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On the cusp form motives in genus 1 and level 1

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Abstract.

We prove that the moduli space of stable n-pointed curves of genus 1 and the projector associated to the alternating representation of the symmetric group on n letters define (for $n > 1$) the Chow motive corresponding to cusp forms of weight $n+1$ for $\mathrm{SL}(2,\mathbb{Z})$. This provides an alternative (in level 1) to the construction of Scholl.

§1. Introduction

In this paper we give an alternative construction of the Chow motives $S[k]$ corresponding to cusp forms of weight k for $\mathrm{SL}(2,\mathbb{Z})$. The Betti cohomology related to these cusp forms was initially studied by Eichler and Shimura, after which Deligne constructed the corresponding ℓ-adic Galois representations. Using the canonical desingularization of the fiber products of the compactified universal elliptic curve constructed by Deligne, Scholl then defined projectors such that the realizations of the associated Chow motives are these parabolic cohomology groups. The smooth projective varieties used in this construction are called Kuga-Sato varieties.

Instead of the Kuga-Sato varieties, we use the spaces $\overline{M}_{1,n}$, the Knudsen-Deligne-Mumford moduli spaces of stable n-pointed curves of genus 1. The symmetric group Σ_n acts naturally on $\overline{M}_{1,n}$, by permuting the n marked points. Let α denote its alternating character. Our main result is that $\overline{M}_{1,n}$ (for $n > 1$) and the projector Π_α corresponding to α define the Chow motive $S[n+1]$. In other words, we have the following result.

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Theorem. *For $n > 1$,*

$$\Pi_\alpha(H^*(\overline{M}_{1,n}, \mathbb{Q})) = \Pi_\alpha(H^n(\overline{M}_{1,n}, \mathbb{Q})) = H^1_!(M_{1,1}, \mathrm{Sym}^{n-1} R^1\pi_*\mathbb{Q}).$$

Here $\pi : \mathcal{E} \to M_{1,1}$ is the universal elliptic curve and $H^i_! = \mathrm{Im}(H^i_c \to H^i)$ denotes the parabolic cohomology.

The cohomology $H^*(\overline{M}_{g,n})$ of the moduli space of stable n-pointed curves of genus g has been studied intensively in recent years, in particular for $n > 0$ through the connection with Gromov-Witten theory. Since $\overline{M}_{g,n}$ is a smooth projective stack over $\mathbb{Z}$, these groups have arithmetic relevance as well. Getzler has initiated the study of the cohomology $H^*(M_{g,n})$ of the moduli space of smooth n-pointed curves of genus g as a representation of Σ_n. Through the theory of modular operads, as developed by Getzler and Kapranov, the Σ_n-equivariant Euler characteristics of the cohomology of the spaces $\overline{M}_{g,n}$ are expressed in the Σ_n-equivariant Euler characteristics of the cohomology of the spaces $M_{g,n}$. The action of Σ_n is crucial here. Another central idea of Getzler is to express the Σ_n-equivariant Euler characteristic of $H^*(M_{g,n})$ in terms of the Euler characteristics of the cohomology of irreducible symplectic local systems on M_g. Since these local systems are pulled back from the moduli space A_g of principally polarized abelian varieties of dimension g, this provides a connection with genus g Siegel modular forms.

In genus 1, this connection is given by Eichler-Shimura theory. In higher genus, despite the very important work of Faltings and Chai, much less is known. Van der Geer and the second author have obtained an explicit conjectural formula for the motivic Euler characteristics of these local systems in genus 2.

Our work is motivated by the desire to understand the motives underlying Siegel modular forms and the cohomology of the corresponding local systems. We expect that the results proved in this paper for genus 1, when suitably generalized, will provide a major step towards this goal.

Unbeknownst to us, Manin had suggested in [Ma1], 0.2 and 2.5, that it would be desirable to replace the Kuga-Sato varieties by moduli spaces of curves of genus 1 with marked points and a level structure. Cf. [Ma2], 3.6.2.

In section 2, we determine the alternating part of the Σ_n-equivariant Euler characteristic of $M_{1,n}$. Section 3 deals with the Σ_n-equivariant cohomology of $M_{0,n}$; some of the results obtained here may be of independent interest. The theory of modular operads and the results of section 3 are used in section 4 to determine the alternating part of the Euler characteristic of $\overline{M}_{1,n} \setminus M_{1,n}$. In section 5 we combine the results of sections 2 and 4 and prove our main theorem.

§2. The contribution of the interior

In this section we determine the contribution of $M_{1,n}$, i.e., we determine

$$\langle s_{1^n}, e_c^{\Sigma_n}(M_{1,n})\rangle,$$

the alternating part of the Σ_n-equivariant Euler characteristic of the compactly supported cohomology of $M_{1,n}$. Here, for a partition λ of n, the notation s_λ is used for the Schur function corresponding to the irreducible representation of Σ_n indexed by λ, and $\langle\, ,\, \rangle$ stands for the standard inner product on the ring of symmetric functions, for which the s_λ form an orthonormal basis. We will usually not make a notational distinction between a (possibly virtual) Σ_n-representation V and its characteristic $\mathrm{ch}_n(V)$, the symmetric function corresponding to it ([GK], 7.1). The Euler characteristic is taken in K_0 of a convenient category, such as the category of mixed Hodge structures or of ℓ-adic Galois representations.

Let $E = (E, 0)$ be an elliptic curve. We may think of the points of E^{n-1} as n-tuples

$$(0, x_2, \ldots, x_n)$$

(with $x_1 = 0$) and by doing so we find a natural action of Σ_n on E^{n-1} (combine the effect of a permutation σ with a translation of each coordinate over $-x_{\sigma^{-1}(1)}$). We are interested in the subspace of $H^\bullet(E^{n-1}) = H^\bullet(E)^{\otimes(n-1)}$ where the induced action of Σ_n is via the alternating representation.

Let $\Sigma_{n-1} \subset \Sigma_n$ be the subgroup permuting the last $n-1$ entries.

Lemma 1. *The subspace of $H^\bullet(E)^{\otimes(n-1)}$ where the induced action of Σ_{n-1} is via the alternating representation is isomorphic to*

$$\oplus_{k=0}^{n-1} \wedge^k H^{\mathrm{even}}(E) \otimes \mathrm{Sym}^{n-1-k} H^1(E).$$

Proof. The subspace of $V = H^\bullet(E)^{\otimes(n-1)}$ where Σ_{n-1} acts alternatingly is generated by sums

$$\sum_{\sigma\in\Sigma_{n-1}} (-1)^{\mathrm{sgn}(\sigma)}\sigma^*(v)$$

with $v \in V$. Clearly, we may restrict ourselves to pure tensors v such that the first k factors are in $H^{\mathrm{even}}(E)$ and the remaining $n-1-k$ factors are in $H^1(E)$, for some k. Fix k. It suffices now to consider the action of $\Sigma_k \times \Sigma_{n-1-k}$ on such v. This leads to the claimed isomorphism. Q.E.D.

Note that only the terms with $k \leq 2$ in the direct sum above are nonzero. Thus it is concentrated in degrees $n-2$, $n-1$, and n.

Proposition 1. *The subspace of $H^\bullet(E)^{\otimes(n-1)}$ where the induced action of Σ_n is via the alternating representation is* $\mathrm{Sym}^{n-1}H^1(E)$.

Proof. Let $\tau \in \Sigma_n$ be the transposition (12). We need to show that $\tau^*(\gamma) = -\gamma$ for all $\gamma \in \mathrm{Sym}^{n-1}H^1(E)$, but that none of the Σ_{n-1}-alternating vectors coming from elements of $H^{\mathrm{even}}(E) \otimes \mathrm{Sym}^{n-2}H^1(E)$ and $\wedge^2 H^{\mathrm{even}}(E) \otimes \mathrm{Sym}^{n-3}H^1(E)$ have this property.

As an example, consider the case $n = 2$. Note that $\tau(0, x_2) = (x_2, 0) = (0, -x_2)$. Thus $\tau = -1_E$ and the (-1)-eigenspace of τ^* on $H^\bullet(E)$ is $H^1(E)$.

In the general case,

$$\tau(0, x_2, x_3, \ldots, x_n) = (0, -x_2, x_3 - x_2, \ldots, x_n - x_2).$$

Denote by $\mathrm{pr}_i : E^{n-1} \to E$ the projection onto the ith factor (with $2 \le i \le n$) and by τ_i the composition $\mathrm{pr}_i \circ \tau$. Then

$$\tau^*(\gamma_2 \otimes \cdots \otimes \gamma_n) = \tau_2^*(\gamma_2) \cdot \ldots \cdot \tau_n^*(\gamma_n).$$

For $k \ge 3$ we have $\tau_k = m \circ ((-\mathrm{pr}_2) \times \mathrm{pr}_k)$, where $m : E \times E \to E$ denotes the group law. Observe now that

$$\tau_k^*(\zeta) = -\mathrm{pr}_2^*(\zeta) + \mathrm{pr}_k^*(\zeta)$$

for $\zeta \in H^1(E)$ and $k \ge 3$.

Denote by $p_i : E^{n-1} \to E^{n-2}$ the projection forgetting the ith factor $(2 \le i \le n)$. Let $\gamma \in \mathrm{Sym}^{n-2}H^1(E)$ and denote by

$$\Gamma = \sum_{i=2}^{n} (-1)^i p_i^* \gamma$$

the Σ_{n-1}-alternating vector corresponding to $1 \otimes \gamma$. Let I be the ideal $\mathrm{pr}_2^*(H^1(E) \oplus H^2(E))$. Note that $\Gamma \equiv p_2^*\gamma = 1 \otimes \gamma \mod I$. But $\tau^*(1 \otimes \gamma) \equiv 1 \otimes \gamma \mod I$ by the above. Thus $\tau^*\Gamma = -\Gamma$ implies $\gamma = 0$.

This shows that the Σ_{n-1}-alternating vectors corresponding to elements of $H^0(E) \otimes \mathrm{Sym}^{n-2}H^1(E)$ are not Σ_n-alternating. We conclude that the alternating representation of Σ_n does not occur in degree $n-2$. By duality, it does not occur in degree n either.

Denote by $p_{ij} : E^{n-1} \to E^{n-3}$ the projection forgetting the ith and jth factors $(2 \le i < j \le n)$. Let $\gamma \in \mathrm{Sym}^{n-3}H^1(E)$ and denote by

$$\Xi = \sum_{i=2}^{n-1} \sum_{j=i+1}^{n} (-1)^{i+j} p_{ij}^* \gamma \cdot (\mathrm{pr}_i^* p - \mathrm{pr}_j^* p)$$

the Σ_{n-1}-alternating vector corresponding to $(1 \wedge p) \otimes \gamma$ (here p is the class of a point). Then

$$(-1)^{n+1} p_{n*} \Xi = p_{n*}\Big(\sum_{i=2}^{n-1} (-1)^i p_{in}^* \gamma \cdot \mathrm{pr}_n^* p\Big) = \sum_{i=2}^{n-1} (-1)^i p_i^* \gamma,$$

the Σ_{n-2}-alternating vector in $H^\bullet(E)^{\otimes(n-2)}$ corresponding to $1 \otimes \gamma$. Using that $p_n \circ \tau = \tau \circ p_n$, one shows that $\tau^* \Xi = -\Xi$ implies $\gamma = 0$. Thus the alternating representation of Σ_n can occur only in $\mathrm{Sym}^{n-1} H^1(E)$.

To conclude, we show that these vectors are indeed Σ_n-alternating. Choose α and β in $H^1(E)$ with $\alpha \cdot \beta = p$. Fix k and l with sum $n-1$ and let $\gamma = \gamma_2 \otimes \cdots \otimes \gamma_n$ with k of the factors equal to α and the remaining l equal to β. If $\gamma_2 = \alpha$, then $\gamma + \tau^* \gamma$ is a sum of l terms; each term arises from γ by replacing γ_2 by p and one of the β's by 1. If $\gamma_2 = \beta$, then $\gamma + \tau^* \gamma$ is a sum of k terms; each term arises from γ by replacing γ_2 by $-p$ and one of the α's by 1. It is now easy to see that the symmetric tensor Γ that is the sum of all γ satisfies $\tau^* \Gamma = -\Gamma$. This finishes the proof. Q.E.D.

We may think of the fiber of $M_{1,n}$ over $[E]$ as the open subset D_n° of E^{n-1} where the n points $0, x_2, \ldots, x_n$ are mutually distinct, i.e., the complement of the $n-1$ zero sections $x_i = 0$ (with $2 \leq i \leq n$) and the diagonals $x_i = x_j$ (with $2 \leq i < j \leq n$). Clearly this open subset is Σ_n-invariant.

Lemma 2. *The subspace of $H_c^\bullet(D_n^\circ)$ where the induced action of Σ_n is via the alternating representation is canonically isomorphic to the corresponding subspace of $H^\bullet(E^{n-1})$, thus to* $\mathrm{Sym}^{n-1} H^1(E)$.

Proof. Write D_k for the closed subset of E^{n-1} where $\{0, x_2, \ldots, x_n\}$ has cardinality at most k and $D_k^\circ = D_k \setminus D_{k-1}$ for its open subset where $\{0, x_2, \ldots, x_n\}$ has cardinality k. The subsets D_k and D_k° are Σ_n-invariant. By induction on k we show that $H_c^\bullet(D_k)$ does not contain a copy of the alternating representation for $k \leq n-1$. Note that D_1 is a point. We may assume $n > 2$. We have exact sequences

$$H_c^{i-1}(D_{k-1}) \to H_c^i(D_k^\circ) \to H_c^i(D_k) \to H_c^i(D_{k-1})$$

of Σ_n-representations. By induction, the outer terms do not contain alternating representations. Consider D_k° for $k \leq n-1$. For every connected component, there exists a transposition in Σ_n acting on it as the identity. This shows that $H_c^\bullet(D_k^\circ)$ does not contain an alternating representation, and the same holds for $H_c^\bullet(D_k)$. The exact sequence above, with $k = n$, now gives the result. Q.E.D.

For a variety X with Σ_n-action, denote $\langle s_{1^n}, e_c^{\Sigma_n}(X)\rangle$ by $A_c(X)$. Clearly, we have

$$A_c(E^{n-1}) = A_c(D_n^\circ) = (-1)^{n-1}\mathrm{Sym}^{n-1}H^1(E).$$

Let $\pi : \mathcal{E} \to S$ be a relative elliptic curve. We may consider the Σ_n-action on the relative spaces $\mathcal{E}^{n-1}/S$ and $\mathcal{D}_n^\circ/S$ and obtain

$$A_c(\mathcal{E}^{n-1}/S) = A_c(\mathcal{D}_n^\circ/S) = (-1)^{n-1}\mathrm{Sym}^{n-1}R^1\pi_*\mathbb{Q}$$

and similarly with $\mathbb{Q}_\ell$-coefficients. The Leray spectral sequence gives then immediately

$$A_c(\mathcal{E}^{n-1}) = A_c(\mathcal{D}_n^\circ) = (-1)^{n-1}e_c(S, \mathrm{Sym}^{n-1}R^1\pi_*\mathbb{Q}).$$

Applying this to the universal elliptic curve, we obtain in particular

$$A_c(M_{1,n}) = (-1)^{n-1}e_c(M_{1,1}, \mathrm{Sym}^{n-1}R^1\pi_*\mathbb{Q}).$$

Let $n > 1$. Then $H_c^i(M_{1,1}, \mathrm{Sym}^{n-1}R^1\pi_*\mathbb{Q}) = 0$ when $i \neq 1$ or n even. For n odd,

$$H_c^1(M_{1,1}, \mathrm{Sym}^{n-1}R^1\pi_*\mathbb{Q}) = S[n+1] + 1,$$

cf. [Ge4], Thm. 5.3 and below. Here we have written 1 for the trivial Hodge structure $\mathbb{Q}$ (or the corresponding ℓ-adic Galois representation) and $S[n+1]$ for Getzler's S_{n+1}; this is an equality in the Grothendieck group of our category. We have proved the following result.

Theorem 1. *The alternating part of the Σ_n-equivariant Euler characteristic of the compactly supported cohomology of $M_{1,n}$ is given by the following formula:*

$$A_c(M_{1,n}) = \begin{cases} -S[n+1] - 1, & n > 1 \text{ odd;} \\ 0, & n \text{ even.} \end{cases}$$

Here $S[n+1] = H_!^1(M_{1,1}, \mathrm{Sym}^{n-1}R^1\pi_*\mathbb{Q})$, the parabolic cohomology of the local system $\mathrm{Sym}^{n-1}R^1\pi_*\mathbb{Q}$, is the part of the cohomology of $M_{1,n}$ corresponding to cusp forms of weight $n+1$.

Of course $A_c(M_{1,1}) = L$, the Hodge structure $\mathbb{Q}(-1)$. If we formally define $S[2] = -L - 1$, then the formula above holds for $n = 1$ as well.

§3. Cohomology of genus 0 moduli spaces and representations of symmetric groups

In this section we study the cohomology groups $H^i(M_{0,n})$ as representations of the symmetric group Σ_n. One of our main tools is the following. Let X be an algebraic variety, let $Y \subset X$ be a closed subvariety, and let $U = X \setminus Y$ denote the complement. Then the long exact sequence of compactly supported cohomology

$$\cdots \to H_c^k(U) \to H_c^k(X) \to H_c^k(Y) \to H_c^{k+1}(U) \to \dots$$

is a sequence of mixed Hodge structures. See [DK], p. 282.

Lemma 3. (Getzler) *The mixed Hodge structure on $H^i(M_{0,n})$ is pure of weight $2i$.*

Proof. This is Lemma 3.12 in [Ge1]. We wish to give a different proof here. The case $n = 3$ is trivial. For $n = 4$, we use the sequence above, with $X = \mathbf{P}^1$, $Y = \{0, 1, \infty\}$, and $U = M_{0,4}$. The sequence reads

$$0 \to H_c^0(M_{0,4}) \to H_c^0(\mathbf{P}^1) \to H_c^0(\{0,1,\infty\}) \to H_c^1(M_{0,4}) \to 0 \to 0 \to$$
$$\to H_c^2(M_{0,4}) \to H_c^2(\mathbf{P}^1) \to 0.$$

Note first that $H_c^0(M_{0,4}) = 0$. Clearly, $H_c^1(M_{0,4})$ has weight 0 and $H_c^2(M_{0,4})$ has weight 2. The statement follows by duality:

$$H_c^k(V)^\vee \cong H^{2m-k}(V)(m)$$

as mixed Hodge structures, for V a nonsingular irreducible variety of dimension m.

For $n > 4$ we have that $U = M_{0,n}$ is isomorphic to the complement in $X = M_{0,n-1} \times M_{0,4}$ of the disjoint union

$$Y = \coprod_{i=4}^{n-1} \{x_i = x_n\},$$

where we think of a k-pointed curve of genus 0 as given by a k-tuple $(0, 1, \infty, x_4, \dots, x_k)$ on $\mathbf{P}^1$. Thus,

$$H_c^{k-1}(Y) \to H_c^k(U) \to H_c^k(X)$$

is an exact sequence of mixed Hodge structures. By dualizing and applying a Tate twist, the same holds for

$$H^i(X) \to H^i(U) \to H^{i-1}(Y)(-1)$$

(with $i = 2(n-3) - k$). By the Künneth formula and induction on n, the terms on the left and right have pure Hodge structures of weight $2i$. Hence the same holds for the term in the middle. Q.E.D.

For $k \geq 0$, denote by Δ_k the closed part of $\overline{M}_{0,n}$ corresponding to stable curves with at least k nodes and denote by Δ_k° the open part $\Delta_k \setminus \Delta_{k+1}$ corresponding to stable curves with exactly k nodes. Put $d = n-3$. Clearly, $\Delta_k \neq \emptyset$ for $0 \leq k \leq d$. In general, Δ_k is singular, with nonsingular irreducible components, all of codimension k. But $\Delta_0 = \overline{M}_{0,n}$ and Δ_d (a collection of points) are nonsingular. All Δ_k° are nonsingular. Of course $\Delta_0^\circ = M_{0,n}$ and $\Delta_d^\circ = \Delta_d$. We have the long exact sequence

$$\cdots \to H_c^{a-1}(\Delta_{k+1}) \to H_c^a(\Delta_k^\circ) \to H_c^a(\Delta_k) \to H_c^a(\Delta_{k+1}) \to \dots$$

of mixed Hodge structures. Since the Δ_k are invariant for the natural action of Σ_n, it is also a sequence of Σ_n-representations.

Lemma 4. *The cohomology groups $H^i(M_{0,n})$ vanish for $i > n-3$. For $0 \leq i \leq n-3$, the irreducible representations of Σ_n occurring in $H^i(M_{0,n})$ have Young diagrams with at most $i+1$ rows. In particular, the irreducible representations of Σ_n occurring in $H^\bullet(M_{0,n})$ have Young diagrams with at most $n-2$ rows.*

Proof. The claimed vanishing is immediate. Let us abbreviate the rest of the statement by "$H^i(M_{0,n})$ has $\leq i+1$ rows". We prove it by induction on n. The case $n=3$ is trivial. Assume $n > 3$. Recall that $d = n-3$. We require an analysis of the boundary strata:

Claim. Assume $d - b > 0$. Then $H_c^a(\Delta_{d-b})$ has $\leq d+1+b-a$ rows.

We prove the claim by induction on b. We begin with the case $b = 0$. Since Δ_d is a collection of points, $a = 0$ may be assumed. Each point corresponds to a stable curve with d nodes, hence with $d+1$ components. Each component has exactly three special points (nodes or marked points). Let n_j be the number of marked points on the jth component, for some numbering of the components. By permuting the n marked points on the stable curve, we obtain a Σ_n-representation R, which is a direct summand of $H^0(\Delta_d)$. Note that R is a subrepresentation of the induced representation

$$\mathrm{Ind}_{\prod_{j=1}^{d+1} \Sigma_{n_j}}^{\Sigma_n} \mathbf{1}.$$

The induced representation has $\leq d+1$ rows, hence R does. Now $H^0(\Delta_d)$ is a direct sum of representations analogous to R, thus it has $\leq d+1$ rows as well. This proves the claim in the case $b = 0$.

Assume $b > 0$. Observe that $H_c^a(\Delta_k^\circ) \cong H^{2(d-k)-a}(\Delta_k^\circ)$ as Σ_n-representations. Also, each connected component of Δ_k° is for $k \geq 1$ a product of $k+1$ spaces M_{0,m_j}, with $m_j < n$. By induction on n and the Künneth formula, $H_c^a(\Delta_k^\circ)$ has $\leq 2(d-k)-a+k+1 = 2d-k-a+1$ rows, for $k \geq 1$. Putting $k = d-b$, we find that $H_c^a(\Delta_{d-b}^\circ)$ has $\leq d+1+b-a$ rows.

By induction on b, we have that $H_c^a(\Delta_{d-b+1})$ has $\leq d+b-a$ rows. From the long exact sequence, we find that $H_c^a(\Delta_{d-b})$ has $\leq d+1+b-a$ rows. This proves the claim.

In particular, $H_c^a(\Delta_1)$ has $\leq 2d-a$ rows. Consider the exact sequence

$$H_c^{k-1}(\Delta_1) \to H_c^k(M_{0,n}) \xrightarrow{\alpha} H_c^k(\overline{M}_{0,n}).$$

From Lemma 3 we know that $H_c^k(M_{0,n})$ has weight $2k-2d$. But $H_c^k(\overline{M}_{0,n})$ has weight k. Thus $\alpha = 0$ for $k < 2d$. Hence $H_c^k(M_{0,n})$ has $\leq 2d+1-k$ rows for $k < 2d$. Thus $H^i(M_{0,n})$ has $\leq i+1$ rows for $i > 0$. But it is obviously true for $i = 0$ as well. This finishes the proof. Q.E.D.

§4. The contribution of the boundary

In this section we determine the contribution of the boundary

$$\partial M_{1,n} = \overline{M}_{1,n} \setminus M_{1,n},$$

i.e., we determine

$$\langle s_{1^n}, e_c^{\Sigma_n}(\partial M_{1,n})\rangle.$$

We use the main result of [Ge2]. To state it, we introduce the following notations:

$$\mathbf{a}_g := \sum_{n>2-2g} \mathrm{ch}_n(e_c^{\Sigma_n}(M_{g,n})), \quad \text{and}$$
$$\mathbf{b}_g := \sum_{n>2-2g} \mathrm{ch}_n(e_c^{\Sigma_n}(\overline{M}_{g,n})).$$

Here ch_n denotes the characteristic of a finite-dimensional Σ_n-representation ([GK], 7.1) and its extension by linearity to virtual representations. For a (formal) symmetric function f (such as $\mathbf{a}_g$ and $\mathbf{b}_g$), we also write

$$f' = \frac{\partial f}{\partial p_1} = p_1^{\perp} f,$$
$$\dot{f} = \frac{\partial f}{\partial p_2} = \frac{1}{2} p_2^{\perp} f,$$
$$\psi_i(f) = p_i \circ f.$$

Here p_i is the symmetric function equal to the sum of the ith powers of the variables, $p_i^{\perp}$ is the adjoint of multiplication with p_i with respect to the standard inner product, and $\circ$ is the plethysm of symmetric functions ([GK], 7.2). We will denote the ith complete symmetric function by h_i and the ith elementary symmetric function by e_i.

We can now state Getzler's result (Theorem 2.5 in [Ge2]):

$$\mathbf{b}_1 = \left(\mathbf{a}_1 - \frac{1}{2} \sum_{n=1}^{\infty} \frac{\phi(n)}{n} \log(1 - \psi_n(\mathbf{a}_0'')) + \frac{\dot{\mathbf{a}}_0^2 + \dot{\mathbf{a}}_0 + \frac{1}{4}\psi_2(\mathbf{a}_0'')}{1 - \psi_2(\mathbf{a}_0'')} \right) \circ (h_1 + \mathbf{b}_0').$$

The numerator of the third term inside the big parentheses on the right-hand side has been corrected here; there is a minor computational mistake in the derivation of the theorem in line 4 on page 487, which affects the result (but not Corollary 2.8).

As Getzler remarks, the term $\mathbf{a}_1 \circ (h_1 + \mathbf{b}_0')$ corresponds to the sum over graphs obtained by attaching a forest whose vertices have genus 0 to a vertex of genus 1; in particular, $\mathbf{a}_1 \circ h_1 = \mathbf{a}_1$, the contribution of smooth curves, is part of this term, corresponding to graphs consisting of a single vertex of genus 1. The remainder of this term, corresponding to graphs where at least one vertex of genus 0 has been attached to a vertex of genus 1, is part of the contribution of the boundary. We show that the alternating representation does not occur here. For a symmetric function

$$f = \sum_{n=0}^{\infty} f_n,$$

we write

$$\mathrm{Alt}(f) = \sum_{n=0}^{\infty} \langle s_{1^n}, f_n \rangle t^n.$$

Lemma 5. *The alternating representation does not occur in the contribution of the part of the boundary of $M_{1,n}$ corresponding to graphs where at least one vertex of genus 0 has been attached to a vertex of genus 1. In terms of the notation introduced above:*

$$\mathrm{Alt}(\mathbf{a}_1 \circ (h_1 + \mathbf{b}_0')) = \mathrm{Alt}(\mathbf{a}_1).$$

Proof. We choose to give a somewhat geometric proof instead of a proof using mostly the language of symmetric functions.

Observe first that a boundary stratum corresponding to a graph with a genus 1 vertex is isomorphic to a product

$$M_{1,m} \times \prod_i M_{0,n_i};$$

i.e., it is not necessary to take the quotient by a finite group. (The corresponding graph has no automorphisms: there is a unique shortest path from each of the n legs to the vertex of genus 1, and every vertex and every edge lie on such a path.) By the Künneth formula, the cohomology of such a product is isomorphic to the tensor product of the cohomologies of the factors.

Consider the Σ_n-orbit of such a stratum. The direct sum of the cohomologies of the strata in the orbit forms a Σ_n-representation V. It is induced from the cohomology of a single stratum, considered as a representation W of the stabilizer G in Σ_n of the stratum. By Frobenius Reciprocity, V contains a copy of the alternating representation if and only if W contains a copy of the restriction of the alternating representation to G.

To each vertex of the graph, one associates the symmetric group corresponding to the legs attached to the vertex. The product over the vertices of these symmetric groups is a subgroup H of G and the further restriction of the alternating representation to H is the tensor product over the vertices of the alternating representations of these symmetric groups.

Consider a moduli space $M_{0,k}$ corresponding to an extremal vertex of the graph corresponding to a boundary stratum as above. The symmetric group associated to this vertex is a standard subgroup $\Sigma_{k-1} \subset \Sigma_k$, permuting the $k-1$ legs attached to the vertex and leaving the unique half-edge fixed. By Lemma 4, the irreducible Σ_k-representations occurring in $H^\bullet(M_{0,k})$ have Young diagrams with at most $k-2$ rows. The Young diagrams of the irreducible representations occurring in the restriction to Σ_{k-1} also have at most $k-2$ rows, as they are obtained by removing one box. Therefore the alternating representation does not occur here. It follows that V does not contain a copy of the alternating representation either. Q.E.D.

We return to Getzler's result. We need to evaluate

$$\mathrm{Alt}\Bigg(\Bigg(-\frac{1}{2}\sum_{n=1}^{\infty}\frac{\phi(n)}{n}\log(1-\psi_n(\mathbf{a}_0''))+\frac{\dot{\mathbf{a}}_0^2+\dot{\mathbf{a}}_0+\frac{1}{4}\psi_2(\mathbf{a}_0'')}{1-\psi_2(\mathbf{a}_0'')}\Bigg)\circ(h_1+\mathbf{b}_0')\Bigg).$$

Getzler remarks that the two terms inside the big inner parentheses may be thought of as a sum over necklaces (graphs consisting of a single circuit) and a correction term, taking into account the fact that necklaces of 1 or 2 vertices have non-trivial involutions (while those with more vertices do not). The plethysm with $h_1 + \mathbf{b}_0'$ stands again for attaching a forest whose vertices have genus 0. We begin with the analogue of Lemma 5.

Lemma 6. *The alternating representation does not occur in the contribution of the part of the boundary of $M_{1,n}$ corresponding to graphs where at least one vertex of genus* 0 *has been attached to a necklace. In terms of the notation introduced above:*

$$\mathrm{Alt}\Bigg(\bigg(-\frac{1}{2}\sum_{n=1}^{\infty}\frac{\phi(n)}{n}\log(1-\psi_n(\mathbf{a}_0'')) + \frac{\dot{\mathbf{a}}_0^2 + \dot{\mathbf{a}}_0 + \frac{1}{4}\psi_2(\mathbf{a}_0'')}{1-\psi_2(\mathbf{a}_0'')}\bigg)\circ(h_1+\mathbf{b}_0')\Bigg)$$
$$= \mathrm{Alt}\bigg(-\frac{1}{2}\sum_{n=1}^{\infty}\frac{\phi(n)}{n}\log(1-\psi_n(\mathbf{a}_0'')) + \frac{\dot{\mathbf{a}}_0^2 + \dot{\mathbf{a}}_0 + \frac{1}{4}\psi_2(\mathbf{a}_0'')}{1-\psi_2(\mathbf{a}_0'')}\bigg).$$

Proof. In this case, each boundary stratum is isomorphic to a product

$$\Big(\prod_{v\in\text{necklace}} M_{0,n(v)}\Big)/I \times \prod_{v\notin\text{necklace}} M_{0,n(v)}.$$

The finite group I is trivial when the necklace has at least 3 vertices. It has 2 elements when the necklace has 1 resp. 2 vertices and acts by reversing the edge in the necklace resp. by interchanging the two edges of the necklace. In particular, I acts trivially on the moduli spaces corresponding to the vertices of the forest.

Just as in the proof of Lemma 5, the alternating representation does not occur in the cohomology of a moduli space $M_{0,k}$ corresponding to an extremal vertex of one of the trees of the forest. It follows that the alternating representation does not occur in the cohomology of a Σ_n-orbit of boundary strata as soon as the forest is nonempty. Q.E.D.

In order to determine the contribution of the part of the boundary of $M_{1,n}$ corresponding to necklaces without attached trees, we need several lemmas.

Lemma 7. *The restriction of the Σ_n-representation $H^\bullet(M_{0,n})$ to the standard subgroup Σ_{n-2} contains the alternating representation exactly once. In terms of the notation introduced above:*

$$\mathrm{Alt}(\mathbf{a}_0'') = \frac{t}{1+t}.$$

Proof. The Young diagrams corresponding to the irreducible representations of Σ_{n-2} occurring in $\mathbf{a}_0''$ are obtained by removing 2 boxes from a Young diagram occurring in $\mathbf{a}_0$. To obtain a copy of the alternating representation of Σ_{n-2}, one needs to start with a Young diagram with at least $n-2$ rows. From Lemma 4, only the top cohomology $H^{n-3}(M_{0,n})$ can contribute. Observe that $H^{n-3}(M_{0,n}) \cong H_c^{n-3}(M_{0,n}) \cong H_{n-3}(M_{0,n})$ as Σ_n-representations. Thus

$$\begin{aligned} \mathrm{Alt}(\mathbf{a}_0'') &= \mathrm{Alt}\left(\frac{\partial^2}{\partial p_1^2}\sum_{n=3}^{\infty}\mathrm{ch}_n\left(e_c^{\Sigma_n}(M_{0,n})\right)\right) \\ &= \mathrm{Alt}\left(\frac{\partial^2}{\partial p_1^2}\sum_{n=3}^{\infty}(-1)^{n-3}\mathrm{ch}_n\left(H_c^{n-3}(M_{0,n})\right)\right). \end{aligned}$$

Getzler shows in [Ge1], p. 213, l. 3 that

$$H_c^{n-3}(M_{0,n}) \cong \mathrm{sgn}_n \otimes \mathcal{L}ie(\!(n)\!).$$

Here sgn_n denotes the alternating representation and $\mathcal{L}ie(\!(n)\!)$ the Σ_n-representation that is part of the cyclic Lie operad. Getzler and Kapranov show in [GK], Example 7.24 that

$$\mathrm{Ch}(\mathcal{L}ie) := \sum_{n=3}^{\infty}\mathrm{ch}_n(\mathcal{L}ie(\!(n)\!)) = (1-p_1)\sum_{n=1}^{\infty}\frac{\mu(n)}{n}\log(1-p_n) + h_1 - h_2,$$

where $\mu(n)$ is the Möbius function. Hence

$$\sum_{n=3}^{\infty}(-1)^{n-3}\mathrm{ch}_n(\mathrm{sgn}_n\otimes\mathcal{L}ie(\!(n)\!)) = -(1+p_1)\sum_{n=1}^{\infty}\frac{\mu(n)}{n}\log(1+p_n)+h_1+e_2$$

and

$$\begin{aligned} &\frac{\partial^2}{\partial p_1^2}\left(\sum_{n=3}^{\infty}(-1)^{n-3}\mathrm{ch}_n(\mathrm{sgn}_n\otimes\mathcal{L}ie(\!(n)\!))\right) \\ &\quad = \frac{\partial}{\partial p_1}\left(-\sum_{n=1}^{\infty}\frac{\mu(n)}{n}\log(1+p_n) - (1+p_1)\frac{1}{1+p_1}\right) + 1 \\ &\quad = 1 - \frac{1}{1+p_1} = \frac{p_1}{1+p_1}. \end{aligned}$$

But

$$\mathrm{Alt}\left(\frac{p_1}{1+p_1}\right) = \frac{t}{1+t},$$

since $\langle p_1^n, s_{1^n}\rangle = 1$. Q.E.D.

Lemma 8. *Let f_n be a symmetric function of degree n. Assume that $\langle s_{1^n}, f_n\rangle = 0$. Then $\langle s_{1^{nk}}, p_k \circ f_n\rangle = 0$.*

Proof. Write $e(\lambda)$ resp. $o(\lambda)$ for the number of even resp. odd parts of a partition λ. For λ a partition of n,

$$o(\lambda) \equiv n \pmod 2 \qquad \text{and} \qquad \langle s_{1^n}, p_\lambda\rangle = (-1)^{e(\lambda)}.$$

Here $p_\lambda = \prod_i p_{\lambda_i}$ is the symmetric function of degree n that is the product of the power sums corresponding to the parts of λ. If $f_n = \sum_\lambda a_\lambda p_\lambda$, then $p_k \circ f_n = \sum_\lambda a_\lambda p_{k\lambda}$, where $k\lambda$ is the partition of kn obtained from λ by multiplying all parts with k. For k odd,

$$\sum_\lambda a_\lambda (-1)^{e(\lambda)} = \sum_\lambda a_\lambda (-1)^{e(k\lambda)},$$

whereas for k even,

$$e(k\lambda) = o(\lambda) + e(\lambda) \equiv n + e(\lambda) \pmod 2,$$

so that

$$\sum_\lambda a_\lambda (-1)^{e(k\lambda)} = (-1)^n \sum_\lambda a_\lambda (-1)^{e(\lambda)}.$$

The result follows. Q.E.D.

Lemma 9. *The occurrence of the alternating representation observed in Lemma 7 is stable under plethysm with p_k. In terms of the notation introduced above:*

$$\mathrm{Alt}(\psi_k(\mathbf{a}_0'')) = \frac{-(-t)^k}{1-(-t)^k}\,.$$

Proof. From Lemma 7,

$$\mathrm{Alt}(\mathbf{a}_0'') = \mathrm{Alt}\left(\frac{p_1}{1+p_1}\right).$$

Applying Lemma 8 and using that $\langle s_{1^{kn}}, p_k^n\rangle = (-1)^{(k-1)n}$, we find

$$\mathrm{Alt}(\psi_k(\mathbf{a}_0'')) = \mathrm{Alt}\left(p_k \circ \left(\frac{p_1}{1+p_1}\right)\right) = \mathrm{Alt}\left(\frac{p_k}{1+p_k}\right)$$

$$= \mathrm{Alt}\left(\sum_{n=1}^{\infty} (-1)^{n-1} p_k^n\right) = \sum_{n=1}^{\infty} (-1)^{n-1} (-1)^{(k-1)n} t^{kn}$$

$$= -\sum_{n=1}^{\infty}((-t)^k)^n = \frac{-(-t)^k}{1-(-t)^k}.$$

Q.E.D.

For two symmetric functions f and g, we have

$$\mathrm{Alt}(fg) = \mathrm{Alt}(f)\mathrm{Alt}(g).$$

This follows immediately from the Littlewood-Richardson rule (cf. [FH], p. 456). One may use this identity to shorten the proof of Lemma 9. Similarly,

$$\mathrm{Alt}(\log(1-\psi_k(\mathbf{a}_0''))) = \log\left(1 - \frac{-(-t)^k}{1-(-t)^k}\right) = -\log(1-(-t)^k)$$

and

$$\mathrm{Alt}\left(-\sum_{n=1}^{\infty}\frac{\phi(n)}{n}\log(1-\psi_n(\mathbf{a}_0''))\right) = \sum_{n=1}^{\infty}\frac{\phi(n)}{n}\log(1-(-t)^n)$$

$$= -\sum_{n=1}^{\infty}\frac{\phi(n)}{n}\sum_{k=1}^{\infty}\frac{(-t)^{nk}}{k} = -\sum_{m=1}^{\infty}\frac{(-t)^m}{m}\sum_{d|m}\phi(d)$$

$$= -\sum_{m=1}^{\infty}(-t)^m = \frac{t}{1+t}.$$

It remains to evaluate the contribution of the correction term,

$$\mathrm{Alt}\left(\frac{\dot{\mathbf{a}}_0^2 + \dot{\mathbf{a}}_0 + \frac{1}{4}\psi_2(\mathbf{a}_0'')}{1-\psi_2(\mathbf{a}_0'')}\right).$$

We need one more lemma.

Lemma 10. *The alternating part of the formal symmetric function*

$$\dot{\mathbf{a}}_0 = \frac{\partial \mathbf{a}_0}{\partial p_2}$$

is given by the following formula:

$$\mathrm{Alt}(\dot{\mathbf{a}}_0) = \frac{1}{2}\frac{t}{1-t}.$$

Proof. In terms of Young diagrams, multiplication by s_2 is the operation of adding two boxes, not in the same column, and multiplication by s_{1^2} is the operation of adding two boxes, not in the same row. Now $p_2 = s_2 - s_{1^2}$ and $\frac{\partial}{\partial p_2} = \frac{1}{2}p_2^{\perp}$, where $p_2^{\perp}$ is the adjoint of multiplication by p_2. Thus, to obtain a copy of the alternating representation of Σ_{n-2} in a term of $\dot{\mathbf{a}}_0$, one needs to start with a Young diagram with at least $n-2$ rows, just as in the proof of Lemma 7. So

$$\mathrm{Alt}(\dot{\mathbf{a}}_0) = \mathrm{Alt}\bigg(\frac{\partial}{\partial p_2}\sum_{n=3}^{\infty}(-1)^{n-3}\mathrm{ch}_n\big(H_c^{n-3}(M_{0,n})\big)\bigg).$$

We now find

$$\frac{\partial}{\partial p_2}\bigg(\sum_{n=3}^{\infty}(-1)^{n-3}\mathrm{ch}_n(\mathrm{sgn}_n \otimes \mathcal{L}ie((n)))\bigg) = \frac{1}{2}\frac{1+p_1}{1+p_2} - \frac{1}{2} = \frac{1}{2}\frac{p_1-p_2}{1+p_2}.$$

But

$$\mathrm{Alt}\bigg(\frac{1}{2}\frac{p_1-p_2}{1+p_2}\bigg) = \frac{1}{2}\frac{t+t^2}{1-t^2} = \frac{1}{2}\frac{t}{1-t}.$$

Q.E.D.

An easy calculation combining Lemmas 9 and 10 gives

$$\mathrm{Alt}\bigg(\frac{\dot{\mathbf{a}}_0^2 + \dot{\mathbf{a}}_0 + \frac{1}{4}\psi_2(\mathbf{a}_0'')}{1-\psi_2(\mathbf{a}_0'')}\bigg) = \frac{1}{2}\frac{t}{1-t}.$$

The contribution from the necklaces becomes then

$$\frac{1}{2}\frac{t}{1+t} + \frac{1}{2}\frac{t}{1-t} = \frac{t}{1-t^2},$$

i.e., 1 for n odd and 0 for n even. We have proved the following result.

Theorem 2. *The alternating part of the Σ_n-equivariant Euler characteristic of the cohomology of $\partial M_{1,n}$ is given by the following formula:*

$$\langle s_{1^n}, e_c^{\Sigma_n}(\partial M_{1,n})\rangle = \begin{cases} 1, & n \text{ odd;} \\ 0, & n \text{ even.} \end{cases}$$

§5. The construction of the motive

The main result of Section 4 (Theorem 2) is

$$A_c(\partial M_{1,n}) = \begin{cases} 1, & n \text{ odd;} \\ 0, & n \text{ even.} \end{cases}$$

Combining this with Theorem 1, we immediately obtain

$$A_c(\overline{M}_{1,n}) = \begin{cases} -S[n+1], & n \text{ odd}; \\ 0, & n \text{ even}. \end{cases}$$

Let $n > 1$ be an odd integer. The pair consisting of $\overline{M}_{1,n}$ and the projector

$$\frac{1}{n!} \sum_{\sigma \in \Sigma_n} (-1)^{\mathrm{sgn}(\sigma)} \sigma_*$$

defines a Chow motive, since $\overline{M}_{1,n}$ is the quotient of a smooth projective variety by a finite group [BP]. We wish to show that it is pure of degree n. This will conclude our construction of the motive $S[n+1]$, an alternative (in level 1 only) to Scholl's construction. The arguments below are similar to those in [Sc], 1.3.4.

First, $\langle s_{1^n}, H_c^i(M_{1,n})\rangle = 0$ for $i \neq n$. As in [De], proof of 5.3, the degeneration of the Leray spectral sequence at E_2 due to Lieberman's trick implies that $\langle s_{1^n}, H_c^i(\mathcal{E}^{n-1})\rangle = 0$ when $i \neq n$ for a relative elliptic curve $\mathcal{E} \to S$ and this implies the statement.

Thus we have an exact sequence

$$0 \to H_c^{n-1}(\overline{M}_{1,n})(\alpha) \to H_c^{n-1}(\partial M_{1,n})(\alpha) \to H_c^n(M_{1,n})(\alpha) \to$$

$$\to H_c^n(\overline{M}_{1,n})(\alpha) \to H_c^n(\partial M_{1,n})(\alpha) \to 0$$

and isomorphisms $H^i(\overline{M}_{1,n})(\alpha) \cong H^i(\partial M_{1,n})(\alpha)$ for $i \notin \{n-1, n\}$. Here $V(\alpha)$ denotes the alternating part of a Σ_n-representation V. Therefore $H^i(\partial M_{1,n})(\alpha)$ is pure of weight i for $i > n$. But then all these spaces vanish, since $H^i(\partial M_{1,n})$ has weight $\leq i$ for all i and since $A_c(\partial M_{1,n})$ has weight 0. Hence $H^i(\overline{M}_{1,n})(\alpha) = 0$ for $i > n$ and then by duality for $i < n$ as well. This shows that $H^n(\overline{M}_{1,n})(\alpha) = S[n+1]$ and concludes the alternative construction of these motives.

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