundle and either $L = 0$ or L is a line bundle with $\deg L \geq -1$. Then the subsheaves $(\mathcal{O}_{\mathbf{P}^1}(1), L)$ breaks the stability of $(E_1, E_2, \phi, \nabla, \varphi, \{l_i\})$.

If $d_1 = 0$ and $d_2 = 1$, then the composite $E_1 \xrightarrow{\phi} E_2 \to \mathcal{O}_{\mathbf{P}^1}(-2)$ must be zero and the composite $f : E_1 \xrightarrow{\nabla} E_2 \otimes \Omega^1_{\mathbf{P}^1}(D(\mathbf{t})) \to \mathcal{O}_{\mathbf{P}^1}(-2) \otimes \Omega^1_{\mathbf{P}^1}(D(\mathbf{t}))$ becomes a homomorphism. Put $L := \ker f$. Then we have either $L = E_1$ or L is a line bundle such that $\deg L \geq -1$. Then the subbundles $(L, \mathcal{O}_{\mathbf{P}^1}(1))$ breaks the stability of $(E_1, E_2, \phi, \nabla, \varphi, \{l_i\})$.

Hence we have $d_1 = d_2 = 0$ and $E_1 \cong E_2 \cong \mathcal{O}_{\mathbf{P}^1} \oplus \mathcal{O}_{\mathbf{P}^1}(-1)$.

Q.E.D.

Lemma 4.1. *For any* $(E_1, E_2, \phi, \nabla, \varphi, \{l_i\}) \in \overline{M_4^{\alpha'}}(\mathbf{t}, \boldsymbol{\lambda}, -1)$, *the homomorphism* u *defined in (37) is injective.*

Proof. Assume that $u = 0$. Then the subbundles $(L_1^{(0)}, L_2^{(0)})$ breaks the stability of $(E_1, E_2, \phi, \nabla, \varphi, \{l_i\})$. Thus $u \neq 0$ and u is injective. Q.E.D.

Lemma 4.2. *Assume* $\alpha'_{2i} - \alpha'_{2i-1} < \sum_{j\neq i}(\alpha'_{2j} - \alpha'_{2j-1})$ *for any* $1 \leq i \leq n$. *Then the homomorphism* ι *defined above is injective.*

Proof. If ϕ is isomorphic, then $h_2 : (L_1^{(-1)})_q \to (L_2^{(-1)})_q$ is isomorphic, and so ι is injective. So we assume that ϕ is not isomorphic, that is, $\wedge^2\phi = 0$.

First consider the case $\operatorname{rank}\phi = 1$. Take decompositions $E_1 = \mathcal{O}_{\mathbf{P}^1} \oplus \mathcal{O}_{\mathbf{P}^1}(-1)$, $E_2 = \mathcal{O}_{\mathbf{P}^1} \oplus \mathcal{O}_{\mathbf{P}^1}(-1)$. Then the homomorphism ϕ can be represented by a matrix

$$\begin{pmatrix} \phi_1 & \phi_3 \\ 0 & \phi_2 \end{pmatrix} \quad (\phi_1, \phi_2 \in H^0(\mathcal{O}_{\mathbf{P}^1}), \phi_3 \in H^0(\mathcal{O}_{\mathbf{P}^1}(1))),$$

where the composite $E_1 \xrightarrow{\phi} E_2 \xrightarrow{p_2} \mathcal{O}_{\mathbf{P}^1}(-1)$ is represented by $(0, \phi_2)$ and $E_1 \xrightarrow{\phi} E_2 \to \mathcal{O}_{\mathbf{P}^1}$ by (ϕ_1, ϕ_3).

Now assume that $p_2 \circ \phi = 0$. Then $\phi_2 = 0$. If moreover $\phi_1 = 0$, then $\phi_3 \neq 0$ since $\operatorname{rank}\phi = 1$. Take local bases e_1 of $\mathcal{O}_{\mathbf{P}^1} \subset E_1$ and e_2 of $\mathcal{O}_{\mathbf{P}^1}(-1) \subset E_1$. Then the condition $\nabla(e_1) \wedge \phi(e_2) + \phi(e_1) \wedge \nabla(e_2) = 0$ implies that $\nabla(e_1) \in \mathcal{O}_{\mathbf{P}^1} \otimes \Omega_{\mathbf{P}^1}(D(\mathbf{t}))$, which contradicts the result of Lemma 4.1. Thus we have $\phi_1 \neq 0$. Then, by multiplying an automorphism of E_1 given by

$$\begin{pmatrix} c_1 & c_3 \\ 0 & c_2 \end{pmatrix} \quad \left(c_1, c_2 \in H^0(\mathcal{O}_{\mathbf{P}^1}^{\times}), c_3 \in H^0(\mathcal{O}_{\mathbf{P}^1}(1))\right),$$

the matrix representing ϕ changes into the form

$$\begin{pmatrix} \phi_1 & \phi_3 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} c_1 & c_3 \\ 0 & c_2 \end{pmatrix} = \begin{pmatrix} c_1\phi_1 & c_3\phi_1 + c_2\phi_3 \\ 0 & 0 \end{pmatrix}.$$

For a suitable choice of c_1, c_2 and c_3, we have $c_1\phi_1 = 1$ and $c_3\phi_1 + c_2\phi_3 = 0$. So we may assume without loss of generality that $\phi_3 = 0$ and $\phi_1 = 1$.

The homomorphism $B : E_1 \to L_2^{(-1)} \otimes \Omega^1_{\mathbf{P}^1}(D(\mathbf{t})) = \Omega^1_{\mathbf{P}^1}(D(\mathbf{t}))(-1)$ defined by $B(a) := (p_2 \otimes \mathrm{id})\nabla(a) - d(p_2\phi(a))$ for $a \in E_1$ can be represented by a matrix (ω_3, ω_4) where $\omega_3 \in H^0(\Omega^1_{\mathbf{P}^1}(D(\mathbf{t}))(-1))$ and $\omega_4 \in H^0(\Omega^1_{\mathbf{P}^1}(D(\mathbf{t})))$. Define a homomorphism $A : E_1 \to \Omega^1_{\mathbf{P}^1}(D(\mathbf{t}))$ by $A(a) := (q_2 \otimes \mathrm{id})\nabla(a) - d(q_2\phi(a))$ for $a \in E_1$, where $q_2 : E_2 \to \mathcal{O}_{\mathbf{P}^1}$

is the projection with respect to the given decomposition of E_2 and d is the trivial connection on $\mathcal{O}_{\mathbf{P}^1}$. Then A can be represented by a matrix (ω_1, ω_2), where $\omega_1 \in H^0(\Omega^1_{\mathbf{P}^1}(D(\mathbf{t})))$ and $\omega_2 \in H^0(\Omega^1_{\mathbf{P}^1}(D(\mathbf{t}))(1))$. Roughly speaking ∇ is represented by the matrix

$$\begin{pmatrix} \omega_1 & \omega_2 \\ \omega_3 & \omega_4 \end{pmatrix}.$$

Since $\phi(e_2) = 0$ and $\phi(e_1) \in \mathcal{O}_{\mathbf{P}^1}$, the condition $\nabla(e_1) \wedge \phi(e_2) + \phi(e_1) \wedge \nabla(e_2) = 0$ implies that $\nabla(e_2) \in \mathcal{O}_{\mathbf{P}^1} \otimes \Omega^1_{\mathbf{P}^1}(D(\mathbf{t}))$. Thus we have $\omega_4 = 0$. Take a nonzero vector $v^{(i)} \in l_i \subset (E_1)_{t_i}$. Then we must have

$$(\mathrm{res}_{t_i} \nabla)(v^{(i)}) = \lambda_i \phi_{t_i}(v^{(i)}). \tag{40}$$

Since $E_1 = \mathcal{O}_{\mathbf{P}^1} \oplus \mathcal{O}_{\mathbf{P}^1}(-1)$, we can write $v^{(i)} = \begin{pmatrix} v_1^{(i)} \\ v_2^{(i)} \end{pmatrix}$ with $v_1^{(i)} \in (\mathcal{O}_{\mathbf{P}^1})_{t_i}$ and $v_2^{(i)} \in (\mathcal{O}_{\mathbf{P}^1}(-1))_{t_i}$. Then we have

$$(\mathrm{res}_{t_i} \nabla) \begin{pmatrix} v_1^{(i)} \\ v_2^{(i)} \end{pmatrix} = \begin{pmatrix} \mathrm{res}_{t_i}(\omega_1) v_1^{(i)} + \mathrm{res}_{t_i}(\omega_2) v_2^{(i)} \\ \mathrm{res}_{t_i}(\omega_3) v_1^{(i)} \end{pmatrix},$$

$$\phi_{t_i} \begin{pmatrix} v_1^{(i)} \\ v_2^{(i)} \end{pmatrix} = \begin{pmatrix} v_1^{(i)} \\ 0 \end{pmatrix}$$

Thus the equality (40) is equivalent to the equalities

$$\mathrm{res}_{t_i}(\omega_1) v_1^{(i)} + \mathrm{res}_{t_i}(\omega_2) v_2^{(i)} = \lambda_i v_1^{(i)}, \quad \mathrm{res}_{t_i}(\omega_3) v_1^{(i)} = 0.$$

Since u is injective by Lemma 4.1, $\omega_3 \neq 0$. So there is at most one point t_i which satisfies $\mathrm{res}_{t_i}(\omega_3) = 0$, because $\omega_3 \in H^0(\Omega^1_{\mathbf{P}^1}(D(\mathbf{t}))(-1)) \cong H^0(\mathcal{O}_{\mathbf{P}^1}(1))$. Thus, for some i, we have $\mathrm{res}_{t_j}(\omega_3) \neq 0$ for $j \neq i$. Then we have $v_1^{(j)} = 0$ for $j \neq i$. So we have $l_j \subset (\mathcal{O}_{\mathbf{P}^1}(-1))_{t_j}$ for $j \neq i$. Recall that the image of $\nabla|_{\mathcal{O}_{\mathbf{P}^1}(-1)}$ is contained in $\mathcal{O}_{\mathbf{P}^1} \otimes \Omega^1_{\mathbf{P}^1}(D(\mathbf{t}))$ because $\omega_4 = 0$. Let $F_*(E_1)$ be the filtration of E_1 corresponding to $\{l_j\}$. Then $(\mathcal{O}_{\mathbf{P}^1}(-1), \mathcal{O}_{\mathbf{P}^1}, \Phi|_{\mathcal{O}_{\mathbf{P}^1}(-1)}, F_*(E_1) \cap \mathcal{O}_{\mathbf{P}^1}(-1))$ is a parabolic ϕ-subconnection of $(E_1, E_2, \Phi, F_*(E_1))$. Since $2(\alpha'_{2i-1} + \sum_{j \neq i} \alpha'_{2j}) > \sum_{j=1}^{8} \alpha'_j$ by the assumption of the lemma, we have

$$\begin{aligned}
&\mu((\mathcal{O}_{\mathbf{P}^1}(-1), \mathcal{O}_{\mathbf{P}^1}, \Phi|_{\mathcal{O}_{\mathbf{P}^1}(-1)}, F_*(E_1) \cap \mathcal{O}_{\mathbf{P}^1}(-1))) \\
&\geq \frac{-1-4-1-\gamma+\alpha'_{2i-1}+\sum_{j \neq i} \alpha'_{2j}}{2} \\
&> \frac{-2-8-2-2\gamma+\sum_{j=1}^{4}(\alpha'_{2j-1}+\alpha'_{2j})}{4} = \mu((E_1, E_2, \Phi, F_*(E_1))),
\end{aligned}$$

which breaks the stability of $(E_1, E_2, \Phi, F_*(E_1))$. Therefore $p_2 \circ \phi \neq 0$ and the homomorphism $L_1^{(-1)} \to L_2^{(-1)}$ induced by ϕ is an isomorphism. Hence $h_2 : (L_1^{(-1)})_q \to (L_2^{(-1)})_q$ is bijective and so ι is injective.

Next consider the case $\phi = 0$. In this case, $\nabla : E_1 \to E_2 \otimes \Omega^1_{\mathbf{P}^1}(D(\mathbf{t}))$ is a homomorphism. If we choose a decomposition $E_1 = \mathcal{O}_{\mathbf{P}^1} \oplus \mathcal{O}_{\mathbf{P}^1}(-1)$, $E_2 = \mathcal{O}_{\mathbf{P}^1} \oplus \mathcal{O}_{\mathbf{P}^1}(-1)$, ∇ is represented by a matrix

$$\begin{pmatrix} \omega_1 & \omega_2 \\ \omega_3 & \omega_4 \end{pmatrix} \quad \begin{cases} \omega_1, \omega_4 \in H^0(\Omega^1_{\mathbf{P}^1}(D(\mathbf{t}))), \\ \omega_2 \in H^0(\Omega^1_{\mathbf{P}^1}(D(\mathbf{t}))(1)), \\ \omega_3 \in H^0(\Omega^1_{\mathbf{P}^1}(D(\mathbf{t}))(-1)). \end{cases}$$

Notice that ω_3 corresponds to the homomorphism $u : L_1^{(0)} \to E_2/L_2^{(0)} \otimes \Omega_{\mathbf{P}^1}(D(\mathbf{t}))$ and so $\omega_3 \neq 0$. Let q be the point of $\mathbf{P}^1$ satisfying $\omega_3(q) = 0$. Assume that $\omega_4(q) = 0$. Multiplying an automorphism of E_1 given by

$$\begin{pmatrix} c_1 & c_3 \\ 0 & c_2 \end{pmatrix} \quad \left(c_1, c_2 \in H^0(\mathcal{O}^\times_{\mathbf{P}^1}), c_3 \in H^0(\mathcal{O}_{\mathbf{P}^1}(1))\right),$$

the matrix representing ∇ changes into the form

$$\begin{pmatrix} \omega_1 & \omega_2 \\ \omega_3 & \omega_4 \end{pmatrix} \begin{pmatrix} c_1 & c_3 \\ 0 & c_2 \end{pmatrix} = \begin{pmatrix} c_1\omega_1 & c_3\omega_1 + c_2\omega_2 \\ c_1\omega_3 & c_3\omega_3 + c_2\omega_4 \end{pmatrix}.$$

For a suitable choice of c_2, c_3, we have $c_3\omega_3 + c_2\omega_4 = 0$. So we may assume without loss of generality that $\omega_4 = 0$. Take a nonzero element $v^{(i)}$ of $l_i \subset (E_1)_{t_i}$. We can write $v^{(i)} = \begin{pmatrix} v_1^{(i)} \\ v_2^{(i)} \end{pmatrix}$ with $v_1^{(i)} \in (\mathcal{O}_{\mathbf{P}^1})_{t_i}$ and $v_2^{(i)} \in (\mathcal{O}_{\mathbf{P}^1}(-1))_{t_i}$. Then we have

$$(\mathrm{res}_{t_i} \nabla)(v^{(i)}) = (\mathrm{res}_{t_i} \nabla) \begin{pmatrix} v_1^{(i)} \\ v_2^{(i)} \end{pmatrix} = \begin{pmatrix} \mathrm{res}_{t_i}(\omega_1)v_1^{(i)} + \mathrm{res}_{t_i}(\omega_2)v_2^{(i)} \\ \mathrm{res}_{t_i}(\omega_3)v_1^{(i)} \end{pmatrix}$$

Since $(\mathrm{res}_{t_i} \nabla)(v^{(i)}) = \lambda_i \phi_{t_i}(v^{(i)}) = 0$, we have $\mathrm{res}_{t_i}(\omega_3)v_1^{(i)} = 0$ for $i = 1, \dots, 4$. There is at most one i satisfying $\mathrm{res}_{t_i}(\omega_3) = 0$ because $\omega_3 \in H^0(\Omega^1_{\mathbf{P}^1}(D(\mathbf{t}))(-1))$. So we may assume that for some i, $\omega_3(t_j) \neq 0$ for $j \neq i$. Then we have $v_1^{(j)} = 0$ for $j \neq i$ and $l_j \subset \mathcal{O}_{\mathbf{P}^1}(-1)_{t_j}$ for $j \neq i$. Since $\omega_4 = 0$, $\nabla(\mathcal{O}_{\mathbf{P}^1}(-1)) \subset \mathcal{O}_{\mathbf{P}^1} \otimes \Omega^1_{\mathbf{P}^1}(D(\mathbf{t}))$. If $F_*(E_1)$ is the filtration of E_1 corresponding to $\{l_j\}$, then $(\mathcal{O}_{\mathbf{P}^1}(-1), \mathcal{O}_{\mathbf{P}^1}, \Phi|_{\mathcal{O}_{\mathbf{P}^1}(-1)}, F_*(E_1) \cap$

$\mathcal{O}_{\mathbf{P}^1}(-1))$ is a parabolic ϕ-subconnection of $(E_1, E_2, \Phi, F_*(E_1))$ and

$$\begin{aligned}&\mu(\mathcal{O}_{\mathbf{P}^1}(-1), \mathcal{O}_{\mathbf{P}^1}, \Phi|_{\mathcal{O}_{\mathbf{P}^1}(-1)}, F_*(E_1) \cap \mathcal{O}_{\mathbf{P}^1}(-1)) \\ &\geq \frac{-1-4-1-\gamma+\alpha'_{2i-1}+\sum_{j\neq i}\alpha'_{2j}}{2} \\ &> \frac{-2-8-2-2\gamma+\sum_{j=1}^{4}(\alpha'_{2j-1}+\alpha'_{2j})}{4} = \mu(E_1, E_2, \Phi, F_*(E_1))\end{aligned}$$

which contradicts the stability of $(E_1, E_2, \Phi, F_*(E_1))$. Therefore we have $\omega_4(q) \neq 0$, which means that h_1 is bijective and so ι is injective. Q.E.D.

4.3. Smoothness of $\overline{M_4^{\boldsymbol{\alpha}'}}(\mathbf{t}, \boldsymbol{\lambda}, -1)$

Let $\mathcal{Y}$ be the closed subscheme of $\overline{M_4^{\boldsymbol{\alpha}'}}(-1)$ defined by the condition $\wedge^2\phi = 0$ and $Y(\mathbf{t}, \boldsymbol{\lambda})$ be the fiber of $\mathcal{Y}$ over $(\mathbf{t}, \boldsymbol{\lambda})$.

Proposition 4.2. *Under the assumption of Lemma 4.2, the restriction* $Y(\mathbf{t}, \boldsymbol{\lambda}) \xrightarrow{p} \mathbf{P}_*\left(\Omega^1_{\mathbf{P}^1}(D(\mathbf{t})) \oplus \mathcal{O}_{\mathbf{P}^1}\right)$ *of the morphism* p *defined above is injective.*

Proof. Let D_0 be the section of $\mathbf{P}_*\left(\Omega^1_{\mathbf{P}^1}(D(\mathbf{t})) \oplus \mathcal{O}_{\mathbf{P}^1}\right)$ over $\mathbf{P}^1$ defined by the injection $\Omega^1_{\mathbf{P}^1}(D(\mathbf{t})) \hookrightarrow \Omega^1_{\mathbf{P}^1}(D(\mathbf{t})) \oplus \mathcal{O}_{\mathbf{P}^1}$. Take any point $(E_1, E_2, \phi, \nabla, \varphi, \{l_i\}) \in Y(\mathbf{t}, \boldsymbol{\lambda})$. From the proof of Lemma 4.2, we can see that $p((E_1, E_2, \phi, \nabla, \varphi, \{l_i\}) \in D_0$ if and only if $\phi = 0$.

First assume that $\operatorname{rank}\phi = 1$. As in the proof of Lemma 4.2, We take decompositions $E_1 = \mathcal{O}_{\mathbf{P}^1} \oplus \mathcal{O}_{\mathbf{P}^1}(-1)$, $E_2 = \mathcal{O}_{\mathbf{P}^1} \oplus \mathcal{O}_{\mathbf{P}^1}(-1)$ and represent ϕ by a matrix

$$\begin{pmatrix} \phi_1 & \phi_3 \\ 0 & \phi_2 \end{pmatrix} \quad (\phi_1, \phi_2 \in H^0(\mathcal{O}_{\mathbf{P}^1}), \phi_3 \in H^0(\mathcal{O}_{\mathbf{P}^1}(1))).$$

By the proof of Lemma 4.2, $\phi_2 \neq 0$. Multiplying a certain automorphism of E_2, we may assume that $\phi_3 = 0$ and $\phi_2 = 1$. Since $\operatorname{rank}\phi = 1$, we have $\phi_1 = 0$. Consider the homomorphism $B : E_1 \to \mathcal{O}_{\mathbf{P}^1}(-1) \otimes \Omega^1_{\mathbf{P}^1}(D(\mathbf{t}))$ defined by $B(a) = p_2\nabla(a) - d(p_2\phi(a))$. Let (ω_3, ω_4) $(\omega_3 \in H^0(\Omega^1_{\mathbf{P}^1}(D(\mathbf{t}))(-1)), \omega_4 \in H^0(\Omega^1_{\mathbf{P}^1}(D(\mathbf{t}))))$ be the matrix which represents B. Since $\phi_1 = 0$, $\phi_3 = 0$, the composite $E_1 \xrightarrow{\nabla} E_2 \otimes \Omega^1_{\mathbf{P}^1}(\mathbf{t}) \xrightarrow{q_2 \otimes 1} \mathcal{O}_{\mathbf{P}^1} \otimes \Omega^1_{\mathbf{P}^1}(\mathbf{t})$ becomes a homomorphism, which can be represented by a matrix (ω_1, ω_2) with $\omega_1 \in H^0(\Omega^1_{\mathbf{P}^1}(\mathbf{t})), \omega_2 \in H^0(\Omega^1_{\mathbf{P}^1}(\mathbf{t})(1))$. Roughly speaking, ∇ is represented by the matrix

$$\begin{pmatrix} \omega_1 & \omega_2 \\ \omega_3 & \omega_4 \end{pmatrix}.$$

We use the same notation as in the proof of Lemma 4.2. Then we have $\nabla(e_1) \wedge \phi(e_2) + \phi(e_1) \wedge \nabla(e_2) = 0$. Since $\phi(e_1) = 0$ and $\phi(e_2) \in$

$\mathcal{O}_{\mathbf{P}^1}(-1)$, we have $\nabla(e_1) \in \mathcal{O}_{\mathbf{P}^1}(-1) \otimes \Omega^1_{\mathbf{P}^1}(D(\mathbf{t}))$ and so $\omega_1 = 0$. Take a nonzero element $v^{(i)}$ of $l_i \subset (E_1)_{t_i}$ and write $v^{(i)} = \begin{pmatrix} v_1^{(i)} \\ v_2^{(i)} \end{pmatrix}$ where $v_1^{(i)} \in (\mathcal{O}_{\mathbf{P}^1})_{t_i}$ and $v_2^{(i)} \in \mathcal{O}_{\mathbf{P}^1}(-1)_{t_i}$. Then we have

$$(\mathrm{res}_{t_i} \nabla)(v^{(i)}) = (\mathrm{res}_{t_i} \nabla) \begin{pmatrix} v_1^{(i)} \\ v_2^{(i)} \end{pmatrix}$$
$$= \begin{pmatrix} \mathrm{res}_{t_i}(\omega_2) v_2^{(i)} \\ \mathrm{res}_{t_i}(\omega_3) v_1^{(i)} + \mathrm{res}_{t_i}(\omega_4) v_2^{(i)} + \mathrm{res}_{t_i}\left(\frac{dz}{z-t_4}\right) v_2^{(i)} \end{pmatrix},$$

$$\phi_{t_i}(v^{(i)}) = \phi_{t_i} \begin{pmatrix} v_1^{(i)} \\ v_2^{(i)} \end{pmatrix} = \begin{pmatrix} 0 \\ v_2^{(i)} \end{pmatrix}$$

Since $(\mathrm{res}_{t_i} \nabla)(v^{(i)}) = \lambda_i \phi_{t_i}(v^{(i)})$, we have

$$\mathrm{res}_{t_i}(\omega_2) v_2^{(i)} = 0,$$
$$\mathrm{res}_{t_i}(\omega_3) v_1^{(i)} + \mathrm{res}_{t_i}(\omega_4) v_2^{(i)} + \mathrm{res}_{t_i}\left(\frac{dz}{z-t_4}\right) v_2^{(i)} = \lambda_i v_2^{(i)}.$$

If $\omega_2(t_i) = 0$ for any i, then $\omega_2 = 0$ because $\omega_2 \in H^0(\Omega^1_{\mathbf{P}^1}(D(\mathbf{t}))(1)) \cong H^0(\mathcal{O}_{\mathbf{P}^1}(3))$ and there is a decomposition

$$(E_1, E_2, \phi, \nabla, \{l_i\}) = (E_1, \mathcal{O}_{\mathbf{P}^1}(-1), \phi, \nabla, \{l_i\}) \oplus (0, \mathcal{O}_{\mathbf{P}^1}, 0, 0, \{0\}),$$

which contradicts the stability of $(E_1, E_2, \phi, \nabla, \varphi, \{l_i\})$. On the other hand, if $\omega_2(t_i) \neq 0$, then $v_2^{(i)} = 0$, $v_1^{(i)} \neq 0$ and $\omega_3(t_i) = 0$. However, there is at most one i which satisfies $\omega_3(t_i) = 0$ because $\omega_3 \in H^0(\Omega^1_{\mathbf{P}^1}(D(\mathbf{t}))(-1)) \cong H^0(\mathcal{O}_{\mathbf{P}^1}(1))$. Therefore there is only one i which satisfies $\omega_2(t_i) \neq 0$. In this case, $\omega_3(t_i) = 0$ and so $q = t_i$, which means that the image $p(E_1, E_2, \phi, \nabla, \varphi, \{l_j\})$ is contained in the fiber D_i of $\mathbf{P}_*\left(\Omega^1_{\mathbf{P}^1}(D(\mathbf{t})) \oplus \mathcal{O}_{\mathbf{P}^1}\right)$ over t_i. Applying certain automorphisms of E_1 and E_2 represented by a matrix of the form

$$\begin{pmatrix} c & 0 \\ 0 & 1 \end{pmatrix} \quad (c \in H^0(\mathcal{O}^{\times}_{\mathbf{P}^1})),$$

we may assume that

$$\omega_2 = \frac{\prod_{j \neq i}(z - t_j)}{\prod_{j=1}^4 (z - t_j)} dz, \quad \omega_3 = \frac{z - t_i}{\prod_{j=1}^4 (z - t_j)} dz,$$

where z is a fixed inhomogeneous coordinate of $\mathbf{P}^1$. Then giving a value $\mathrm{res}_{t_i}(\omega_4)$ is equivalent to giving a point $p(E_1, E_2, \phi, \nabla, \varphi, \{l_i\})$ in the fiber D_i. Applying an automorphism of E_1 represented by a matrix of the form

$$\begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix} \quad (c \in H^0(\mathcal{O}_{\mathbf{P}^1}(1))),$$

we may assume that ω_4 is of the form

$$\omega_4 = \frac{a dz}{\prod_{j=1}^4 (z - t_j)}$$

with $a \in \mathbf{C}$. a is determined by the value $\mathrm{res}_{t_i}(\omega_4)$. Thus the matrices representing ϕ and ∇ are determined uniquely, up to automorphisms of E_1 and E_2, by the point $p(E_1, E_2, \phi, \nabla, \varphi, \{l_j\})$. Recall that $v_1^{(i)} \neq 0$, $v_2^{(i)} = 0$ and $\mathrm{res}_{t_j}(\omega_3) v_1^{(j)} + \mathrm{res}_{t_j}(\omega_4) v_2^{(j)} + \mathrm{res}_{t_j}(\frac{dz}{z - t_4}) v_2^{(j)} = \lambda_j v_2^{(j)}$ for $j \neq i$. Since $\mathrm{res}_{t_j}(\omega_3) \neq 0$ for $j \neq i$, every $v^{(j)}$ (including $v^{(i)}$) is uniquely determined up to a scalar multiplication. Thus the parabolic structure is determined by ϕ, ∇. Hence $(E_1, E_2, \phi, \nabla, \varphi, \{l_j\})$ is uniquely determined by the point $p(E_1, E_2, \phi, \nabla, \varphi, \{l_j\})$.

Next we assume that $\phi = 0$. Let

$$\begin{pmatrix} \omega_1 & \omega_2 \\ \omega_3 & \omega_4 \end{pmatrix}, \quad \begin{cases} \omega_1, \omega_4 \in H^0(\Omega^1_{\mathbf{P}^1}(D(\mathbf{t}))), \\ \omega_2 \in H^0(\Omega^1_{\mathbf{P}^1}(D(\mathbf{t}))(1)), \\ \omega_3 \in H^0(\Omega^1_{\mathbf{P}^1}(D(\mathbf{t}))(-1)). \end{cases}$$

be a matrix representing ∇. Let q be the point of $\mathbf{P}^1$ satisfying $\omega_3(q) = 0$. We may assume without loss of generality that $q \neq t_i$ for $i = 1, 2, 3$. From the proof of Lemma 4.2, we have $\omega_4(q) \neq 0$. Applying an automorphism of E_1, we may assume

$$\omega_4 = \frac{(z - t_1)(z - t_2)}{\prod_{j=1}^4 (z - t_j)} dz, \quad \omega_3 = \frac{z - q}{\prod_{j=1}^4 (z - t_j)} dz.$$

For a nonzero element $v^{(i)} \in l_i$, we have $(\mathrm{res}_{t_i} \nabla)(v^{(i)}) = \lambda_i \phi_{t_i}(v^{(i)}) = 0$ for $i = 1, \ldots, 4$. Thus $\det(\nabla_{t_i}) = \omega_1(t_i)\omega_4(t_i) - \omega_2(t_i)\omega_3(t_i) = 0$ for $i = 1, \ldots, 4$. Since $\omega_3(t_i) \neq 0$ for $i = 1, 2$, we have $\omega_2(t_i) = 0$ for $i = 1, 2$. We write

$$\omega_2 = \frac{(z - t_1)(z - t_2) u}{\prod_{j=1}^4 (z - t_j)} dz$$

with u a polynomial in z of degree less than or equal to 1. Applying a certain automorphism of E_2 of the form

$$\begin{pmatrix} c_1 & c_2 \\ 0 & 1 \end{pmatrix} \quad (c_1 \in H^0(\mathcal{O}^{\times}_{\mathbf{P}^1}), c_2 \in H^0(\mathcal{O}_{\mathbf{P}^1}(1))),$$

we may assume that $u = z - t_3$. Note that ∇ is of the form

$$\frac{dz}{\prod_{j=1}^{4}(z-t_j)}\begin{pmatrix} \alpha & (z-t_1)(z-t_2)(z-t_3) \\ z-q & (z-t_1)(z-t_2) \end{pmatrix} \quad (\alpha \in H^0(\mathcal{O}_{\mathbf{P}^1}(2)))$$

Since $\det(\nabla_{t_3}) = 0$, we have $\alpha(t_3) = 0$. The condition $\det(\nabla_{t_4}) = 0$ implies that α is of the form $\alpha = (z-t_3)(c(z-t_4)+t_4-q)$, where $c \in \mathbf{C}$. If $c = 1$, we have $\nabla(E_1) \subset \mathcal{O}_{\mathbf{P}^1}(-1) \otimes \Omega^1_{\mathbf{P}^1}(D(\mathbf{t}))$ after applying a certain automorphism of E_2. Then there is a decomposition $(E_1, E_2, \phi, \nabla, \{l_i\}) = (E_1, \mathcal{O}_{\mathbf{P}^1}(-1), \phi, \nabla, \{l_i\}) \oplus (0, \mathcal{O}_{\mathbf{P}^1}, 0, 0, \{0\})$, which contradicts the stability of $(E_1, E_2, \phi, \nabla, \varphi, \{l_i\})$. Thus we have $c \neq 1$. Applying a certain automorphism of E_2 of the form

$$\begin{pmatrix} t & (1-t)(z-t_3) \\ 0 & 1 \end{pmatrix} \quad (t \in H^0(\mathcal{O}^{\times}_{\mathbf{P}^1})),$$

we may assume that $c = 0$. Since $\nabla_{t_i} \neq 0$, $\ker(\nabla_{t_i}) = l_i$ for $i = 1, \ldots, 4$. Hence $(E_1, E_2, \phi, \nabla, \varphi, \{l_i\})$ is uniquely determined by q and it is determined by the point $p(E_1, E_2, \phi, \nabla, \varphi, \{l_i\})$. Q.E.D.

Proposition 4.3. *Under the assumption of Lemma 4.2, $\overline{M_4^{\boldsymbol{\alpha}'}}(-1)$ is smooth over $T_4 \times \Lambda_4$.*

Proof. Let A be an artinian local ring over $T_4 \times \Lambda_4$ with residue field $A/m = k$ and I be an ideal of A such that $mI = 0$. It is sufficient to show that

$$\overline{M_4^{\boldsymbol{\alpha}'}}(-1)(A) \longrightarrow \overline{M_4^{\boldsymbol{\alpha}'}}(-1)(A/I)$$

is surjective. Take any member

$$(E_1, E_2, \phi, \nabla, \varphi, \{l_i\}) \in \overline{M_4^{\boldsymbol{\alpha}'}}(-1)(A/I).$$

Note that $E_1 \cong \mathcal{O}_{\mathbf{P}^1_{A/I}} \oplus \mathcal{O}_{\mathbf{P}^1_{A/I}}(-1)$ and $E_2 \cong \mathcal{O}_{\mathbf{P}^1_{A/I}} \oplus \mathcal{O}_{\mathbf{P}^1_{A/I}}(-1)$. Then the homomorphism $\phi : E_1 \to E_2$ can be represented by a matrix of the form

$$\begin{pmatrix} \phi_1 & \phi_3 \\ 0 & \phi_2 \end{pmatrix} \quad (\phi_1, \phi_2 \in A/I, \phi_3 \in H^0(\mathcal{O}_{\mathbf{P}^1_{A/I}}(1))).$$

As in the proof of Proposition 4.2, we may assume that $\phi_3 \in m \otimes H^0(\mathcal{O}_{\mathbf{P}^1_{A/I}}(1))$. Put

$$A := (q_2 \otimes 1) \circ \nabla - d \circ q_2 \circ \phi : E_1 \longrightarrow \mathcal{O}_{\mathbf{P}^1_{A/I}} \otimes \Omega^1_{\mathbf{P}^1}(D(\mathbf{t})) \cong \mathcal{O}_{\mathbf{P}^1_{A/I}}(2),$$

$$B := (p_2 \otimes 1) \circ \nabla - d \circ p_2 \circ \phi : E_1 \longrightarrow \mathcal{O}_{\mathbf{P}^1_{A/I}}(-1) \otimes \Omega^1_{\mathbf{P}^1}(D(\mathbf{t})) \cong \mathcal{O}_{\mathbf{P}^1_{A/I}}(1),$$

where $q_2 : E_2 \to \mathcal{O}_{\mathbf{P}^1_{A/I}}$, $p_2 : E_2 \to \mathcal{O}_{\mathbf{P}^1_{A/I}}(-1)$ are projections with respect to the decomposition of E_2. Let (ω_1, ω_2) and (ω_3, ω_4) be the matrices representing A and B, respectively. We can see that the condition

$$(\varphi \otimes 1)(\nabla(s_1) \wedge \phi(s_2) + \phi(s_1) \wedge \nabla(s_2)) = d(\varphi(\phi(s_1) \wedge \phi(s_2))) \quad (s_1, s_2 \in E_1)$$

is equivalent to the equality

$$\omega_1\phi_2 - \omega_3\phi_3 + \omega_4\phi_1 = 0.$$

Let $(t_1, \ldots, t_4) \in \mathbf{P}^1(A) \times \cdots \times \mathbf{P}^1(A)$, $(\lambda_1, \ldots, \lambda_4) \in A \times \cdots \times A$ be the data corresponding to the structure morphism $\operatorname{Spec} A \to T_4 \times \Lambda_4$. Let $v^{(i)}$ be a basis of l_i. Then we can write $v^{(i)} = \begin{pmatrix} v_1^{(i)} \\ v_2^{(i)} \end{pmatrix}$ with $v_1^{(i)} \in \mathcal{O}_{\mathbf{P}^1_{A/I}}|_{t_i}$ and $v_2^{(i)} \in \mathcal{O}_{\mathbf{P}^1_{A/I}}(-1)|_{t_i}$ We must find lifts

$$\tilde{\phi}_1, \tilde{\phi}_2, \tilde{\phi}_3, \tilde{\omega}_1, \tilde{\omega}_2, \tilde{\omega}_3, \tilde{\omega}_4, \begin{pmatrix} v_1^{(i)} \\ v_2^{(i)} \end{pmatrix}_{i=1,\ldots,4}$$

over A of $\phi_1, \phi_2, \phi_3, \omega_1, \omega_2, \omega_3, \omega_4, \begin{pmatrix} v_1^{(i)} \\ v_2^{(i)} \end{pmatrix}_{i=1,\ldots,4}$ satisfying the following conditions:

$$\begin{cases} \tilde{\omega}_1\tilde{\phi}_2 - \tilde{\omega}_3\tilde{\phi}_3 + \tilde{\omega}_4\tilde{\phi}_1 = 0, \\ (\mathsf{res}_{t_i}(\tilde{\omega}_1) - \lambda_i\tilde{\phi}_1)\tilde{v}_1^{(i)} + (\mathsf{res}_{t_i}(\tilde{\omega}_2) - \lambda_i\tilde{\phi}_3(t_i))\tilde{v}_2^{(i)} = 0, \\ \mathsf{res}_{t_i}(\tilde{\omega}_3)\tilde{v}_1^{(i)} + \left(\mathsf{res}_{t_i}(\tilde{\omega}_4) + \left(\mathsf{res}_{t_i}\left(\frac{dz}{z-t_4}\right) - \lambda_i\right)\tilde{\phi}_2\right)\tilde{v}_2^{(i)} = 0, \\ \text{for } i = 1, \ldots, 4. \end{cases}$$

Since we have already proved the smoothness of $M_4^{\boldsymbol{\alpha}/2}(-1)$ over $T_4 \times \Lambda_4$, we may assume that $\wedge^2\phi \in mA/I$.

Assume that $\phi_1 \in mA/I$ and $\phi_2 \in (A/I)^\times$. Still we may assume that $\phi_3 = 0$. In this case we can see from the proof of Proposition 4.2 that $\mathsf{res}_{t_i}(\omega_3) \in mA/I$ and $\mathsf{res}_{t_i}(\omega_2) \in (A/I)^\times$ for some i. Take lifts $\tilde{\omega}_2^{(i)} \in \Omega^1_{\mathbf{P}^1_A}(D(\mathbf{t}))(1)_{t_i}$, $\tilde{\omega}_4 \in H^0(\Omega^1_{\mathbf{P}^1_A}(D(\mathbf{t})))$, $\tilde{\phi}_1 \in A$ and $\tilde{\phi}_2 \in A$ of $\omega_2(t_i)$, ω_4, ϕ_1 and ϕ_2, respectively. Put $\tilde{\omega}_1 := -\tilde{\omega}_4\tilde{\phi}_1\tilde{\phi}_2^{-1}$. Then we can find a lift $\tilde{\omega}_3 \in H^0(\Omega^1_{\mathbf{P}^1_A}(D(\mathbf{t}))(-1))$ of ω_3 satisfying

$$\begin{aligned} &(\mathsf{res}_{t_i}(\tilde{\omega}_1) - \lambda_i\tilde{\phi}_1)\left(\mathsf{res}_{t_i}(\tilde{\omega}_4) + \left(\mathsf{res}_{t_i}\left(\frac{dz}{z-t_4}\right) - \lambda_i\right)\tilde{\phi}_2\right) \\ &\quad - \mathsf{res}_{t_i}(\tilde{\omega}_2^{(i)})\,\mathsf{res}_{t_i}(\tilde{\omega}_3) = 0. \end{aligned}$$

Let $\tilde{\omega}_2$ be the element of $H^0(\Omega^1_{\mathbf{P}^1_A}(D(\mathbf{t}))(1))$ satisfying

$$(\mathrm{res}_{t_j}(\tilde{\omega}_1) - \lambda_j\tilde{\phi}_1)\left(\mathrm{res}_{t_j}(\tilde{\omega}_4) + \left(\mathrm{res}_{t_j}\left(\tfrac{dz}{z-t_4}\right) - \lambda_j\right)\tilde{\phi}_2\right) \\ - \mathrm{res}_{t_j}(\tilde{\omega}_2)\,\mathrm{res}_{t_j}(\tilde{\omega}_3) = 0$$

for $j \neq i$ and $\tilde{\omega}_2(t_i) = \tilde{\omega}_2^{(i)}$. For $j = 1, \ldots, 4$, we can take lifts $\tilde{v}_1^{(j)} \in \mathcal{O}_{\mathbf{P}^1_A}|_{t_j}$, $\tilde{v}_2^{(j)} \in \mathcal{O}_{\mathbf{P}^1_A}(-1)|_{t_j}$ of $v_1^{(j)}, v_2^{(j)}$ satisfying

$$(\mathrm{res}_{t_i}(\tilde{\omega}_1) - \lambda_i\tilde{\phi}_1)\tilde{v}_1^{(i)} + \mathrm{res}_{t_i}(\tilde{\omega}_2)\tilde{v}_2^{(i)} = 0.$$

and

$$\mathrm{res}_{t_j}(\tilde{\omega}_3)\tilde{v}_1^{(j)} + \left(\mathrm{res}_{t_j}(\tilde{\omega}_4) + \left(\mathrm{res}_{t_j}\left(\frac{dz}{z-t_4}\right) - \lambda_j\right)\tilde{\phi}_2\right)\tilde{v}_2^{(j)} = 0.$$

for $j \neq i$. Put $\tilde{\phi}_3 := 0$. Then $\tilde{\phi}_1, \tilde{\phi}_2, \tilde{\phi}_3, \tilde{\omega}_1, \tilde{\omega}_2, \tilde{\omega}_3, \tilde{\omega}_4, (\tilde{v}_1^{(j)}, \tilde{v}_2^{(j)})_{j=1}^4$ are desired lifts.

Next assume that $\phi_2 \in m/I$. In this case, we can see from the proof of Proposition 4.2 that $\phi_1 \in m/I$ and $\phi_2 \in mH^0(\mathcal{O}_{\mathbf{P}^1_{A/I}}(1))$. Take a lift $\tilde{\omega}_3 \in H^0(\Omega^1_{\mathbf{P}^1_A}(D(\mathbf{t}))(-1))$ of ω_3 and let $q \in \mathbf{P}^1(A)$ be the zero point of $\tilde{\omega}_3$. There exists $i \in \{1, \ldots, 4\}$ such that $\mathrm{res}_{t_j}(\tilde{\omega}_3) \in A^\times$ for $j \neq i$. Applying a certain auotomorphism of E_1, we may assume that $\mathrm{res}_{t_i}(\omega_4) \in (A/I)^\times$. Take lifts $\tilde{\omega}_4 \in H^0(\Omega_{\mathbf{P}^1_A}(D(\mathbf{t})))$, $\tilde{\omega}_2^{(i)} \in \Omega_{\mathbf{P}^1_A}(D(\mathbf{t})(1))_{t_i}$ and $\tilde{\phi}_2 \in A$ of ω_4, $\omega_2(t_i)$ and ϕ_2, respectively. We can see from Lemma 4.2 that $\tilde{\omega}_4(q)$ is a basis of $\Omega^1_{\mathbf{P}^1_A}(D(\mathbf{t}))|_q$. Then we can find an element $\tilde{\omega}_1 \in H^0(\Omega^1_{\mathbf{P}^1_A}(D(\mathbf{t})))$ such that

$$\left(\mathrm{res}_{t_i}(\tilde{\omega}_1)\tilde{\omega}_4(q) + \lambda_i\tilde{\omega}_1(q)\tilde{\phi}_2\right)\left(\mathrm{res}_{t_i}(\tilde{\omega}_4) + \left(\mathrm{res}_{t_i}\left(\frac{dz}{z-t_4}\right) - \lambda_i\right)\tilde{\phi}_2\right) \\ = \mathrm{res}_{t_i}(\tilde{\omega}_3)\,\mathrm{res}_{t_i}(\tilde{\omega}_2^{(i)})\tilde{\omega}_4(q) - \lambda_i\left(\mathrm{res}_{t_i}(\tilde{\omega}_1)\tilde{\phi}_2\tilde{\omega}_4(q) - \mathrm{res}_{t_i}(\tilde{\omega}_4)\tilde{\omega}_1(q)\tilde{\phi}_2\right).$$

We can take an element $\tilde{\phi}_1$ of A such that $\tilde{\phi}_2\tilde{\omega}_1(q) + \tilde{\phi}_1\tilde{\omega}_4(q) = 0$. Then there is an element $\tilde{\phi}_3 \in H^0(\mathcal{O}_{\mathbf{P}^1_A}(1))$ such that

$$\tilde{\omega}_1\tilde{\phi}_2 - \tilde{\omega}_3\tilde{\phi}_3 + \tilde{\omega}_4\tilde{\phi}_1 = 0.$$

Let $\tilde{\omega}_2$ be the element of $H^0(\Omega^1_{\mathbf{P}^1_A}(D(\mathbf{t}))(1))$ satisfying $\tilde{\omega}_2(t_i) = \tilde{\omega}_2^{(i)}$ and

$$(\mathrm{res}_{t_j}(\tilde{\omega}_1) - \lambda_j\tilde{\phi}_1)\left(\mathrm{res}_{t_j}(\tilde{\omega}_4) + \left(\mathrm{res}_{t_j}\left(\tfrac{dz}{z-t_4}\right) - \lambda_j\right)\tilde{\phi}_2\right) \\ = \mathrm{res}_{t_j}(\tilde{\omega}_3)(\mathrm{res}_{t_j}(\tilde{\omega}_2) - \lambda_j\tilde{\phi}_3(t_j))$$

for $j \neq i$. We can take lifts $\tilde{v}_1^{(j)} \in \mathcal{O}_{\mathbf{P}^1_A}|_{t_j}$, $\tilde{v}_2^{(j)} \in \mathcal{O}_{\mathbf{P}^1_A}(-1)|_{t_j}$ of $v_1^{(j)}, v_2^{(j)}$ such that

$$\mathrm{res}_{t_j}(\tilde{\omega}_3)\tilde{v}_1^{(j)} + \left(\mathrm{res}_{t_j}(\tilde{\omega}_4) + \left(\mathrm{res}_{t_j}\left(\frac{dz}{z-t_4}\right) - \lambda_j\right)\tilde{\phi}_2\right)\tilde{v}_2^{(j)} = 0$$

for $j = 1, \dots, 4$. Then $\tilde{\phi}_1, \tilde{\phi}_2, \tilde{\phi}_3, \tilde{\omega}_1, \tilde{\omega}_2, \tilde{\omega}_3, \tilde{\omega}_4, (\tilde{v}_1^{(j)}, \tilde{v}_2^{(j)})_{j=1}^4$ are desired lifts. Q.E.D.

4.4. Proof of Theorem 4.1

We put $\lambda_i^+ := \lambda_i$ for $i = 1, \dots, 4$, $\lambda_i^- := -\lambda_i$ for $i = 1, \dots, 3$ and $\lambda_4^- := 1 - \lambda_4$. Let D_i be the fiber of $\mathbf{P}_*(\Omega^1_{\mathbf{P}^1}(D(\mathbf{t})) \oplus \mathcal{O}_{\mathbf{P}^1})$ over $t_i \in \mathbf{P}^1$ and b_i^+ (resp. b_i^-) be the point of D_i corresponding to λ_i^+ (resp. λ_i^-). Put $Z := \{b_1^+, \dots, b_4^+, b_1^-, \dots, b_4^-\}$.

Proposition 4.4. *Under the above notation,*

$$\overline{M_4^{\boldsymbol{\alpha}'}}(\mathbf{t}, \boldsymbol{\lambda}, -1) \setminus p^{-1}(Z) \xrightarrow{p} \mathbf{P}_*(\Omega^1_{\mathbf{P}^1}(D(\mathbf{t})) \oplus \mathcal{O}_{\mathbf{P}^1}) \setminus Z \tag{41}$$

is an isomorphism.

Proof. Let D_0 be the section of $\mathbf{P}_*(\Omega^1_{\mathbf{P}^1}(D(\mathbf{t})) \oplus \mathcal{O}_{\mathbf{P}^1})$ over $\mathbf{P}^1$ defined by the injection $\Omega^1_{\mathbf{P}^1}(D(\mathbf{t})) \hookrightarrow \Omega^1_{\mathbf{P}^1}(D(\mathbf{t})) \oplus \mathcal{O}_{\mathbf{P}^1}$. First we will show that

$$\overline{M_4^{\boldsymbol{\alpha}}}(\mathbf{t}, \boldsymbol{\lambda}, -1) \setminus \bigcup_{i=0}^{4} p^{-1}(D_i) \longrightarrow \mathbf{P}_*(\Omega^1_{\mathbf{P}^1}(D(\mathbf{t})) \oplus \mathcal{O}_{\mathbf{P}^1}) \setminus \bigcup_{i=0}^{4} D_i \tag{42}$$

is an isomorphism. Fix a section

$$\tau : (\pi_2)_*(\pi_1^*\Omega^1_{\mathbf{P}^1}(D(\mathbf{t}))|_{\Delta}) \longrightarrow (\pi_2)_*(\pi_1^*\Omega^1_{\mathbf{P}^1}(D(\mathbf{t})))$$

of the canonical homomorphism

$$(\pi_2)_*(\pi_1^*\Omega^1_{\mathbf{P}^1}(D(\mathbf{t}))) \longrightarrow (\pi_2)_*(\pi_1^*\Omega^1_{\mathbf{P}^1}(D(\mathbf{t}))|_{\Delta}),$$

where

$$\pi_1 : \mathbf{P}^1 \times (\mathbf{P}^1 \setminus D(\mathbf{t})) \to \mathbf{P}^1, \quad \pi_2 : \mathbf{P}^1 \times (\mathbf{P}^1 \setminus D(\mathbf{t})) \to \mathbf{P}^1 \setminus D(\mathbf{t})$$

are projections and $\Delta \subset \mathbf{P}^1 \times (\mathbf{P}^1 \setminus D(\mathbf{t}))$ is the diagonal. Take a point s of $\mathbf{P}_*(\Omega^1_{\mathbf{P}^1}(D(\mathbf{t})) \oplus \mathcal{O}_{\mathbf{P}^1}) \setminus \bigcup_{i=0}^4 D_i$, which is given by $q \in \mathbf{P}^1$ and an injection $(-h_1, h_2) : \mathbf{C} \hookrightarrow \Omega^1_{\mathbf{P}^1}(D(\mathbf{t}))|_q \oplus \mathcal{O}_{\mathbf{P}^1}|_q$. We may assume that $h_2 = 1$. We put

$$\begin{aligned} \omega_4 &:= \tau_q(h_1) \in H^0(\Omega^1_{\mathbf{P}^1}(D(\mathbf{t}))), \\ \omega_3 &:= \frac{z-q}{(t_4-q)\prod_{j=1}^4 (z-t_j)} dz \in H^0(\Omega^1_{\mathbf{P}^1}(D(\mathbf{t}))(-1)), \end{aligned}$$

where z is a fixed inhomogeneous coordinate of $\mathbf{P}^1$. Let ω_2 be the element of $H^0(\Omega^1_{\mathbf{P}^1}(D(\mathbf{t}))(1))$ determined by

$$(\mathrm{res}_{t_i}(\omega_4)+\lambda_i)\left(\mathrm{res}_{t_i}(\omega_4)+\mathrm{res}_{t_i}\left(\frac{dz}{z-t_4}\right)-\lambda_i\right)+\mathrm{res}_{t_i}(\omega_2)\,\mathrm{res}_{t_i}(\omega_3)=0$$

for $i=1,\dots,4$. Define a rational connection ∇ on $\mathcal{O}_{\mathbf{P}^1}\oplus\mathcal{O}_{\mathbf{P}^1}(-1)$ by

$$\nabla\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} := \begin{pmatrix} df_1 \\ df_2 \end{pmatrix} + \begin{pmatrix} -f_1\omega_4+f_2\omega_2 \\ f_1\omega_3+f_2\omega_4 \end{pmatrix}$$

for $f_1\in\mathcal{O}_{\mathbf{P}^1}$ and $f_2\in\mathcal{O}_{\mathbf{P}^1}(-1)$. Then $s\mapsto(\mathcal{O}_{\mathbf{P}^1}\oplus\mathcal{O}_{\mathbf{P}^1}(-1),\nabla)$ determines a morphism

$$\mathbf{P}_*(\Omega^1_{\mathbf{P}^1}(D(\mathbf{t}))\oplus\mathcal{O}_{\mathbf{P}^1})\setminus\bigcup_{i=0}^4 D_i \longrightarrow \overline{M_4^{\boldsymbol{\alpha}'}}(\mathbf{t},\boldsymbol{\lambda},-1)\setminus\bigcup_{i=0}^4 p^{-1}(D_i),$$

which is just the inverse of the morphism (42). Then the morphism (41) is surjective, since it is proper and dominant. The morphism (41) is also injective by the above argument and Proposition 4.2. Thus, by Zariski's Main Theorem, the morphism (41) is an isomorphism. Q.E.D.

Proposition 4.5. *If $\lambda_i^+\neq\lambda_i^-$, then $p^{-1}(b_i^+)\cong\mathbf{P}^1$, $p^{-1}(b_i^-)\cong\mathbf{P}^1$ and these are (-1)-curves.*

Proof. We can see that $p^{-1}(b_i^+)$ is just the moduli space of $(\mathbf{t},\boldsymbol{\lambda})$-parabolic ϕ-connections $(\mathcal{O}_{\mathbf{P}^1}\oplus\mathcal{O}_{\mathbf{P}^1}(-1),\mathcal{O}_{\mathbf{P}^1}\oplus\mathcal{O}_{\mathbf{P}^1}(-1),\phi,\nabla,\varphi,\{l_j\})$ satisfying

$$\phi\begin{pmatrix} s_1 \\ s_2 \end{pmatrix} = \begin{pmatrix} \phi_1 s_1 \\ s_2 \end{pmatrix}$$

$$\nabla\begin{pmatrix} s_1 \\ s_2 \end{pmatrix} = \begin{pmatrix} \phi_1 s_1 \\ s_2 \end{pmatrix} + \begin{pmatrix} s_1\phi_1\frac{\lambda_i^+\prod_{j\neq i}(t_i-t_j)}{\prod_{j=1}^4(z-t_j)}dz+s_2\omega_2 \\ s_1\frac{(z-t_i)dz}{\prod_{j=1}^4(z-t_j)}-s_2\frac{\lambda_i^+\prod_{j\neq i}(t_i-t_j)}{\prod_{j=1}^4(z-t_j)}dz \end{pmatrix}$$

for $s_1\in\mathcal{O}_{\mathbf{P}^1}$ and $s_2\in\mathcal{O}_{\mathbf{P}^1}(-1)$, where $\phi_1\in\mathbf{C}$, $l_j=\ker(\mathrm{res}_{t_j}(\nabla)-\lambda_j^+\phi|_{t_j})$ for $j=1,\dots,4$ and $\omega_2\in H^0(\Omega^1_{\mathbf{P}^1}(D(\mathbf{t}))(1))$ satisfies the condition

$$\phi_1\left(\mathrm{res}_{t_k}\left(\frac{\lambda_i^+\prod_{j\neq i}(t_i-t_j)}{\prod_{j=1}^4(z-t_j)}dz\right)-\lambda_k^+\right)\left(\mathrm{res}_{t_k}\left(\frac{dz}{z-t_4}-\frac{\lambda_i^+\prod_{j\neq i}(t_i-t_j)}{\prod_{j=1}^4(z-t_j)}dz\right)-\lambda_k^+\right)$$
$$-\mathrm{res}_{t_k}\left(\frac{(z-t_i)dz}{\prod_{j=1}^4(z-t_j)}\right)\mathrm{res}_{t_k}(\omega_2)=0.$$

for $k \neq i$. Then we can define a mapping

$$\begin{array}{ccc} p^{-1}(b_i^+) & \longrightarrow & \mathbf{P}^1 \\ (\mathcal{O}_{\mathbf{P}^1} \oplus \mathcal{O}_{\mathbf{P}^1}(-1), \mathcal{O}_{\mathbf{P}^1} \oplus \mathcal{O}_{\mathbf{P}^1}(-1), \phi, \nabla, \varphi, \{l_j\}) & \mapsto & [\phi_1 : \mathrm{res}_{t_i}(\omega_2)] \end{array}$$

which is an isomorphism.

Similarly we can see that $p^{-1}(b_i^-) \cong \mathbf{P}^1$. Since $\overline{M_4^{\boldsymbol{\alpha}'}}(\mathbf{t}, \boldsymbol{\lambda}, -1)$ and $\mathbf{P}_*(\Omega^1_{\mathbf{P}^1}(D(\mathbf{t}))\oplus\mathcal{O}_{\mathbf{P}^1})$ are smooth, $p^{-1}(b_i^+), p^{-1}(b_i^-)$ must be (-1)-curves.

Q.E.D.

Proposition 4.6. *Assume that* $\lambda_i^+ = \lambda_i^-$. *Put*

$$C_1 := \left\{(E_1, E_2, \phi, \nabla, \varphi, \{l_j\}) \in p^{-1}(b_i^+) \,\middle|\, l_i = L_1^{(0)}|_{t_i} \right\},$$
$$C_2 := \left\{ (E_1, E_2, \phi, \nabla, \varphi, \{l_j\}) \in p^{-1}(b_i^+) \middle| \mathrm{res}_{t_i}(\nabla) = \lambda_i \phi_{t_i} \right\}.$$

Then $C_1 \cong \mathbf{P}^1$, $C_2 \cong \mathbf{P}^1$, $C_1 \cap C_2 = \{$*one point*$\}$, $C_1 \cap Y(\mathbf{t}, \boldsymbol{\lambda}) = \{$*one point*$\}$, $C_2 \subset M_4^{\boldsymbol{\alpha}}(\mathbf{t}, \boldsymbol{\lambda}, -1)$, $(C_1)^2 = -1$, $(C_2)^2 = -2$ *and* $p^{-1}(b_i^+) = C_1 \cup C_2$.

Proof. $p^{-1}(b_i^+)$ is the moduli space of the objects

$$(\mathcal{O}_{\mathbf{P}^1} \oplus \mathcal{O}_{\mathbf{P}^1}(-1), \mathcal{O}_{\mathbf{P}^1} \oplus \mathcal{O}_{\mathbf{P}^1}(-1), \phi, \nabla, \varphi, \{l_j\})$$

satisfying

$$\phi\begin{pmatrix} s_1 \\ s_2 \end{pmatrix} = \begin{pmatrix} \phi s_1 \\ s_2 \end{pmatrix}$$
$$\nabla\begin{pmatrix} s_1 \\ s_2 \end{pmatrix} = \begin{pmatrix} \phi d s_1 \\ d s_2 \end{pmatrix} + \begin{pmatrix} s_1 \phi_1 \frac{\lambda_i^+ \prod_{j\neq i}(t_i - t_j)}{\prod_{j=1}^4 (z-t_j)} dz + s_2 \omega_2 \\ s_1 \frac{(z-t_i)dz}{\prod_{j=1}^4(z-t_j)} - s_2 \frac{\lambda_i^+ \prod_{j\neq i}(t_i-t_j)}{\prod_{j=1}^4(z-t_j)} dz \end{pmatrix}$$

for $s_1 \in \mathcal{O}_{\mathbf{P}^1}$ and $s_2 \in \mathcal{O}_{\mathbf{P}^1}(-1)$, where $\phi_1 \in \mathbf{C}$, $l_k = \ker(\mathrm{res}_{t_k}(\nabla) - \lambda_k \phi|_{t_k})$ for $k \neq i$ and ω_2 satisfies the condition

$$\phi_1 \left(\mathrm{res}_{t_k}\left(\frac{\lambda_i^+ \prod_{j\neq i}(t_i-t_j)}{\prod_{j=1}^4(z-t_j)} dz\right) - \lambda_k^+\right)\left(\mathrm{res}_{t_k}\left(\frac{dz}{z-t_4} - \frac{\lambda_i^+ \prod_{j\neq i}(t_i-t_j)}{\prod_{j=1}^4(z-t_j)}dz\right) - \lambda_k^+\right)$$
$$- \mathrm{res}_{t_k}\left(\frac{(z-t_i)dz}{\prod_{j=1}^4(z-t_j)}\right) \mathrm{res}_{t_k}(\omega_2) = 0.$$

for $k \neq i$. If $v^{(i)} = \begin{pmatrix} v_1^{(i)} \\ v_2^{(i)} \end{pmatrix}$ is a basis of l_i, $\mathrm{res}_{t_i}(\omega_2) v_2^{(i)} = 0$. Thus we have

$$p^{-1}(b_i^+) = \left(\{v_2^{(i)} = 0\} \cap p^{-1}(b_i^+)\right) \cup \left(\{\omega_2(t_i) = 0\} \cap p^{-1}(b_i^+)\right).$$

We can see that $\{v_2^{(i)} = 0\} \cap p^{-1}(b_i^+) = C_1$ and $\{\omega_2(t_i) = 0\} \cap p^{-1}(b_i^+) = C_2$. From the proof of Proposition 4.2, we can see that the objects of C_2 satisfies the condition $\phi_1 \neq 0$. Thus we have $C_2 \cap Y(\mathbf{t}, \boldsymbol{\lambda}) = \emptyset$. We can also see that $C_1 \cap C_2$ consists of one point corresponding to the object of $p^{-1}(b_i^+)$ satisfying $\omega_2(t_i) = 0$, $\phi_1 = 1$ and $l_i = L_1^{(0)}|_{t_i}$. $C_1 \cap Y(\mathbf{t}, \boldsymbol{\lambda})$ consists of one point corresponding to the object of C_1 satisfying $\phi_1 = 0$. We have $C_1 \cong \mathbf{P}^1$ by the same proof as Proposition 4.5. ϕ, ∇, φ and l_k for $k \neq i$ are all constant on C_2. So C_2 is just the moduli of lines $l_i \subset \mathcal{O}_{\mathbf{P}^1}|_{t_i} \oplus \mathcal{O}_{\mathbf{P}^1}(-1)|_{t_i}$, which is isomorphic to $\mathbf{P}^1$.

Let $N_4(\mathbf{t}, \boldsymbol{\lambda}, -1)$ be the moduli space of rank 2 bundles E with a connection $\nabla : E \to E \otimes \Omega^1_{\mathbf{P}^1}(D(\mathbf{t}))$ and a horizontal isomorphism $\varphi : \bigwedge^2 E \xrightarrow{\sim} \mathcal{O}_{\mathbf{P}^1}(-x_4)$ satisfying

(1) $\det(\mathrm{res}_{t_i}(\nabla) - \lambda_i \mathrm{id}_{E|_{t_i}}) = 0$ for $i = 1, \dots, 4$ and

(2) (E, ∇) is stable in the sense of Simpson [Sim].

Then there is a canonical morphism

$$M_4^{\boldsymbol{\alpha}}(\mathbf{t}, \boldsymbol{\lambda}, -1) \longrightarrow N_4(\mathbf{t}, \boldsymbol{\lambda}, -1),$$

which is obtained by forgetting parabolic structure. We can see that the image of C_2 in $N_4(\mathbf{t}, \boldsymbol{\lambda}, -1)$ is a singular point with A_1-singularity. Thus C_2 is a (-2)-curve and we can see that C_1 is a (-1)-curve. Q.E.D.

The morphism $p : \overline{M_4^{\boldsymbol{\alpha}'}}(\mathbf{t}, \boldsymbol{\lambda}, \mathcal{O}_{\mathbf{P}^1}(-t_4)) \to \mathbf{P}(\Omega^1_{\mathbf{P}^1}(D(\mathbf{t})) \oplus \mathcal{O}_{\mathbf{P}^1})$ defined in (39) extends to the morphism

$$p : \overline{M_4^{\boldsymbol{\alpha}'}}(\mathcal{O}_{\mathbf{P}^1 \times T_4 \times \Lambda_4}(-\tilde{t}_4)) \longrightarrow \mathbf{P}\left(\Omega^1_{\mathbf{P}^1 \times T_4 \times \Lambda_4 / T_4 \times \Lambda_4}(D(\tilde{\mathbf{t}})) \oplus \mathcal{O}_{\mathbf{P}^1 \times T_4 \times \Lambda_4}\right).$$

We can check that the inverse image $p^{-1}(B^+)$ is a Cartier divisor on $\overline{M_4^{\boldsymbol{\alpha}'}}(\mathbf{t}, \boldsymbol{\lambda}, \mathcal{O}_{\mathbf{P}^1}(-t_4))$. Since Z is a blow up of

$$\mathbf{P}\left(\Omega^1_{\mathbf{P}^1 \times T_4 \times \Lambda_4 / T_4 \times \Lambda_4}(D(\tilde{\mathbf{t}})) \oplus \mathcal{O}_{\mathbf{P}^1 \times T_4 \times \Lambda_4}\right)$$

along B^+, p induces a morphism

$$f : \overline{M_4^{\boldsymbol{\alpha}'}}(\mathbf{t}, \boldsymbol{\lambda}, \mathcal{O}_{\mathbf{P}^1}(-t_4)) \longrightarrow Z.$$

We can also check that $f^{-1}(g^{-1}(B)) = p^{-1}(B)$ is a Cartier divisor on $\overline{M_4^{\boldsymbol{\alpha}'}}(\mathbf{t}, \boldsymbol{\lambda}, \mathcal{O}_{\mathbf{P}^1}(-t_4))$. Since $\overline{S}$ is a blow up of Z along $g^{-1}(B)$, f induces a morphism

$$f' : \overline{M_4^{\boldsymbol{\alpha}'}}(\mathbf{t}, \boldsymbol{\lambda}, \mathcal{O}_{\mathbf{P}^1}(-t_4)) \longrightarrow \overline{S}.$$

We can see by Proposition 4.4, Proposition 4.5 and Proposition 4.6 that each fiber of f' over $T_4 \times \Lambda_4$ is an isomorphism. Thus f' is an isomorphism and Theorem 4.1 (1) is proved.

Theorem 4.1 (2) is easy. It is well-known that $K_{\overline{S}_{(\mathbf{t},\boldsymbol{\lambda})}} \equiv -(2D_0 + D_1 + D_2 + D_3 + D_4)$. So it is sufficient to prove the following proposition in order to prove Theorem 4.1 (3).

Proposition 4.7. *$\mathcal{Y}$ is a Cartier divisor on $\overline{M_4^{\boldsymbol{\alpha}'}}(-1)$ flat over $T_4 \times \Lambda_4$ and the divisor $Y(\mathbf{t},\boldsymbol{\lambda})$ on $\overline{M_4^{\boldsymbol{\alpha}'}}(\mathbf{t},\boldsymbol{\lambda},-\mathcal{O}_{\mathbf{P}^1}(-t_4))$ has multiplicity 2 along $(p|_{\mathcal{Y}(\mathbf{t},\boldsymbol{\lambda})})^{-1}(D_0)$ and 1 along $(p|_{Y(\mathbf{t},\boldsymbol{\lambda})})^{-1}(D_i)$ for $i = 1,\dots,4$.*

Proof. Let $(\mathcal{E}_1, \mathcal{E}_2, \tilde{\phi}, \tilde{\nabla}, \tilde{\varphi}, \{\tilde{l}_i\})$ be a universal family on $\mathbf{P}^1 \times \overline{M_4^{\boldsymbol{\alpha}'}}(-1)$. Then $\tilde{\phi} : \mathcal{E}_1 \to \mathcal{E}_2$ determines a section f of $(\pi_{M_4^{\boldsymbol{\alpha}}})_*(\det(\mathcal{E}_1)^{-1} \otimes \det(\mathcal{E}_2))$, whose zero scheme is $\mathcal{Y}$. Since $(\pi_{M_4^{\boldsymbol{\alpha}}})_*(\det(\mathcal{E}_1)^{-1} \otimes \det(\mathcal{E}_2))$ is a line bundle on $\overline{M_4^{\boldsymbol{\alpha}'}}(-1)$, $\mathcal{Y}$ is a Cartier divisor on $\overline{M_4^{\boldsymbol{\alpha}'}}(-1)$. $Y(\mathbf{t},\boldsymbol{\lambda})$ is also a Cartier divisor on $\overline{M_4^{\boldsymbol{\alpha}'}}(\mathbf{t},\boldsymbol{\lambda},-1)$ and so $\mathcal{Y}$ is flat over $T_4 \times \Lambda_4$.

Let U_i be the open subscheme of $Y(\mathbf{t},\boldsymbol{\lambda})$ whose underlying space is $(p|_{Y(\mathbf{t},\boldsymbol{\lambda})})^{-1}(D_i \setminus (D_0 \cap D_i))$. Then U_i is just the moduli space of the objects $(\mathcal{O}_{\mathbf{P}^1} \oplus \mathcal{O}_{\mathbf{P}^1}(-1), \mathcal{O}_{\mathbf{P}^1} \oplus \mathcal{O}_{\mathbf{P}^1}(-1), \phi, \nabla, \varphi, \{l_j\})$ satisfying

$$\phi\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \begin{pmatrix} 0 \\ f_2 \end{pmatrix},$$

$$\nabla\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \begin{pmatrix} 0 \\ df_2 \end{pmatrix} + \begin{pmatrix} f_2 \frac{\prod_{j\neq i}(z-t_j)}{\prod_{j=1}^4 (z-t_j)} dz \\ f_1 \frac{(z-t_i)dz}{\prod_{j=1}^4 (z-t_j)} + f_2 \frac{a dz}{\prod_{j=1}^4 (z-t_j)} \end{pmatrix}$$

for $f_1 \in \mathcal{O}_{\mathbf{P}^1}$ and $f_2 \in \mathcal{O}_{\mathbf{P}^1}(-1)$, where $a \in \mathbf{C}$ and $l_j = \ker(\mathrm{res}_{t_j}(\nabla) - \lambda_j \phi_{t_j})$ for $j = 1,\dots,4$. Thus $U_i \cong \mathbf{A}^1$ and U_i is reduced.

Let U_0 be the open subscheme of $Y(\mathbf{t},\boldsymbol{\lambda})$ such that $p(U_0) = D_0 \setminus \bigcup_{j=1}^4 D_j$ as sets. U_0 is the moduli space of the objects $(\mathcal{O}_{\mathbf{P}^1} \oplus \mathcal{O}_{\mathbf{P}^1}(-1), \mathcal{O}_{\mathbf{P}^1} \oplus \mathcal{O}_{\mathbf{P}^1}(-1), \phi, \nabla, \varphi, \{l_j\})$ satisfying

$$\phi\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \begin{pmatrix} f_1\phi_1 + f_2\phi_3 \\ f_2\phi_2 \end{pmatrix}$$

$$\nabla\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \begin{pmatrix} \phi_1 df_1 + \phi_3 df_2 \\ \phi_2 df_2 \end{pmatrix} + \begin{pmatrix} \omega_1 f_1 \\ \omega_3 f_1 + \omega_4 f_2 \end{pmatrix}$$

for $f_1 \in \mathcal{O}_{\mathbf{P}^1}$ and $f_2 \in \mathcal{O}_{\mathbf{P}^1}(-1)$ with the conditions $\phi_1\phi_2 = 0$ and $\omega_1\phi_2 - \omega_3\phi_3 + \omega_4\phi_1 = 0$, where $q \in \mathbf{P}^1 \setminus \{t_1,\dots,t_4\}$, $l_j = \ker(\mathrm{res}_{t_j}(\nabla) - \lambda_j\phi_{t_j})$ for $j = 1,\dots,4$ and

$$\omega_1 = \frac{\prod_{k=3}^4 (z - t_k + (t_k - t_1)(t_k - t_2)\lambda_k\phi_1)}{\prod_{j=1}^4 (z-t_j)} dz, \quad \omega_3 = \frac{(z-q)dz}{(t_4 - q)\prod_{j=1}^4 (z-t_j)}$$

$$\omega_4 = \frac{\prod_{k=1}^2 (z - t_k + (t_k - t_3)(t_k - t_4)\lambda_k\phi_2)}{\prod_{j=1}^4 (z-t_j)} dz.$$

ϕ_2 and ϕ_3 are determined by ϕ_1 and the conditions

$$\omega_1(q)\phi_2 + \omega_4(q)\phi_1 = 0, \quad \omega_3(t_j)\phi_3(t_j) = \omega_1(t_j)\phi_2 + \omega_4(t_j)\phi_1 \quad (j = 1, 2)$$

and ϕ_2 must satisfy the condition $\phi_1^2 = 0$. Thus $U_0 \cong \mathbf{P}^1 \setminus \{t_1, \ldots, t_4\} \times \operatorname{Spec} \mathbf{C}[\phi_1]/(\phi_1^2)$ and $Y(\mathbf{t}, \boldsymbol{\lambda})$ has multiplicity 2 along $(p|_{Y(\mathbf{t},\boldsymbol{\lambda})})^{-1}(D_0)$.
Q.E.D.

§5. Moduli of stable parabolic connections in general case

In this section, we will formulate the general moduli theory of $\boldsymbol{\alpha}$-stable parabolic connections over a curve and state the existence theorem of the coarse moduli scheme due to Inaba [Ina]. We fix integers g, d, r, n with $g \geq 0, r > 0, n > 0$ and let $(C, \mathbf{t}) = (C, t_1, \ldots, t_n)$ be an n-pointed smooth projective curve of genus g, which consists of a smooth projective curve C and a set of n-distinct points $\mathbf{t} = \{t_i\}_{1 \leq i \leq n}$ on C. We denote by $D(\mathbf{t}) = t_1 + \cdots + t_n$ the divisor associated to $\mathbf{t}$. Define the set of exponents as

(43)
$$\Lambda_r^n(d) := \left\{ \boldsymbol{\lambda} = (\lambda_j^{(i)})_{0 \leq j \leq r-1}^{1 \leq i \leq n} \in \mathbf{C}^{nr} \;\middle|\; d + \sum_{1 \leq i \leq n,\ 0 \leq j \leq r-1} \lambda_j^{(i)} = 0 \right\}.$$

Definition 5.1. A $(\mathbf{t}, \boldsymbol{\lambda})$-*parabolic connection of rank* r on C is a collection of data $(E, \nabla, \{l_*^{(i)}\}_{1 \leq i \leq n})$ consisting of:

(1) a vector bundle E of rank r on C,
(2) a logarithmic connection $\nabla : E \longrightarrow E \otimes \Omega_C^1(D(\mathbf{t}))$,
(3) and a filtration $l_*^{(i)} : E_{|t_i} = l_0^{(i)} \supset l_1^{(i)} \supset \cdots \supset l_{r-1}^{(i)} \supset l_r^{(i)} = 0$ for each $i, 1 \leq i \leq n$ such that $\dim(l_j^{(i)}/l_{j+1}^{(i)}) = 1$ and $(\mathrm{res}_{t_i}(\nabla) - \lambda_j^{(i)})(l_j^{(i)}) \subset l_{j+1}^{(i)}$ for $j = 0, 1, \cdots, r-1$.

We set $\deg E = \deg(\wedge^r E)$ as usual.

Take a sequence of rational numbers $\boldsymbol{\alpha} = (\alpha_j^{(i)})_{1 \leq j \leq r}^{1 \leq i \leq n}$ such that

$$0 < \alpha_1^{(i)} < \alpha_2^{(i)} < \cdots < \alpha_r^{(i)} < 1 \tag{44}$$

for $i = 1, \ldots, n$ and $\alpha_j^{(i)} \neq \alpha_{j'}^{(i')}$ for $(i, j) \neq (i', j')$. We choose $\boldsymbol{\alpha} = (\alpha_j^{(i)})$ sufficiently generic. Let $(E, \nabla, \{l_*^{(i)}\}_{1 \leq i \leq n})$ be a $(\mathbf{t}, \boldsymbol{\lambda})$-parabolic connection, and $F \subset E$ a nonzero subbundle satisfying $\nabla(F) \subset F \otimes \Omega_C^1(D(\mathbf{t}))$. We define integers $\mathrm{len}(F)_j^{(i)}$ by

$$\mathrm{len}(F)_j^{(i)} = \dim(F|_{t_i} \cap l_{j-1}^{(i)})/(F|_{t_i} \cap l_j^{(i)}). \tag{45}$$

Note that $\operatorname{len}(E)^{(i)}_j = \dim(l^{(i)}_{j-1}/l^{(i)}_j) = 1$ for $1 \leq j \leq r$.

Definition 5.2. A parabolic connection $(E, \nabla, \{l^{(i)}_*\}_{1\leq i\leq n})$ is $\boldsymbol{\alpha}$-stable if for any proper nonzero subbundle $F \subsetneqq E$ satisfying $\nabla(F) \subset F \otimes \Omega^1_C(D(\mathbf{t}))$, the inequality

(46)
$$\frac{\deg F + \sum_{i=1}^m \sum_{j=1}^r \alpha^{(i)}_j \operatorname{len}(F)^{(i)}_j}{\operatorname{rank} F} < \frac{\deg E + \sum_{i=1}^n \sum_{j=1}^r \alpha^{(i)}_j \operatorname{len}(E)^{(i)}_j}{\operatorname{rank} E}$$

holds.

For a fixed $(C, \mathbf{t})$ and $\boldsymbol{\lambda}$, let us define the coarse moduli space by

(47)
$$\mathcal{M}^{\boldsymbol{\alpha}}_{((C,\mathbf{t}),\boldsymbol{\lambda})}(r,n,d) = \left\{(E, \nabla, \{l^{(i)}_*\}_{1\leq i\leq n}) \;\middle|\; \begin{array}{l}\text{an } \boldsymbol{\alpha}\text{-stable } (\mathbf{t},\boldsymbol{\lambda})\text{-parabolic connection} \\ \text{of rank } r \text{ and degree } d \text{ over } C\end{array}\right\}/\simeq .$$

Varying $(C, \mathbf{t})$ and $\boldsymbol{\lambda}$, we can also consider the moduli space in relative setting. Let $\mathcal{M}_{g,n}$ be the coarse moduli space of n-pointed curves of genus g. Here we assume that every point of $\mathcal{M}_{g,n}$ corresponds to an n-pointed smooth curve $(C, \mathbf{t})$ such that $\mathbf{t} = (t_1, \ldots, t_n)$ is a set of n-distinct points on C. We consider a finite covering $\mathcal{M}'_{g,n} \to \mathcal{M}_{g,n}$ where $\mathcal{M}'_{g,n}$ is the coarse moduli space of n-pointed curves of genus g with a suitable level structure so that there exists the universal family $(\mathcal{C}, \tilde{\mathbf{t}}) = (\mathcal{C}, \tilde{t}_1, \ldots, \tilde{t}_n)$ of n-pointed curves (with a level structure). From now on, for simplicity, we set

$$T = \mathcal{M}'_{g,n} \tag{48}$$

and let

$$(\mathcal{C}, \tilde{\mathbf{t}}) \longrightarrow T = \mathcal{M}'_{g,n} \tag{49}$$

be the universal family.

We can show the existence theorem of moduli space as a smooth quasi-projective algebraic scheme (cf. [IIS1], [Ina]).

Theorem 5.1. (*Cf.* [IIS1], [Ina]). *Assume that r, n, d are positive integers. There exists a relative moduli scheme*

$$\varphi_{r,n,d} : \mathcal{M}^{\boldsymbol{\alpha}}_{(\mathcal{C},\tilde{\mathbf{t}})/T}(r,n,d) \longrightarrow T \times \Lambda^{(n)}_r(d) \tag{50}$$

of $\boldsymbol{\alpha}$-stable parabolic connections of rank r and degree d, which is smooth and quasi-projective over $T \times \Lambda^{(n)}_r(d)$. Moreover the fiber $\mathcal{M}^{\boldsymbol{\alpha}}_{((C,\mathbf{t}),\boldsymbol{\lambda})}(r,n,d)$ of $\varphi_{r,n,d}$ over $((C,\mathbf{t}),\boldsymbol{\lambda}) \in T \times \Lambda^{(n)}_r(d)$ is the moduli space of $\boldsymbol{\alpha}$-stable

$(\mathbf{t}, \boldsymbol{\lambda})$*-parabolic connections over* C*, which is a* smooth algebraic scheme *and*

$$\dim \mathcal{M}^{\boldsymbol{\alpha}}_{((C,\mathbf{t}),\boldsymbol{\lambda})}(r,n,d) = 2r^2(g-1) + nr(r-1) + 2. \tag{51}$$

Remark 5.1.

(1) When $C = \mathbf{P}^1$ and $r = 2$, Theorem 5.1 is proved in [IIS1].
(2) Inaba [Ina] showed that the moduli space $\mathcal{M}^{\boldsymbol{\alpha}}_{((C,\mathbf{t}),\boldsymbol{\lambda})}(r,n,d)$ is irreducible in the following cases:
   (a) $g \geq 2, n \geq 1$,
   (b) $g = 1, n \geq 2$,
   (c) $g = 0, r \geq 2, rn - 2r - 2 > 0$

5.1. The moduli space of representations

For each n-pointed curve $(C, \mathbf{t}) = (C, t_1, \cdots, t_n) \in T = \mathcal{M}'_{g,n}$ $(g \geq 0, n \geq 1)$, set $D(\mathbf{t}) = t_1 + \cdots + t_n$. By abuse of notation, we denote by $\pi_1(C \setminus D(\mathbf{t})*)$ the fundamental group of $C \setminus \{t_1, \cdots, t_n\}$. The set

$$\mathrm{Hom}(\pi_1(C \setminus D(\mathbf{t}), *), GL_r(\mathbf{C})) \tag{52}$$

of $GL_r(\mathbf{C})$-representations of $\pi_1(C \setminus D(\mathbf{t}), *)$ is an affine variety, and $GL_r(\mathbf{C})$ naturally acts on this space by the adjoint action.

We define the moduli space by

$$\mathcal{RP}^r_{(C,\mathbf{t})} = \mathrm{Hom}(\pi_1(C \setminus D(\mathbf{t}), *), GL_r(\mathbf{C}))//Ad(GL_r(\mathbf{C})). \tag{53}$$

Here the quotient // means the categorical quotient ([Mum]). More precisely, it is known that $\pi_1(C \setminus D(\mathbf{t}), *)$ is generated by $(2g+n)$-elements $\alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g, \gamma_1, \ldots, \gamma_n$ with one relation

$$\prod_{i=1}^{g} [\alpha_i, \beta_i] \gamma_1 \cdots \gamma_n = 1.$$

Therefore if we denote by R the ring of invariants of the simultaneous adjoint action of $GL_r(\mathbf{C})$ on the coordinate ring of $GL_r(\mathbf{C})^{2g+n-1}$, then we have an isomorphism

$$\mathcal{RP}^r_{(C,\mathbf{t})} \simeq \mathrm{Spec}(R). \tag{54}$$

Hence the moduli space $\mathcal{RP}^r_{(C,\mathbf{t})}$ becomes an affine algebraic scheme. Furthermore, each closed point of $\mathcal{RP}^r_{(C,\mathbf{t})}$ corresponds to a Jordan equivalence class of a representation (cf. [Section 4, [IIS1]]).

Let us set

$$(55)\quad \mathcal{A}_r^{(n)} := \left\{ \mathbf{a} = (a_j^{(i)})_{0\le j\le r-1}^{1\le i\le n} \in \mathbf{C}^{nr} \,\middle|\, a_0^{(1)} a_0^{(2)} \cdots a_0^{(n)} = (-1)^{rn} \right\}.$$

For each $\mathbf{a} = (a_j^{(i)}) \in \mathcal{A}_r^{(n)}$ and $i, 1 \le i \le n$, we set $\mathbf{a}^{(i)} = (a_0^{(i)}, \cdots, a_{r-1}^{(i)})$ and define

$$(56)\qquad \chi_{\mathbf{a}^{(i)}}(s) = s^r + a_{r-1}^{(i)} s^{r-1} + \cdots + a_0^{(i)}.$$

Moreover we define a morphism

$$(57)\qquad \phi^r_{(C,\mathbf{t})} : \mathcal{RP}^r_{(C,\mathbf{t})} \longrightarrow \mathcal{A}_r^{(n)}$$

by the relation

$$(58)\qquad \det(sI_r - \rho(\gamma_i)) = \chi_{\mathbf{a}^{(i)}}(s)$$

where $[\rho] \in \mathcal{RP}^r_{(C,\mathbf{t})}$ and γ_i is a counterclockwise loop around t_i.

For $\mathbf{a} = (a_j^{(i)}) \in \mathcal{A}_r^{(n)}$, we denote by $\mathcal{RP}^r_{(C,\mathbf{t}),\mathbf{a}}$ the fiber of $\phi^r_{(C,\mathbf{t})}$ over $\mathbf{a}$, that is,

(59)

$$\mathcal{RP}^r_{(C,\mathbf{t}),\mathbf{a}} = \{[\rho] \in \mathcal{RP}^r_{(C,\mathbf{t})} | \det(sI_r - \rho(\gamma_i)) = \chi_{\mathbf{a}^{(i)}}(s), 1 \le i \le n\}.$$

For any covering $T' \to T$, we can define a relative moduli space $\mathcal{RP}^r_{n,T'} = \coprod_{(C,\mathbf{t})\in T'} \mathcal{RP}^r_{(C,\mathbf{t})}$ of representations with the natural morphism

$$(60)\qquad \mathcal{RP}^r_{n,T'} \longrightarrow T'.$$

As in Section 4, [IIS1], there exists a finite covering $T' \longrightarrow T$ with the morphism

$$(61)\qquad \phi^r_n : \mathcal{RP}^r_{n,T'} \longrightarrow T' \times \mathcal{A}_r^{(n)},$$

such that

$$(\phi^r_n)^{-1}((C,\mathbf{t}),\mathbf{a}) = \mathcal{RP}^r_{(C,\mathbf{t}),\mathbf{a}}.$$

§6. The Riemann-Hilbert correspondence

Next we define the Riemann-Hilbert correspondence from the moduli space of $\boldsymbol{\alpha}$-stable parabolic connections to the moduli space of the representations.

Let us fix positive integers r, d, $\boldsymbol{\alpha} = (\alpha_j^{(i)})$ as in (44), and $(C,\mathbf{t}) \in T' = \mathcal{M}'_{g,n}$. For simplicity, we set $\mathcal{M}^{\boldsymbol{\alpha}}_{((C,\mathbf{t}),\boldsymbol{\lambda})} = \mathcal{M}^{\boldsymbol{\alpha}}_{((C,\mathbf{t}),\boldsymbol{\lambda})}(r,n,d)$ (cf. (47)).

We define a morphism

$$rh : \Lambda_r^{(n)}(d) \longrightarrow \mathcal{A}_r^{(n)}, \quad rh(\boldsymbol{\lambda}) = \mathbf{a} \tag{62}$$

by the relation

$$\prod_{j=0}^{r-1}(s - \exp(-2\pi\sqrt{-1}\lambda_j^{(i)})) = s^r + a_{r-1}^{(i)}s^{r-1} + \cdots + a_0^{(i)}. \tag{63}$$

For each member $(E, \nabla, \{l_j^{(i)}\}) \in \mathcal{M}^{\boldsymbol{\alpha}}_{(C,\mathbf{t}),\boldsymbol{\lambda}}$, the solution subsheaf of E^{an}

$$\ker(\nabla^{an}|_{C\setminus D(\mathbf{t})}) \subset E^{an} \tag{64}$$

becomes a local system on $C \setminus D(\mathbf{t})$ and corresponds to a representation

$$\rho : \pi_1(C \setminus \{\mathbf{t}\}, *) \longrightarrow GL_r(\mathbf{C}). \tag{65}$$

Since the eigenvalues of the residue matrix of ∇^{an} at t_i are $\lambda_j^{(i)}$, $0 \leq j \leq r-1$, considering the local fundamental solutions of $\nabla^{an} = 0$ near t_i, the monodromy matrix of $\rho(\gamma_i)$ has eigenvalues $\exp(-2\pi\sqrt{-1}\lambda_j^{(i)})$, $0 \leq j \leq r-1$. Hence under the relation (63), or $\mathbf{a} = rh(\boldsymbol{\lambda})$, we can define a morphism

$$\mathbf{RH}_{(C,\mathbf{t}),\boldsymbol{\lambda}} : \mathcal{M}^{\boldsymbol{\alpha}}_{((C,\mathbf{t}),\boldsymbol{\lambda})} \longrightarrow \mathcal{RP}^r_{(C,\mathbf{t}),\mathbf{a}}. \tag{66}$$

Replacing $T = \mathcal{M}'_{g,n}$ by a certain finite étale covering $u : T' \longrightarrow T$ and varying $((C, \mathbf{t}), \boldsymbol{\lambda}) \in T' \times \Lambda_r^{(n)}(d)$ we can define a morphism

$$\mathbf{RH} : \mathcal{M}^{\boldsymbol{\alpha}}_{(\mathcal{C},\mathbf{t})/T'}(r, n, d) \longrightarrow \mathcal{RP}^r_{n,T'} \tag{67}$$

which makes the diagram

$$\begin{array}{ccc} \mathcal{M}^{\boldsymbol{\alpha}}_{(\mathcal{C},\tilde{\mathbf{t}})/T'}(r, n, d) & \xrightarrow{\mathbf{RH}} & \mathcal{RP}^r_{n,T'} \\ {\scriptstyle \varphi_{r,n,d}}\downarrow & & \downarrow{\scriptstyle \phi^r_n} \\ T' \times \Lambda_r^{(n)}(d) & \xrightarrow{Id \times rh} & T' \times \mathcal{A}_r^{(n)} \end{array} \tag{68}$$

commute. The following result is proved in [Ina].

Theorem 6.1. ([Theorem 2.2, [Ina]]). *Assume that $\boldsymbol{\alpha}$ is so generic that $\boldsymbol{\alpha}$-stable $\Leftrightarrow$ $\boldsymbol{\alpha}$-semistable. Moreover we assume that $r \geq 2, rn - 2r - 2 > 0$ if $g = 0$, $n \geq 2$ if $g = 1$ and $n \geq 1$ if $g \geq 2$. Then the morphism*

$$\mathbf{RH} : \mathcal{M}^{\boldsymbol{\alpha}}_{(\mathcal{C},\tilde{\mathbf{t}})/T'}(r, n, d) \longrightarrow \mathcal{RP}^r_{n,T'} \times_{\mathcal{A}_r^{(n)}} \Lambda_r^{(n)} \tag{69}$$

induced by (67) is a proper surjective bimeromorphic analytic *morphism. In particular, for each* $((C,\mathbf{t}),\boldsymbol{\lambda}) \in T' \times \Lambda_r^{(n)}(d)$*, the restricted morphism*

$$\mathbf{RH}_{((C,\mathbf{t}),\boldsymbol{\lambda})} : \mathcal{M}^{\boldsymbol{\alpha}}_{((C,\mathbf{t}),\boldsymbol{\lambda})}(r,n,d) \longrightarrow \mathcal{RP}^r_{(C,\mathbf{t}),\mathbf{a}} \tag{70}$$

gives an analytic resolution of singularities of $\mathcal{RP}^r_{(C,\mathbf{t}),\mathbf{a}}$ *where* $\mathbf{a} = rh(\boldsymbol{\lambda})$.

Remark 6.1. Take $\boldsymbol{\lambda} \in \Lambda_r^{(n)}$ such that $rh(\boldsymbol{\lambda}) = \mathbf{a}$. A representation ρ such that $[\rho] \in \mathcal{RP}^r_{(C,\mathbf{t}),\mathbf{a}}$ is said to be *resonant* if

$$\dim(\ker(\rho(\gamma_i) - \exp(-2\pi\sqrt{-1}\lambda_j^{(i)}))) \geq 2 \text{ for some } i, j. \tag{71}$$

The singular locus of $\mathcal{RP}^r_{(C,\mathbf{t}),\mathbf{a}}$ is given by the set

$$\left(\mathcal{RP}^r_{(C,\mathbf{t}),\mathbf{a}}\right)^{sing} := \left\{ [\rho] \in \mathcal{RP}^r_{(C,\mathbf{t}),\mathbf{a}} \;\middle|\; \begin{array}{c} \rho \text{ is reducible or} \\ \text{resonant} \end{array} \right\}. \tag{72}$$

Moreover we denote the smooth part of $\mathcal{RP}^r_{(C,\mathbf{t}),\mathbf{a}}$ by

$$\left(\mathcal{RP}^r_{(C,\mathbf{t}),\mathbf{a}}\right)^{\sharp} = \mathcal{RP}^r_{(C,\mathbf{t}),\mathbf{a}} \setminus \left(\mathcal{RP}^r_{(C,\mathbf{t}),\mathbf{a}}\right)^{sing}. \tag{73}$$

Theorem 6.1 implies that the restriction

$$\mathbf{RH}_{((C,\mathbf{t}),\boldsymbol{\lambda})|\left(\mathcal{M}^{\boldsymbol{\alpha}}_{(C,\mathbf{t}),\boldsymbol{\lambda}}\right)^{\sharp}} : \left(\mathcal{M}^{\boldsymbol{\alpha}}_{(C,\mathbf{t}),\boldsymbol{\lambda}}\right)^{\sharp} \xrightarrow{\simeq} \left(\mathcal{RP}^r_{(C,\mathbf{t}),\mathbf{a}}\right)^{\sharp} \tag{74}$$

is an analytic isomorphism, where

$$\left(\mathcal{M}^{\boldsymbol{\alpha}}_{(C,\mathbf{t}),\boldsymbol{\lambda}}\right)^{\sharp} = \mathbf{RH}^{-1}_{((C,\mathbf{t}),\boldsymbol{\lambda})}(\left(\mathcal{RP}^r_{(C,\mathbf{t}),\mathbf{a}}\right)^{\sharp}).$$

§7. Isomonodromic flows and Differential systems of Painlevé type

Consider the family of the moduli spaces of $\boldsymbol{\alpha}$-stable parabolic connections

$$\varphi_{r,n,d} : \mathcal{M}^{\boldsymbol{\alpha}}_{(C,\mathbf{t})/T}(r,d,n) \longrightarrow T \times \Lambda_r^{(n)}(d) \tag{75}$$

where $T = \mathcal{M}'_{g,n}$ as in (48).

Fix $((C_0,\mathbf{t}_0),\boldsymbol{\lambda}_0) \in T \times \Lambda_r^{(n)}(d)$ and take an $\boldsymbol{\alpha}$-stable parabolic connection $\mathbf{x} = (E,\nabla,\{l_*^{(i)}\}_{1\leq i\leq n}) \in \mathcal{M}^{\boldsymbol{\alpha}}_{((C_0,\mathbf{t}_0),\boldsymbol{\lambda}_0)}(r,d,n)$. Let $\Delta = \{t \in \mathbf{C} | |t| < 1\}$ be the unit disc and let $h : \Delta \longrightarrow T$ be a holomorphic embedding such that $h(0) = (C_0, \mathbf{t}_0)$. Then pulling back the universal family, we obtain the family of n-pointed curves $f : (\mathcal{C}, \mathbf{t}) \longrightarrow \Delta$ with the central fiber $f^{-1}(0) = (C_0, \mathbf{t}_0)$. An $\boldsymbol{\alpha}$-stable parabolic connection $(\mathcal{E}, \nabla, l)$ on the family of n-pointed curves $(\mathcal{C}, \mathbf{t})$ over Δ is called a *(1-parameter) deformation* of $(E, \nabla, \{l_*^{(i)}\}_{1 \leq i \leq n})$ if we have an isomorphism $(\mathcal{E}, \nabla, l)_{|(C_0, \mathbf{t}_0)} \simeq (E, \nabla, \{l_*^{(i)}\}_{1 \leq i \leq n})$. Restricting the $\boldsymbol{\alpha}$-stable parabolic connection $(\mathcal{E}, \nabla, l)$ to each fiber $(\mathcal{C}_t, \mathbf{t}_t)$, we have a family of $\boldsymbol{\alpha}$-stable parabolic connections $(\mathcal{E}_t, \nabla_t, l_t)$ over $(\mathcal{C}_t, \mathbf{t}_t)$ which are automatically flat in the direction of each fiber. If the connection ∇ on $\mathcal{E}$ is flat on the total space $\mathcal{C}$, which means that the curvature 2-form of ∇ vanishes over the total space $\mathcal{C}$, the associated representations $\rho_t : \pi_1(\mathcal{C}_t \setminus \{\mathbf{t}_t\}, *) \longrightarrow GL_r(\mathbf{C})$ is constant with respect to $t \in \Delta$. Moreover the converse is also true. Therefore such a deformation $(\mathcal{E}, \nabla, l)$ over $\mathcal{C} \longrightarrow \Delta$ is called *an isomonodromic deformation* of a $\boldsymbol{\alpha}$-stable parabolic connection. Under an isomonodromic deformation, local exponents $\boldsymbol{\lambda}_t$ of the connection $(\mathcal{E}_t, \nabla_t, l_t)$ are also constant, so we have $\boldsymbol{\lambda}_t = \boldsymbol{\lambda}_0$. Therefore an isomonodromic deformation determines a holomorphic map $\tilde{h} : \Delta \longrightarrow \mathcal{M}^{\boldsymbol{\alpha}}_{(\mathcal{C},\mathbf{t}),\boldsymbol{\lambda}_0/T}(r, d, n)$ which is a lift of $h : \Delta \longrightarrow T$ such that $\tilde{h}(0) = \mathbf{x} \in \mathcal{M}^{\boldsymbol{\alpha}}_{((C_0,\mathbf{t}_0),\boldsymbol{\lambda}_0)}(r, d, n)$.

$$\begin{array}{ccc} & & \mathcal{M}^{\boldsymbol{\alpha}}_{(\mathcal{C},\mathbf{t}),\boldsymbol{\lambda}_0/T}(r, n, d) \\ & \tilde{h} \nearrow & \downarrow \varphi_{r,n,d,\boldsymbol{\lambda}_0} \\ \Delta & \xrightarrow{h} & T \times \{\boldsymbol{\lambda}_0\} \end{array} .$$

Next we will define a global foliation $\mathcal{IF}$ on the total space of $\mathcal{M}^{\boldsymbol{\alpha}}_{(\mathcal{C},\mathbf{t})/T}(r, d, n)$ from isomonodromic deformations of the $\boldsymbol{\alpha}$-stable parabolic connections. We mean that a foliation $\mathcal{IF}$ is a subsheaf of the tangent sheaf $\Theta_{\mathcal{M}^{\boldsymbol{\alpha}}_{(\mathcal{C},\mathbf{t})/T}(r,d,n)}$. We will show that the global foliation $\mathcal{IF}$ coming from isomonodromic deformations has the Painlevé property, whose precise meaning will be defined in Theorem 7.1.

Let us consider the universal covering map $u : \tilde{T} \to T = \mathcal{M}'_{g,n}$. Note that u factors thorough the morphism $u' : \tilde{T} \to T'$. Pulling back the fibration $\phi^r_n : \mathcal{RP}^r_{n,T'} \longrightarrow T' \times \mathcal{A}^{(n)}_r$ in (61) by u', we obtain the fibration $\mathcal{RP}^r_{n,T'} \times_{T'} \tilde{T} \longrightarrow \tilde{T}$, which becomes a trivial fibration as explained in Section 4 in [IIS1]. This means that if we fix a point $(C_0, \mathbf{t}_0) \in T$ there exists an isomorphism

$$\pi : \mathcal{RP}^r_{n,T'} \times_{T'} \tilde{T} \xrightarrow{\simeq} \mathcal{RP}^r_{(C_0,\mathbf{t}_0)} \times \tilde{T} \tag{76}$$

which makes the following diagram commute.

$$(77)\qquad \begin{array}{ccc} \mathcal{RP}^r_{n,T'} \times_{T'} \tilde{T} & \xrightarrow[\simeq]{\pi} & \mathcal{RP}^r_{(C_0,\mathbf{t}_0)} \times \tilde{T} \\ \widetilde{\phi^r_n}\big\downarrow & & \big\downarrow p_2\times\phi^r_{(C_0,\mathbf{t}_0)} \\ \tilde{T}\times\mathcal{A}^{(n)}_r & \longrightarrow & \tilde{T}\times\mathcal{A}^{(n)}_r. \end{array}$$

Fixing $\mathbf{a} \in \mathcal{A}^{(n)}_r$, we set $\mathcal{RP}^r_{n,T',\mathbf{a}} = (\phi^r_n)^{-1}(T' \times \{\mathbf{a}\})$. From the morphisms (57) and (61), we also have the following commutative diagram:

$$(78)\qquad \begin{array}{ccc} \mathcal{RP}^r_{n,T',\mathbf{a}} \times_{T'} \tilde{T} & \xrightarrow[\simeq]{\pi_{\mathbf{a}}} & \mathcal{RP}^r_{(C_0,\mathbf{t}_0),\mathbf{a}} \times \tilde{T} \\ \widetilde{\phi^r_{n,\mathbf{a}}}\big\downarrow & & \big\downarrow p_2 \\ \tilde{T}\times\{\mathbf{a}\} & \xrightarrow{\simeq} & \tilde{T}. \end{array}$$

By using the isomorphism (78) we can define the smooth part of $\mathcal{RP}^r_{n,T',\mathbf{a}} \times_{T'} \tilde{T}$ by

$$\left(\mathcal{RP}^r_{n,T',\mathbf{a}} \times_{T'} \tilde{T}\right)^\sharp = \pi_{\mathbf{a}}^{-1}\left(\left(\mathcal{RP}^r_{(C_0,\mathbf{t}_0),\mathbf{a}}\right)^\sharp \times \tilde{T}\right)$$

where $\left(\mathcal{RP}^r_{(C_0,\mathbf{t}_0),\mathbf{a}}\right)^\sharp$ is the smooth locus of $\mathcal{RP}^r_{(C_0,\mathbf{t}_0),\mathbf{a}}$ (cf. (73)). Note that for generic $\mathbf{a}$ the variety $\mathcal{RP}^r_{(C_0,\mathbf{t}_0),\mathbf{a}}$ is non-singular, but for special $\mathbf{a}$, $\mathcal{RP}^r_{(C_0,\mathbf{t}_0),\mathbf{a}}$ does have singularities (cf. [(72), Remark 6.1]).

We also have the following commutative diagram

$$(79)\qquad \begin{array}{ccc} \left(\mathcal{RP}^r_{n,T',\mathbf{a}} \times_{T'} \tilde{T}\right)^\sharp & \xrightarrow[\simeq]{\pi_{\mathbf{a}}} & \left(\mathcal{RP}^r_{(C_0,\mathbf{t}_0),\mathbf{a}}\right)^\sharp \times \tilde{T} \\ \big\downarrow & & \big\downarrow p_2 \\ \tilde{T}\times\{\mathbf{a}\} & \xrightarrow{\simeq} & \tilde{T} \end{array} .$$

By using this isomorphism, for any fixed $\mathbf{a} \in \mathcal{A}^{(n)}_r$, we define the set of constant sections

(80)

$$\mathrm{Isomd}(\tilde{T}, \left(\mathcal{RP}^r_{n,T',\mathbf{a}} \times_{T'} \tilde{T}\right)^\sharp) = \left\{\sigma : \tilde{T} \to \left(\mathcal{RP}^r_{n,T',\mathbf{a}} \times_{T'} \tilde{T}\right)^\sharp, \text{constant}\right\}.$$

Note that by using the isomorphism (79), we have a natural isomorphism

$$(81)\qquad \mathrm{Isomd}(\tilde{T}, \left(\mathcal{RP}^r_{n,T',\mathbf{a}} \times_{T'} \tilde{T}\right)^\sharp) \simeq \left(\mathcal{RP}^r_{(C_0,\mathbf{t}_0),\mathbf{a}}\right)^\sharp.$$

A section $\sigma \in \mathrm{Isomd}(\tilde{T}, \left(\mathcal{RP}^r_{n,T',\mathbf{a}} \times_{T'} \tilde{T}\right)^\sharp)$ is called an *isomonodromic section* by trivial reason and its image $\sigma(\tilde{T})$ is called an *isomonodromic flow.*

Next, considering the pullback of $\varphi_{r,n,d}$ in (50) by $\tilde{T} \longrightarrow T$, we can obtain the family of moduli spaces of $\boldsymbol{\alpha}$-stable parabolic connections

$$\widetilde{\varphi_{r,n,d}} : \mathcal{M}^{\boldsymbol{\alpha}}_{(\mathcal{C},\mathbf{t})/\tilde{T}} \longrightarrow \tilde{T} \times \Lambda^{(n)}_r(d). \tag{82}$$

Fixing $\boldsymbol{\lambda} \in \Lambda$ such that $rh(\boldsymbol{\lambda}) = \mathbf{a}$, we also obtain the restricted family over $\tilde{T} \times \{\boldsymbol{\lambda}\}$

$$\widetilde{\varphi_{r,n,d,\boldsymbol{\lambda}}} : \mathcal{M}^{\boldsymbol{\alpha}}_{((\mathcal{C},\mathbf{t}),\boldsymbol{\lambda})/\tilde{T}} \longrightarrow \tilde{T} \times \{\boldsymbol{\lambda}\}. \tag{83}$$

Restricting the Riemann-Hilbert correspondence (68) to this space, we obtain the following commutative diagram

$$\begin{array}{ccc} \mathcal{M}^{\boldsymbol{\alpha}}_{((\mathcal{C},\tilde{\mathbf{t}}),\boldsymbol{\lambda})/\tilde{T}}(r,n,d) & \xrightarrow{\mathbf{RH}_{\boldsymbol{\lambda}}} & \mathcal{RP}^r_{n,T,\mathbf{a}} \times_T \tilde{T} \\ {\scriptstyle \widetilde{\varphi_{r,n,d,\boldsymbol{\lambda}}}}\downarrow & & \downarrow{\scriptstyle \widetilde{\phi^r_{n,\mathbf{a}}}} \\ \tilde{T} \times \{\boldsymbol{\lambda}\} & \xrightarrow{Id \times rh} & \tilde{T} \times \{\mathbf{a}\} \end{array} . \tag{84}$$

Note that by Theorem 6.1 the morphism $\mathbf{RH}_{\boldsymbol{\lambda}}$ gives an analytic resolution of singularities. Set

$$\left(\mathcal{M}^{\boldsymbol{\alpha}}_{((\mathcal{C},\tilde{\mathbf{t}}),\boldsymbol{\lambda})/\tilde{T}}(r,n,d)\right)^\sharp = \mathbf{RH}^{-1}_{\boldsymbol{\lambda}}((\mathcal{RP}^r_{n,T,\mathbf{a}} \times_T \tilde{T})^\sharp), \tag{85}$$

and

$$\left(\mathcal{M}^{\boldsymbol{\alpha}}_{((\mathcal{C},\tilde{\mathbf{t}}),\boldsymbol{\lambda})/\tilde{T}}(r,n,d)\right)^{sing} = \mathbf{RH}^{-1}_{\boldsymbol{\lambda}}((\mathcal{RP}^r_{n,T,\mathbf{a}} \times_T \tilde{T})^{sing}). \tag{86}$$

(Cf. (72), (73)). Then we have an analytic isomorphism

$$(\mathbf{RH}_{\boldsymbol{\lambda}})^\sharp : \left(\mathcal{M}^{\boldsymbol{\alpha}}_{((\mathcal{C},\tilde{\mathbf{t}}),\boldsymbol{\lambda})/\tilde{T}}(r,n,d)\right)^\sharp \xrightarrow{\simeq} (\mathcal{RP}^r_{n,T,\mathbf{a}} \times_T \tilde{T})^\sharp.$$

Now we define:
(87)

$$\mathrm{Isomd}(\tilde{T}, \left(\mathcal{M}^{\boldsymbol{\alpha}}_{((\mathcal{C},\tilde{\mathbf{t}}),\boldsymbol{\lambda})/\tilde{T}}(r,n,d)\right)^\sharp) = \mathbf{RH}^{-1}_{\boldsymbol{\lambda}}(\mathrm{Isomd}(\tilde{T}, (\mathcal{RP}^r_{n,T,\mathbf{a}} \times_T \tilde{T})^\sharp)).$$

Each section $\sigma \in \mathrm{Isomd}(\tilde{T}, \left(\mathcal{M}^{\boldsymbol{\alpha}}_{((\mathcal{C},\tilde{\mathbf{t}}),\boldsymbol{\lambda})/\tilde{T}}(r,n,d)\right)^\sharp)$ is called an *isomonodromic section* on $\left(\mathcal{M}^{\boldsymbol{\alpha}}_{((\mathcal{C},\tilde{\mathbf{t}}),\boldsymbol{\lambda})/\tilde{T}}(r,n,d)\right)^\sharp$ and its image

$$\sigma(\tilde{T}) \subset \left(\mathcal{M}^{\boldsymbol{\alpha}}_{((\mathcal{C},\tilde{\mathbf{t}}),\boldsymbol{\lambda})/\tilde{T}}(r,n,d)\right)^\sharp$$

is called an *isomonodromic flow.* Note that since the Riemann-Hilbert correspondence $(\mathbf{RH}_{\boldsymbol{\lambda}})^{\sharp}$ is a highly non-trivial analytic isomorphism, isomonodromic flows $\{\sigma(\tilde{T})\}$ are not constant any more and it is known that they define highly transcendental analytic functions.

From the morphism (83) restricted to $\left(\mathcal{M}^{\boldsymbol{\alpha}}_{((C,\tilde{\mathbf{t}}),\boldsymbol{\lambda})/\tilde{T}}(r,n,d)\right)^{\sharp}$, we obtain the natural sheaf homomorphism

$$\Theta_{\left(\mathcal{M}^{\boldsymbol{\alpha}}_{((C,\tilde{\mathbf{t}}),\boldsymbol{\lambda})/\tilde{T}}(r,n,d)\right)^{\sharp}} \xrightarrow{\widetilde{\varphi_{r,n,d,\boldsymbol{\lambda}}}^{*}} \widetilde{\varphi_{r,n,d,\boldsymbol{\lambda}}}^{*}(\Theta_{\tilde{T}})_{|\left(\mathcal{M}^{\boldsymbol{\alpha}}_{((C,\tilde{\mathbf{t}}),\boldsymbol{\lambda})/\tilde{T}}(r,n,d)\right)^{\sharp}} \longrightarrow 0.$$

Then the set of all isomonodromic sections defines a sheaf homomorphism

$$\text{(88)}\quad \mathcal{V}_{\boldsymbol{\lambda}} : \widetilde{\varphi_{r,n,d,\boldsymbol{\lambda}}}^{*}(\Theta_{\tilde{T}})_{|\left(\mathcal{M}^{\boldsymbol{\alpha}}_{((C,\tilde{\mathbf{t}}),\boldsymbol{\lambda})/\tilde{T}}(r,n,d)\right)^{\sharp}} \longrightarrow \Theta_{\left(\mathcal{M}^{\boldsymbol{\alpha}}_{((C,\tilde{\mathbf{t}}),\boldsymbol{\lambda})/\tilde{T}}(r,n,d)\right)^{\sharp}}$$

which gives a splitting of the homomorphism $\widetilde{\varphi_{r,n,d,\boldsymbol{\lambda}}}^{*}$. The splitting (88) is algebraic, because the condition of isomonodromic flows given by the vanishing of the curvature 2-forms of the associated universal connections. Since the exceptional locus for $\mathbf{RH} = \cup_{\boldsymbol{\lambda}} \mathbf{RH}_{\boldsymbol{\lambda}}$ has codimension at least 2, by Hartogs' theorem, it is easy to see that this algebraic splitting (88) can be extend to the whole family of moduli spaces, and we obtain an extended homomorphism

$$\text{(89)}\qquad \mathcal{V}_{\boldsymbol{\lambda}} : \widetilde{\varphi_{r,n,d,\boldsymbol{\lambda}}}^{*}(\Theta_{\tilde{T}}) \longrightarrow \Theta_{\mathcal{M}^{\boldsymbol{\alpha}}_{((C,\tilde{\mathbf{t}}),\boldsymbol{\lambda})/\tilde{T}}(r,n,d)}.$$

Under the notation above, we have the following

Definition 7.1. (1) The foliation $\mathcal{IF}_{\boldsymbol{\lambda}}$ defined by the subsheaf

$$\text{(90)}\qquad \mathcal{IF}_{\boldsymbol{\lambda}} = \mathcal{V}_{\boldsymbol{\lambda}}(\widetilde{\varphi_{r,n,d,\boldsymbol{\lambda}}}^{*}(\Theta_{\tilde{T}})) \subset \Theta_{\mathcal{M}^{\boldsymbol{\alpha}}_{((C,\tilde{\mathbf{t}}),\boldsymbol{\lambda})/\tilde{T}}(r,n,d)}$$

is called *an isomonodromic foliation on* $\mathcal{M}^{\boldsymbol{\alpha}}_{((C,\tilde{\mathbf{t}}),\boldsymbol{\lambda})/\tilde{T}}(r,n,d)$.

(2) Let $h : \Delta \longrightarrow \tilde{T}$ be a holomorphic embedding such that $h(t) = (C_t, \mathbf{t}_t)$ for $t \in \Delta$. A holomorphic map $\tilde{h} : \Delta \longrightarrow \mathcal{M}^{\boldsymbol{\alpha}}_{((C,\tilde{\mathbf{t}}),\boldsymbol{\lambda})/\tilde{T}}(r,n,d)$ such that $\widetilde{\varphi_{r,n,d,\boldsymbol{\lambda}}} \circ \tilde{h} = h$ is called a $\mathcal{IF}_{\boldsymbol{\lambda}}$-lift of h if $\tilde{h}$ is tangent to $\mathcal{IF}_{\boldsymbol{\lambda}}$, that is, $\tilde{h}_{*}(\Theta_{\Delta}) \subset \mathcal{IF}_{\boldsymbol{\lambda}}$.

Lemma 7.1. *Let $h : \Delta \longrightarrow \tilde{T}$ be a holomorphic embedding and $\tilde{h} : \Delta \longrightarrow \mathcal{M}^{\boldsymbol{\alpha}}_{((C,\tilde{\mathbf{t}}),\boldsymbol{\lambda})/\tilde{T}}(r,n,d)$ a $\mathcal{IF}_{\boldsymbol{\lambda}}$-lift of h. Then the image of* $\mathbf{RH}_{\boldsymbol{\lambda}} \circ \tilde{h}$ *lies in the image of a constant section* $\sigma \in \mathrm{Isomd}(\tilde{T}, \left(\mathcal{RP}^{r}_{n,T',\mathbf{a}} \times_{T'} \tilde{T}\right))$.

Proof. Note that a lift $\tilde{h}$ of h corresponds to a 1-parameter deformation of $\boldsymbol{\alpha}$-stable parabolic connection under a deformation of n-pointed curves associated to $h : \Delta \longrightarrow \tilde{T}$. Since $\left(\mathcal{M}^{\boldsymbol{\alpha}}_{((\mathcal{C},\tilde{\mathbf{t}}),\boldsymbol{\lambda})/\tilde{T}}(r,n,d)\right)^{\sharp}$ is a Zariski dense open subset of $\left(\mathcal{M}^{\boldsymbol{\alpha}}_{((\mathcal{C},\tilde{\mathbf{t}}),\boldsymbol{\lambda})/\tilde{T}}(r,n,d)\right)$, we see that the curvature form vanishes on the $\mathcal{IF}$-foliation defined on the total space $\left(\mathcal{M}^{\boldsymbol{\alpha}}_{((\mathcal{C},\tilde{\mathbf{t}}),\boldsymbol{\lambda})/\tilde{T}}(r,n,d)\right)$. Therefore if $\tilde{h}$ is a $\mathcal{IF}$-lift of h, we can conclude that the deformation of connections is isomonodromic. Hence the associated representations of the fundamental group of $\mathcal{C}_t \setminus \{\mathbf{t}_t\}$ are constant, which means that $\mathbf{RH}_{\boldsymbol{\lambda}}(\tilde{h}(\Delta))$ is contained in the image of a constant section of $\left(\mathcal{RP}^r_{n,T',\mathbf{a}} \times_{T'} \tilde{T}\right) \longrightarrow \tilde{T}$. Q.E.D.

Now, we can show that the isomonodromic foliation is a differential system satisfying the Painlevé property (cf. [Mal], [Miwa] and [IIS3]).

Theorem 7.1. *For any $\boldsymbol{\lambda} \in \Lambda^{(n)}_r(d)$, the isomonodromic foliation $\mathcal{IF}_{\boldsymbol{\lambda}}$ defined on $\mathcal{M}^{\boldsymbol{\alpha}}_{((\mathcal{C},\tilde{\mathbf{t}}),\boldsymbol{\lambda})/\tilde{T}}(r,n,d)$ has Painlevé property. That is, for any holomorphic embedding $h : \Delta \longrightarrow \tilde{T}$ of the unit disc $\Delta = \{t \in \mathbf{C} | |t| < 1\}$ such that $h(0) = (C,\mathbf{t})$ and $\mathbf{x} = (E,\nabla,\{l^{(i)}_*\}_{1\leq i\leq n}) \in \mathcal{M}^{\boldsymbol{\alpha}}_{((C,\mathbf{t}),\boldsymbol{\lambda})}(r,n,d)$, there exists the unique $\mathcal{IF}_{\boldsymbol{\lambda}}$-lift*

$$\tilde{h} : \Delta \longrightarrow \mathcal{M}^{\boldsymbol{\alpha}}_{((\mathcal{C},\tilde{\mathbf{t}}),\boldsymbol{\lambda})/\tilde{T}}(r,n,d)$$

of h such that $\tilde{h}(0) = \mathbf{x}$.

Proof. If $\mathbf{x} \in \left(\mathcal{M}^{\boldsymbol{\alpha}}_{((C,\mathbf{t}),\boldsymbol{\lambda})}(r,n,d)\right)^{\sharp}$, there is a unique isomonodromic section $\sigma : \tilde{T} \longrightarrow \left(\mathcal{M}^{\boldsymbol{\alpha}}_{((\mathcal{C},\tilde{\mathbf{t}}),\boldsymbol{\lambda})/\tilde{T}}(r,n,d)\right)^{\sharp}$ such that $\sigma((C,\mathbf{t})) = \mathbf{x}$. The holomorphic map $\tilde{h} = \sigma \circ h : \Delta \longrightarrow \left(\mathcal{M}^{\boldsymbol{\alpha}}_{((\mathcal{C},\tilde{\mathbf{t}}),\boldsymbol{\lambda})/\tilde{T}}(r,n,d)\right)^{\sharp}$ is the unique $\mathcal{IF}_{\boldsymbol{\lambda}}$-lift of h.

Let us consider the case when $\mathbf{x} \in \left(\mathcal{M}^{\boldsymbol{\alpha}}_{((C,\mathbf{t}),\boldsymbol{\lambda})}(r,n,d)\right)^{sing}$. Pulling back the commutative diagrams (84) and (78) via the embedding $h : \Delta \longrightarrow \tilde{T}$, we obtain the commutative diagram

$$\begin{array}{ccc} \mathcal{M}^{\boldsymbol{\alpha}}_{((\mathcal{C},\tilde{\mathbf{t}}),\boldsymbol{\lambda})/\Delta}(r,n,d) & \xrightarrow{\pi_{\mathbf{a}}\circ\mathbf{RH}_{\boldsymbol{\lambda}}} & \mathcal{RP}^r_{(C_0,\mathbf{t}_0),\mathbf{a}} \times \Delta \\ \widetilde{\varphi_{\Delta}} \downarrow & & \downarrow p_2 \\ \Delta & \xrightarrow{Id} & \Delta \end{array} \tag{91}$$

The restriction of the foliation $\mathcal{IF}_{\boldsymbol{\lambda}}$ to $\mathcal{M}^{\boldsymbol{\alpha}}_{((\mathcal{C},\tilde{\mathbf{t}}),\boldsymbol{\lambda})/\Delta}(r,n,d)$ determines a vector field $v_{\boldsymbol{\lambda}}$ on $\mathcal{M}^{\boldsymbol{\alpha}}_{((\mathcal{C},\tilde{\mathbf{t}}),\boldsymbol{\lambda})/\Delta}(r,n,d)$ such that $\widetilde{\varphi_{\Delta}}_*(v_{\boldsymbol{\lambda}}) = \frac{\partial}{\partial t}$ where t is a coordinate of Δ. We will show that there exist a unique section $\tilde{h} : \Delta \longrightarrow \mathcal{M}^{\boldsymbol{\alpha}}_{((\mathcal{C},\tilde{\mathbf{t}}),\boldsymbol{\lambda})/\Delta}(r,n,d)$ such that $\tilde{h}(0) = \mathbf{x}$ and $\tilde{h}_*(\frac{\partial}{\partial t}) = v_{\boldsymbol{\lambda}}$, which gives a $\mathcal{IF}_{\boldsymbol{\lambda}}$-lift of h. Such a section $\tilde{h}$ can be locally given by an analytic solution of the Cauchy problem of an ordinary differential equation associated to the vector field $v_{\boldsymbol{\lambda}}$. Such an analytic solution can be locally given by holomorphic functions of t on $\Delta_\epsilon = \{t \in \mathbf{C} \mid |t| < \epsilon\}$ for some $0 < \epsilon < 1$. This gives a section $\tilde{h}_\epsilon : \Delta_\epsilon \longrightarrow \mathcal{M}^{\boldsymbol{\alpha}}_{((\mathcal{C},\tilde{\mathbf{t}}),\boldsymbol{\lambda})/\Delta_\epsilon}(r,n,d)$ which is a $\mathcal{IF}_{\boldsymbol{\lambda}}$-lift of $h_\epsilon = h_{|\Delta_\epsilon}$. Let ϵ_1 be the supremum of ϵ such that a $\mathcal{IF}_\lambda$ lift of h_ϵ exists. The above argument shows that $\epsilon_1 > 0$. Now we will show that $\epsilon_1 = 1$. Assume the contrary, that is, $\epsilon_1 < 1$, and let $\tilde{h}_{\epsilon_1} : \Delta_{\epsilon_1} \longrightarrow \mathcal{M}^{\boldsymbol{\alpha}}_{((\mathcal{C},\tilde{\mathbf{t}}),\boldsymbol{\lambda})/\Delta_{\epsilon_1}}(r,n,d)$ be the section over Δ_{ϵ_1}.

Let $p_1 : \mathcal{RP}^r_{(C_0,\mathbf{t}_0),\mathbf{a}} \times \Delta \longrightarrow \mathcal{RP}^r_{(C_0,\mathbf{t}_0),\mathbf{a}}$ be the first projection and consider the morphism

$$p_1 \circ \pi_a \circ \mathbf{RH}_{\boldsymbol{\lambda}} : \mathcal{M}^{\boldsymbol{\alpha}}_{((\mathcal{C},\tilde{\mathbf{t}}),\boldsymbol{\lambda})/\Delta}(r,n,d) \longrightarrow \mathcal{RP}^r_{(C_0,\mathbf{t}_0),\mathbf{a}}.$$

By definition of $\left(\mathcal{M}^{\boldsymbol{\alpha}}_{((C,\mathbf{t}),\boldsymbol{\lambda})}(r,n,d)\right)^{sing}$, the point $\mathbf{y} = p_1 \circ \pi_a \circ \mathbf{RH}_{\boldsymbol{\lambda}}(\mathbf{x})$ is a singular point of $\mathcal{RP}^r_{(C_0,\mathbf{t}_0),\mathbf{a}}$ and let

$$\mathcal{K}_{\Delta,\mathbf{y}} = (\pi_a \circ \mathbf{RH}_{\boldsymbol{\lambda}})^{-1}(\{\mathbf{y}\} \times \Delta) \subset \left(\mathcal{M}^{\boldsymbol{\alpha}}_{((\mathcal{C},\tilde{\mathbf{t}}),\boldsymbol{\lambda})/\Delta}(r,n,d)\right)^{sing}$$

denote the exceptional locus dominated over $\{\mathbf{y}\} \times \Delta$. Then restricting (91) to $\mathcal{K}_{\Delta,\mathbf{y}}$, we have the following commutative diagram:

(92)
$$\begin{array}{ccc} \mathcal{K}_{\Delta,\mathbf{y}} & \xrightarrow{\pi_{\mathbf{a}} \circ \mathbf{RH}_\lambda} & \{\mathbf{y}\} \times \Delta \\ \widetilde{\varphi_{\Delta,\mathbf{y}}} \downarrow & & \downarrow p_2 \\ \Delta & \xrightarrow{Id} & \Delta. \end{array}$$

From Theorem 6.1, we see that $\pi_{\mathbf{a}} \circ \mathbf{RH}_{\boldsymbol{\lambda}}$ is a resolution of singularity of $\mathcal{RP}^r_{(C_0,\mathbf{t}_0),\mathbf{a}} \times \Delta$, hence each fiber of $\widetilde{\varphi_{\Delta,\mathbf{y}}} : \mathcal{K}_{\Delta,\mathbf{y}} \longrightarrow \Delta$ is compact. Now from Lemma 7.1, we see that $\tilde{h}_{\epsilon_1}(\Delta_\epsilon) \subset \mathcal{K}_{\Delta_{\epsilon_1},\mathbf{y}}$. Moreover since $\widetilde{\varphi_{\Delta,\mathbf{y}}}$ is proper, we see that $\tilde{h}_{\epsilon_1}(\overline{\Delta_{\epsilon_1}}) \subset \mathcal{K}_{\overline{\Delta_{\epsilon_1}},\mathbf{y}}$ where $\overline{\Delta_{\epsilon_1}} = \{t, |t| \leq \epsilon_1\}$. Take and fix $t = b$ such that $|b| = \epsilon_1$. Then

$$\tilde{h}_{\epsilon_1}(b) = \mathbf{y}_b \in \mathcal{K}_{\overline{\Delta_{\epsilon_1}},\mathbf{y}} \subset \mathcal{M}^{\boldsymbol{\alpha}}_{((\mathcal{C},\tilde{\mathbf{t}}),\boldsymbol{\lambda})/\overline{\Delta_{\epsilon_1}}}(r,n,d)$$

Starting from $t = b$ and $\mathbf{y}_b$, we can extend the section $\tilde{h}_{\epsilon_1}$ over $\Delta(b,\epsilon_b) = \{t \in \Delta \mid |t-b| < \epsilon_b\}$ with $0 < \epsilon_b \leq 1 - \epsilon_1$. Again, from the compactness of the fiber of $\widetilde{\varphi_{\Delta,\mathbf{y}}} : \mathcal{K}_{\Delta,\mathbf{y}} \longrightarrow \Delta$, we can show that the minimum ϵ_0 of ϵ_b for $|b| = \epsilon_1$ is positive, hence for $\epsilon = \epsilon_1 + \epsilon_0$ the section $\tilde{h}_\epsilon$ exists and this contradicts to the fact that ϵ_1 is the supremum and $\epsilon_1 < \epsilon$. Q.E.D.

Remark 7.1. Let us remark that the isomonodromic foliation $\mathcal{IF}_{\boldsymbol{\lambda}}$ on $\mathcal{M}^{\boldsymbol{\alpha}}_{((\mathcal{C},\tilde{\mathbf{t}}),\boldsymbol{\lambda})/\tilde{T}}(r,n,d)$ descends to a foliation on $\mathcal{M}^{\boldsymbol{\alpha}}_{((\mathcal{C},\tilde{\mathbf{t}}),\boldsymbol{\lambda})/T'}(r,n,d)$ under the covering map $\tilde{T} \longrightarrow T'$, which we also denote by $\mathcal{IF}_{\boldsymbol{\lambda}}$. Recall that the isomonodromic section (81) is the constant section with respect to the isomorphism (76). Moreover, when the base point $* \in T'$ corresponds to $(C_0, \mathbf{t}_0)$, the fundamental group $\pi_1(T', *)$ acts on the moduli space $\mathcal{RP}^r_{(C_0,\mathbf{t}_0)}$ via the action to the generators of $\pi_1(C_0 \setminus D(\mathbf{t}_0), *')$. Therefore, we can define the local isomonodromic sections for $\mathcal{RP}^r_{n,T',\mathbf{a}'} \longrightarrow T'$, which also defines a local isomonodromic sections for $\left(\mathcal{M}^{\boldsymbol{\alpha}}_{((\mathcal{C},\mathbf{t}),\boldsymbol{\lambda})/T'}\right)^{\sharp} \longrightarrow T'$. Now the set of local isomonodromic sections determines a splitting homomorphism $\mathcal{V}_{\boldsymbol{\lambda}}$ like (89), and it defines an isomonodromic foliation

$$\mathcal{IF}_{\boldsymbol{\lambda}} = \mathcal{V}_{\boldsymbol{\lambda}}(\Theta_{T'}) \subset \Theta_{\mathcal{M}^{\boldsymbol{\alpha}}_{((\mathcal{C},\mathbf{t}),\boldsymbol{\lambda})/T'}}$$

which is obviously the descent of the original isomonodromic foliation on $\mathcal{M}^{\boldsymbol{\alpha}}_{((\mathcal{C},\mathbf{t}),\boldsymbol{\lambda})/\tilde{T}'}$

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