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Vector bundles on curves and theta functions

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Abstract.

This is a survey lecture on the "theta map" from the moduli space of SL_r-bundles on a curve C to the projective space of r-th order theta functions on JC. Some recent results and a few open problems about that map are discussed.

Introduction

These notes survey the relation between the moduli spaces of vector bundles on a curve C and the spaces of (classical) theta functions on the Jacobian J of C. The connection appears when one tries to describe the moduli space $\mathcal{M}_r$ of rank r vector bundles with trivial determinant as a projective variety in an explicit way (as opposed to the somewhat non-constructive way provided by GIT). The Picard group of the moduli space is infinite cyclic, generated by the *determinant line bundle* $\mathcal{L}$; thus the natural maps from $\mathcal{M}_r$ to projective spaces are those defined by the linear systems $|\mathcal{L}^k|$, and in the first instance the map $\varphi_{\mathcal{L}} : \mathcal{M}_r \dashrightarrow |\mathcal{L}|^*$. The key point is that this map can be identified with the *theta map*

$$\theta : \mathcal{M}_r \dashrightarrow |r\Theta|$$

which associates to a general bundle $E \in \mathcal{M}_r$ its *theta divisor* Θ_E, an element of the linear system $|r\Theta|$ on J – we will recall the precise definitions below. This description turns out to be sufficiently manageable to get some information on the behaviour of this map, at least when r or g are small.

We will describe the results which have been obtained so far – most of them fairly recently. Thus these notes can be viewed as a sequel to [B2], though with a more precise focus on the theta map. For the convenience of the reader we have made this paper independent of [B2], by recalling in §1 the necessary definitions. Then we discuss the indeterminacy locus of θ (§2), the case $r = 2$ (§3), the case $g = 2$ (§4), and the

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higher rank case (§5). Finally, as in [B2] we will propose a small list of questions and conjectures related to the topic (§6).

§1. The moduli space $\mathcal{M}_r$ and the theta map

(1.1) Throughout this paper C will be a complex curve of genus $g \geq 2$. We denote by J its Jacobian variety, and by J^k the variety (isomorphic to $J = J^0$) parametrizing line bundles of degree k on C.

For $r \geq 2$, we denote by $\mathcal{M}_r$ the moduli space of semi-stable vector bundles of rank r and trivial determinant on C. It is a normal, projective, unirational variety, of dimension $(r^2-1)(g-1)$. The points of $\mathcal{M}_r$ correspond to isomorphism classes of vector bundles with trivial determinant which are direct sums of stable vector bundles of degree zero. The singular locus consists precisely of those bundles which are decomposable (with the exception of $\mathcal{M}_2$ in genus 2, which is smooth). The corresponding singularities are rational Gorenstein – that is, reasonably mild.

(1.2) The Picard group of $\mathcal{M}_r$ has been thoroughly studied in [D-N]; let us recall the main results. Fix some $L \in J^{g-1}$, and consider the reduced subvariety

$$\Delta_L := \{E \in \mathcal{M}_r \mid H^0(C, E \otimes L) \neq 0\} .$$

Then Δ_L is a Cartier divisor in $\mathcal{M}_r$; the line bundle $\mathcal{L} := \mathcal{O}_{\mathcal{M}_r}(\Delta_L)$, called the determinant bundle, is independent of the choice of L and generates $\operatorname{Pic}(\mathcal{M}_r)$. The canonical bundle of $\mathcal{M}_r$ is $\mathcal{L}^{-2r}$.

(1.3) To study the rational map $\varphi_{\mathcal{L}} : \mathcal{M}_r \dashrightarrow |\mathcal{L}|^*$ associated to the determinant line bundle, the following construction is crucial. For a vector bundle $E \in \mathcal{M}_r$, consider the locus

$$\Theta_E := \{L \in J^{g-1} \mid H^0(C, E \otimes L) \neq 0\} .$$

Since $\chi(E \otimes L) = 0$ for L in J^{g-1}, it is readily seen that Θ_E is in a natural way a divisor in J^{g-1} – unless it is equal to J^{g-1}. The latter case (which may occur only for special bundles) is a serious source of trouble – see §2 below. In the former case we say that E admits a theta divisor; this divisor belongs to the linear system $|r\Theta|$, where Θ is the canonical Theta divisor in J^{g-1}. In this way we get a rational map

$$\theta : \mathcal{M}_r \dashrightarrow |r\Theta| .$$

Proposition 1.4. [BNR] *There is a canonical isomorphism* $|\mathcal{L}|^* \xrightarrow{\sim} |r\Theta|$ *which identifies* $\varphi_{\mathcal{L}}$ *to* θ.

As a consequence, the base locus of $|\mathcal{L}|$ is the locus of bundles E in $\mathcal{M}_r$ such that $H^0(C, E \otimes L) \neq 0$ for all $L \in J^{g-1}$. This is also the indeterminacy locus of θ (because $|\mathcal{L}|$ cannot have a fixed component).

(1.5) The r-torsion subgroup $J[r]$ of J acts on $\mathcal{M}_r$ by tensor product; it also acts on $|r\Theta|$ by translation, and the map θ is equivariant with respect to these actions. In particular, the image of θ is $J[r]$-invariant.

(1.6) The case when θ is a morphism is much easier to analyze: we know then that it is finite (since $|\mathcal{L}|$ is ample, θ cannot contract any curve), we know its degree by the Verlinde formula, etc. Unfortunately there are few cases where this is known to happen:

Proposition 1.6. *The base locus of $|\mathcal{L}|$ is empty in the following cases:*
a) $r = 2$;
b) $r = 3$, $g = 2$ *or* 3 ;
c) $r = 3$, C *is generic.*

All these results except the case $r = g = 3$ are due to Raynaud [R]. While a) and the first part of b) are easy, c) and the second part of b) are much more involved. We will discuss the latter in §5 below. The proof of c) is reduced, through a degeneration argument, to an analogous statement for torsion-free sheaves on a rational curve with g nodes.

§2. Base locus

(2.1) Recall that the *slope* of a vector bundle E of rank r and degree d is the rational number $\mu = d/r$. It is convenient to extend the definition of the theta divisor to vector bundles E with *integral* slope μ, by putting $\Theta_E := \{L \in J^{g-1-\mu} \mid H^0(C, E \otimes L) \neq 0\}$. If δ is a line bundle such that $\delta^{\otimes r} \cong \det E$, the vector bundle $E_0 := E \otimes \delta^{-1}$ has trivial determinant and $\Theta_{E_0} \subset J^{g-1}$ is the translate by δ of $\Theta_E \subset J^{g-1-\mu}$.

(2.2) We have the following relations between stability and existence of the theta divisor:
(2.2 a) If E admits a theta divisor, it is semi-stable;
(2.2 b) If moreover Θ_E is a prime divisor, E is stable.
Indeed let F be a proper subbundle of E. If $\mu(F) > \mu(E)$, the Riemann-Roch theorem implies $H^0(C, F \otimes L) \neq 0$, and therefore $H^0(C, E \otimes L) \neq 0$, for all L in $J^{g-1-\mu}$. If $\mu(F) = \mu(E)$, one has $\Theta_E = \Theta_F + \Theta_{E/F}$, so that Θ_E is not prime.

(2.3) The converse of these assertions do not hold. We will see in (2.6) examples of stable bundles with a reducible theta divisor. The first examples of stable bundles with no theta divisor are due to Raynaud [R].

They are restrictions of projectively flat vector bundles on J. Choose a theta divisor Θ on J. The line bundle $\mathcal{O}_J(n\Theta)$ is invariant under the n-torsion subgroup $J[n]$ of J. The action of $J[n]$ does *not* lift to $\mathcal{O}_J(n\Theta)$, but it does lift to the vector bundle $H^0(J, \mathcal{O}_J(n\Theta))^* \otimes_{\mathbb{C}} \mathcal{O}_J(n\Theta)$. Thus this vector bundle is the pull back under the multiplication $n_J : J \to J$ of a vector bundle E_n on J. Restricting E_n to the curve C embedded in J by an Abel-Jacobi mapping gives the Raynaud bundle R_n. It is well defined up to a twist by an element of J, has rank n^g and slope $\frac{g}{n}$. It has the property that $H^0(C, R_n \otimes \alpha) \neq 0$ for all $\alpha \in J$. Thus if $n \mid g$ R_n has integral slope and no theta divisor. More generally, Schneider has shown that a general vector bundle on C of rank n^g, slope $g-1$ and containing R_n is still stable [S2]. This gives a very large dimension for the base locus of $|\mathcal{L}|$, approximately $(1-\frac{1}{n}) \dim \mathcal{M}_r$ if $r = n^g$. Some related results are discussed in [A].

(2.4) Another series of examples have been constructed by Popa [P]. Let L be a line bundle on C spanned by its global sections. The *evaluation bundle* E_L is defined by the exact sequence

$$0 \to E_L^* \longrightarrow H^0(L) \otimes_{\mathbb{C}} \mathcal{O}_C \xrightarrow{ev} L \to 0 \;;$$

it has the same degree as L and rank $h^0(L)-1$. In particular, if we choose $\deg L = g + r$ with $r \geq g+2$, E_L has rank r and slope $\mu = 1 + \frac{g}{r}$. Then, *for all p such that $2 \leq p \leq r-2$ and $p\mu \in \mathbb{Z}$, the vector bundle $\mathsf{\Lambda}^p E_L$ does not admit a theta divisor* (see [S1]). For instance, when $r = 2g$, $\mathsf{\Lambda}^2 E_L$ gives a base point of $|\mathcal{L}|$ in $\mathcal{M}_{g(2g-1)}$.

(2.5) An interesting limit case of this construction is when $\mu = 2$; this occurs when $L = K_C$, or $r = g$. The first case has been studied in [FMP]. It turns out that the vector bundle $\mathsf{\Lambda}^p E_K$ has a theta divisor, equal to $C_{g-p-1} - C_p$ (here C_k denotes the locus of effective divisor classes in J^k). While the proof is elementary for $p = 1$, it is extremely involved for the higher exterior powers: it requires going to the moduli space of curves and computing various divisor classes in the Picard group of this moduli space. It remains a challenge to find a direct proof.

(2.6) The case $\deg L = 2g$ is treated in [B4], building on the results of [FMP]. Here again $\mathsf{\Lambda}^p E_L$ admits a theta divisor, at least if L is general enough; it has two components, namely $C_{g-p-1} - C_p$ and the translate of $C_{g-p} - C_{p-1}$ by the class $[K \otimes L^{-1}]$. These are the first examples defined on a general curve of stable bundles with a reducible theta divisor.

(2.7) Since $|\mathcal{L}|$ has usually a large base locus, it is natural to look at the systems $|\mathcal{L}^k|$ to improve the situation. There has been much progress on this question in the recent years:

Proposition 2.7. (i) [P-R] *$|\mathcal{L}^k|$ is base point free on $\mathcal{M}_r$ for $k \geq [\frac{r^2}{4}]$.*

(ii) [E-P] *For $k \geq r^2 + r$, the linear system $|\mathcal{L}^k|$ defines an injective morphism of $\mathcal{M}_r$ into $|\mathcal{L}^k|^*$, which is an embedding on the stable locus.*

On the other hand Popa [P] has observed that one should not be too optimistic, at least if one believes in the *strange duality* conjecture (see [B2]): this conjecture implies that for $n \mid g$ the Raynaud bundle R_n, twisted by an appropriate line bundle, is a base point of $|\mathcal{L}^k|$ when $k \leq n(1 - \frac{n}{g})$.

§3. Rank 2

(3.1) In rank 2 the situation is now well understood. As pointed out in (1.6), $\theta : \mathcal{M}_2 \longrightarrow |2\Theta|$ is a finite morphism. In genus 2, θ is actually an *isomorphism* onto $\mathbb{P}^3$ [N-R1]. If C is hyperelliptic of genus $g \geq 3$, it follows from [D-R] and [B1] that θ factors through the involution ι^* induced by the hyperelliptic involution and embeds $\mathcal{M}_2/\langle \iota^* \rangle$ into $|2\Theta|$; moreover the image admits an explicit geometric description [D-R].

(3.2) In the non-hyperelliptic case, after much effort we have now a complete answer, which is certainly one of the highlights of the subject:

Theorem 3.2. *If C is not hyperelliptic, $\theta : \mathcal{M}_2 \hookrightarrow |2\Theta|$ is an embedding.*

The fact that θ embeds the stable locus of $\mathcal{M}_2$ is proved in [B-V1], and the remaining part in [vG-I]. Both parts are highly nontrivial, and involve some beautiful geometric constructions.

(3.3) Thus we can identify $\mathcal{M}_2$ with a subvariety of $|2\Theta| \cong \mathbb{P}^{2^g-1}$, canonically associated to C, of dimension $3g - 3$ (1.1). This variety is invariant under the natural action of $J[2]$ on $|2\Theta|$ (1.5). Its degree can be computed from the Verlinde formula (see e.g. [Z], Thm. 1(*iii*)):

$$\deg \mathcal{M}_2 = (3g-3)!\, 2^g (2\pi)^{2-2g} \zeta(2g-2) \ ,$$

which gives $\deg \mathcal{M}_2 = 1$ for $g = 2$, 4 for $g = 3$, 96 for $g = 4$, etc.

The singular locus $\operatorname{Sing} \mathcal{M}_2$ is the locus of decomposable bundles in $\mathcal{M}_2$ (1.1), which are of the form $\alpha \oplus \alpha^{-1}$, for $\alpha \in J$; the map $\alpha \mapsto \alpha \oplus \alpha^{-1}$ identifies $\operatorname{Sing} \mathcal{M}_2$ to the *Kummer variety* $\mathcal{K}$ of J – that is, the quotient of J by the involution $\alpha \mapsto \alpha^{-1}$. The restriction of θ to $\mathcal{K} = \operatorname{Sing} \mathcal{M}_2$ is the classical embedding of $\mathcal{K}$ in $|2\Theta|$, deduced from the map $\alpha \mapsto \Theta_\alpha + \Theta_{-\alpha}$ from J to $|2\Theta|$.

(3.4) The case $g = 3$, which had been treated previously in [N-R2], is particularly interesting: we obtain a hypersurface in $|2\Theta|$, of degree 4, which is $J[2]$-invariant and singular along the Kummer variety. Now Coble shows in [C2] that there is a unique such quartic (the $J[2]$-invariance is actually superfluous, see [B5]). Thus in genus 3, the theta map identifies $\mathcal{M}_2$ with the Coble quartic hypersurface.

Coble gives an explicit equation for this hypersurface, which we now express in modern terms. Recall that Mumford's theory of the Heisenberg group allows us to find canonical coordinates $(X_v)_{v\in V}$ in the projective space $|2\Theta|$, where V is a 3-dimensional vector space over $\mathbb{F}_2$. Then Coble equation reads:

$$\alpha \sum_{u\in V} X_u^4 + \sum_{\ell=\{u,v\}} \alpha_{d(\ell)}\, X_u^2 X_v^2 + \sum_{P=\{t,u,v,w\}} \alpha_{d(P)} X_t X_u X_v X_w = 0$$

where the second sum (resp. the third) is taken over the set of affine lines (resp. planes) in V, and $d(\ell) \in \mathbb{P}(V)$ (resp. $d(P) \in \mathbb{P}(V^*)$) denotes the direction of the line ℓ (resp. of the plane P).

In many ways the Coble quartic $\mathcal{Q} \subset \mathbb{P}^7$ can be seen as an analogue of the Kummer quartic surface in $\mathbb{P}^3$. Pauly has proved that $\mathcal{Q}$ shares a famous property of the Kummer surface, the *self-duality* : the dual hypersurface $\mathcal{Q}^* \subset (\mathbb{P}^7)^*$ is isomorphic to $\mathcal{Q}$ [Pa]. The proof is geometric, and includes several beautiful geometric constructions along the way.

(3.5) In genus 4, $\mathcal{M}_2$ is a variety of dimension 9 and degree 96 in $\mathbb{P}^{15}$. Oxbury and Pauly have observed that there exists a unique $J[2]$-invariant quartic hypersurface singular along $\mathcal{M}_2$ [O-P]. A geometric interpretation of this quartic is not known.

(3.6) In arbitrary genus, the quartic hypersurfaces in $|2\Theta|$ containing $\mathcal{M}_2$ have been studied in [vG] and [vG-P]. Here is one sample of their results:

Proposition 3.6. *Assume that C has no vanishing thetanull. A $J[2]$-invariant quartic form F on $|2\Theta|$ vanishes on $\mathcal{M}_2$ if and only if the hypersurface $F = 0$ is singular along $\mathcal{K}$.*

(Note that though the action of $J[2]$ on $|2\Theta|$ does not come from a linear action, it *does* lift to the space of quartic forms on $|2\Theta|$. Requiring the invariance of F is stronger than the invariance of the corresponding hypersurface.)

Van Geemen and Previato also describe the quartics containing $\mathcal{M}_2$ in terms of the Prym varieties associated to C – this is related to the Schottky-Jung configuration studied by Mumford.

§4. Genus 2

(4.1) Going to higher rank, it is natural to look first at the genus 2 case. There a curious numerical coincidence occurs, namely

$$\dim \mathcal{M}_r = \dim |r\Theta| = r^2 - 1 \ .$$

Recall that θ is a finite morphism for $r = 2, 3$ (1.6). However already for $r = 4$ it is only a rational map: the Raynaud bundle R_2 has rank 4 and slope 1 (2.3), so once twisted by appropriate line bundles of degree -1 it provides finitely many (actually 16) base points of $|\mathcal{L}|$.

We have seen that θ is an isomorphism in rank 2. In rank 3 there is again a beautiful story, surprisingly analogous to the rank 2, genus 3 case. Indeed the Coble quartic has a companion, the *Coble cubic* : this is the unique cubic hypersurface $\mathcal{C} \subset |3\Theta|^*$ singular along J^1 embedded in $|3\Theta|^*$ by the linear system $|3\Theta|$ (this is implicit in Coble [C1]; see [B5] for a modern explanation).

Theorem 4.2. *The map $\theta : \mathcal{M}_3 \to |3\Theta|$ is a double covering; the corresponding involution of $\mathcal{M}_3$ is $E \mapsto \iota^* E^*$, where ι is the hyperelliptic involution. The branch locus $\mathcal{S} \subset |3\Theta|$ of θ is a sextic hypersurface, which is the dual of the Coble cubic $\mathcal{C} \subset |3\Theta|^*$.*

This is fairly straightforward (see [O]) except for the duality statement, which was conjectured by Dolgachev and proved in [O] (a different proof appears in [N]).

(4.3) Like for the Coble quartic we get an explicit equation for $\mathcal{C}$ by choosing a level 3 structure on C, which provides canonical coordinates $(X_v)_{v\in V}$ on $|3\Theta|^*$, where V is a 2-dimensional vector space over $\mathbb{F}_3$. Then from [C1] we get the following equation for $\mathcal{C}$:

$$\alpha_0 \sum_{v\in V} X_v^3 \ + 6 \sum_{\ell=\{u,v,w\}} \alpha_{d(\ell)} X_u X_v X_w = 0 \ ,$$

where the second sum is taken over the set of affine lines in V, and $d(\ell) \in \mathbb{P}(V)$ is the direction of the line ℓ. The 5 coefficients (α_i) satisfy the Burkhardt equation

$$\alpha_0^4 - \alpha_0 \sum_{p\in\mathbb{P}(V)} \alpha_p^3 + 3 \prod_{p\in\mathbb{P}(V)} \alpha_p = 0$$

(see [H], 5.3).

(4.4) In rank $r \geq 4$ we start getting base points, and this causes a lot of trouble – since θ is only rational, we cannot compute its degree using intersection theory. However we still have:

Proposition 4.5. [B6] *The rational map* $\theta : \mathcal{M}_r \dashrightarrow |r\Theta|$ *is generically finite* (or, equivalently, dominant).

The idea is to prove the finiteness of $\theta^{-1}(\Theta + \Delta)$, where Δ is a general element of $|(r-1)\Theta|$. Any decomposable bundle in that fibre must be of the form $\mathcal{O}_C \oplus F$ for some $F \in \mathcal{M}_{r-1}$ with $\Theta_F = \Delta$; reasoning by induction on r we can assume that there are finitely many such F. Thus the whole point is to control the *stable* bundles E with $\Theta_E = \Theta + \Delta$. Now the condition $\Theta_E \supset \Theta$ means by definition $H^0(C, E(p)) \neq 0$ for all $p \in C$, or equivalently $H^0(C, E'(-p)) \neq 0$ for all $p \in C$, where $E' := E^* \otimes K_C^{-1}$ is the Serre dual of E. Since $h^0(E') = r$ by stability of E, this implies that the global sections of E' generate a subbundle of rank $< r$. A precise analysis of this situation allows us to prove that there are only finitely many such bundles E with $\Theta_E = \Theta + \Delta$.

(4.6) The map θ is no longer finite in rank $r \geq 4$, in fact it admits some fibres of dimension $\geq [\frac{r}{2}] - 1$ [B6]. When r is even, this is seen by restricting θ to the moduli space of *symplectic* bundles: the corresponding moduli space has dimension $\frac{1}{2}r(r+1)$, but its image under θ lands in the subspace $|r\Theta|^+$ of $|r\Theta|$ corresponding to even theta functions of order r, which has dimension $\frac{r^2}{2} + 1$. For r odd one considers bundles of the form $\mathcal{O}_C \oplus F$ with F symplectic.

(4.7) It would be interesting to find the *degree* of θ, which is unknown already in genus 4. For trivial reasons it has to grow exponentially with g (see [B6], 2.3). Brivio and Verra have found a nice geometric interpretation of the generic fibre of θ which might lead at least to a good estimate for $\deg \theta$ [B-V2].

§5. Higher rank and genus

Not much is known here. We already mentioned the following result proved in [B6]:

Proposition 5.1. *In genus* 3 *the map* $\theta : \mathcal{M}_3 \to |3\Theta|$ *is a finite morphism.*

The proof is rather roundabout, and gives actually a more interesting result: the complete list of stable vector bundles E of rank 3 and degree 0 such that $\Theta_E \supset \Theta$. It turns out that each bundle in this list admits a theta divisor. Since $\Theta_E = J$ implies $\Theta_E \supset \Theta$, Proposition 5.1 follows.

(5.2) The idea for establishing that list is to translate the problem into a classical question of projective geometry. Similarly to the genus 2 case, the condition $\Theta_E \supset \Theta$ means $H^0(E(p+q)) \neq 0$ for all p, q in C, or equivalently $H^0(E'(-p-q)) \neq 0$, where $E' := E^* \otimes K_C^{-1}$ is the Serre dual of E. One checks that stability implies $h^0(E') = 6$ and $h^0(E'(-p)) = 3$ for p general in C. This gives a family of 2-planes in $\mathbb{P}(H^0(E')) \cong \mathbb{P}^5$, parametrized by C, such that any two planes of the family intersect. It turns out that the maximal such families have been classified in a beautiful paper by Morin [M]: there are three families given by linear algebra (like the 2-planes contained in a given hyperplane), and three coming from geometry: the 2-planes contained in a smooth quadric, the tangent planes to the Veronese surface, and the planes intersecting the Veronese surface along a conic. Translating back this result in terms of vector bundles gives the list we were looking for.

(5.3) This list also shows that $\theta^{-1}(\Theta + \Theta_F) = \{\mathcal{O}_C \oplus F\}$ for F general in $\mathcal{M}_2$. This might indicate that θ has degree one; it would follow if we could prove the injectivity of its tangent map at $\mathcal{O}_C \oplus E$ for some E in $\mathcal{M}_2$, perhaps in the spirit of [vG-I].

§6. Questions and conjectures

The list of results ends at this point, but let me finish with a (small) list of open problems. About the general behaviour of the theta map, the most optimistic statement would be:

Speculation 6.1. *For $g \geq 3$, θ is generically injective if C is not hyperelliptic, and generically two-to-one onto its image if C is hyperelliptic.*

Note that in the hyperelliptic case θ factors as in thm. 4.2 through the non-trivial involution $E \mapsto \iota^* E^*$. Admittedly the evidence for 6.1 is very weak: the only case where it is known is in rank 2.

As for base points, Proposition 1.6 leads naturally to:

Conjecture 6.2. *Every bundle $E \in \mathcal{M}_3$ has a theta divisor.*

(6.3) There exists an integer $r(C)$ such that θ is a morphism for $r < r(C)$ but only a rational map for $r \geq r(C)$ (observe that if $E \in \mathcal{M}_r$ has no theta divisor, so does $E \oplus F$ for any F in $\mathcal{M}_s$, $s \geq 1$). We know very little about this integer: we have $r(C) = 4$ for $g = 2$, $4 \leq r(C) \leq 8$ for $g = 3$, and $r(C) \leq \frac{1}{2}(g+1)(g+2)$ [A].

Questions 6.4. a) *Does $r(C)$ depend only on g?*

b) *Put $r(g) := \min r(C)$ for all curves C of genus g. Is $r(g)$ an increasing function of g?*

The next question does not involve directly the theta map, but it is related to several questions about the existence of theta divisors.

Conjecture 6.5. *Let $\pi : C' \to C$ be a finite morphism between smooth projective curves of genus ≥ 2. The direct image π_*L of a general vector bundle L on C' is stable.*

One reduces readily to the case when L is a line bundle. The problem depends in a crucial way on the degree of L: one can prove for instance that π_*L is stable (for L generic) if $|\chi(L)| < g + \frac{g^2}{r}$, where r is the degree of π and g the genus of C (see [B3]).

One of the relations between this conjecture and the existence of theta divisors is the following: the conjecture for a general line bundle L of degree d is implied by the existence of *a vector bundle E of rank r and degree $g(C') - 1 - d$ such that π^*E admits a prime theta divisor.* Indeed we have $\Theta_{\pi_*L\otimes E} = (\pi^*)^{-1}(\Theta_{L\otimes\pi^*E})$; if Θ_{π^*E} is prime, so is $\Theta_{\pi_*L\otimes E}$ for general L, and as in (2.2) this implies that π_*L is stable.

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