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# Birational geometry of symplectic resolutions of nilpotent orbits

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***Dedicated to Professor Masaki Maruyama on his sixtieth birthday***

**Abstract.**

We shall give a complete description of the relatively ample cones and the relatively movable cones of symplectic resolutions of the closures of the nilpotent orbits in complex simple Lie algebras. Moreover, we shall prove that all symplectic resolutions of such nilpotent orbit closures are connected by finite numbers of Mukai flops of type $A$, $D$ and $E_6$.

## §1. Introduction

Let $G$ be a complex simple Lie group and let $\mathfrak{g}$ be its Lie algebra. Then $G$ has the adjoint action on $\mathfrak{g}$. The orbit $\mathcal{O}_x$ of a nilpotent element $x \in \mathfrak{g}$ is called a nilpotent orbit. A nilpotent orbit $\mathcal{O}_x$ admits a non-degenerate closed 2-form $\omega$ called the Kostant-Kirillov symplectic form. The closure $\bar{\mathcal{O}}_x$ of $\mathcal{O}_x$ then becomes a symplectic singularity. In other words, the 2-form $\omega$ extends to a holomorphic 2-form on a resolution of $\bar{\mathcal{O}}_x$. A resolution of $\bar{\mathcal{O}}_x$ is called a symplectic resolution if this extended form is everywhere non-degenerate on the resolution. For a parabolic subgroup $P$ of $G$, one can find a unique nilpotent orbit $\mathcal{O}$ such that $\mathcal{O} \cap n(\mathfrak{p})$ is an open dense subset of $n(\mathfrak{p})$. Here $n(\mathfrak{p})$ is the nil-radical of $\mathfrak{p} := \mathrm{Lie}(P)$. This orbit is called the Richardson orbit for $P$. Conversely, $P$ is called a polarization of $\mathcal{O}$. We then have a generically finite proper surjective map

$$\mu : T^*(G/P) \to \bar{\mathcal{O}}.$$

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Here $T^*(G/P)$ is the cotangent bundle of the homogenous space $G/P$. When $\deg(\mu) = 1$, $\mu$ becomes a symplectic resolution of $\bar{\mathcal{O}}$. We call it a *Springer resolution.* Recently, Fu [Fu 1] (see also some corrections in its e-print version) has shown that, if a nilpotent orbit $\bar{\mathcal{O}}$ has a (projective) symplectic resolution $f$, then $\mathcal{O}$ has a polarization $P$ such that $f$ coincides with the Springer resolution for $P$. However, there is a nilpotent orbit with no polarizations. Moreover, even if $\mathcal{O}$ has a polarization, it is not unique and we may possibly have $\deg(\mu) > 1$. Spaltenstein [S2] and Hesselink [He] obtained a necessary and sufficient condition for $\bar{\mathcal{O}}$ to have a Springer resolution when $\mathfrak{g}$ is a classical simple Lie algebra. Moreover, [He] gave an explicit number of such parabolics $P$ up to conjugacy class that give Springer resolutions of $\bar{\mathcal{O}}_x$ (cf. §4). In this paper we shall deal with an arbitrary simple Lie algebra. First we introduce an equivalence relation in the set of parabolic subgroups of $G$ in terms of marked Dynkin diagrams (Definition 1, §5). The following is one of main results of this paper.

**Theorem**(cf. Theorem 6.1): *Let $\mathcal{O}$ be a nilpotent orbit of a complex simple Lie algebra. Assume that $\bar{\mathcal{O}}$ has a Springer resolution $Y_{P_0} := T^*(G/P_0)$. Then, for any parabolic subgroup $P$ equivalent to $P_0$, $Y_P := T^*(G/P)$ is a Springer resolution of $\bar{\mathcal{O}}$. Moreover, any projective symplectic resolution of $\bar{\mathcal{O}}$ has this form. All $Y_P$ $(P \sim P_0)$ are connected by Mukai flops of type A, D, and $E_6$.*

A *Mukai flop of type A* is a kind of Springer resolutions; let $x \in \mathfrak{sl}(n)$ be a nilpotent element of Jordan type $[2^k, 1^{n-2k}]$ with $2k < n$. Then a Mukai flop of type A is the diagram of two Springer resolutions of $\bar{\mathcal{O}}_x$:

$$T^*G(k,n) \to \bar{\mathcal{O}}_x \leftarrow T^*G(n-k,n)$$

where $G(k,n)$ (resp. $G(n-k,n)$) is the Grassmannian which parametrizes $k$-dimensional (resp. $n-k$-dimensional) subspaces of $\mathbf{C}^n$. This flop naturally appears in the wall-crossing of the moduli spaces of various objects (eg. stable sheaves on K3 surfaces, quiver varieties and so on). On the other hand, a *Mukai flop of type D* comes from an orbit of a simple Lie algebra of type D. Let $x \in \mathfrak{so}(2k)$ be a nilpotent element of type $[2^{k-1}, 1^2]$, where $k$ is an odd integer with $k \geq 3$. Then $\bar{\mathcal{O}}_x$ admits two Springer resolutions

$$T^*G^+_{iso}(k,2k) \to \bar{\mathcal{O}}_x \leftarrow T^*G^-_{iso}(k,2k)$$

where $G^+_{iso}(k,2k)$ and $G^-_{iso}(k,2k)$ are two connected components of the orthogonal Grassmannian $G_{iso}(k,2k)$. Finally, there are two Mukai flops of type $E_6$. We call them of type $E_{6,I}$ and of type $E_{6,II}$. The Mukai flop of type $E_{6,I}$ (resp. $E_{6,II}$) consists of two resolutions of the nilpotent orbit closure $\bar{\mathcal{O}}_{2A_1}$ (resp. $\bar{\mathcal{O}}_{A_2+2A_1}$) in $E_6$. For details on these flops, see §5. Let us consider a family of Mukai flops parametrized by a variety $T$: $Y \to W \leftarrow Y'$. By definition, there is a bundle map $W \to T$ with a typical fiber $\bar{\mathcal{O}}_x$ such that, for each $t \in T$, $Y_t \to \bar{\mathcal{O}}_x \leftarrow Y'_t$ is a Mukai flop. A flop

$$Z \to X \leftarrow Z'$$

is called a *locally trivial family of Mukai flop* if there is a smooth surjective map $X \to W$ and it is the pull-back by this map of the family of Mukai flops above. The last statement of Theorem claims that, for any two $Y_P$ and $Y_{P'}$, the birational map $Y_P - - \to Y_{P'}$ is decomposed into diagrams $Y_i \to X_i \leftarrow Y_{i+1}$ $(i = 1, ..., m-1)$ with $Y_1 = Y_P$ and $Y_m = Y_{P'}$ so that each diagram is a locally trivial family of Mukai flops.

In the course of the proof of Theorem, we describe the ample cones and movable cones of symplectic resolutions of $\bar{\mathcal{O}}$. Even when $\mathfrak{g}$ is classical, it would clarify the geometric meaning of the results of Spaltenstein and Hesselink. To illustrate these, three examples will be given (see Examples 6.7, 6.8, 6.9).

Another purpose of this paper is to give an affirmative answer to the following conjecture in the case of (the normalization of) a nilpotent orbit closure in a simple Lie algebra (Theorem 7.9).

**Conjecture**([F-N]): *Let $W$ be a normal symplectic singularity. Then for any two symplectic resolutions $f_i : X_i \to W$, $i = 1, 2$, there are deformations $\mathcal{X}_i \xrightarrow{F_i} \mathcal{W}$ of $f_i$ over a parameter space $S$ such that, for $s \in S - \{0\}$, $F_{i,s} : \mathcal{X}_{i,s} \to \mathcal{W}_s$ are isomorphisms. In particular, $X_1$ and $X_2$ are deformation equivalent.*

This conjecture is already proved in [F-N] when $W$ is a nilpotent orbit closure in $\mathfrak{sl}(n)$. On the other hand, a weaker version of this conjecture is proved in [Fu 2] when $W$ is the normalization of a nilpotent orbit closure in a classical simple Lie algebra. According to the idea of Borho and Kraft [B-K], we shall define a deformation of $\bar{\mathcal{O}}_x$ by using a *Dixmier sheet.* Corresonding to each parabolic subgroup $P$, this deformation has a simultaneous resolution. These simultaneous resolutions would give the desired deformations of the conjecture. Details on the construction of them can be found in §7.

The content of this paper is as follows. Main body of the paper are §§. 5,6,7. The first three sections §§. 2,3,4 are preliminaries for the later sections. In the proof of Proposition 5.1, we shall use Springer's correspondence (cf. Theorem 3.1, Proposition 4.3) to calculate the dimension of fibers of Springer maps. The proofs of Theorem 6.1 are written in an abstract way so that they are valid for exceptional Lie algebras. One can, however, find a more explicit treatment in Example 6.5 when $\mathfrak{g}$ is classical.

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**Notation.** (1) A partition $\mathbf{d}$ of $n$ is a set of positive integers $[d_1, ..., d_k]$ such that $\Sigma d_i = n$ and $d_1 \geq d_2 \geq ... \geq d_k$. We mean by $[d_1^{j_1}, ..., d_k^{j_k}]$ the partition where $d_i$ appear in $j_i$ multiplicity. If $(p_1, ..., p_s)$ is a sequence of positive integers, then we define the partition $\mathbf{d} = \mathrm{ord}(p_1, ..., p_s)$ by $d_i := \sharp\{j; p_j \geq i\}$. In particular, for a partition $\mathbf{d}$, ${}^t\mathbf{d} := \mathrm{ord}(d_1, ..., d_k)$ is called the dual partition of $\mathbf{d}$. We define $d^i := ({}^t d)_i$.

(2) For a proper birational map $f$ of algebraic varieties, we say that $f$ is *divisorial* if $\mathrm{Exc}(f)$ contains a divisor, and otherwise, we say that $f$ is *small*. Note that the terminology of "small" is, for example, different from that in [B-M].

## §2. Classification of nilpotent orbits

Let $G$ be a complex simple Lie group and let $\mathfrak{g}$ be its Lie algebra. $G$ has the adjoint action on $\mathfrak{g}$. The orbit $\mathcal{O}_x$ of a nilpotent element $x \in \mathfrak{g}$ for this action is called a nilpotent orbit. This orbit carries a natural closed non-degenerate 2-form (Kostant-Kirillov form) $\omega$ (cf. [C-G], Prop. 1.1.5, [C-M], 1.3), and its closure $\bar{\mathcal{O}}_x$ becomes a *symplectic* singularity, that is, the symplectic 2-form $\omega$ extends to a holomorphic 2-form on a resolution $Y$ of $\bar{\mathcal{O}}_x$. When $\mathfrak{g}$ is classical, $\mathfrak{g}$ is naturally a Lie subalgebra of $\mathrm{End}(V)$ for a complex vector space $V$. Then we can attach a partition $\mathbf{d}$ of $n := \dim V$ to each orbit as the Jordan type of an element contained in the orbit. Here a partition $\mathbf{d} := [d_1, d_2, ..., d_k]$ of $n$ is a set of positive integers with $\Sigma d_i = n$ and $d_1 \geq d_2 \geq ... \geq d_k$. When a number $e$ appears in the partition $\mathbf{d}$, we say that $e$ is a *part* of $\mathbf{d}$. We call $\mathbf{d}$ *very even* when $\mathbf{d}$ consists with only even parts, each having even multiplicity. The following result can be found, for example, in [C-M, §5].

**Proposition 2.1.** *Let $\mathcal{N}o(\mathfrak{g})$ be the set of nilpotent orbits of $\mathfrak{g}$.*

*(1)($A_{n-1}$): When $\mathfrak{g} = \mathfrak{sl}(n)$, there is a bijection between $\mathcal{N}o(\mathfrak{g})$ and the set of partitions $\mathbf{d}$ of $n$.*

*(2)($B_n$): When $\mathfrak{g} = \mathfrak{so}(2n+1)$, there is a bijection between $\mathcal{N}o(\mathfrak{g})$ and the set of partitions $\mathbf{d}$ of $2n+1$ such that even parts occur with even multiplicity.*

*(3)($C_n$): When $\mathfrak{g} = \mathfrak{sp}(2n)$, there is a bijection between $\mathcal{N}o(\mathfrak{g})$ and the set of partitions $\mathbf{d}$ of $2n$ such that odd parts occur with even multiplicity*

*(4)($D_n$): When $\mathfrak{g} = \mathfrak{so}(2n)$, there is a surjection $f$ from $\mathcal{N}o(\mathfrak{g})$ to the set of partitions $\mathbf{d}$ of $2n$ such that even parts occur with even multiplicity. For a partition $\mathbf{d}$ which is not very even, $f^{-1}(\mathbf{d})$ consists of exactly one orbit, but, for very even $\mathbf{d}$, $f^{-1}(\mathbf{d})$ consists of exactly two different orbits.*

When $\mathfrak{g}$ is of exceptional type, we need different methods to classify nilpotent orbits. Dynkin [D] associates a *weighted Dynkin diagram* with each nilpotent orbit. The weighted Dynkin diagram uniquely determines a nilpotent orbit. However, all weighted Dynkin diagrams do not come from nilpotent orbits. Bala and Carter [B-L] has classified which weighted Dynkin diagram is realized, and they give a label (Bala-Carter label) to each nilpotent orbit. We shall use these labels to indicate nilpotent orbits in an exceptional Lie algebra $\mathfrak{g}$ (cf. [B-C],[C-M]).

## §3. Springer's correspondence

Let $G$ be a complex simple Lie group and let $B$ be a Borel subgroup of $G$. Let $\mathfrak{g}$ (resp. $\mathfrak{b}$) be the Lie algebra of $G$ (resp. $B$). The set of nilpotent elements $\mathcal{N}$ of $\mathfrak{g}$ is called the nilpotent variety. It coincides with the closure of the regular nilpotent orbit in $\mathfrak{g}$. The (original) Springer resolution

$$\pi : T^*(G/B) \to \mathcal{N}$$

is constructed as follows. Let $n(\mathfrak{b})$ be the nil-radical of $\mathfrak{b}$. Then the cotangent bundle $T^*(G/B)$ of $G/B$ is identified with $G \times^B n(\mathfrak{b})$, which is, by definition, the quotient space of $G \times n(\mathfrak{b})$ by the equivalence relation $\sim$. Here $(g, x) \sim (g', x')$ if $g' = gb$ and $x' = Ad_{b^{-1}}(x)$ for some $b \in B$. Then we define $\pi([g, x]) := Ad_g(x)$. According to Borho-MacPherson [B-M], we shall briefly review Springer's correspondence [Sp]. The nilpotent variety $\mathcal{N}$ is decomposed into the disjoint union of nilpotent orbits $\mathcal{O}_x$, where $x$ is a distinguished base point of the orbit $\mathcal{O}_x$. We put $d_x := \dim \pi^{-1}(x)$. Now $\pi_1(\mathcal{O}_x)$ acts on $H^{2d_x}(\pi^{-1}(x), \mathbf{Q})$ by monodromy. Decompose $H^{2d_x}(\pi^{-1}(x), \mathbf{Q})$ into irreducible representations of $\pi_1(\mathcal{O}_x)$:

$$H^{2d_x}(\pi^{-1}(x), \mathbf{Q}) = \oplus_\phi (V_\phi \otimes V_{(x,\phi)}),$$

where $\phi : \pi_1(\mathcal{O}_x) \to \mathrm{End}(V_\phi)$ are irreducible representations and $V_{(x,\phi)} = \mathrm{Hom}_{\pi_1(\mathcal{O}_x)}(V_\phi, H^{2d_x}(\pi^{-1}(x), \mathbf{Q}))$. By definition, $\dim V_{(x,\phi)}$ coincides with the multiplicity of $\phi$ in $H^{2d_x}(\pi^{-1}(x), \mathbf{Q})$. We call $(x, \phi)$ is *$\pi$-relevant* if $V_{(x,\phi)} \neq 0$. Fix a maximal torus $T$ in $B$, and let $W$ be the Weyl group relative to $T$. Then there is a natural action of $W$ on $H^{2d_x}(\pi^{-1}(x), \mathbf{Q})$ commuting with the action of $\pi_1(\mathcal{O}_x)$. Each factor $V_\phi \otimes V_{(x,\phi)}$ becomes a $W$-module, where $W$ acts trivially on $V_\phi$ and $V_{(x,\phi)}$ is an irreducible representation of $W$. These representations were originally constructed by Springer. In [B-M], they are given in terms of the decomposition theorem of intersection cohomology by Beilinson, Bernstein, Deligne and Gabber. The following theorem is called Springer's correspondence:

**Theorem 3.1.** *Any irreducible representaion of $W$ is isomorphic to $V_{(x,\phi)}$ for a unique $\pi$-relevant pair $(x, \phi)$.*

One can find the tables on Springer's correspondence in [C, 13.3] for each simple Lie group (see also [A-L], [B-L]).

## §4. Parabolic subgroups and Springer maps

Let $G$ be a complex reductive Lie group and let $\mathfrak{g}$ be its Lie algebra. Fix a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ and let

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha$$

be the root space decomposition. Let $\Delta \subset \Phi$ be a base of $\Phi$ and denote by $\Phi^+$ (resp. $\Phi^-$) the set of positive roots (resp. negative root). We define a Borel subalgebra $\mathfrak{b}$ of $\mathfrak{g}$ as

$$\mathfrak{b} := \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi^+} \mathfrak{g}_\alpha.$$

For a subset $\Theta \subset \Delta$, let $< \Theta >$ be the sub-root system generated by $\Theta$. We put $< \Theta >^+ := < \Theta > \cap \Phi^+$ and $< \Theta >^- := < \Theta > \cap \Phi^-$. We define

$$\mathfrak{p}_\Theta := \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi^+} \mathfrak{g}_\alpha \oplus \bigoplus_{\alpha \in < \Theta >^-} \mathfrak{g}_\alpha.$$

By definition, $\mathfrak{p}_\Theta$ is a parabolic subalgebra containing $\mathfrak{b}$. Moreover, any parabolic subalgebra $\mathfrak{p}$ of $\mathfrak{g}$ is $G$-conjugate to $\mathfrak{p}_\Theta$ for some $\Theta \subset \Delta$. $\mathfrak{p}_\Theta$ and $\mathfrak{p}_{\Theta'}$ are $G$-conjugate if and only if $\Theta = \Theta'$. Therefore, there is a one-to-one correspondence between subsets of $\Delta$ and the conjugacy classes of parabolic subalgebras of $\mathfrak{g}$. An element of $\Delta$ is called a simple root, which corresponds to a vertex of the Dynkin diagram attached to $\mathfrak{g}$. A Dynkin diagram with some vertices being marked is called a *marked Dynkin diagram.* If $\Theta \subset \Delta$ is given, we have a marked Dynkin diagram by marking the vertices which correspond to $\Delta \setminus \Theta$. A marked Dynkin diagram with only one marked vertex is called a *single marked Dynkin diagram.* A conjugacy class of parabolic subgroups $P \subset G$ with $b_2(G/P) = 1$ corresponds to a single marked Dynkin diagram.

**Example 4.1.** *When $G = SL(n)$, the parabolic subgroup of flag type $(k, n-k)$ corresponds to the marked Dynkin diagram*

$$\circ\!\!-\!\!-\!\!- \; - - - \; -\!\!\underset{k}{\bullet}\!\!- \; - - - \; -\!\!-\!\!-\!\!-\!\!\circ .$$

**Example 4.2.** *Let $\epsilon$ denote the number 0 or 1. Assume that $V$ is a* **C***-vector space equipped with a non-degenerate bilinear form $<,>$ such that*

$$< v, w >= (-1)^{\epsilon} < w, v >, (v, w \in V).$$

*When $\epsilon = 0$ (resp. $\epsilon = 1$), this means that the bilinear form is symmetric (resp. skew-symmetric). We shall describe parabolic subgroups of $SO(V)$ and $Sp(V)$. We put*

$$H := \{x \in GL(V); < xv, xw >=< v, w >, (v, w \in V)\},$$

*and*

$$G := \{x \in H; \det(x) = 1\}.$$

*Note that*

$$H = \begin{cases} O(V) & (\epsilon = 0) \\ Sp(V) & (\epsilon = 1) \end{cases}$$

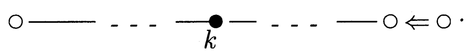
*and*

$$G = \begin{cases} SO(V) & (\epsilon = 0) \\ Sp(V) & (\epsilon = 1) \end{cases}$$

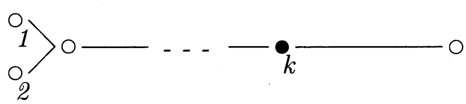
*A flag $F := \{F_i\}_{1 \leq i \leq s}$ of $V$ is called* isotropic *if $F_i^{\perp} = F_{s-i}$ for $1 \leq i \leq s$. An isotropic flag $F$ is* admissible *if the stabilizer group $P$ of $F$ has no finner stabilized flag than $F$. In other words, let $P_F := \{g \in G; gF_i \subset F_i \; \forall i\}$. Then, for any $i$, there is no $P_F$-invariant subspace $F'_i$ such that $F_i \subset F'_i \subset F_{i+1}$ with $F'_i \neq F_i, F_{i+1}$. When the length $s$ of an isotropic flag $F$ is even, one can write the type of $F$ as $(p_1, ..., p_k, p_k, ..., p_1)$ with $k = s/2$. On the other hand, when $s$ is odd, one can write the type of $F$ as $(p_1, ..., p_k, q, p_k, ..., p_1)$ with $k = (s-1)/2$. For the consistency, we shall write the flag type of $F$ as $(p_1, ..., p_k, 0, p_k, ..., p_1)$ when $s$ is even. An isotropic flag $F$ is not admissible when $\epsilon = 0$ and $q = 2$. In fact, one can always find a $P_F$-invariant subspace $F'_k$ such that $F_k \subset F'_k \subset F_{k+1}$ and $\dim(F'_k/F_k) = 1$. This is the only case where an isotropic* *flag is not admissible. The stabilizer group of an admissible isotropic flag becomes a parabolic subgroup of $G$. If a parabolic subgroup of $G$ has a stabilized (admissible) flag $F$ of type $(p_1, ..., p_k, q, p_k, ..., p_1)$, then $\pi := \mathrm{ord}(p_1, ..., p_k, q, p_k, ..., p_1)$ is called the* Levi type *of $P$.*

*When $G = SO(2n+1)$, the parabolic subgroup of flag type $(k, 2n-2k+1, k)$ corresponds to the marked Dynkin diagram*

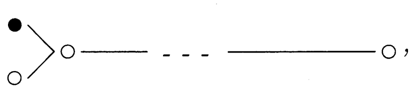
*When $G = Sp(2n)$, the parabolic subgroup of flag type $(k, 2n-2k, k)$ corresponds to the marked Dynkin diagram*

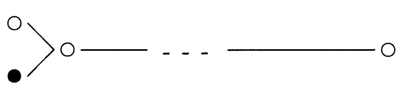


*Finally, assume that $G = SO(2n)$. Then the parabolic subgroup corresponding to the marked Dynkin diagram ($k \geq 3$)*



*has flag type $(n-k+1, 2k-2, n-k+1)$. On the other hand, two marked Dynkin diagrams*





*both give parabolic subgroups of flag type $(n, 0, n)$ which are not $G$-conjugate.*

For a parabolic subgroup $P$ of $G$, let $\mathfrak{p}$ be its Lie algebra and let $n(\mathfrak{p})$ be the nil-radical of $\mathfrak{p}$. There is a unique nilpotent orbit $\mathcal{O} \subset \mathfrak{g}$ such that $\mathcal{O} \cap n(\mathfrak{p})$ is an open dense subset of $n(\mathfrak{p})$. This nilpotent orbit is called the *Richardson orbit* for $P$. Conversely, such parabolic subgroup $P$ is called a *polarization* of $\mathcal{O}$. When $x \in n(\mathfrak{p})$ and $P$ is a polarization of $\mathcal{O}_x$, we call $P$ a polarization of $x$. A parabolic subgroup $P$ is a polarization of $x$ if and only if $x \in n(\mathfrak{p})$ and $\dim \mathcal{O}_x = 2\dim(G/P)$ (cf.

[He]). The cotangent bundle $T^*(G/P)$ of the homogenous space $G/P$ is naturally isomorphic to $G \times^P n(\mathfrak{p})$, which is the quotient space of $G \times n(\mathfrak{p})$ by the equivalence relation $\sim$. Here $(g,x) \sim (g',x')$ if $g' = gp$ and $x' = Ad_{p^{-1}}(x)$ for some $p \in P$. The *Springer map*

$$\mu : T^*(G/P) \to \overline{\mathcal{O}}$$

is defined as $\mu([g,x]) = Ad_g(x)$. The Springer map $\mu$ is a generically finite surjective proper map. When $\deg\mu = 1$, it is called a Springer resolution. For a nilpotent orbit $\mathcal{O}_x \subset \overline{\mathcal{O}}$, we call $\mathcal{O}_x$ is $\mu$-relevant if

$$\dim \mu^{-1}(x) = \mathrm{codim}(\mathcal{O}_x \subset \overline{\mathcal{O}})/2.$$

From now on, we assume that $\mathfrak{g}$ is a simple Lie algebra. For the Springer resolution $\pi$ for a Borel subgroup $B$, every nilpotent orbit is $\pi$-relevant. However, this is not the case for a general parabolic subgroup $P$. The $\mu$-relevancy is closely related to Springer's correspondence. In order to state the result, we shall prepare some terminology. Let $L$ be a Levi subgroup of $P$. Fix a maximal torus $T$ of $L$. Then $T$ is also a maximal torus of $G$. Let $W(L)$ be the Weyl group for $L$ relative to $T$ and let $W$ be the Weyl group for $G$ relative to $T$. Now we have a natural inclusion $W(L) \subset W$. Let $\epsilon_{W(L)}$ be the sign representation of $W(L)$. Denote by $\epsilon^W_{W(L)}$ the induced representation of $\epsilon_{W(L)}$ to $W$. By Theorem 3.1, every irreducible representation of $W$ has the form $V_{(x,\phi)}$ for a $\pi$-relevant pair $(x,\phi)$. Recall that $\phi$ is an irreducible representation of $\pi_1(\mathcal{O}_x)$. Denote by 1 the trivial representation. Then $(x,1)$ is a $\pi$-relevant pair (cf. [B-M, Lemma 1.2]).

**Proposition 4.3.** *A nilpotent orbit $\mathcal{O}_x \subset \overline{\mathcal{O}}$ is $\mu$-relevant if and only if $V_{(x,1)}$ occurs in $\epsilon^W_{W(L)}$.*

*Proof.* See [B-M, Collorary 3.5, (b)].

In the remainder of this section we shall review some results on Richardson orbits and polarizations when $\mathfrak{g}$ is a complex classical Lie algebra. Let $x \in \mathfrak{g}$ be a nilpotent element and denote by $\mathrm{Pol}(x)$ the set of polarizations of $x$.

**Theorem 4.4.** *Let $x \in \mathfrak{sl}(n)$ be a nilpotent element. Then* $\mathrm{Pol}(x) \neq \emptyset$. *Assume that $x$ is of type $\mathbf{d} = [d_1, ..., d_k]$. Then $P \in \mathrm{Pol}(x)$ has the flag type $(p_1, ..., p_s)$ such that* $\mathrm{ord}(p_1,...,p_s) = \mathbf{d}$. *Conversely, for any sequence $(p_1,...,p_s)$ with* $\mathrm{ord}(p_1,...,p_s) = \mathbf{d}$, *there is a unique polarization $P \in \mathrm{Pol}(x)$ which has the flag type $(p_1,...,p_s)$.*

*Proof.* We shall construct a flag $F$ of type $(p_1,...,p_s)$ such that $xF_i \subset F_{i-1}$ for all $i$. We identify the partition $\mathbf{d}$ with a Young table consisting of $n$ boxes, where the $i$-th row consists of $d_i$ boxes for each $i$. We denote by $(i,j)$ the box of $\mathbf{d}$ lying on the $i$-th row and on the $j$-th column. Let $e(i,j)$, $(i,j) \in \mathbf{d}$ be a *Jordan basis* of $V := \mathbf{C}^n$ such that $xe(i,j) = e(i-1,j)$. We consruct a flag by the induction on $n$. Define first $F_1 := \Sigma_{1 \leq j \leq p_1} \mathbf{C}e(1,j)$. Then $x$ induces a nilpotent endomorphism $\bar{x}$ of $V/F_1$. The Jordan type of $\bar{x}$ is $[d_1 - 1, ..., d_{p_1} - 1, d_{p_1+1}, ..., d_k]$. Note that this coincides with $\mathrm{ord}(p_2, ..., p_k)$. By the induction hypothesis, we already have a flag of type $(p_2, ..., p_k)$ on $V/F_1$ stabilized by $\bar{x}$; hence we have a desired flag $F$. Let $P$ be the stabilizer group of $F$. Then it is clear that $x \in \mathfrak{n}(P)$. By an explicit calculation $\dim \mathcal{O}_x = 2\dim G/P$. Q.E.D.

Next consider simple Lie algebras of type $B$, $C$ or $D$. Let $V$ be an $n$ dimensional $\mathbf{C}$-vector space with a non-degenerate symmetric (skew-symmetric) form. As in Example 4.2, $\epsilon = 0$ when this form is symmetric and $\epsilon = 1$ when this form is skew-symmetric. Let $P_\epsilon(n)$ be the set of partitions $\mathbf{d}$ of $n$ such that $\sharp\{i; d_i = m\}$ is even for every integer $m$ with $m \equiv \epsilon$ (mod 2). Note that these partitions are nothing but those which appear as the Jordan types of nilpotent elements of $\mathfrak{so}(n)$ or of $\mathfrak{sp}(n)$. Next, let $q$ be a non-negative integer and assume moreover that $q \neq 2$ when $\epsilon = 0$. We define $\mathrm{Pai}(n,q)$ to be the set of partitions $\pi$ of $n$ such that $\pi_i \equiv 1$ (mod 2) if $i \leq q$ and $\pi_i \equiv 0$ (mod 2) if $i > q$. Note that, if $(p_1, ..., p_k, q, p_k, ..., p_1)$ is the type of an admissible flag of $V$, then $\mathrm{ord}(p_1, ..., p_k, q, p_k, ..., p_1) \in \mathrm{Pai}(n,q)$. Now we shall define the *Spaltenstein map* $S$ from $\mathrm{Pai}(n,q)$ to $P_\epsilon(n)$. For $\pi \in \mathrm{Pai}(n,q)$, let

$$I(\pi) := \{j \in \mathbf{N} | j \not\equiv n \,(\mathrm{mod}\, 2), \pi_j \equiv \epsilon \,(\mathrm{mod}\, 2), \pi_j \geq \pi_{j+1} + 2\}.$$

Then the Spaltenstein map (cf. [He])

$$S : \mathrm{Pai}(n,q) \to P_\epsilon(n)$$

is defined as

$$S(\pi)_j := \begin{cases} \pi_j - 1 & (j \in I(\pi)) \\ \pi_j + 1 & (j - 1 \in I(\pi)) \\ \pi_j & (otherwise) \end{cases}$$

**Theorem 4.5.** *Let $G$ be $SO(V)$ or $Sp(V)$ according as $\epsilon = 0$ or $\epsilon = 1$. Let $x \in \mathfrak{g}$ be a nilpotent element of type $\mathbf{d} \in P_\epsilon(n)$. For $\pi \in$* $\mathrm{Pai}(n,q)$ *, define* $\mathrm{Pol}(x,\pi)$ *to be the set of polarizations of $x$ with Levi type $\pi$ (cf. Example 4.2). Then* $\mathrm{Pol}(x,\pi) \neq \emptyset$ *if and only if $S(\pi) = \mathbf{d}$.*

*Proof.* The proof of this theorem can be found in [He], Theorem 7.1, (a). But we prove here that $\mathrm{Pol}(x,\pi) \neq \emptyset$ if $S(\pi) = \mathbf{d}$ because we will later use this argument. There is a basis $\{e(i,j)\}$ of $V$ indexed by the Young diagram $\mathbf{d}$ with the following properties (cf. [S-S], p.259, see also [C-M], 5.1.)

(i) $\{e(i,j)\}$ is a Jordan basis of $x$, that is, $xe(i,j) = e(i-1,j)$ for $(i,j) \in \mathbf{d}$.

(ii) $< e(i,j), e(p,q) > \neq 0$ if and only if $p = d_j - i + 1$ and $q = \beta(j)$, where $\beta$ is a permutation of $\{1, 2, ..., d^1\}$ which satiesfies: $\beta^2 = id$, $d_{\beta(j)} = d_j$, and $\beta(j) \not\equiv j \pmod 2$ if $d_j \not\equiv \epsilon \pmod 2$. One can choose an arbitrary $\beta$ within these restrictions.

For a sequence $(p_1, ..., p_s)$ with $\pi = \mathrm{ord}(p_1, ..., p_s)$ and $p_i = p_{s+1-i}$, $(1 \leq i \leq s)$, we shall construct an admissible flag $F$ of type $(p_1, ..., p_s)$ such that $xF_i \subset F_{i-1}$ for all $i$. We proceed by the induction on $s$. When $s = 1$, $\pi = [1^n]$ and $\pi = \mathbf{d}$. In this case, $x = 0$ and $F$ is a trivial flag $F_1 = V$. When $s > 1$, we shall construct an isotropic flag $0 \subset F_1 \subset F_{s-1} \subset V$. Put $p := p_1 (= p_s)$ and let $\rho := \mathrm{ord}(p_2, ..., p_{s-1}) \in \mathrm{Pai}(n-2p, q)$. Then we have

$$\rho_j := \begin{cases} \pi_j - 2 & (j \leq p) \\ \pi_j & (j > p) \end{cases}$$

Let

$$S' : \mathrm{Pai}(n-2p, q) \to P_\epsilon(n-2p)$$

be the Spatenstein map and we put $\mu := S'(\rho)$. There are two cases (A) and (B). The first case (A) is when $i(\pi) = \{p\} \cup I(\rho)$ and $p \notin I(\rho)$. In this case, $p \not\equiv n \pmod 2$, $\pi_p \equiv \epsilon \pmod 2$ and $\pi_p = \pi_{p+1} + 2$. Now we have

$$\mu_j = d_j - 2, (j < p),$$
$$\mu_p = d_p - 1,$$
$$\mu_{p+1} = d_{p+1} - 1,$$
$$\mu_j = d_j, (j > p+1),$$

where $d_p = d_{p+1}$. The second case is exactly when (A) does not occur. In this case, $I(\pi) = I(\rho)$ and

$$\mu_j = d_j - 2, (j \leq p),$$
$$\mu_j = d_j, (j > p).$$

Let us assume that the case (A) occurs. We choose the basis $e(i,j)$ of $V$ in such a way that the permutaion $\beta$ satisfies $\beta(p) = p+1$. There are two choices for $F_1$. The first one is to put

$$F_1 = \Sigma_{1 \leq j \leq p} \mathbf{C} e(1,j).$$

The second one is to put

$$F_1 = \Sigma_{1 \le j \le p+1, j \ne p} \mathbf{C} e(1, j).$$

In any case, we put $F_{s-1} = F_1^{\perp}$. Then $x$ induces a nilpotent endomorphism of $F_{s-1}/F_1$ of type $\mu$. Next assume that the case (B) occurs. In this case, we put

$$F_1 = \Sigma_{1 \le j \le p} \mathbf{C} e(1, j)$$

and $F_{s-1} = F_1^{\perp}$. Then $x$ induces a nilpotent endomorphism of $F_{s-1}/F_1$ of type $\mu$. By the induction on $s$, we have an admissible filtration $0 \subset F_1 \subset \ldots \subset F_{s-1} \subset V$ with desired properties. Let $P$ be the stabilizer group of $F$. Then it is clear that $x \in \mathfrak{n}(P)$. By an explicit calculation $\dim \mathcal{O}_x = 2 \dim G/P$.

**Theorem 4.6.** *Let $G$ and $\mathfrak{g}$ be the same as Theorem 4.5. Let $x \in \mathfrak{g}$ be a nilpotent element of type $\mathbf{d}$ and denote by $\mathcal{O}$ the orbit containing $x$. Assume that $P$ is a polarization of $x$ with Levi type $\pi$. Let*

$$\mu : T^*(G/P) \to \bar{\mathcal{O}}$$

*be the Springer map. Then*

$$\deg(\mu) := \begin{cases} 2^{\sharp I(\pi) - 1} & (q = \epsilon = 0, \pi^i \not\equiv 0 \,(\mathrm{mod}\, 2)\, \exists i) \\ 2^{\sharp I(\pi)} & (q + \epsilon \ge 1 \,\mathrm{or}\, q = \epsilon = 0,\ \pi^i \equiv 0 \,(\mathrm{mod}\, 2)\, \forall i) \end{cases}$$

*Moreover, if $\deg(\mu) = 1$, then the Levi type of $P$ is unique. In other words, if two polarizations of $x$ respectively give Springer resolutions of $\bar{\mathcal{O}}$, then they have the same Levi type.*

*Proof.* The first part is [He], Theorem 7.1, (d) (cf. [He], §1). The proof of the second part is rather technical, but for the completeness, we include it here. Let

$$B(\mathbf{d}) = \{j \in \mathbf{N}; d_j > d_{j+1}, d_j \not\equiv \epsilon \,(\mathrm{mod}\, 2)\}.$$

Note that $S(\pi) = \mathbf{d}$, where $S$ is the Spaltenstein map. When $\epsilon = 0$, $B(\mathbf{d}) = \emptyset$ if and only if $q = 0$ and $d^i \equiv 0$ (mod 2) for all $i$. Assume that $B(\mathbf{d}) = \emptyset$. Since $\deg(\mu) = 1$, by the first part of our theorem, $\sharp I(\pi) = 0$. Then $\pi = \mathbf{d}$. Assume that $B(\mathbf{d}) \ne \emptyset$. If $q \ne 0$ for our $\pi$ or $\epsilon = 1$, then $\sharp I(\pi) = 0$; hence $\pi = \mathbf{d}$. If $\epsilon = 0$ and $q = 0$ for $\pi$, then $\sharp I(\pi) = 1$. Since $\sharp I(\pi) = 1/2 \,\sharp\{j; d_j \equiv 1 \,(\mathrm{mod}\, 2)\}$ by [He], Lemma 6.3, (b). This implies that $\sharp\{j; d_j \equiv 1 \,(\mathrm{mod}\, 2)\} = 2$. Note that $\pi$ with $q = 0$ is uniquely determined by $\mathbf{d}$ because the Spaltenstein map is injective ([He], Prop. 6.5, (a)).

Now let us prove the second part of our theorem. When $\epsilon = 1$, we should have $\pi = \mathbf{d}$ by the argument above. Next consider the case where $\epsilon = 0$. Assume that there exist two polarizations $P_1$ and $P_2$ giving Springer resolutions. Let $\pi_1$ and $\pi_2$ be their Levi types. Assume that $\pi_1 \in \mathrm{Pai}(n, 0)$ and $\pi_2 \in \mathrm{Pai}(n, q_2)$ with $q_2 > 0$. By the argument above, we see that $\sharp\{j; d_j \equiv 1 \,(\mathrm{mod}\, 2)\} = 2$. On the other hand, since $q_2 > 0$, $\pi_2 = \mathbf{d}$. This shows that $q_2 = 2$; but, when $\epsilon = 0$, $q_2 \neq 2$ by Example 4.2, which is a contradiction. Hence, in this case, $\pi$ is also uniquely determined by $\mathbf{d}$.

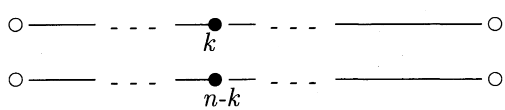
## §5. Equivalence relation in the set of parabolic subgroups

**Proposition 5.1.** *Let $G$ be a complex simple Lie group. Assume that $b_2(G/P) = 1$. Then the following are equivalent.*

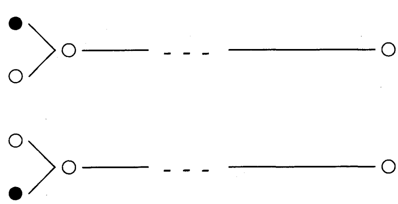
*(i)* $\deg\mu = 1$ *and* $\mathrm{Codim}(\mathrm{Exc}(\mu)) \geq 2$,

*(ii) The single marked Dynkin diagram associated with $P$ is one of the following:*

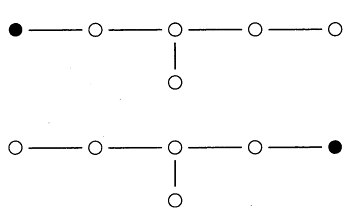
$A_{n-1}$ $(k < n/2)$



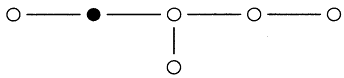

$D_n$ $(n : \text{odd} \geq 4)$

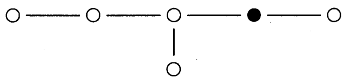


$E_{6,I}$:



$E_{6,II}$:





**Remark 5.2.** *In (ii) there are exactly two different markings for each Dynkin diagram $A_{n-1}$ with $k < n/2$, $D_n$, $E_{6,I}$ or $E_{6,II}$. They are called* dual *marked Dynkin diagrams. Let $P$ and $P'$ be the corresponding (conjugacy classes of) parabolic subgroups of $G$. Then $\mathfrak{p}$ and $\mathfrak{p}'$ have conjugate Levi factors by Proposition 6.3 of [B-C]. This implies that $P$ and $P'$ have the same Richardson orbit.*

*Proof of Proposition 5.1.* Assume that the single marked Dynkin diagram is one of first two series in (ii). For this case we shall prove in Lemmas 5.4 and 5.6, that the Springer map $\mu : T^*(G/P) \to \bar{\mathcal{O}}$ becomes a small resolution (cf. Notation (2)). If the single marked diagram is of type $E_{6,I}$, then the Richardson orbit $\mathcal{O}$ of $P$ coincides with orbit $\mathcal{O}_{2A_1}$ in the list of [C-M],p.129, which has dimension 32. The maximal orbit contained in $\bar{\mathcal{O}}_{2A_1} - \mathcal{O}_{2A_1}$ is $\mathcal{O}_{A_1}$, which has dimension 22. This shows that $\mathrm{Sing}(\bar{\mathcal{O}})$ has codimension $\geq 10$ in $\bar{\mathcal{O}}$. On the other hand, since $\pi_1(\mathcal{O}_{2A_1}) = 1$ (cf. [C-M], p.129), $\deg(\mu) = 1$. If $\mu$ is a divisorial birational contraction, then $\mathrm{Codim}(\mathrm{Sing}(\bar{\mathcal{O}}) \subset \bar{\mathcal{O}}) = 2$ (cf. [Na 1, Cor. 1.5]), which is absurd. Hence $\mu$ should be a small resolution. If the single marked diagram is of type $E_{6,II}$, then the Richardson orbit $\mathcal{O}$ of $P$ coincides with the orbit $\mathcal{O}_{A_2+2A_1}$ in the list of [C-M], p.129, which has dimension 50. Moreover, $\pi_1(\mathcal{O}_{A_2+2A_1}) = 1$. By looking at the closure ordering of $E_6$ orbits (cf. [C], p.441), we see that the maximal orbit contained in $\bar{\mathcal{O}}_{A_2+2A_1} - \mathcal{O}_{A_2+2A_1}$ is the orbit $\mathcal{O}_{A_2+A_1}$, which has dimension 46. By the same argument as above, $\mu$ becomes a small resolution.

To prove the implication (i) $\Rightarrow$ (ii), let us assume that the single marked Dynkin diagram is not contained in the list of (ii). Let $\mathcal{O}$ be the corresponding Richardson orbit. We shall first prove that $\bar{\mathcal{O}}$ contains a nilpotent orbit $\mathcal{O}'$ of codimension 2 (STEP 1). Next we shall prove that $\mathcal{O}'$ is $\mu$-relevant(STEP 2). These imply that $\mu$ is a divisorial birational contraction map if $\deg(\mu) = 1$.

STEP 1: Assume that $\mathfrak{g}$ is classical. If $\mathfrak{g}$ is of type $A_{n-1}$, then we must look at the single marked Dynkin diagram with $k = n/2$. In this case, we will see in Remark 5.5 that $\mu$ is a divisorial birational contraction map.

When $\mathfrak{g}$ is of type $B_n$, $C_n$ or $D_n$, the parabolic subgroup $P$ is a stabilizer group of an admissible isotropic flag. Its flag type is written as $(k, q, k)$. When $\mathfrak{g}$ is of type $B_n$, we have $k > 0$, $q > 0$ and $2k+q = 2n+1$. When $\mathfrak{g}$ is of type $C_n$ or of type $D_n$, we have $k > 0$, $q \geq 0$ and $2k+q = 2n$. Denote by $\pi$ the dual partition of $\mathrm{ord}(k, q, k)$ and call $\pi$ the Levi type of $P$.

Assume that $\mathfrak{g}$ is of type $B_n$. The Levi type of $P$ is given by

$$\pi := \begin{cases} [3^{2n+1-2k}, 2^{3k-2n-1}] & (k > (2n+1)/3) \\ [3^k, 1^{2n-3k+1}] & (k \leq (2n+1)/3) \end{cases}$$

When $k > (2n+1)/3$, $k$ must be an odd number. In fact, if $k$ is even, then $I(\pi) \neq \emptyset$ and $\deg(\mu) > 1$ (cf. Theorem 4.6). Recall that the Richardson orbit $\mathcal{O}$ of $P$ has the Jordan type $S(\pi)$, where $S$ is the Spaltenstein map (cf. Theorem 4.5). Since now $I(\pi) = \emptyset$, $S(\pi) = \pi$. Let us consider the nilpotent orbit $\mathcal{O}'$ of the Jordan type $[3^{2n+1-2k}, 2^{3k-2n-3}, 1^4]$ (resp. $[3^{k-1}, 2^2, 1^{2n-3k}]$, $[3^{k-1}, 1^3]$) when $k > (2n+1)/3$ (resp. $k < (2n+1)/3$, $k = (2n+1)/3$). In any case, we have $\mathcal{O}' \subset \bar{\mathcal{O}}$. By the dimension formula of nilpotent orbits ([C-M, Corollary 6.1.4]), we see that $\dim \mathcal{O}' = \dim \mathcal{O} - 2$.

Assume that $\mathfrak{g}$ is of type $C_n$. The Levi type of $P$ is given by

$$\pi := \begin{cases} [3^{2n-2k}, 2^{3k-2n}] & (k > 2n/3) \\ [3^k, 1^{2n-3k}] & (k \leq 2n/3) \end{cases}$$

When $k \leq 2n/3$, $k$ must be an even number. In fact, if $k$ is odd, then $I(\pi) \neq \emptyset$ and $\deg(\mu) > 1$ (cf. Theorem 4.6). The Richardson orbit $\mathcal{O}$ has the Jordan type $\pi$. Let us consider the nilpotent orbit $\mathcal{O}'$ of the Jordan type $[3^{2n-2k}, 2^{3k-2n-1}, 1^2]$ (resp. $[3^{k-2}, 2^4, 1^{2n-3k-2}]$, $[3^{k-2}, 2^3]$) when $k > 2n/3$ (resp. $k < 2n/3$, $k = 2n/3$). In any case, we have $\mathcal{O}' \subset \bar{\mathcal{O}}$. By the dimension formula of nilpotent orbits ([C-M, Corollary 6.1.4]), we see that $\dim \mathcal{O}' = \dim \mathcal{O} - 2$.

Assume that $\mathfrak{g}$ is of type $D_n$. First assume that the Levi type of $P$ is $[2^k]$. The single marked Dynkin diagram is not contained in the list of (ii) exactly when $k$ is even. In this case, we will see in Remark 5.7 that $\mu$ is a divisorial birational contraction map. We next assume $k < n$. In this case, the Levi type of $P$ is given by

$$\pi := \begin{cases} [3^{2n-2k}, 2^{3k-2n}] & (n > k > 2n/3) \\ [3^k, 1^{2n-3k}] & (k \leq 2n/3) \end{cases}$$

When $k > 2n/3$, $k$ must be an even number. In fact, if $k$ is odd, then $I(\pi) \neq \emptyset$ and $\deg(\mu) > 1$ (cf. Theorem 4.6). Recall that the Richardson orbit $\mathcal{O}$ of $P$ has the Jordan type $S(\pi)$, where $S$ is the Spaltenstein map (cf. Theorem 4.5). Since now $I(\pi) = \emptyset$, $S(\pi) = \pi$. Let us consider the nilpotent orbit $\mathcal{O}'$ of the Jordan type $[3^{2n-2k}, 2^{3k-2n-2}, 1^4]$ (resp. $[3^{k-1}, 2^2, 1^{2n-3k-1}]$, $[3^{k-1}, 1^3]$) when $k > 2n/3$ (resp. $k < 2n/3$, $k = 2n/3$). In any case, we have $\mathcal{O}' \subset \bar{\mathcal{O}}$. By the dimension formula of nilpotent orbits ([C-M, Corollary 6.1.4]), we see that $\dim \mathcal{O}' = \dim \mathcal{O} - 2$.

When $\mathfrak{g}$ is of type $G_2$, there are exactly two single marked Dynkin diagrams. In the table of $G_2$ nilpotent orbits in [C-M, p.128], $\mathcal{O}_{G_2(a_1)}$ is the Richardson orbit of the parabolic subgroups corresonding to these diagrams. The orbit $\mathcal{O}_{\tilde{A}_1}$ is contained in $\bar{\mathcal{O}}_{G_2(a_1)}$. Note that $\dim \mathcal{O}_{G_2(a_1)} = 10$ and $\dim \mathcal{O}_{\tilde{A}_1} = 8$.

When $\mathfrak{g}$ is of type $F_4$, there are exactly four single marked Dynkin diagrams. Richardson orbits of the parabolic subgroups corresponding to them are $\mathcal{O}_{A_2}$, $\mathcal{O}_{\tilde{A}_2}$, $\mathcal{O}_{F_4(a_3)}$ in the table of [C-M, p.128]. Note that two non-conjugate parabolic subgroups have the same Richardson orbit $\mathcal{O}_{F_4(a_3)}$. By looking at the closure ordering of $F_4$ orbits [C, p.440], we see that the closure of each orbit contain a codimension 2 orbit.

When $\mathfrak{g}$ is of type $E_6$, there are exactly 6 single marked Dynkin diagrams. Four of them are already contained in the list of (ii). The Richardson orbits corresponding to other diagrams are $\mathcal{O}_{A_2}$ and $\mathcal{O}_{D_4(a_1)}$ in the list of $E_6$ nilpotent orbits in [C-M, p.129]. $\bar{\mathcal{O}}_{A_2}$ contains a codimension 2 orbit $\mathcal{O}_{3A_1}$. $\bar{\mathcal{O}}_{D_4(a_1)}$ contains a codimension 2 orbit $\mathcal{O}_{A_3+A_1}$.

When $\mathfrak{g}$ is of type $E_7$, there are exactly 7 single marked Dynkin diagrams. Richardson orbits of the parabolic subgroups corresponding to them are $\mathcal{O}_{(3A_1)''}$, $\mathcal{O}_{A_2}$, $\mathcal{O}_{2A_2}$, $\mathcal{O}_{A_2+3A_1}$, $\mathcal{O}_{D_4(a_1)}$, $\mathcal{O}_{A_3+A_2+A_1}$ and $\mathcal{O}_{A_4+A_2}$ in the table of [C-M, p.130-p.131]. By looking at the closure ordering of $E_7$ orbits [C, p.442], we see that the closure of each orbit contains a codimension 2 orbit.

When $\mathfrak{g}$ is of type $E_8$, there are exactly 8 single marked Dynkin diagrams. In the table of [C-M, p.132-p.134], Richardson orbits of the parabolic subgroups corresponding to them are $\mathcal{O}_{A_2}$, $\mathcal{O}_{2A_2}$, $\mathcal{O}_{D_4(a_1)}$, $\mathcal{O}_{D_4(a_1)+A_2}$, $\mathcal{O}_{A_4+A_2}$, $\mathcal{O}_{A_4+A_2+A_1}$, $\mathcal{O}_{E_8(a_7)}$ and $\mathcal{O}_{A_6+A_1}$. By looking at the closure ordering of $E_8$ orbits, we see that the closure of each orbit contains a codimension 2 orbit.

STEP 2: Assume that $\mathfrak{g}$ is classical. Let $f : \tilde{\mathcal{O}} \to \bar{\mathcal{O}}$ be the normalization map. By STEP 1 we may assume that $\bar{\mathcal{O}}$ contains a codimension 2 orbit $\mathcal{O}'$. In the classical case, by [K-P, 14], we see that $\tilde{\mathcal{O}}$ has actually singularities along $f^{-1}(\mathcal{O}')$. The Springer map $\mu$ is factorized as

$$T^*(G/P) \xrightarrow{\mu'} \tilde{\mathcal{O}} \xrightarrow{f} \bar{\mathcal{O}}.$$

If $\deg(\mu) = 1$, then $\mu'$ is a birational maps of normal varieties. Then, by Zariski's main theorem, $\mu'$ must have a positive dimensional fiber over a point of $f^{-1}(\mathcal{O}')$. This implies that $\mu$ is a divisorial birational map.

Assume that $\mathfrak{g}$ is of exceptional type. As explained above, the codimension 2 orbit $\mathcal{O}'$ of $\bar{\mathcal{O}}$ can be specified. It is enough to show that $\mathcal{O}'$ is $\mu$-relevant. By the previous proposition, we have to check that $V_{(x,1)}$ occurs in $\epsilon^W_{W(L)}$ for $x \in \mathcal{O}'$. In [Al], Alvis describes an irreducible decomposition of the induced representation $\mathrm{Ind}^W_{W(L)}(\rho)$ for any irreducible representation $\rho$ of $W(L)$. Hence, this can be done by using the tables of [Al] (see also the tables in [A-L], [B-L] and [C, 13.3]). Note that Spaltenstein [S1] (cf. the footnote of p.68, [B-M]) has already checked that a special orbit is $\mu$-relevant by using these tables. Hence it is enough to check for non-special orbits $\mathcal{O}'$. One can find which orbits are non-special in the tables of [C-M, 8.4]. Q.E.D.

**Example 5.3.** (**Mukai flops**): *Let $P$ and $P'$ be two parabolic subgroups of $G$ which correspond to dual marked Dynkin diagrams in the proposition above. Let $\mathcal{O}$ be the Richardson orbit of them. Then we have a diagram*

$$T^*(G/P) \xrightarrow{\mu} \bar{\mathcal{O}} \xleftarrow{\mu'} T^*(G/P').$$

*The birational maps $\mu$ and $\mu'$ are both small by the proposition, Lemmas 5.4 and 5.6. Moreover, $T^*(G/P) - - \to T^*(G/P')$ is not an isomorphism. In fact, $T^*(G/P)$, $T^*(G/P')$ and $\bar{\mathcal{O}}$ all have $G$ actions, and $\mu$ and $\mu'$ are $G$-equivariant. If the birational map is an isomorphism, this would become a $G$-equivariant isomorphism. This implies that $G/P$ and $G/P'$ are isomorphic as $G$-varieties. In particular, $P$ and $P'$ are $G$-conjugate, which is absurd. Since the relative Picard numbers $\rho(T^*(G/P)/\bar{\mathcal{O}})$ and $\rho(T^*(G/P')/\bar{\mathcal{O}})$ equal 1, we see that the diagram above is a* flop. *The diagram is called a* Mukai flop *of type $A_{n-1,k}$ (resp. $D_n$, $E_{6,I}$, $E_{6,II}$) according to the type of the corresponding marked Dynkin diagram.*

We shall describe Mukai flops of type A and D in terms of flags.

**Mukai flop of type A**. Let $x \in \mathfrak{sl}(n)$ be a nilpotent element of type $[2^k, 1^{n-2k}]$ and let $\mathcal{O}$ be the nilpotent orbit containing $x$. By Theorem 4.4, there are two polarizations $P$ and $P'$ of $x$, where $P$ has the flag type $(k, n-k)$ and $P'$ has the flag type $(n-k, k)$. The closure $\bar{\mathcal{O}}$ of $\mathcal{O}$ admits two Springer resolutions

$$T^*(SL(n)/P) \xrightarrow{\pi} \bar{\mathcal{O}} \xleftarrow{\pi'} T^*(SL(n)/P').$$

Note that $SL(n)/P$ is isomorphic to the Grassmannian $G(k,n)$ and $SL(n)/P'$ is isomorphic to $G(n-k,n)$.

**Lemma 5.4.** *When $k < n/2$, $\pi$ and $\pi'$ are both small birational maps and the diagram becomes a flop.*

*Proof.* The closure $\bar{\mathcal{O}}$ consists of finite number of orbits $\{\mathcal{O}_{[2^i,1^{n-2i}]}\}_{0\leq i\leq k}$. The main orbit $\mathcal{O}_{[2^k,1^{n-2k}]}$ is an open set of $\bar{\mathcal{O}}$. A fiber of $\pi$ (resp. $\pi'$) over a point of $\mathcal{O}_{[2^i,1^{n-2i}]}$ is isomorphic to the Grassmannian $G(k-i,n-2i)$ (resp. $G(n-i-k,n-2i)$). By a simple dimension count, if $k < n/2$, then $\pi$ and $\pi'$ are both small birational maps. Next let us prove that the diagram is a flop. This is already proved in Example 5.3. But, we shall give here a more explicit proof. Let $\tau \subset \mathcal{O}^{\oplus n}_{G(k,n)}$ (resp. $\tau' \subset \mathcal{O}^{\oplus n}_{G(n-k,n)}$) be the universal subbundle. Denote by $T$ (resp. $T'$) the pull-back of $\tau$ (resp. $\tau'$) by the projection $T^*G(k,n) \to G(k,n)$ (resp. $T^*G(n-k,n) \to G(n-k,n)$). We shall describe the strict transform of $\wedge^k T$ by the birational map $T^*G(k,n) - - \to T^*G(n-k,n)$. Take a point $y \in \mathcal{O}_{[2^k,1^{n-2k}]}$. Note that $T^*G(k,n)$ is naturally embedded in $G(k,n) \times \bar{\mathcal{O}}$. Then the fiber $\pi^{-1}(y)$ consists of one point $([\mathrm{Im}(y)], y) \in G(k,n) \times \bar{\mathcal{O}}$. The fiber $T_{\pi^{-1}(y)}$ of the vector bundle $T$ over $\pi^{-1}(y)$ coincides with the vector space $\mathrm{Im}(y)$. Hence $(\wedge^k T)_{\pi^{-1}(y)}$ is isomorphic to $\wedge^k \mathrm{Im}(y)$. Now let $L$ be the strict transform of $\wedge^k T$ by $T^*G(k,n) - - \to T^*G(n-k,n)$. First note that $(\pi')^{-1}(y)$ also consists of one point $([\mathrm{Ker}(y)], y) \in G(n-k,n) \times \bar{\mathcal{O}}$. Then, by definition, $L_{(\pi')^{-1}(y)} = \wedge^k \mathrm{Im}(y)$. Since $\wedge^k \mathrm{Im}(y) \cong (\wedge^{n-k}\mathrm{Ker}(y))^*$, we see that $L \cong (\wedge^{n-k}T')^{-1}$. Now $\wedge^k T$ is a negative line bundle. On the other hand, its strict transform $L$ becomes an ample line bundle. This implies that our diagram is a flop. Q.E.D.

**Remark 5.5.** *When $k = n/2$, $\pi$ and $\pi'$ are both divisorial birational contraction maps. Moreover, two resolutions are isomorphic.*

**Mukai flop of type D**. Assume that $k$ is an odd integer with $k \geq 3$. Let $V$ be a $\mathbf{C}$-vector space of $\dim 2k$ with a non-degenerate symmetric form $<,>$. Let $x \in \mathfrak{so}(V)$ be a nilpotent element of type $[2^{k-1}, 1^2]$ and let $\mathcal{O}$ be the nilpotent orbit containing $x$. Let $S : \mathrm{Pai}(2k,0) \to P_\epsilon(2k)$ be the Spaltenstein map, where $\epsilon = 0$ in our case. Then, for $\pi := (2^k) \in \mathrm{Pai}(2k,0)$, $S(\pi) = [2^{k-1}, 1^2]$. Let us recall the construction of the stabilized flags by the polarizations of $x$ in the proof of Theorem 4.5. Since $I(\pi) = \{k\}$, the case (A) occurs (cf. the proof of Theorem 4.5); hence there are two choices of the flags. We denote by $P^+$ the stabilizer subgroup of $SO(V)$ of one flag, and denote by $P^-$ the stabilizer subgroup of another one. Let $G_{iso}(k,V)$ be the orthogonal Grassmannian which parametrizes $k$ dimensional isotropic subspaces of $V$. $G_{iso}(k,V)$

has two connected components $G^{+}{}_{iso}(k,V)$ and $G^{-}{}_{iso}(k,V)$. Note that $SO(V)/P^{+} \cong G^{+}{}_{iso}(k,V)$ and $SO(V)/P^{-} \cong G^{-}{}_{iso}(k,V)$. The closure $\bar{\mathcal{O}}$ of $\mathcal{O}$ admits two Springer resolutions

$$T^{*}(SO(V)/P^{+}) \overset{\pi^{+}}{\to} \bar{\mathcal{O}} \overset{\pi^{-}}{\leftarrow} T^{*}(SO(V)/P^{-}).$$

**Lemma 5.6.** *$\pi^{+}$ and $\pi^{-}$ are both small birational maps and the diagram becomes a flop.*

*Proof.* The closure $\bar{\mathcal{O}}$ consists of the orbits $\{\mathcal{O}_{[2^{k-2i-1},1^{4i+2}]}\}_{1\leq i\leq 1/2(k-1)}$. The main orbit is an open set of $\bar{\mathcal{O}}$. A fiber of $\pi^{+}$ (resp. $\pi^{-}$) over a point of $\mathcal{O}_{[2^{k-2i-1},1^{4i+2}]}$ is isomorphic to $G^{+}{}_{iso}(2i+1,4i+2)$ (resp. $G^{-}{}_{iso}(2i+1,4i+2)$). By dimension counts of each orbit and of each fiber, we see that $\pi^{+}$ and $\pi^{-}$ are both small birational maps. Next let us prove that the diagram is a flop. This is already proved in Example 5.3. But, we shall give here a more explicit proof. Let $\tau^{+} \subset \mathcal{O}^{\oplus 2k}_{G^{+}{}_{iso}(k,V)}$ (resp. $\tau^{-} \subset \mathcal{O}^{\oplus 2k}_{G^{-}{}_{iso}(k,V)}$) be the universal subbundle. Denote by $T^{+}$ (resp. $T^{-}$) the pull-back of $\tau^{+}$ (resp. $\tau^{-}$) by the projection $T^{*}(G^{+}{}_{iso}(k,V)) \to G^{+}{}_{iso}(k,V)$ (resp. $T^{*}(G^{-}{}_{iso}(k,V)) \to G^{-}{}_{iso}(k,V)$). We shall describe the strict transform of $\wedge^{k}T^{-}$ by the birational map $T^{*}(G^{-}{}_{iso}(k,V)) - - \to T^{*}(G^{+}{}_{iso}(k,V))$. Take a point $y \in \mathcal{O}_{[2^{k-1},1^{2}]}$. Let $g \in SO(V)$ be an element such that $gxg^{-1} = y$. Note that $T^{*}(G^{+}{}_{iso}(k,V))$ (resp. $T^{*}(G^{-}{}_{iso}(k,V))$) is naturally embedded in $G^{+}{}_{iso}(k,V) \times \bar{\mathcal{O}}$ (resp. $G^{-}{}_{iso}(k,V)\times\bar{\mathcal{O}}$). Then the fiber $(\pi^{+})^{-1}(y)$ (resp. $(\pi^{-})^{-1}(y)$) consists of one point $([F_{y}^{+}], y) \in G^{+}{}_{iso}(k,V)\times\bar{\mathcal{O}}$ (resp. $([F_{y}^{-}], y) \in G^{-}{}_{iso}(k,V)\times\bar{\mathcal{O}}$) where $F_{y}^{+} \subset V$ (resp. $F_{y}^{-} \subset V$) is the flag stabilized by $gP^{+}g^{-1}$ (resp. $gP^{-}g^{-1}$). Note that $gP^{+}g^{-1}$ and $gP^{-}g^{-1}$ are both polarizations of $y$. Let us recall the construction of flags in the proof of Theorem 4.5. For $y$ we choose a Jordan basis $\{e(i,j)\}$ of $V$ as in the proof of Theorem 4.5. Since $\mathbf{d} = [2^{k-1},1^{2}]$, $\beta$ is a permutation of $\{1,2,...,k,k+1\}$. But it preserves the subsets $\{1,2,...,k-1\}$ and $\{k,k+1\}$ respectively. We assume that $\beta(k) = k+1$ and $\beta(k+1) = k$. In our situation, the case (A) occurs. There are two choices of the flags:

$$\Sigma_{1\leq j\leq k-1}\mathbf{C}e(1,j) + \mathbf{C}e(1,k)$$

and

$$\Sigma_{1\leq j\leq k-1}\mathbf{C}e(1,j) + \mathbf{C}e(1,k+1).$$

Note that one of these is stabilized by $gP^{+}g^{-1}$ and another one is stabilized by $gP^{-}g^{-1}$. We may assume that

$$F_{y}^{+} = \Sigma_{1\leq j\leq k-1}\mathbf{C}e(1,j) + \mathbf{C}e(1,k),$$

and

$$F_y^- = \Sigma_{1 \le j \le k-1} \mathbf{C}e(1,j) + \mathbf{C}e(1,k+1).$$

Since $\mathrm{Ker}(y) = \Sigma_{1\le j\le k+1} e(1,j)$ and $\mathrm{Im}(y) = \Sigma_{1 \le j \le k-1} e(1,j)$, we have two exact sequences

$$0 \to \mathrm{Ker}(y)/F_y^+ \to V/F_y^+ \to \mathrm{Im}(y) \to 0,$$

and

$$0 \to \mathrm{Im}(y) \to F_y^- \to F_y^-/\mathrm{Im}(y) \to 0.$$

Since $F_y^-/\mathrm{Im}(y) \cong \mathrm{Ker}(y)/F_y^+$, we conclude that

$$\wedge^k F_y^- \cong \wedge^k (V/F_y^+).$$

Let $L$ be the strict transform of $\wedge^k T^-$ by the birational map $T^*(G^-{}_{iso}(k,V)) - - \to T^*(G^+{}_{iso}(k,V))$. The fiber $T^-_{(\pi^-)^{-1}(y)}$ of the vector bundle $T^-$ is isomorphic to the vector space $\wedge^k F_y^-$. Hence, by the definition of $L$, $L_{(\pi^+)^{-1}(y)} = \wedge^k F_y^-$. By the observation above, we see that $L_{(\pi^+)^{-1}(y)} = \wedge^k(V/F_y^+)$. This shows that $L \cong (\wedge^k T^+)^{-1}$. Now $\wedge^k T^-$ is a negative line bundle. On the other hand, its strict transform $L$ is an ample line bundle. This implies that our diagram is a flop. Q.E.D.

**Remark 5.7.** *When $k$ is an even integer with $k \ge 2$, there are two nilpotent orbits $\mathcal{O}^+$ and $\mathcal{O}^-$ with Jordan type $[2^k]$. They have Springer resolutions*

$$T^*(G^+{}_{iso}(k,2k)) \to \bar{\mathcal{O}}^+,$$

*and*

$$T^*(G^-{}_{iso}(k,2k)) \to \bar{\mathcal{O}}^-.$$

*These resolutions are both divisorial birational contraction maps. When $k = 1$, three varieties $T^*(G^+{}_{iso}(1,2))$, $T^*(G^-{}_{iso}(1,2))$ and $\bar{\mathcal{O}}$ are all isomorphic.*

Let us return to the general situation. The following notion will play important roles in the later section.

**Definition 1.** *(i) Let $\mathcal{D}$ be a marked Dynkin diagram with exactly $l$ marked vertices. Choose $l-1$ marked vertices from them. Making the remained one vertex unmarked, we have a new marked Dynkin diagram $\bar{\mathcal{D}}$. This procedure is called a* contraction *of a marked Dynkin diagram. Next remove from $\mathcal{D}$ these $l-1$ vertices and edges touching these vertices. We then have a (non-connected) diagram; one of its connected component is a single marked Dynkin diagram. Assume that this* *single marked Dynkin diagram is one of those listed in Proposition 5.1. Replace this single marked Dynkin diagram by its dual and leave other components untouched. Connecting again removed edges and vertices as before, we obtain a new marked Dynkin diagram $\mathcal{D}'$. Note that $\mathcal{D}'$ (resp. $\bar{\mathcal{D}}$) has exactly $l$ (resp. $l-1$) marked vertices. Now we say that $\mathcal{D}'$ is* adjacent *to $\mathcal{D}$ by means of $\bar{\mathcal{D}}$.*

*(ii) Two marked Dynkin diagrams $\mathcal{D}$ and $\mathcal{D}'$ are called* equivalent *and are written as $\mathcal{D} \sim \mathcal{D}'$ if there is a finite chain of adjacent diagrams connecting $\mathcal{D}$ and $\mathcal{D}'$.*

*(iii) Let $P$ be a parabolic subgroup of $G$ and let $\mathcal{D}_P$ be the corresponding marked Dynkin diagram. Two parabolic subgroups $P$ and $P'$ of $G$ are called* equivalent *and are written as $P \sim P'$ if $\mathcal{D}_P \sim \mathcal{D}_{P'}$.*

**Example 5.8.** *Let us consider the marked Dynkin diagram*

$$\mathcal{D}: \quad \circ - \underset{2}{\bullet} - \underset{3}{\bullet} \Rightarrow \circ$$

*where vertices 2 and 3 are marked. We choose the vertex 3. Making the remained one vertex (= the vertex 2) unmarked, we have a marked Dynkin diagram*

$$\bar{\mathcal{D}}: \quad \circ - \underset{2}{\circ} - \underset{3}{\bullet} \Rightarrow \circ$$

*Now the following marked Dynkin diagram $\mathcal{D}'$ is adjacent to $\mathcal{D}$ by $\bar{\mathcal{D}}$.*

$$\mathcal{D}': \quad \bullet - \underset{2}{\circ} - \underset{3}{\bullet} \Rightarrow \circ$$

## §6. Main Theorem

The following is our main theorem. For the notion of a relative ample cone and a relative movable cone, see [Ka 1], where some elementary roles of these cones in birational geometry are discussed.

**Theorem 6.1.** *Let $\mathcal{O} \subset \mathfrak{g}$ be a nilpotent orbit of a complex simple Lie algebra $\mathfrak{g}$. Assume that its closure $\bar{\mathcal{O}}$ has a Springer resolution $\mu_{P_0} : T^*(G/P_0) \to \bar{\mathcal{O}}$. Then the following hold.*

*(i) For a parabolic subgroup $P$ of $G$ such that $P \sim P_0$, $Y_P := T^*(G/P)$ gives a symplectic resolution of $\bar{O}$. Conversely, any symplectic resolution is a Springer resolution of this form.*

*(ii) The closure $\overline{\mathrm{Amp}}(Y_P/\bar{O})$ of the relative ample cone is a simplicial polyhedral cone.*

*(iii) $\overline{\mathrm{Mov}}(Y_{P_0}/\overline{O}) = \cup_{P\sim P_0}\overline{\mathrm{Amp}}(Y_P/\overline{O})$, where $\overline{\mathrm{Mov}}(Y_{P_0}/\overline{O})$ is the closure of the relative movable cone of $Y_{P_0}$ over $\overline{O}$.*

*(iv) A codimension 1 face of $\overline{\mathrm{Amp}}(Y_P/\overline{O})$ corresponds to a small birational contraction map when it is a face of another ample cone, and corresponds to a divisorial contraction map when it is not a face of any other ample cone.*

*(v) $\{Y_P\}_{P\sim P_0}$ are connected by Mukai flops of type $A$, $D$, $E_{6,I}$ and $E_{6,II}$.*

**Remark 6.2.** *For a classical complex Lie algebra, it is already known which nilpotent orbit closure has a Springer resolution (cf. Theorems 4.5 and 4.6). When $\mathfrak{g}$ is $G_2$, there are exactly 2 nilpotent orbits $O_{G_2}$ and $O_{G_2(a_1)}$ whose closures admit Springer resolutions. When $\mathfrak{g}$ is $F_4$, such orbits are $O_{A_2}$, $O_{\tilde{A}_2}$, $O_{F_4(a_3)}$, $O_{B_3}$, $O_{C_3}$, $O_{F_4(a_2)}$, $O_{F_4(a_1)}$ and $O_{F_4}$. When $\mathfrak{g}$ is $E_6$, such orbits are $O_{2A_1}$, $O_{A_2}$, $O_{2A_2}$, $O_{A_2+2A_1}$, $O_{A_3}$, $O_{D_4(a_1)}$, $O_{A_4}$, $O_{D_4}$, $O_{A_4+A_1}$, $O_{D_5(a_1)}$, $O_{E_6(a_3)}$, $O_{D_5}$, $O_{E_6(a_1)}$, and $O_{E_6}$.*

The statement (ii) of Theorem 6.1 follows from the next Lemma.

**Lemma 6.3.** *Let $G$ be a complex simple Lie group and let $P$ be a parabolic subgroup. Let $\hat{O}$ be the Stein factorization of a Springer map $\mu : Y_P := T^*(G/P) \to \bar{O}$. Then $\overline{Amp}(Y_P/\hat{O})$ is a simplicial polyhedral cone.*

*Proof.* Let $\mathcal{D}$ be the marked Dynkin diagram corresponding to $P$. Assume that $\mathcal{D}$ has $k$ marked vertices, say, $v_1$, ..., $v_k$. Then $b_2(G/P) = k$. Choose $l$ vertices $v_{i_1}, ..., v_{i_l}$, $1 \leq i_1 < ... < i_l \leq k$ and let $\mathcal{D}_{i_1,...,i_l}$ be the marked Dynkin diagram such that exactly these $l$ vertices are marked and its underlying diagram is the same as $\mathcal{D}$. We denote by $X_{i_1,...,i_l}$ the image of $Y_P \subset G/P \times \bar{O}$ by the projection

$$G/P \times \bar{O} \to G/P_{1_1,...,i_l} \times \bar{O}.$$

Let

$$\nu_{i_1,...,i_l} : Y_P \to X_{i_1,...,i_l}$$

be the induced map. Then the Stein factorization of $\nu_{i_1,...,i_l}$ is a birational contraction map, which corresponds to a codimension $k-l$ face of $\overline{Amp}(Y_P/\hat{O})$. We shall denote by $F_{i_1,...,i_l}$ this face. Then $\overline{\mathrm{Amp}}(Y_P/\hat{O})$

is a simplicial polyhedral cone generated by $F_1$, $F_2$, ..., and $F_k$. In fact, any $l$ dimensional face generated by $F_{i_1}$, ..., $F_{i_l}$ corresponds to the Stein factorization of $\nu_{i_1,...,i_l}$, which is not an isomorphism. Q.E.D.

Next assume that two marked Dynkin diagrams $\mathcal{D}$ and $\mathcal{D}'$ are adjacent by means of $\bar{\mathcal{D}}$. We have three parabolic subgroups $P$, $P'$ and $\bar{P}$ of $G$ corresponding to $\mathcal{D}$, $\mathcal{D}'$ and $\bar{\mathcal{D}}$ respectively. One can assume that these subgroups contain the same Borel subgroup $B$ of $G$ and $\bar{P}$ contains both $P$ and $P'$. Let $\mu : T^*(G/P) \to \mathfrak{g}$ and $\mu' : T^*(G/P') \to \mathfrak{g}$ be the Springer maps.

**Proposition 6.4.** *(i) The Richardson orbits $\mathcal{O}$ of $P$ is the Richardson orbit of $P'$*

*(ii) Let $\nu$ be the composed map*

$$T^*(G/P) \to G/P \times \bar{\mathcal{O}} \to G/\bar{P} \times \bar{\mathcal{O}}$$

*and let $\nu'$ be the composed map*

$$T^*(G/P') \to G/P' \times \bar{\mathcal{O}} \to G/\bar{P} \times \bar{\mathcal{O}}.$$

*Then* $\mathrm{Im}(\nu) = \mathrm{Im}(\nu')$.

*(iii) If we put $X := \mathrm{Im}(\nu)$, then*

$$T^*(G/P) \to X \leftarrow T^*(G/P')$$

*is a locally trivial family of Mukai flops of type $A$, $D$, $E_{6,I}$ or $E_{6,II}$. In particular, $\nu$ and $\nu'$ are both small birational maps. If* $\deg(\mu) = 1$, *then* $\deg(\mu') = 1$.

*Proof.* (i): Take a Levi decomposition

$$\bar{\mathfrak{p}} = l(\bar{\mathfrak{p}}) \oplus n(\bar{\mathfrak{p}}).$$

In the reductive Lie algebra $l(\bar{\mathfrak{p}})$, $\mathfrak{p} \cap l(\bar{\mathfrak{p}})$ and $\mathfrak{p}' \cap l(\bar{\mathfrak{p}})$ are parabolic subalgebras corresponding to dual marked Dynkin diagrams in Proposition 5.1. Hence they have conjugate Levi factors by Remark 5.2. On the other hand, we have

$$l(\mathfrak{p}) = l(\mathfrak{p} \cap l(\bar{\mathfrak{p}})),$$

and

$$l(\mathfrak{p}') = l(\mathfrak{p}' \cap l(\bar{\mathfrak{p}})).$$

Therefore, $l(\mathfrak{p})$ and $l(\mathfrak{p}')$ are conjugate. Since $\mathfrak{p}$ and $\mathfrak{p}'$ have conjugate Levi factors, their Richardson orbits coincide (cf. [C-M, Theorem 7.1.3]).

(ii): Let $\mathcal{O}$ be the Richardson orbit of $\mathfrak{p}$ and $\mathfrak{p}'$. Springer maps $\mu : T^*(G/P) \to \bar{\mathcal{O}}$ and $\mu' : T^*(G/P') \to \bar{\mathcal{O}}$ are both $G$-equivariant with respect to natural $G$-actions. Then $U := \mu^{-1}(\mathcal{O})$ and $U' := (\mu')^{-1}(\mathcal{O})$ are open dense orbits of $T^*(G/P)$ and $T^*(G/P')$ respectively. Since $\nu$ and $\nu'$ are proper maps, $\mathrm{Im}(\nu) = \overline{\nu(U)}$ and $\mathrm{Im}(\nu') = \overline{\nu'(U')}$. In the following we shall prove that $\nu(U) = \nu'(U')$.

(ii-1): We regard $T^*(G/P)$ (resp. $T^*(G/P')$) as a closed subvariety of $G/P \times \bar{\mathcal{O}}$ (resp. $G/P' \times \bar{\mathcal{O}}$). By replacing $P'$ by a suitable conjugate in $\bar{P}$, we may assume that there exists an element $x \in \mathcal{O}$ such that $([P], x) \in U$ and $([P'], x) \in U'$. In fact, for a Levi decomposition

$$\bar{\mathfrak{p}} = l(\bar{\mathfrak{p}}) \oplus n(\bar{\mathfrak{p}}),$$

we have a direct sum decomposition

$$n(\mathfrak{p}) = n(\mathfrak{p} \cap l(\bar{\mathfrak{p}})) \oplus n(\bar{\mathfrak{p}}).$$

Let $p_1 : n(\mathfrak{p}) \to n(\mathfrak{p} \cap l(\bar{\mathfrak{p}}))$ be the 1-st projection. Let $\mathcal{O}' \subset l(\bar{\mathfrak{p}})$ be the Richardson orbit of the parabolic subalgebra $\mathfrak{p} \cap l(\bar{\mathfrak{p}})$ of $l(\bar{\mathfrak{p}})$. Since $p_1^{-1}(n(\mathfrak{p}) \cap \mathcal{O}')$ and $n(\mathfrak{p}) \cap \mathcal{O}$ are both Zariski open subsets of $n(\mathfrak{p})$, we can take an element

$$x \in p_1^{-1}(n(\mathfrak{p}) \cap \mathcal{O}') \cap (n(\mathfrak{p}) \cap \mathcal{O}).$$

Since $x \in n(\mathfrak{p}) \cap \mathcal{O}$, we have $([P], x) \in U$. Decompose $x = x_1 + x_2$ according to the direct sum decomposition. Then $x_1 \in \mathcal{O}'$. The orbit $\mathcal{O}'$ is also the Richardson orbit of $\mathfrak{p}' \cap l(\bar{\mathfrak{p}})$. Therefore, for some $g \in L(\bar{P})$ (the Levi factor of $\bar{P}$ corresponding to $l(\bar{P})$),

$$x_1 \in n(Ad_g(\mathfrak{p}' \cap l(\bar{\mathfrak{p}}))).$$

The Levi decomposition of $\bar{\mathfrak{p}}$ induces a direct sum decomposition

$$n(Ad_g(\mathfrak{p}')) = n(Ad_g(\mathfrak{p}') \cap l(\bar{\mathfrak{p}})) \oplus n(\bar{\mathfrak{p}}).$$

Note that $Ad_g(\mathfrak{p}') \cap l(\bar{\mathfrak{p}}) = Ad_g(\mathfrak{p}' \cap l(\bar{\mathfrak{p}}))$. Hence we see that $x_1 + x_2 \in n(Ad_g(\mathfrak{p}'))$. Now, for $Ad_g(P') \subset \bar{P}$, we have $([Ad_g(P')], x) \in U'$.

(ii-2): Any element of $U$ can be written as $([gP], Ad_g(x))$ for some $g \in G$. Then

$$\nu([gP], Ad_g(x)) = ([g\bar{P}], Ad_g(x)).$$

For the same $g \in G$, we have $([gP'], Ad_g(x)) \in U'$ and

$$\nu'([gP'], Ad_g(x)) = ([g\bar{P}], Ad_g(x)).$$

Therefore, $\nu(U) \subset \nu'(U')$. By the same argument, we also have $\nu'(U') \subset \nu(U)$.

(iii): For $g \in G$, $Ad_g(n(\bar{\mathfrak{p}}))$ is the nil-radical of $Ad_g(\bar{\mathfrak{p}})$. Since $Ad_g(\bar{\mathfrak{p}})$ depends only on the class $[g] \in G/\bar{P}$, $Ad_g(n(\bar{\mathfrak{p}}))$ also depends on the class $[g] \in G/\bar{P}$. We denote by $Ad_g(l(\bar{\mathfrak{p}}))$ the quotient of $Ad_g(\bar{\mathfrak{p}})$ by its nil-radical $Ad_g(n(\bar{\mathfrak{p}}))$. Let us consider the vector bundle over $G/\bar{P}$

$$\cup_{[g]\in G/\bar{P}} Ad_g(\bar{\mathfrak{p}}) \to G/\bar{P}.$$

Let $\mathcal{L}$ be its quotient bundle whose fiber over $[g] \in G/\bar{P}$ is $Ad_g(l(\bar{\mathfrak{p}}))$. We call $\mathcal{L}$ the Levi bundle. Let $\mathcal{O}'$ be the Richardson orbit of the parabolic subalgebra $\mathfrak{p} \cap l(\bar{\mathfrak{p}})$ of $l(\bar{\mathfrak{p}})$. Note that $\mathcal{O}'$ is also the Richardson orbit of $\mathfrak{p}' \cap l(\bar{\mathfrak{p}})$. In $\mathcal{L}$, we consider the fiber bundle

$$W := \cup_{[g]\in G/\bar{P}} Ad_g(\bar{\mathcal{O}}')$$

whose fiber over $[g] \in G/\bar{P}$ is $Ad_g(\bar{\mathcal{O}}')$. Put $X := \mathrm{Im}(\nu)$. Define a map

$$f : X \to W$$

as $f([g], x) := ([g], x_1)$, where $x_1$ is the first factor of $x$ under the direct sum decomposition

$$Ad_g(\bar{\mathfrak{p}}) = Ad_g(l(\bar{\mathfrak{p}})) \oplus n(Ad_g(\bar{\mathfrak{p}})).$$

Note that $x_1 \in Ad_g(\bar{\mathcal{O}}')$. In fact, in the direct sum decomposition, we have

$$n(Ad_g(\mathfrak{p})) = n(Ad_g(\mathfrak{p}) \cap Ad_g(l(\bar{\mathfrak{p}}))) \oplus n(Ad_g(\bar{\mathfrak{p}})).$$

Therefore

$$x_1 \in n(Ad_g(\mathfrak{p}) \cap Ad_g(l(\bar{\mathfrak{p}}))) \subset Ad_g(\bar{\mathcal{O}}').$$

Since $W \to G/\bar{P}$ is an $\bar{\mathcal{O}}'$ bundle, we have a family of Mukai flops parametrized by $G/\bar{P}$:

$$Y \to W \leftarrow Y'.$$

By pulling back this diagram by $f : X \to W$, we have the diagram

$$T^*(G/P) \to X \leftarrow T^*(G/P').$$

Q.E.D.

**Example 6.5.** *Let $\mathfrak{g}$ be a simple Lie algebra of type $B$, $C$ or $D$. This long example will explain what actually goes on in the proof of Proposition 6.4. The example consists of two claims.*

**Claim 6.5.1.** Let $V$ be a $\mathbf{C}$-vector space of $\dim n$ with a non-degenerate bilinear form such that $< v, w >= (-1)^\epsilon < w, v >$ for all $v, w \in V$. Let $\mathfrak{g}$ be the Lie algebra $\mathfrak{so}(V)$ or $\mathfrak{sp}(V)$ according as $\epsilon = 0$ or $\epsilon = 1$. Let $x \in \mathfrak{g}$ be a nilpotent element of type $\mathbf{d}$. Suppose that for $\pi \in \mathrm{Pai}(n, q)$, $\mathbf{d} = S(\pi)$ where $S$ is the Spaltenstein map. Let $(p_1, ..., p_k, q, p_k, ..., p_1)$ be a sequence of integers such that $\pi = \mathrm{ord}(p_1, ..., p_k, q, p_k, ..., p_1)$. Fix an admissible flag $F$ of type $(p_1, ..., p_k, q, p_k, ..., p_1)$ such that $xF_i \subset F_{i-1}$ for all $i$.

*(i)* Assume that $p_{j-1} \neq p_j$ for an index $1 \leq j \leq k$. Then we obtain a new flag $F'$ of type $(p_1, ..., p_j, p_{j-1}, ..., p_k, q, p_k, ..., p_{j-1}, p_j, ..., p_1)$ from $F$ such that $xF'_i \subset F'_{i-1}$ for all $i$ by the following operation.

*(The case where $p_{j-1} < p_j$):* $x$ induces an endomorphism $\bar{x} \in \mathrm{End}(F_j/F_{j-2})$. For the projection $\phi : F_j \to F_j/F_{j-2}$, we put $F'_{j-1} := \phi^{-1}(\mathrm{Ker}(\bar{x}))$. We then put

$$F'_i := \begin{cases} F_i & (i \neq j-1, 2k+2-j) \\ F'_{j-1} & (i = j-1) \\ (F'_{j-1})^\perp & (i = 2k+2-j) \end{cases}$$

*(The case where $p_{j-1} > p_j$):* $x$ induces an endomorphism $\bar{x} \in \mathrm{End}(F_j/F_{j-2})$. For the projection $\phi : F_j \to F_j/F_{j-2}$, we put $F'_{j-1} := \phi^{-1}(\mathrm{Im}(\bar{x}))$. We then put

$$F'_i := \begin{cases} F_i & (i \neq j-1, 2k+2-j) \\ F'_{j-1} & (i = j-1) \\ (F'_{j-1})^\perp & (i = 2k+2-j) \end{cases}$$

*(ii)* Assume that $q = 0$ and $p_k$ is odd. Then there is an admissible flag $F'$ of $V$ of type $(p_1, ..., p_k, p_k, ..., p_1)$ such that

$xF'_i \subset F'_{i-1}$ *for all* $i$,

$F'_i = F_i$ *for* $i \neq k$ *and*

$F'_k \neq F_k$.

Proof. *(i): When* $p_{j-1} < p_j$, $\mathrm{rank}(\bar{x}) = p_{j-1}$ *for* $\bar{x} \in \mathrm{End}(F_j/F_{j-2})$. *In fact, since* $xF_j \subset F_{j-1}$, $\mathrm{rank}(\bar{x}) \leq p_{j-1}$. *Assume that* $\mathrm{rank}(\bar{x}) < p_{j-1}$. *Then we can construct a new flag from* $F$ *by replacing* $F_{j-1}$ *with a subspace* $F'_{j-1}$ *containing* $F_{j-2}$ *such that*

$$\mathrm{Im}(\bar{x}) \subset F'_{j-1}/F_{j-2} \subset \mathrm{Ker}(\bar{x})$$

*and* $\dim F'_{j-1}/F_{j-2} = p_{j-1}$. *The new flag satisfies* $xF'_i \subset F'_{i-1}$ *for all* $i$ *and it has the same flag type as* $F$. *Since there are infinitely many* *choices of* $F'_{j-1}$*, we have infinitely many such* $F'$*. This contradicts the fact that* $x$ *has only finite polarizations. Hence,* $\mathrm{rank}(\bar{x}) = p_{j-1}$*. Then the flag* $F'$ *in our Lemma satisfies the desired properties. When* $p_{j-1} > p_j$*, we see that* $\dim \mathrm{Ker}(\bar{x}) = p_{j-1}$ *by a similar way. Then the latter argument is the same as when* $p_{j-1} < p_j$*.*

*(ii): According to the proof of Theorem 4.5. we construct a flag* $F$ *such that* $xF_i \subset F_{i-1}$*. Since* $q = 0$ *and* $p_k$ *is odd, we have the case (A) in the last step. As a consequence, we have two choices of the flags. One of them is* $F$ *and another one is* $F'$*.* *Q.E.D.*

*Let* $F$ *be the flag in Claim 6.5.1, (i) or (ii). In the claim, we have constructed another flag* $F'$*. Let* $G$ *be the complex Lie group* $Sp(V)$ *or* $SO(V)$ *according as* $V$ *is a* **C***-vector with a non-degenerate skew-symmetric form or with a non-degenerate symmetric form. Let* $P \subset G$ *(resp.* $P' \subset G$*) be the stabilizer group of the flag* $F$ *(resp.* $F'$*). Then* $P$ *and* $P'$ *are both polarizations of* $x \in \mathfrak{g}$*. Let* $\mathcal{O} \subset \mathfrak{g}$ *be the nilpotent orbit containing* $x$*. Let us consider two Springer maps*

$$T^*(G/P) \xrightarrow{\mu} \bar{\mathcal{O}} \xleftarrow{\mu'} T^*(G/P').$$

*Note that* $T^*(G/P)$ *(resp.* $T^*(G/P')$*) is embedded in* $G/P \times \bar{\mathcal{O}}$ *(resp.* $G/P' \times \bar{\mathcal{O}}$*). The variety* $G/P$ *(resp.* $G/P'$*) is identified with the set of parabolic subgroups of* $G$ *which are conjugate to* $P$ *(resp.* $P'$*). Assume that*

$$\mu^{-1}(x) = \{(P_i, x)\}_{1 \leq i \leq m},$$

*where* $\deg(\mu) = m$ *and* $P_1 = P$*. Fix stabilized flags* $F^{(i)}$ *of* $P_i$*. Here* $F^{(1)} = F$*. For each* $F^{(i)}$*, we make a flag* $(F^{(i)})'$ *by Claim 6.5.1. Thus we have* $\deg(\mu') = m$*. An element* $y \in \mathcal{O}$ *can be written as* $y = gxg^{-1}$ *for some* $g \in G$*. Then we have*

$$\mu^{-1}(y) = \{([g(F^{(i)})], y)\}_{1 \leq i \leq m},$$

*and*

$$(\mu')^{-1}(y) = \{([g((F^{(i)})')], y)\}_{1 \leq i \leq m}.$$

*Here we identify a flag with the parabolic subgroup stabilizing it. We define the flag* $\bar{F}$ *in the following manner. If* $F$ *is the flag in Claim 6.5.1, (i), then* $\bar{F}$ *is the flag obtained from* $F$ *by deleting subspaces* $F_{j-1}$ *and* $F_{2k+2-j}$*. Finally, if* $F$ *is the flag in Claim 6.5.1, (ii), then* $\bar{F}$ *is the flag obtained from* $F$ *by deleting* $F_k$*. Note that* $\bar{F}$ *is also obtained from* $F'$ *by the same manner. Let* $\bar{P} \subset G$ *be the stabilizer group of the flag* $\bar{F}$*. We then have two projections*

$$G/P \xrightarrow{p} G/\bar{P} \xleftarrow{p'} G/P'.$$

*By two projections*

$$G/P \times \bar{O} \overset{p\times id}{\rightarrow} G/\bar{P} \times \bar{O} \overset{p'\times id}{\leftarrow} G/P' \times \bar{O},$$

*$T^*(G/P)$ and $T^*(G/P')$ have the same image $X$ in $G/\bar{P}\times\bar{O}$. Since p and p′ are both proper maps, $X$ is a closed subvariety of $G/\bar{P}\times\bar{O}$. The following diagram has been obtained as a consequence:*

$$T^*(G/P) \to X \leftarrow T^*(G/P').$$

**Claim 6.5.2**. When $F$ is the flag in Claim 6.5.1, (i), the diagram

$$T^*(G/P) \xrightarrow{f} X \xleftarrow{f'} T^*(G/P')$$

is locally a trivial family of Mukai flops of type A. When $F$ is the flag in Claim 6.5.1, (ii), the diagram is locally a trivial family of Mukai flops of type D.

Proof. *Consider the situation in Claim 6.5.1, (i). A point of $G/\bar{P}$ corresponds to an isotropic flag $\bar{F}$ of $V$ of type $(p_1, ..., p_{j-1}+p_j, ..., p_k, q, p_k, ..., p_{j-1}+p_j, ..., p_1)$. Let*

$$0 \subset \bar{\mathcal{F}}_1 \subset ... \subset \bar{\mathcal{F}}_{2k-1} = (\mathcal{O}_{G/\bar{P}})^n$$

*be the universal subundles on $G/\bar{P}$. Let*

$$W \subset \underline{\mathrm{End}}(\bar{\mathcal{F}}_{j-1}/\bar{\mathcal{F}}_{j-2})$$

*be the subvariety consisting of the points $([\bar{F}],\bar{x})$ where $\bar{x}\in \mathrm{End}(\bar{F}_{j-1}/\bar{F}_{j-2})$, $\bar{x}^2 = 0$ and $\mathrm{rank}(\bar{x}) \leq \min(p_j, p_{j-1})$. If we put $m := p_{j-1}+p_j$ and $r := \min(p_j, p_{j-1})$, then*

$$W \to G/\bar{P}$$

*is an $\bar{O}_{[2^r,1^{m-2r}]}$ bundle over $G/\bar{P}$. Let us recall the definition of $X$.*

$$X \subset G/\bar{P}\times\bar{O}$$

*consists of the points $([\bar{F}],x)$ such that $x\bar{F}_i \subset \bar{F}_{i-1}$ for all $i \neq j-1, 2k-j$ and $x\bar{F}_i \subset \bar{F}_i$ for $i = j-1, 2k-j$. Moreover, the induced endomorphism $\bar{x}\in\mathrm{End}(\bar{F}_{j-1}/\bar{F}_{j-2})$ satisfies $\bar{x}^2 = 0$ and $\mathrm{rank}(\bar{x}) \leq \min(p_{j-1}, p_j)$. Let*

$$\phi : X \to W$$

*be the projection defined by $\phi([\bar{F}],x) = ([\bar{F}],\bar{x})$, where $\bar{x}\in\mathrm{End}(\bar{F}_{j-1}/\bar{F}_{j-2})$ is the induced endomorphism by $x$. It can be checked that $\phi$ is an affine* *bundle. Since* $W$ *is an* $\bar{\mathcal{O}}_{[2^r,1^{m-2r}]}$ *bundle over* $G/\bar{P}$*, there exists a family of Mukai flops of type* $A$*:*

$$Y \to W \leftarrow Y'$$

*parametrized by* $G/\bar{P}$*. The diagram*

$$T^*(G/P) \to X \leftarrow T^*(G/P')$$

*coincides with the pull back of the previous diagram by* $\phi : X \to W$*. Since* $\phi$ *is an affine bundle, this diagram is locally a trivial family of Mukai flops of type* $A$*.*

*Next consider the situation in Claim 6.5.1, (ii). A point of* $G/\bar{P}$ *corresponds to an isotropic flag* $\bar{F}$ *of* $V$ *of type* $(p_1, ..., 2p_k, ..., p_1)$*. Let*

$$0 \subset \bar{\mathcal{F}}_1 \subset ... \subset \bar{\mathcal{F}}_{2k-1} = (\mathcal{O}_{G/\bar{P}})^n$$

*be the universal subundles on* $G/\bar{P}$*. Let*

$$W \subset \underline{\mathrm{End}}(\bar{\mathcal{F}}_k/\bar{\mathcal{F}}_{k-1})$$

*be the subvariety consisting of the points* $([\bar{F}], \bar{x})$ *where*

$$\bar{x} \in \bar{\mathcal{O}}_{[2^{p_k-1},1^2]} \subset \mathfrak{so}(\bar{F}_k/\bar{F}_{k-1}).$$

$W \to G/\bar{P}$ *is an* $\bar{\mathcal{O}}_{[2^{p_k-1},1^2]}$ *bundle over* $G/\bar{P}$*. Let us recall the definition of* $X$*.*

$$X \subset G/\bar{P} \times \bar{\mathcal{O}}$$

*consists of the points* $([\bar{F}], x)$ *such that* $x\bar{F}_i \subset \bar{F}_{i-1}$ *for all* $i \neq k$ *and* $x\bar{F}_k \subset \bar{F}_k$*. Moreover, the induced endomorphism* $\bar{x} \in \mathfrak{so}(\bar{F}_k/\bar{F}_{k-1})$ *is contained in* $\bar{\mathcal{O}}_{[2^{p_k-1},1^2]}$*. Let*

$$\phi : X \to W$$

*be the projection defined by* $\phi([\bar{F}], x) = ([\bar{F}], \bar{x})$*, where* $\bar{x} \in \mathfrak{so}(\bar{F}_k/\bar{F}_{k-1})$ *is the induced endomorphism by* $x$*. It can be checked that* $\phi$ *is an affine bundle. Since* $W$ *is an* $\bar{\mathcal{O}}_{[2^{p_k-1},1^2]}$ *bundle over* $G/\bar{P}$*, there exists a family of Mukai flops of type* $D$*:*

$$Y \to W \leftarrow Y'$$

*parametrized by* $G/\bar{P}$*. The diagram*

$$T^*(G/P) \to X \leftarrow T^*(G/P')$$

*coincides with the pull back of the previous diagram by* $\phi : X \to W$*. Since* $\phi$ *is an affine bundle, this diagram is locally a trivial family of Mukai flops of type* $D$*. Q.E.D.*

Now let us return to the general situation. Let $\mathcal{D}$ be a marked Dynkin diagram and let $\bar{\mathcal{D}}$ be the diagram obtained from $\mathcal{D}$ by a contraction. Let $P$ and $\bar{P}$ be parabolic subgroups of $G$ corresponding to $\mathcal{D}$ and $\bar{\mathcal{D}}$ respectively. One can assume that $\bar{P}$ contains $P$. Let $\mathcal{O}$ be the Richardson orbit of $P$ and let $\nu$ be the compoed map

$$T^*(G/P) \to G/P \times \bar{\mathcal{O}} \to G/\bar{P} \times \bar{\mathcal{O}}.$$

We put $X := \mathrm{Im}(\nu)$. As above, $\mu : T^*(G/P) \to \bar{\mathcal{O}}$ is the Springer map.

**Proposition 6.6.** *Let $\mathfrak{g}$ be a complex simple Lie algebra. Assume that no marked Dynkin diagram is adjacent to $\mathcal{D}$ by means of $\bar{\mathcal{D}}$. If* $\deg(\mu) = 1$, *then $\nu : T^*(G/P) \to X$ is a divisorial birational contraction map.*

*Proof.* As in the proof of Proposition 6.4, (iii), we construct an $\bar{\mathcal{O}}'$ bundle $W$ over $G/\bar{P}$ and define a map $f : X \to W$. There is a family of Springer maps

$$Y \xrightarrow{\sigma} W \to G/\bar{P}.$$

By pulling back $Y \xrightarrow{\sigma} W$ by $f : X \to W$, we have the $\nu : T^*(G/P) \to X$. Since $\deg\mu = 1$, $\nu$ is a birational map. Hence $\sigma$ should be a birational map. Hence $\sigma : Y \to W$ is a family of Springer resolutions. By the assumption, there are no marked Dynkin diagrams adjacent to $\mathcal{D}$ by means of $\bar{\mathcal{D}}$. Now Proposition 5.1 shows that the Springer resolution is divisorial. Therefore, $\nu$ is also divisorial. Q.E.D.

Now let us prove Theorem 6.1. By Proposition 6.4, (iii), $Y_P := T^*(G/P)$ all give symplectic resolutions of $\bar{\mathcal{O}}$ for $P \sim P_0$. Hence the first statement of (i) has been proved. Moreover, $\{Y_P\}$ are connected by Mukai flops, which is nothing but (v). Let us consider $\cup_{P\sim P_0}\overline{Amp}(Y_P/\bar{\mathcal{O}})$ in $N^1(Y_{P_0}/\bar{\mathcal{O}})$. Then (iv) follows from Proposition 6.4, (iii) and Proposition 6.6. For an $\bar{\mathcal{O}}$-movable divisor $D$ on $Y_{P_0}$, a $K_{Y_{P_0}} + D$-extremal contraction is a small birational map. Therefore, the corresponding codimension 1 face of $\overline{\mathrm{Amp}}(Y_{P_0}/\bar{\mathcal{O}})$ becomes a codimension 1 face of another $\overline{\mathrm{Amp}}(Y_P/\bar{\mathcal{O}})$. For this small birational map, there exists a flop. Replace $D$ by its proper transform and continue the same. We shall prove that this procedure ends in finite times. Suppose to the contrary. Since the flops occur between finite number of varieties $\{Y_P\}$, a variety, say $Y_{P_1}$, appears at least twice in the sequence of flops:

$$Y_{P_1} \dashrightarrow Y_{P_2} \dashrightarrow \ldots \dashrightarrow Y_{P_1}.$$

For the first flop

$$Y_{P_1} \xrightarrow{\nu_1} X_1 \leftarrow Y_{P_2},$$

take a discrete valation $v$ of the function field $K(Y_{P_1})$ in such a way that its center is contained in the exceptional locus $\mathrm{Exc}(\nu_1)$ of $\nu_1$. Let $D_i \subset Y_{P_i}$ be the proper transforms of $D$. Then we have inequalities for *discrepancies* (cf. [KMM], Proposition 5-1-11):

$$a(v, D_1) < a(v, D_2) \leq .... \leq a(v, D_1).$$

Here the first inequality is a strict one since the center of $v$ is contained in $\mathrm{Exc}(\nu_1)$. This is absurd. Hence the procedure ends in finite times, which implies that $D \in \overline{\mathrm{Amp}}(Y_P/\bar{O})$ for some $P$. Therefore, (iii) has been proved. The second statement of (i) immediately follows from (iii).

**Example 6.7.** *Assume that* $\mathfrak{g} = \mathfrak{sl}(6)$. *The marked Dynkin diagram* $\mathcal{D}$

$$\bullet - \circ - \bullet - \circ - \circ$$

*gives a parabolic subgroup* $P_{1,2,3} \subset SL(6)$ *of flag type* $(1, 2, 3)$. *We put* $Y_{1,2,3} := T^*(G/P_{1,2,3})$. *There are 5 other marked Dynkin diagrams which are equivalent to* $\mathcal{D}$*:*

$$\bullet - \circ - \circ - \bullet - \circ$$

$$\circ - \circ - \bullet - \bullet - \circ$$

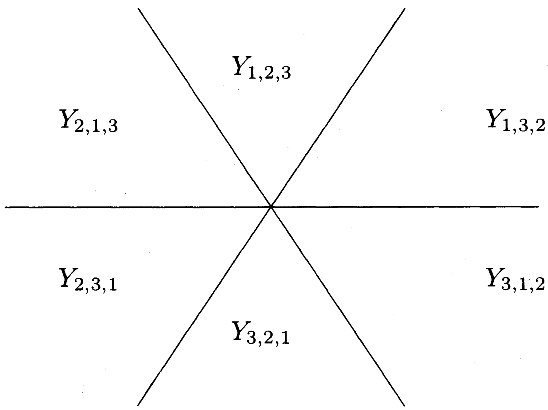
$$\circ - \circ - \bullet - \circ - \bullet$$

$$\circ - \bullet - \circ - \circ - \bullet$$

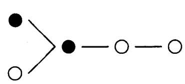
$$\circ - \bullet - \bullet - \circ - \circ$$

*Five parabolic subgroups* $P_{1,3,2}$, $P_{3,1,2}$, $P_{3,2,1}$, $P_{2,3,1}$, $P_{2,1,3}$ *correspond to the marked Dynkin diagrams above respectively. We put* $Y_{i,j,k} :=$

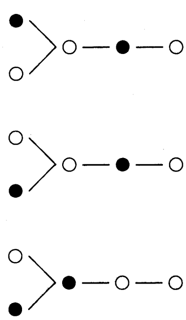
*$T^*(SL(6)/P_{i,j,k})$. Let $\mathcal{O}$ be the Richardson orbit of these parabolic subgroups. Then $\overline{\mathrm{Mov}}(Y_{1,2,3}/\bar{\mathcal{O}}) \cong \mathbf{R}^2$, which is divided into six chambers by the ample cones of $Y_{i,j,k}$ in the following way:*



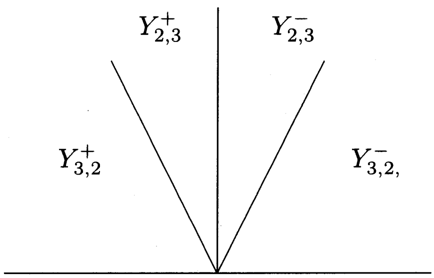

**Example 6.8.** *Assume that $\mathfrak{g} = \mathfrak{so}(10)$. The marked Dynkin diagram*



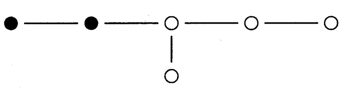
*gives a parabolic subgroup $P^+_{3,2,2,3}$ of flag type $(3, 2, 2, 3)$. There are three marked Dynkin diagrams equivalent to this marked diagram:*



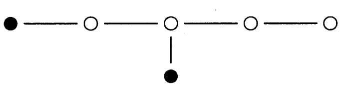
*Three parabolic subgroups* $P^+_{2,3,3,2}$, $P^-_{2,3,3,2}$, $P^-_{3,2,2,3}$ *correspond to these marked Dynkin diagrams respectively. Note that there are exactly two conjugacy classes of parabolic subgroups with the same flag type (cf. Example 4.2). We put* $Y^+_{i,j} := T^*(SO(10)/P^+_{i,j,j,i})$ *and put* $Y^-_{i,j} := T^*(SO(10)/P^-_{i,j,j,i})$. *Let* $\mathcal{O}$ *be the Richardson orbit of these parabolic subgroups. Then* $\overline{\mathrm{Mov}}(Y^+_{3,2}/\bar{\mathcal{O}})$ *is divided into four chambers by the ample cones of* $Y^+_{3,2}$, $Y^+_{2,3}$, $Y^-_{2,3}$, $Y^-_{3,2}$ *in the following way:*

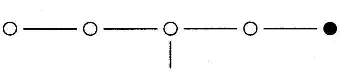


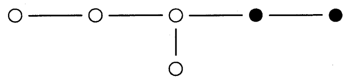

**Example 6.9.** *Assume that* $\mathfrak{g}$ *is of type* $E_6$. *Consider the nilpotent orbit* $\mathcal{O} := \mathcal{O}_{A_3}$ *(cf. [C-M], p.129). This is the unique orbit with dimension 52. By a dimension count, we see that* $\mathcal{O}$ *is the Richardson orbit of the parabolic subgroup* $P_1 \subset G$ *associated with the marked Dynkin diagram*



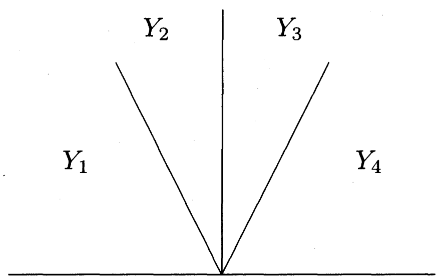
*Since* $\pi_1(\mathcal{O}) = 1$ *([C-M], p.129], the Springer map* $\nu_1 : T^*(G/P_1) \to \bar{\mathcal{O}}$ *has degree 1. The following marked Dynkin diagrams are equivalent to the diagram above:*







*Denote by $P_2$, $P_3$, $P_4$ respectively the parabolic subgroups corresponding to the diagrams above. We put $Y_i := T^*(G/P_i)$ for $i = 1, 2, 3, 4$. Then $\overline{\mathrm{Mov}}(Y_1/\bar{O})$ is divided into four chambers by the ample cones of $Y_i$:*




*$Y_1$ and $Y_2$ are connected by a Mukai flop of type $D_5$ (cf. Proposition 6.4, (iii)). $Y_2$ and $Y_3$ are connected by a Mukai flop of type $A_{5,1}$ (for the notation, see Example 5.3). $Y_3$ and $Y_4$ are connected by a Mukai flop of type $D_5$.*

**Derived categories**: Two smooth quasi-projective varieties $Y$ and $Y'$ are called $D$-equivalent if there is an equivalence between the bounded derived categories of coherent sheaves $D^b(\mathrm{Coh}(Y))$ and $D^b(\mathrm{Coh}(Y'))$. On the other hand, if we can take common resolutions $\mu : Z \to Y$ and $\mu' : Z \to Y'$ in such a way that $\mu^*(K_Y) = \mu'^*(K_{Y'})$, then we say that $Y$ and $Y'$ are $K$-equivalent. The following conjecture is posed by Kawamata [Ka 2].

**Conjecture 1.** *If $Y$ and $Y'$ are $K$-equivalent, then they are $D$-equivalent.*

Assume that $Y$ and $Y'$ are two different symplectic resolutions of a nilpotent orbit closure $\bar{O}$ in a complex simple Lie algebra. Since $\bar{O}$ admits a good $\mathbf{C}^*$-action, the conjecture is true as a special case of a result recently proved by Bezrukavnikov and Kaledin [K]. It would be interesting to know whether the equivalence in Conjecture is realized as Fourier-Mukai functors associated with suitable objects of $D^b(Y \times Y')$. Actually, for the Mukai flop of type $A_{n,1}$ (cf. Example 5.3), the Fourier-Mukai functor induced from the fiber product $Y \times_{\bar{\mathcal{O}}} Y'$ gives an equivalence [Na 2]. However, the same functor is no more an equivalence for the Mukai flop of type $A_{n,k}$ with $k > 1$ ([Na 3]).

## §7. Deformations of nilpotent orbits

Let $x \in \mathfrak{g}$ be a nilpotent element of a Lie algebra attached to a complex simple Lie group $G$. Let $\mathcal{O}$ be the nilpotent orbit containing $x$. In this section, by using an idea of Borho and Kraft [B-K], we shall construct a morphism $f : \mathcal{S} \to \mathfrak{k}$ such that

(i) $f^{-1}(0) = \bar{\mathcal{O}}$ for $0 \in \mathfrak{k}$,

(ii) for any Springer resolution $T^*(G/P) \to \bar{\mathcal{O}}$, there is a smooth morphism $\tau_P : \mathcal{E}_P \to \mathfrak{k}$ with $(\tau_P)^{-1}(0) = T^*(G/P)$ such that there is a proper birational morphism

$$\nu_P : \mathcal{E}_P \to \mathcal{S},$$

and

(iii) the induced map $(\nu_P)_t : (\tau_P)^{-1}(t) \to f^{-1}(t)$ is a resolution for every $t \in \mathfrak{k}$ and it is an isomorphism for a general point $t \in \mathfrak{k}$.

As a corollary of this construction, we can verify Conjecture 2 in [F-N] for the closure of a nilpotent orbit of a simple Lie algebra. Conjecture 2 has already been proved for $\mathfrak{sl}(n)$ in [F-N], Theorem 4.4 in a very explicit form. Note that, a weaker version of this conjecture has been proved by Fu [Fu 2] for the closure of a nilpotent orbit of a classical simple Lie algebra.

The Lie algebra $\mathfrak{g}$ becomes a $G$-variety via the adjoint action. Let $Z \subset \mathfrak{g}$ be a closed subvariety. For $m \in \mathbf{N}$, put

$$Z^{(m)} := \{x \in Z; \dim Gx = m\}.$$

$Z^{(m)}$ becomes a locally closed subset of $Z$. We put $m(Z) := \max\{m; m = \dim Gx, \exists x \in Z\}$. Then $Z^{m(Z)}$ is an open subset of $Z$, which will be denoted by $Z^{\mathrm{reg}}$. A *sheet* of $Z$ is an irreducible component of some $Z^{(m)}$. A sheet of $\mathfrak{g}$ is called a *Dixmier sheet* if it contains a semi-simple element of $\mathfrak{g}$. We fix a maximal torus $H$ of $G$. In the remainder, all parabolic subgroups are assumed to contain $H$. Denote by $\mathfrak{h}$ the Lie algebra of $H$.

Let $P \subset G$ be a parabolic subgroup and let $\mathfrak{p}$ be its Lie algebra. Let $\mathfrak{m}(P)$ be the Levi factor of $\mathfrak{p}$ such that $\mathfrak{h} \subset \mathfrak{m}(P)$. We put $\mathfrak{k}(P) := \mathfrak{g}^{\mathfrak{m}(P)}$

where
$$\mathfrak{g}^{\mathfrak{m}(P)} := \{x \in \mathfrak{g}; [x,y] = 0, \forall y \in \mathfrak{m}(P)\}.$$

Let $\mathfrak{r}(P)$ be the radical of $\mathfrak{p}$.

**Theorem 7.1.** *$G\mathfrak{r}(P) = \overline{G\mathfrak{k}(P)}$ and $G\mathfrak{r}(P)^{\mathrm{reg}}$ $(= \overline{G\mathfrak{k}(P)}^{\mathrm{reg}})$ is a Dixmier sheet.*

*Proof.* See [B-K], Satz 5.6.

Every element $x$ of $\mathfrak{g}$ can be uniquely written as $x = x_n + x_s$ with $x_n$ nilpotent and with $x_s$ semi-simple such that $[x_n, x_s] = 0$. Let $W$ be the Weyl group of $\mathfrak{g}$ with respect to $\mathfrak{h}$. The set of semi-simple orbits is identified with $\mathfrak{h}/W$. Let $\mathfrak{g} \to \mathfrak{h}/W$ be the map defined as $x \to [\mathcal{O}_{x_s}]$. There is a direct sum decomposition

$$\mathfrak{r}(P) = \mathfrak{k}(P) \oplus \mathfrak{n}(P), (x \to x_1 + x_2)$$

where $\mathfrak{n}(P)$ is the nil-radical of $\mathfrak{p}$ (cf. [Slo], 4.3). We have a well-defined map
$$G \times^P \mathfrak{r}(P) \to \mathfrak{k}(P)$$

by sending $[g,x] \in G \times^P \mathfrak{r}(P)$ to $x_1 \in \mathfrak{k}(P)$ and there is a commutative diagram
$$\begin{array}{ccc} G \times^P \mathfrak{r}(P) & \to & G\mathfrak{r}(P) \\ \downarrow & & \downarrow \\ \mathfrak{k}(P) & \to & \mathfrak{h}/W. \end{array}$$

by [Slo], 4.3.

**Lemma 7.2.** *The induced map*

$$G \times^P \mathfrak{r}(P) \xrightarrow{\mu_P} \mathfrak{k}(P) \times_{\mathfrak{h}/W} G\mathfrak{r}(P)$$

*is a birational map.*

*Proof.* Let $h \in \mathfrak{k}(P)^{\mathrm{reg}}$ and denote by $\bar{h} \in \mathfrak{h}/W$ its image by the map $\mathfrak{k}(P) \to \mathfrak{h}/W$. Then the fiber of the map $G\mathfrak{r}(P) \to \mathfrak{h}/W$ over $\bar{h}$ coincides with a semi-simple orbit $\mathcal{O}_h$ of $\mathfrak{g}$ containing $h$. In fact, by Theorem 7.1, the fiber actally contains this orbit. The fiber is closed in $\mathfrak{g}$ because $G\mathfrak{r}(P)$ is closed subset of $\mathfrak{g}$ by Theorem 7.1. Note that a semi-simple orbit of $\mathfrak{g}$ is also closed. Hence if the fiber and $\mathcal{O}_h$ does not coincide, then the fiber contains an orbit with larger dimension than $\dim \mathcal{O}_h$. This contradicts the fact that $\overline{G\mathfrak{k}(P)^{\mathrm{reg}}} = G\mathfrak{r}(P)$. Take a point

$(h, h') \in \mathfrak{k}(P)^{\mathrm{reg}} \times_{\mathfrak{h}/W} G\mathfrak{r}(P)$. Then $h'$ is a semi-simple element $G$-conjugate to $h$. Fix an element $g_0 \in G$ such that $h' = g_0 h (g_0)^{-1}$. We have

$$(\mu_P)^{-1}(h, h') = \{[g, x] \in G \times^P \mathfrak{r}(P);\ x_1 = h,\ gxg^{-1} = h'\}.$$

Since $x = px_1p^{-1}$ for some $p \in P$ and conversely $(px_1p^{-1})_1 = x_1$ for any $p \in P$ (cf. [Slo], Lemma 2, p.48), we have

$$(\mu_P)^{-1}(h, h') = \{[g, php^{-1}] \in G \times^P \mathfrak{r}(P);\ g \in G,\ p \in P,\ (gp)h(gp)^{-1} = h'\}$$

$$= \{[gp, h] \in G \times^P \mathfrak{r}(P);\ g \in G,\ p \in P,\ (gp)h(gp)^{-1} = h'\} =$$

$$\{[g, h] \in G \times^P \mathfrak{r}(P);\ ghg^{-1} = h'\} =$$

$$\{[g_0 g', h] \in G \times^P \mathfrak{r}(P);\ g' \in Z_G(h)\} = g_0(Z_G(h)/Z_P(h)).$$

Here $Z_G(h)$ (resp. $Z_P(h)$) is the centralizer of $h$ in $G$ (resp. $P$). By [Ko], 3.2, Lemma 5, $Z_G(h)$ is connected. Moreover, since $\mathfrak{g}^h \subset \mathfrak{p}$, $\mathrm{Lie}(Z_G(h)) = \mathrm{Lie}(Z_P(h))$. Therefore, $Z_G(h)/Z_P(h) = \{1\}$, and $(\mu_P)^{-1}(h, h')$ consists of one point.

**Lemma 7.3.** *The map*

$$G \times^P \mathfrak{r}(P) \to G\mathfrak{r}(P)$$

*is a proper map.*

*Proof.* As a vector subbundle, we have a closed immersion

$$G \times^P \mathfrak{r}(P) \to G/P \times \mathfrak{g}.$$

This map factors through $G/P \times G\mathfrak{r}(P)$, and hence we have a closed immersion

$$G \times^P \mathfrak{r}(P) \to G/P \times G\mathfrak{r}(P).$$

Our map is the composition of this closed immersion and the projection

$$G/P \times G\mathfrak{r}(P) \to G\mathfrak{r}(P).$$

Since $G/P$ is compact, this projection is a proper map.

**Lemma 7.4.** *Let $\mathfrak{g}$ be a nilpotent orbit of a complex simple Lie algebra and denote by $\mathcal{O}$ a nilpotent orbit. Then the polarizations of $\mathcal{O}$ giving Springer resolutions of $\bar{\mathcal{O}}$ all have conjugate Levi factors.*

*Proof.* This follows from Theorem 6.1 and Proposition 6.4.

**Lemma 7.5.** *Let $\mathcal{O}$ be the same as the previous lemma. Let $P$ and $P'$ be polarizations of $\mathcal{O}$. Assume that they both give Springer resolutions of $\bar{\mathcal{O}}$. Then $\mathfrak{k}(P)$ and $\mathfrak{k}(P')$ are conjugate to each other.*

*Proof.* Let $M_P$ and $M_{P'}$ be Levi factors of $P$ and $P'$ respectively. Then $M_P$ and $M_{P'}$ are conjugate by the previous lemma. Hence their centralizers are also conjugate. The Lie algebras of these centralizers are $\mathfrak{k}(P)$ and $\mathfrak{k}(P')$.

**Corollary 7.6.** *For $P$, $P'$ which give Springer resolutions of $\bar{\mathcal{O}}$, we have $G\mathfrak{r}(P) = G\mathfrak{r}(P')$.*

*Proof.* By Theorem 7.1, $G\mathfrak{r}(P) = \overline{G\mathfrak{k}(P)}$ and $G\mathfrak{r}(P') = \overline{G\mathfrak{k}(P')}$. Since $G\mathfrak{k}(P) = G\mathfrak{k}(P')$, we have the result.

**Lemma 7.7.** *The image of the composed map*

$$G\mathfrak{r}(P) \to \mathfrak{g} \to \mathfrak{h}/W$$

*coincides with $\mathfrak{k}(P)/W_P$, where*

$$W_P = \{w \in W; w(\mathfrak{k}(P)) = \mathfrak{k}(P)\}.$$

*Proof.* By definition, $\mathfrak{k}(P)/W_P \subset \mathfrak{h}/W$, which is a closed subset. Since $G\mathfrak{r}(P) = \overline{G\mathfrak{k}(P)}$, we only have to prove that the image of $G\mathfrak{k}(P)$ by the map $\mathfrak{g} \to \mathfrak{h}/W$ coincides with $\mathfrak{k}(P)/W_P$. Every element of $G\mathfrak{k}(P)$ is semi-simple, and the map $G\mathfrak{k}(P) \to \mathfrak{h}/W$ sends an element of $G\mathfrak{k}(P)$ to its (semi-simple) orbit. Hence the image coincides with $\mathfrak{k}(P)/W_P$.

**Corollary 7.8.** *$\mathfrak{k}(P)$ and $\mathfrak{k}(P')$ are $W$-conjugate in $\mathfrak{h}$.*

*Proof.* Let $q : \mathfrak{h} \to \mathfrak{h}/W$ be the quotient map. Since $G\mathfrak{r}(P) = G\mathfrak{r}(P')$, $q(\mathfrak{k}(P)) = q(\mathfrak{k}(P'))$ by the previous lemma. Put $\mathfrak{k}_\pi := q(\mathfrak{k}(P))$. Then $\mathfrak{k}(P)$ and $\mathfrak{k}(P')$ are both irreducible components of $q^{-1}(\mathfrak{k}_\pi)$. Hence, $\mathfrak{k}(P)$ and $\mathfrak{k}(P')$ are $W$-conjugate in $\mathfrak{h}$.

We fix a polarization of $P_0$ of $\mathcal{O}$ which gives a Springer resolution of $\bar{\mathcal{O}}$. Let $P$ be another such polarization. By the corollary above, $\mathfrak{k}(P)$ and $\mathfrak{k}(P_0)$ are finite coverings of $\mathfrak{k}_\pi$, and there is a $\mathfrak{k}_\pi$-isomorphism $\mathfrak{k}(P) \cong \mathfrak{k}(P_0)$. We fix such an isomorphism. Then it induces an isomorphism

$$\mathfrak{k}(P) \times_{\mathfrak{h}/W} G\mathfrak{r}(P) \overset{\iota_P}{\to} \mathfrak{k}(P_0) \times_{\mathfrak{h}/W} G\mathfrak{r}(P_0).$$

We put $\nu_P := \iota_P \circ \mu_P$, and

$$\mathcal{S} := \mathfrak{k}(P_0) \times_{\mathfrak{h}/W} G\mathfrak{r}(P_0).$$

Denote by $f$ the first projection $\mathcal{S} \to \mathfrak{k}(P_0)$. Then $f^{-1}(0) = \bar{\mathcal{O}}$, and for each polarization $P$ of $x$,

$$G \times^P \mathfrak{r}(P) \stackrel{\nu_P}{\to} \mathcal{S} \to \mathfrak{k}(P_0)$$

gives a simultaneous resolution of $f$. This simultaneous resolution coincides with the Springer resolution $T^*(G/P) \to \bar{\mathcal{O}}$ over $0 \in \mathfrak{k}(P_0)$.

The following conjecture is posed in [F-N].

**Conjecture 2.** *Let $Z$ be a normal symplectic singularity. Then for any two symplectic resolutions $f_i : X_i \to Z$, $i = 1, 2$, there are flat deformations $\mathcal{X}_i \stackrel{F_i}{\to} \mathcal{Z} \to T$ such that, for $t \in T - \{0\}$, $F_{i,t} : \mathcal{X}_{i,t} \to \mathcal{Z}_t$ are isomorphisms.*

**Theorem 7.9.** *Let $\mathfrak{g}$ be a complex simple Lie algebra. Assume a nilpotent orbit closure $\bar{\mathcal{O}} \subset \mathfrak{g}$ admits a Springer resolution. Then the conjecture holds for the normalization $\tilde{\mathcal{O}}$ of $\bar{\mathcal{O}}$.*

*Proof.* By Theorem 6.1, all symplectic resolutions of $\tilde{\mathcal{O}}$ are realized as Springer resolutions. Take a general curve $T \subset \mathfrak{k}(P_0)$ passing through $0 \in \mathfrak{k}(P_0)$, and pull back the family

$$G \times^P \mathfrak{r}(P) \stackrel{\nu_P}{\to} \mathcal{S} \to \mathfrak{k}(P_0)$$

by $T \to \mathfrak{k}(P_0)$. Put $\bar{\mathcal{Z}} := \mathcal{S} \times_{\mathfrak{k}(P_0)} T$. Then, for each $P$, we have a simultaneous resolution of $\bar{\mathcal{Z}} \to T$:

$$\mathcal{X}_P \to \bar{\mathcal{Z}} \to T.$$

Let $\mathcal{Z}$ be the normalization of $\bar{\mathcal{Z}}$. Then the map $\mathcal{X}_P \to \bar{\mathcal{Z}}$ factors through $\mathcal{Z}$. Now

$$\mathcal{X}_P \to \mathcal{Z} \to T$$

gives a desired deformation of the Springer resolution $T^*(G/P) \to \tilde{\mathcal{O}}$.

**Example 7.10.** *Our abstract construction coincides with the explicit construction in [F-N], Theorem 4.4 in the case where $\mathfrak{g} = \mathfrak{sl}(n)$. Let us briefly observe the correspondence between two constructions. Assume that $\mathcal{O}_x \subset \mathfrak{sl}(n)$ is the orbit containing an nilpotent element $x$ of type $\mathbf{d} := [d_1, ..., d_k]$. Let $[s_1, ..., s_m]$ be the dual partition of $\mathbf{d}$ (cf. Notation (1)). By Theorem 4.4, the polarizations of $x$ have the flag type $(s_{\sigma(1)}, \cdots, s_{\sigma(m)})$ with $\sigma \in \Sigma_m$. We denote them by $P_\sigma$. We put $P_0 := P_{id}$. Define $F_\sigma := SL(n)/P_\sigma$. Let*

$$\tau_1 \subset \cdots \subset \tau_{m-1} \subset \mathbb{C}^n \otimes_{\mathbb{C}} \mathcal{O}_{F_\sigma}$$

*be the universal subbundles on $F_\sigma$. A point of $T^*F_\sigma$ is expressed as a pair $(p, \phi)$ of $p \in F_\sigma$ and $\phi \in \mathrm{End}(\mathbb{C}^n)$ such that*

$$\phi(\mathbb{C}^n) \subset \tau_{m-1}(p), \cdots, \phi(\tau_2(p)) \subset \tau_1(p), \phi(\tau_1(p)) = 0.$$

*The Springer resolution*

$$s_\sigma : T^*F_\sigma \to \bar{O}$$

*is defined as $s_\sigma((p,\phi)) := \phi$. In [F-N], Theorem 4.4, we have next defined a vector bundle $\mathcal{E}_\sigma$ with an exact sequence*

$$0 \to T^*F_\sigma \to \mathcal{E}_\sigma \xrightarrow{\eta_\sigma} \mathcal{O}_{F_\sigma}^{m-1} \to 0.$$

*For $p \in F_\sigma$, we can choose a basis of $\mathbf{C}^n$ such that $T^*F_\sigma(p)$ consists of the matrices of the following form*

$$\begin{pmatrix} 0 & * & \cdots & * \\ 0 & 0 & \cdots & * \\ \cdots & & & \cdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}.$$

*Then $\mathcal{E}_\sigma(p)$ is the vector subspace of $\mathfrak{sl}(n)$ consisting of the matrices $A$ of the following form*

$$\begin{pmatrix} a_{\sigma(1)} & * & \cdots & * \\ 0 & a_{\sigma(2)} & \cdots & * \\ \cdots & & & \cdots \\ 0 & 0 & \cdots & a_{\sigma(m)} \end{pmatrix},$$

*where $a_i := a_i I_{s_i}$ and $I_{s_i}$ is the identity matrix of the size $s_i \times s_i$. Since $A \in \mathfrak{sl}(n)$, $\Sigma_i s_i a_i = 0$. Here we define the map $\eta_\sigma(p) : \mathcal{E}_\sigma(p) \to \mathbb{C}^{\oplus m-1}$ as $\eta_\sigma(p)(A) := (a_1, a_2, \cdots, a_{m-1})$. This vector bundle $\mathcal{E}_\sigma$ is nothing but our $SL(n) \times^{P_\sigma} \mathfrak{r}(P_\sigma)$. Moreover, the map*

$$\eta_\sigma : \mathcal{E}_\sigma \to \mathbb{C}^{m-1}$$

*coincides with the map*

$$SL(n) \times^{P_\sigma} \mathfrak{r}(P_\sigma) \to \mathfrak{k}(P_0),$$

*where we identify $\mathfrak{k}(P_\sigma)$ with $\mathfrak{k}(P_0)$ by an $\mathfrak{k}_\pi$-isomorphism. Finally, in [F-N], Theorem 4.4 we have defined $\overline{N} \subset \mathfrak{sl}(n)$ to be the set of all matrices which is conjugate to a matrix of the following form:*

$$\begin{pmatrix} b_1 & * & \cdots & * \\ 0 & b_2 & \cdots & * \\ \cdots & & & \cdots \\ 0 & 0 & \cdots & b_m \end{pmatrix},$$

*where* $b_i = b_i I_{s_i}$ *and* $I_{s_i}$ *is the identity matrix of order* $s_i$. *Furthermore the zero trace condition* $\sum_i s_i b_i = 0$ *was required. For* $A \in \overline{N}$, *let* $\phi_A(x) := \det(xI - A)$ *be the characteristic polynomial of* $A$. *Let* $\phi_i(A)$ *be the coefficient of* $x^{n-i}$ *in* $\phi(A)$. *Here the characteristic map* $ch : \overline{N} \to \mathbb{C}^{n-1}$ *has been defined as* $ch(A) := (\phi_2(A), ..., \phi_n(A))$. *This* $\overline{N}$ *is nothing but our* $SL(n)\mathfrak{r}(P_\sigma)$. *As is proved in Corollary 7.6, this is independent of the choice of* $P_\sigma$. *The characteristic map ch above coincides with the composed map*

$$SL(n)\mathfrak{r}(P_\sigma) \subset \mathfrak{sl}(n) \to \mathfrak{h}/W.$$

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