Advanced Studies in Pure Mathematics 45, 2006
Moduli Spaces and Arithmetic Geometry (Kyoto, 2004)
pp. 43–73

On integral Hodge classes on uniruled or Calabi-Yau threefolds

Claire Voisin

To Masaki Maruyama, on his 60th birthday

§0. Introduction

Let X be a smooth complex projective variety of dimension n. The Hodge conjecture is then true for rational Hodge classes of degree $2n-2$, that is, the degree $2n-2$ rational cohomology classes of X which are of Hodge type $(n-1, n-1)$ are algebraic, which means that they are the cohomology classes of algebraic cycles with $\mathbb{Q}$-coefficients. Indeed, this follows from the hard Lefschetz theorem, which provides an isomorphism:

$$\cup c_1(L)^{n-2} : H^2(X, \mathbb{Q}) \cong H^{2n-2}(X, \mathbb{Q}),$$

from the fact that the isomorphism above sends the space of rational Hodge classes of degree 2 onto the space of rational Hodge classes of degree $2n-2$, and from the Lefschetz theorem on $(1,1)$-classes.

For integral Hodge classes, Kollár [11], (see also [14]) gave examples of smooth complex projective manifolds which do not satisfy the Hodge conjecture for integral degree $2n-2$ Hodge classes, for any $n \geq 3$. The examples are smooth general hypersurfaces X of certain degrees in $\mathbb{P}^{n+1}$. By the Lefschetz restriction theorem, such a variety satisfies

$$H^2(X, \mathbb{Z}) = \mathbb{Z}H,\ H = c_1(\mathcal{O}_X(1)),$$

and

$$H^{2n-2}(X, \mathbb{Z}) = \mathbb{Z}\alpha,\ <\alpha, H>= 1.$$

Plane sections C of X have cohomology class $[C] = d\alpha$, $d = deg\, X$, because

$$deg\, C = d =< [C], H > .$$

Kollár [11] proves the following :

Received December 14, 2004.

Theorem 1. *Consider hypersurfaces $X \subset \mathbb{P}^{n+1}$ of degree d, where $n \geq 4$. Assume d satisfies the property that p^n divides d, for some integer p coprime to $n!$. Then for a general X, any curve C in X has degree divisible by p, hence its cohomology class is a multiple of $p\alpha$. Thus the class α is not algebraic, that is, it is not the cohomology class of an algebraic cycle with integral coefficients.*

The condition on the degree makes the canonical bundle of X very ample, since the smallest possible degree available by this construction is $\geq 2^n$. It is thus natural to try to understand whether this is an artificial consequence of the method of construction, or whether the positivity of the canonical bundle is essential.

Another reason to ask whether one could find examples above with Kodaira dimension equal to $-\infty$ is the remark made in [14] :

Lemma 1. *Let X be a smooth rational complex projective manifold. Then the Hodge conjecture is true for integral Hodge classes of degree $2n-2$.*

(Note that the whole degree $2n-2$ cohomology of such an X is of type $(n-1, n-1)$, so the statement is that classes of curves generate $H^{2n-2}(X, \mathbb{Z})$ for a rational variety X.)

One can thus ask whether this criterion could be used to produce new examples of unirational or rationally connected, but non rational varieties (we refer to [5], [1], [9] for other criteria). Namely, it would suffice to produce a smooth projective rationally connected variety which does not satisfy the Hodge conjecture for degree $2n-2$ integral cohomology classes. The main result of this paper implies that in dimension 3, this cannot be done:

Theorem 2. *Let X be a smooth complex projective threefold which either is uniruled, or satisfies*

$$K_X \cong \mathcal{O}_X,\ H^2(X, \mathcal{O}_X) = 0.$$

Then the Hodge conjecture is true for integral degree 4 Hodge classes on X.

Remark 1. Recall [12] that a complex projective threefold is uniruled, that is swept out by rational curves, if and only if it has Kodaira dimension equal to $-\infty$. Thus our condition is that either $\kappa(X) = -\infty$ or $K_X = \mathcal{O}_X$ and $H^2(X, \mathcal{O}_X) = 0$.

Note that as an obvious corollary, we get the following:

Corollary 1. *Let X be a smooth complex projective n-fold. Assume X contains a subvariety Y which is a smooth 3-dimensional complete intersection of ample divisors, and satisfies one of the conditions in Theorem 2. Then the Hodge conjecture is true for integral degree $2n-2$ Hodge classes on X.*

Indeed, let j be the inclusion of Y into X. By Lefschetz restriction theorem, the map

$$j_* : H^4(Y,\mathbb{Z}) = H_2(Y,\mathbb{Z}) \to H_2(X,\mathbb{Z}) = H^{2n-2}(X,\mathbb{Z})$$

is an isomorphism. Thus the Hodge conjecture for integral Hodge classes of degree 4 on Y implies the Hodge conjecture for integral Hodge classes of degree $2n-2$ on X. ∎

Note that in higher dimensions, there are two possible generalizations of the problem studied above. Namely, one can study the Hodge conjecture for integral Hodge classes in degree 4 or $2n-2$. Both problems are birationally invariant, in the sense that the two groups

$$Hdg^4(X,\mathbb{Z})/ < [Z] >,\ Hdg^{2n-2}(X,\mathbb{Z})/ < [Z] >,$$

where $< [Z] >$ denotes the subgroups generated by classes of algebraic cycles with integral coefficients, are birational invariants of a smooth complex projective manifold X of dimension n (see [14]). For both problems, it is clear that the assumption "uniruled" will not be sufficient in higher dimension to guarantee that the groups above vanish. Indeed, starting from one of Kollár's 3-dimensional example of a pair $X, \alpha \in H^4(X,\mathbb{Z})$, where α is a non-algebraic integral Hodge class (Theorem 1), we can consider the product

$$Y = X \times \mathbb{P}^1,$$

and both classes

$$pr_1^*\alpha,\ pr_1^*\alpha \cup pr_2^*([pt])$$

in degree 4 and $6 = 2n-2$ respectively will give examples of non-algebraic integral Hodge classes.

However, one may wonder if the analogue of Theorem 2 holds for X rationally connected, and for integral Hodge classes of degree 4 or $2n-2$ on X, $n = dim\, X$.

The proof of Theorem 2 uses the Noether-Lefschetz locus for surfaces S in an adequately chosen ample linear system on X. This leads to simple criteria which guarantee that integral degree 2 cohomology classes on a given S are generated over $\mathbb{Z}$ by those which become algebraic on some small deformation S_t of S. The Lefschetz hyperplane section Theorem allows then to conclude.

In section 1, we state this criterion, which is an algebraic criterion concerning the infinitesimal variation of Hodge structure on $H^2(S)$, for varieties X with $H^2(X, \mathcal{O}_X) = 0$. In section 2, we prove that this criterion is satisfied for uniruled or K-trivial varieties with trivial $H^2(X, \mathcal{O}_X)$. In the case of K-trivial varieties, the criterion had been also checked in [15], but the proof given here is substantially simpler. In section 3, a refinement of this criterion for uniruled threefolds with $H^2(X, \mathcal{O}_X) \neq 0$ is given and proven to hold for an adequate choice of linear system.

Thanks. This work was started during the very interesting conference "*Arithmetic Geometry and Moduli Spaces.* It is a pleasure to thank the organizers for the nice atmosphere they succeeded to create. I also wish to thank S. Mori for his help in the proof of Lemma 4 and J. Starr for interesting discussions on related questions.

§1. An infinitesimal criterion

Let X be a smooth complex projective n-fold. Let $j : S \hookrightarrow X$ be a surface which is a smooth complete intersection of ample divisors. Thus by Lefschetz theorem, the Gysin map:

$$j_* : H^2(S, \mathbb{Z}) \to H^{2n-2}(X, \mathbb{Z})$$

is surjective.

We assume that the Hilbert scheme $\mathcal{H}$ of deformations of S in X is smooth near S. This is the case if S is a smooth complete intersection of sufficiently ample divisors. The space $H^0(S, N_{S/X})$ is the tangent space to $\mathcal{H}$ at S. Let $\rho : H^0(S, N_{S/X}) \to H^1(S, T_S)$ be the Kodaira-Spencer map, which is the classifying map for the first order deformations of the complex structure on S induced by the universal family $\pi : \mathcal{S} \to \mathcal{H}$ of surfaces parameterized by $\mathcal{H}$.

For $u \in H^1(S, T_S)$ we have the interior product with u:

$$u \lrcorner : H^1(S, \Omega_S) \to H^2(S, \mathcal{O}_S).$$

The criterion we shall use is the following:

Proposition 1. *Assume there exists a $\lambda \in H^1(S, \Omega_S)$ such that the map*

$$\mu_\lambda : H^0(S, N_{S/X}) \to H^2(S, \mathcal{O}_S), \qquad \mu_\lambda(n) = \rho(n) \lrcorner \lambda, \tag{1.1}$$

is surjective. Then any class $\alpha \in H^{2n-2}(X, \mathbb{Z})$ *is algebraic.*

Remark 2. Our assumptions imply immediately that the cohomology $H^{2n-2}(X, \mathbb{C})$ is of type $(n-1, n-1)$. Indeed, this last fact is equivalent to the vanishing of the space $H^n(X, \Omega_X^{n-2})$. On the other hand, interpreting the map μ_λ above in terms of infinitesimal variations of Hodge structures on the degree 2 cohomology of the surfaces S_t parameterized by $\mathcal{H}$, one sees that $Im\, \mu_\lambda$ is contained in

$$Ker\,(j_* : H^2(S, \mathcal{O}_S) \to H^n(X, \Omega_X^{n-2})).$$

Thus the assumptions imply that this last map j_* is 0, and as it is surjective by Lefschetz theorem, it follows that $H^n(X, \Omega_X^{n-2}) = 0$.

Remark 3. The assumption of Proposition 1 is exactly the assumption of Green's infinitesimal criterion for the density of the Noether-Lefschetz locus (see [19], 5.3.4), which allows to conclude that real degree 2 cohomology classes on S can be approximated by rational algebraic cohomology classes on nearby fibers S_t. It had been already used in [16], [17] to construct interesting algebraic cycles on Calabi-Yau threefolds.

Proof of Proposition. We refer to [19], chapter 5, for more details on infinitesimal variations of Hodge structures. On a simply connected neighborhood U in $\mathcal{H}$ of the point $0 \in \mathcal{H}$ parameterizing $S \subset X$, the restricted family

$$\pi : \mathcal{S}_U \to U$$

is differentiably trivial, and in particular the local system

$$H^2_{\mathbb{Z}} := R^2\pi_*\mathbb{Z}/torsion$$

is trivial. Thus the locally free sheaf

$$\mathcal{H}^2 := H^2_{\mathbb{Z}} \otimes \mathcal{O}_U$$

is canonically trivial, and denoting by H^2 the corresponding vector bundle on U, we get a canonical isomorphism

$$H^2 \cong U \times H^2(S, \mathbb{C}),$$

since the fiber of H^2 at 0 is canonically isomorphic to $H^2(S, \mathbb{C})$. Composing with the second projection gives us a holomorphic map

$$\tau : H^2 \to H^2(S, \mathbb{C}),$$

which on each fiber $H^2_t = H^2(S_t, \mathbb{C})$ is the natural identification

$$H^2(S_t, \mathbb{C}) \cong H^2(S, \mathbb{C}).$$

Next the vector bundle H^2 contains a holomorphic subbundle F^1H^2, which at the point $t \in U$ has for fibre the subspace

$$F^1H^2(S_t) := H^{2,0}(S_t) \oplus H^{1,1}(S_t) \subset H^2(S_t, \mathbb{C}).$$

We shall denote by

$$\tau_1 : F^1H^2 \to H^2(S, \mathbb{C})$$

the restriction of τ to F^1H^2.

The key point is the following fact, for which we refer to [19], 5.3.4 :

Lemma 2. *For $\lambda \in H^1(S_t, \Omega_{S_t})$, choose any lifting $\tilde{\lambda} \in F^1H^2_t$ of λ. Then the surjectivity of the map*

$$\mu_\lambda : H^0(S_t, N_{S_t/X}) \to H^2(S_t, \mathcal{O}_{S_t})$$

is equivalent to the fact that the map τ_1 is a submersion at $\tilde{\lambda}$.

Having this, we conclude as follows: First of all, we observe that the assumption of Proposition 1 is a Zariski open condition on $\lambda \in H^1(S, \Omega_S)$. Now, the space $H^1(S, \Omega_S) = H^{1,1}(S)$ has a real structure, namely

$$H^{1,1}(S) = H^{1,1}(S)_{\mathbb{R}} \otimes \mathbb{C},$$

where $H^{1,1}(S)_{\mathbb{R}} = H^{1,1}(S) \cap H^2(S, \mathbb{R})$. Thus if the assumption is satisfied for one $\lambda \in H^{1,1}(S)$, it is satisfied for one real $\lambda \in H^{1,1}(S)_{\mathbb{R}}$.

In the lemma above, choose for lifting $\tilde{\lambda}$ the class λ itself. Thus $\tilde{\lambda}$ is real, and so is $\tau_1(\tilde{\lambda})$. As the assumption on λ and the lemma imply that τ_1 is a submersion at $\tilde{\lambda}$, so is the restriction

$$\tau_{1,\mathbb{R}} : H^{1,1}_{\mathbb{R}} \to H^2(S, \mathbb{R})$$

of τ_1 to $\tau_1^{-1}(H^2(S, \mathbb{R}))$. Here we identified $\tau_1^{-1}(H^2(S, \mathbb{R}))$ to

$$\cup_{t \in U} F^1H^2(S_t) \cap H^2(S_t, \mathbb{R}) = \cup_{t \in U} H^{1,1}(S_t)_{\mathbb{R}} =: H^{1,1}_{\mathbb{R}}.$$

As $\tau_{1,\mathbb{R}}$ is a submersion at $\tilde{\lambda}$, and $H^{1,1}(S)_{\mathbb{R}}$ is a smooth real manifold, because it is a real vector bundle on U and U is smooth, $Im\, \tau_{1,\mathbb{R}}$ contains a non-empty open set of $H^2(S, \mathbb{R})$. On the other hand $Im\, \tau_{1,\mathbb{R}}$ is a cone. We use now the following elementary lemma:

Lemma 3. *Let $V_{\mathbb{Z}}$ be a lattice, and let C be a non-empty open cone in $V_{\mathbb{R}} := V_{\mathbb{Z}} \otimes \mathbb{R}$. Then $V_{\mathbb{Z}}$ is generated over $\mathbb{Z}$ by the points in $V_{\mathbb{Z}} \cap C$.*

Proof. $V_{\mathbb{Z}} \cap C$ is non-empty because $V_{\mathbb{Q}}$ is dense, and C is a non-empty open cone in $V_{\mathbb{R}}$. Let $u \in V_{\mathbb{Z}}$, and let $u' \in V_{\mathbb{Z}} \cap C$. For q a large integer, we have $\frac{1}{q}u + u' \in C$, because C is open. Then $u + qu' := v' \in V_{\mathbb{Z}} \cap C$. Thus $u = v' - qu'$ is in the sublattice generated over $\mathbb{Z}$ by the points in $V_{\mathbb{Z}} \cap C$. ∎

We apply Lemma 3 to $V_{\mathbb{Z}} = H^2(S,\mathbb{Z})/torsion$ and to C an open cone contained in $Im\,\tau_{1,\mathbb{R}}$. Thus we conclude that $H^2(S,\mathbb{Z})/torsion$ is generated over $\mathbb{Z}$ by classes $\alpha \in Im\,\tau_{1,\mathbb{R}}$. But by definition of τ_1, if an integral cohomology class $\alpha \in H^2(S,\mathbb{Z})/torsion$ is equal to $\tau_{1,\mathbb{R}}(\lambda_t)$, for some

$$\lambda_t \in H^{1,1}(S_t)_{\mathbb{R}} \subset H^2(S_t,\mathbb{R}),$$

the corresponding class

$$\alpha_t \in H^2(S_t,\mathbb{Z})/torsion \subset H^2(S_t,\mathbb{R})$$

is equal to λ_t in $H^2(S_t,\mathbb{R})$. Thus the class

$$\alpha_t = \lambda_t \in H^{1,1}(S_t) \cap H^2(S_t,\mathbb{Z})/torsion$$

is algebraic on S_t by Lefschetz theorem on $(1,1)$-classes.

The conclusion is that, under the assumptions of Proposition 1, the lattice $H^2(S,\mathbb{Z})/torsion$ is generated over $\mathbb{Z}$ by integral classes which become algebraic (i.e. are the class of a divisor) on some nearby fiber S_t. As the torsion of $H^2(S,\mathbb{Z})$ is algebraic, the same conclusion holds for $H^2(S,\mathbb{Z})$.

Finally, as the map $j_* : H^2(S,\mathbb{Z}) \to H^{2n-2}(X,\mathbb{Z})$ is surjective, we conclude that $H^{2n-2}(X,\mathbb{Z})$ is generated over $\mathbb{Z}$ by classes of 1-cycles in X.

■

§2. Proof of Theorem 2 when $H^2(X,\mathcal{O}_X) = 0$

In this section, we assume that $H^2(X,\mathcal{O}_X) = 0$ and X either has trivial canonical bundle or is uniruled.

In case where X is uniruled, we have the following result:

Lemma 4. *Let X be a uniruled threefold. Then a smooth birational model X' of X carries an ample line bundle H such that*

$$H^2K_{X'} < 0.$$

Proof. As X is uniruled, X is birationally equivalent to a $\mathbb{Q}$-Gorenstein threefold Y which is either a singular Fano threefold, or a Del Pezzo fibration over a smooth curve, or a conic bundle over a $\mathbb{Q}$-Gorenstein surface. Let us first prove the existence of an ample line bundle H_Y on Y such that $K_Y H_Y^2 < 0$:

a) If Y is Fano, $-K_Y$ is ample, so we can take for H_Y an integral multiple of $-K_Y$.

b) Otherwise there is a morphism

$$\pi : Y \to B,$$

where B is $\mathbb{Q}$-Gorenstein of dimension 1 or 2, and the relative canonical bundle K_π has the property that $-K_\pi$ is a relatively ample $\mathbb{Q}$-divisor. Let H_B be an ample line bundle on B, and choose for H_Y the $\mathbb{Q}$-divisor

$$H_Y = \pi^* H_B - \epsilon K_\pi,$$

where ϵ is a small rational number. As $-K_\pi$ is relatively ample, H_Y is ample for small enough ϵ. We compute now:

$$H_Y^2 K_Y = (\pi^* H_B - \epsilon K_\pi)^2 (\pi^* K_B + K_\pi)$$

$$= \pi^* H_B^2 K_\pi - 2\epsilon K_\pi \pi^* H_B (\pi^* K_B + K_\pi) + O(\epsilon^2).$$

If $dim\, B = 2$, the term $\pi^* H_B^2 K_\pi$ is negative, so that for small ϵ,

$$H_Y^2 K_Y < 0.$$

If $dim\, B = 1$, the first term vanishes but the second term is equal to $-2\epsilon K_\pi^2 \pi^* H_B$ and this is negative because $-K_\pi$ is relatively ample.

Let now Y, H_Y be as above, and let $\tau : X' \to Y$ be a desingularization of Y. Thus X' is a smooth birational model of X. Then there is a relatively ample divisor E on X' which is supported on the exceptional divisor of τ. Consider the $\mathbb{Q}$-divisor

$$H = \tau^* H_Y + \epsilon E,$$

for ϵ a sufficiently small rational number. Then we have $K_{X'} = \tau^* K_Y + F$ where F is supported on the exceptional divisor of τ. This gives

$$H^2 K_{X'} = (\tau^* H_Y + \epsilon E)^2 (\tau^* K_Y + F).$$

As $\tau^* H_Y^2 F = 0$, the dominating term is equal to

$$\tau^* H_Y^2 \tau^* K_Y = H_Y^2 K_Y < 0.$$

Thus for small ϵ we have $H^2 K_{X'} < 0$.

■

From now on, we will, in the uniruled case, consider X' instead of X, which can be done since the statement of Theorem 2 is invariant under birational equivalence, and we will assume that H satisfies the conclusion of Lemma 4.

Our aim in this section is to prove the following Proposition, which by Proposition 1 implies Theorem 2 for uniruled and Calabi-Yau threefolds X with $H^2(X, \mathcal{O}_X) = 0$.

Proposition 2. *Let X be a smooth projective uniruled or Calabi-Yau threefold such that $H^2(X, \mathcal{O}_X) = 0$. Let H be an ample line bundle on X. In the uniruled case, assume that H satisfies $H^2K_X < 0$. Then for n large enough, and for S a generic surface in $\mid nH \mid$, there is a $\lambda \in H^1(S, \Omega_S)$ which satisfies the property that*

$$\mu_\lambda : H^0(S, \mathcal{O}_S(nH)) \to H^2(S, \mathcal{O}_S)$$

is surjective.

To see that this is a reasonable statement, note that in the K-trivial case, the spaces $H^0(S, \mathcal{O}_S(nH))$ and $H^2(S, \mathcal{O}_S)$ have the same dimension, since, if $S \in \mid nH \mid$, we have by adjunction

$$H^0(S, K_S) = H^0(S, \mathcal{O}_S(S)) = H^0(S, N_{S/X}),$$

with $H^0(S, K_S) = H^2(S, \mathcal{O}_S)^*$. Thus the two spaces involved in Proposition 2 have the same dimension. In the uniruled case, we have:

Lemma 5. *Assume X, H satisfies $H^2K_X < 0$, then for $S \in \mid nH \mid$, we have*

$$h^0(\mathcal{O}_S(S)) = h^0(K_S) + \phi(n),$$

where $\phi(n) = \alpha n^2 + o(n^2)$, $\alpha > 0$.

Proof. We have $K_S = K_X(S)_{|S}$. Thus

$$\chi(\mathcal{O}_S(S)) = \chi(K_S(-K_X))$$

$$= \chi(K_{X|S}) = \chi(\mathcal{O}_S) + \frac{1}{2}(K^2_{X|S} - K_{X|S}(K_{X|S} + nH_{|S}))$$

$$= \chi(K_S) + \frac{1}{2}(nHK_X^2 - nHK_X(nH + K_X)).$$

It follows that

$$\chi(\mathcal{O}_S(S)) - \chi(K_S) = -\frac{1}{2}n^2H^2K_X + \text{affine linear term in } n.$$

On the other hand, for large n, the ranks

$$h^1(\mathcal{O}_S(S)) = h^2(\mathcal{O}_X),\ h^2(\mathcal{O}_S(S)) = h^3(\mathcal{O}_X),$$

$$h^1(K_S) = h^1(\mathcal{O}_S) = h^1(\mathcal{O}_X),\ h^2(K_S) = \mathbb{C}$$

do not depend on n. It follows that we also have

$$h^0(\mathcal{O}_S(S)) - h^0(K_S) = -\frac{1}{2}n^2H^2K_X + \text{affine linear term in } n,$$

which proves the result with $\alpha = -\frac{1}{2}H^2K_X > 0$. ∎

By this Lemma, we conclude that in the K-trivial case and in the uniruled case, we can assume that we have for n large enough, and $S \in \mid nH \mid$,

$$h^0(N_{S/X}) = h^0(S, \mathcal{O}_S(S)) \geq h^0(K_S) = h^2(\mathcal{O}_S).$$

This makes possible the surjectivity of the map

$$\mu_\lambda : H^0(S, N_{S/X}) \to H^2(S, \mathcal{O}_S)$$

of (1.1), and also says that μ_λ is surjective if and only if it has maximal rank.

Another way to see this is to introduce

$$V := H^0(S, K_S),\ V' := H^0(S, N_{S/X}).$$

The bilinear map

$$\mu : V \times V' \to H^1(S, \Omega_S), \qquad \mu(v, v') = \rho(v') \lrcorner v \tag{2.2}$$

and Serre's duality $H^1(S, \Omega_S) \cong H^1(S, \Omega_S)^*$ give a dual map

$$q = \mu^* : H^1(S, \Omega_S) \to (V \otimes V')^* = H^0(\mathbb{P}(V) \times \mathbb{P}(V'), \mathcal{O}(1,1)),$$

given by

$$q(\lambda)(v \otimes v') = < \lambda, \mu(v \times v') > .$$

As we have

$$< \lambda, \rho(v') \lrcorner v >= - < \rho(v') \lrcorner \lambda, v >,$$

where the $<,>$ stand for Serre's duality on $H^1(S, \Omega_S)$ on the left and between $H^0(S, K_S)$ and $H^2(S, \mathcal{O}_S)$ on the right, we see that $q(\lambda)$ identifies to $\mu_\lambda \in Hom\,(V, V'^*)$.

Thus the condition that μ_λ has maximal rank for generic λ is equivalent to the condition that the hypersurface of $\mathbb{P}(V) \times \mathbb{P}(V')$ defined by $q(\lambda)$ is non singular.

We shall use the following criterion:

Lemma 6. *Given μ as in (2.2), the generic hypersurface defined by $q(\lambda)$ is non singular if the following set*

$$Z = \{(v, v') \in \mathbb{P}(V) \times \mathbb{P}(V'),\ \mu(v \times v') = 0 \in H^1(S, \Omega_S)\} \tag{2.3}$$

satisfies

$$dim\, Z < dim\, \mathbb{P}(V').$$

Proof. Assume to the contrary that the generic $q(\lambda)$ is singular. Let

$$Z' \subset \mathbb{P}(H^1(S,\Omega_S)) \times \mathbb{P}(V),$$

$$Z' = \{(\lambda, v),\ q(\lambda) \text{ is singular at } (v, v') \text{ for some } v' \in \mathbb{P}(V')\}.$$

By assumption Z' dominates $\mathbb{P}(H^1(S,\Omega_S))$. Clearly there is only one irreducible component Z'_g of Z' which dominates $\mathbb{P}(H^1(S,\Omega_S))$. Let Z'_1 be the second projection of Z'_g in $\mathbb{P}(V)$.

As Z'_g dominates $\mathbb{P}(H^1(S,\Omega_S))$ we have

$$dim\, Z'_g \geq rk\, H^1(S,\Omega_S) - 1.$$

On the other hand, the fiber of Z'_g over the generic point v_g of Z'_1 is equal to

$$\mu(v_g \times V')^{\perp}.$$

Thus we have

$$dim\, Z'_g = dim\, Z'_1 + rk\, H^1(S,\Omega_S) - 1 - rk\, \mu_{v_g},$$

where $\mu_{v_g} : V' \to H^1(S,\Omega_S)$ is the map $v' \mapsto \mu(v_g \times v')$.

The condition $dim\, Z'_g \geq rk\, H^1(S,\Omega_S) - 1$ is thus equivalent to

$$dim\, Z'_1 \geq rk\, \mu_{v_g}. \tag{2.4}$$

But on the other hand, the unique irreducible component Z_0 of Z which dominates Z'_1 has dimension equal to $dim\, Z'_1 + dim\, \mathbb{P}(V') - rk\, \mu_{v_g}$ and inequality (2.4) implies that this is $\geq dim\, \mathbb{P}(V')$. ∎

Our first task will be thus to study the set Z introduced in (2.3). To this effect, we degenerate the surface $S \in\mid nH \mid$ to a surface with many nodes. The reason for doing that is the following fact (cf [15]):

Lemma 7. *Let $\mathcal{S} \to \Delta$ be a Lefschetz degeneration of surfaces S_t in $\mid nH \mid$, where the central fiber has ordinary double points $x_1, \ldots, x_N$ as singularities. Then the limiting space*

$$\lim_{t\to 0} Im\, (q_t : H^1(S_t,\Omega_{S_t}) \to (H^0(S_t, K_{S_t}) \otimes H^0(S_t, \mathcal{O}_{S_t}(nH)))^*),$$

which is a subspace of $(H^0(S_0, K_{S_0}) \otimes H^0(S_0, \mathcal{O}_{S_0}(nH)))^*$, contains for each $i = 1, \ldots, N$ the multiplication-evaluation map which is the composite:

$$H^0(S_0, K_{S_0}) \otimes H^0(S_0, \mathcal{O}_{S_0}(nH)) \stackrel{mult}{\to} H^0(S_0, K_{S_0}(nH))$$

$$\stackrel{ev_{x_i}}{\to} H^0(K_{S_0}(nH)_{|x_i}).$$

As we want to use this lemma to bound the dimension of the space Z of (2.3) for a generic surface, it is natural to degenerate the generic surface to a surface with many nodes. To get surfaces with many nodes, we use discriminant surfaces as in [2]. We assume here that H is very ample on X, and we consider a generic symmetric n by n matrix A whose entries A_{ij} are in $H^0(X, \mathcal{O}_X(H))$. Let $\sigma_A := discr\, A \in H^0(X, \mathcal{O}_X(nH))$ and S_A be the surface defined by σ_A.

Theorem 3. *(Barth [2]) The surface S_A has N ordinary double points as singularities, with*

$$N = \binom{n+1}{3} H^3.$$

Note that for large n, this grows like $\frac{n^3}{6}H^3$ while both dimensions $h^0(K_S)$, $h^0(\mathcal{O}_S(nH))$ of the spaces V, V' grow like $h^0(\mathcal{O}_X(nH))$, that is like $\frac{n^3}{6}H^3$ by Riemann-Roch.

Next we have the following lemma, which might well be known already, but for which we could not find a reference:

Lemma 8. *Let X be a smooth projective threefold, and H a very ample line bundle on X which satisfies the property that*

$$H^i(X, \mathcal{O}_X(lH)) = 0, \text{ for } i > 0,\, l > 0.$$

Let $S_A \in \mid nH \mid$ be a generic discriminant surface as above, and let $W \subset X$ be its singular set. Then the cohomology group $H^1(X, \mathcal{I}_W((n+2)H))$ vanishes.

Proof. Let $G = Grass(2, n)$ be the Grassmannian of 2-dimensional vector subspaces of $K := \mathbb{C}^n$. The matrix A as above can be seen as a family of quadrics A_x on $\mathbb{P}(K)$ parameterized by $x \in X$, the surface S_A corresponds to singular quadrics and the singular set W parameterizes quadrics of rank $n-2$. Thus W is via the second projection in one-to-one correspondence with the following algebraic set:

$$\widetilde{W} := \{(l, x) \in G \times X, A_x \text{ is singular along } l\}.$$

Let $\mathcal{E}$ be the tautological rank 2 quotient bundle on G, whose fiber at l is $H^0(\mathcal{O}_{\Delta_l}(1))$. $\mathcal{E}$ is a quotient of $K^* \otimes \mathcal{O}_G$, and there is the natural map

$$e : S^2K^* \otimes \mathcal{O}_G \to K^* \otimes \mathcal{E}.$$

Let

$$\mathcal{F} := Im\, e. \tag{2.5}$$

Clearly, a quadric $A \in S^2K^*$ on $\mathbb{P}(K)$ is singular along Δ_l if and only if it vanishes under the map e at the point l. Thus the set $\widetilde{W}$ is the zero locus of a section of the vector bundle

$$\mathcal{F} \boxtimes \mathcal{O}_X(H)$$

which is of rank $2n-1$ on $G \times X$. Note that the cokernel of e identifies to $\bigwedge^2 \mathcal{E} =: \mathcal{L}$ where $\mathcal{L}$ is the Plücker line bundle on G. Thus we have an exact sequence

$$0 \to \mathcal{F} \to K^* \otimes \mathcal{E} \to \mathcal{L} \to 0 \tag{2.6}$$

on G.

As $\widetilde{W}$ is the zero set of a transverse section of a rank $2n-1$ vector bundle on $G \times X$, its ideal sheaf admits the Koszul resolution:

$$0 \to \bigwedge^{2n-1} \mathcal{F}^* \boxtimes \mathcal{O}_X((-2n+1)H) \to \ldots \to \mathcal{F}^* \boxtimes \mathcal{O}_X(-H) \to \mathcal{I}_{\widetilde{W}} \to 0.$$

Thus the space $H^1(X, \mathcal{I}_W((n+2)H)) = H^1(G \times X, \mathcal{I}_{\widetilde{W}} \otimes pr_2^*((n+2)H))$ is the abutment of a spectral sequence whose E_1-term is equal to

$$H^i(G \times X, \bigwedge^i \mathcal{F}^* \boxtimes \mathcal{O}_X((n+2-i)H)),\ i \geq 1.$$

By Künneth decomposition and the vanishing assumptions, these spaces split as:

$$H^i(G, \bigwedge^i \mathcal{F}^*) \otimes H^0(X, (n+2-i)H),\ n+2-i > 0,$$

$$H^i(G, \bigwedge^i \mathcal{F}^*) \otimes H^0(X, \mathcal{O}_X) \oplus H^{i-1}(G, \bigwedge^i \mathcal{F}^*) \otimes H^1(X, \mathcal{O}_X)$$

$$\oplus H^{i-2}(G, \bigwedge^i \mathcal{F}^*) \otimes H^2(X, \mathcal{O}_X) \oplus H^{i-3}(G, \bigwedge^i \mathcal{F}^*) \otimes H^3(X, \mathcal{O}_X),\ i = n+2,$$

$$H^{i-3}(G, \bigwedge^i \mathcal{F}^*) \otimes H^3(X, (n+2-i)H),\ n+2-i < 0.$$

The proof of Lemma 8 is thus concluded by the following lemma, which implies that the E_1-terms of the spectral sequence above all vanish.

Lemma 9. *On the Grassmannian $G = Grass(2,n)$, the bundle $\mathcal{F}$ being defined as in (2.5), we have the vanishings:*

(1) $H^i(G, \bigwedge^i \mathcal{F}^*) = 0,\ n+2-i \geq 0,\ i \geq 1$

(2) $H^{i-1}(G, \bigwedge^i \mathcal{F}^*) = 0,\ n+2-i = 0.$

(3) $H^{i-2}(G, \bigwedge^i \mathcal{F}^*) = 0,\ n+2-i = 0.$
(4) $H^{i-3}(G, \bigwedge^i \mathcal{F}^*) = 0,\ n+2-i \leq 0,\ i \geq 1.$

The proof of this last lemma is postponed to an Appendix. ■

As an immediate corollary, we get the following:

Corollary 2. *Under the same assumptions as in Lemma 8, the numbers*

$$rk\, H^1(X, K_X \otimes \mathcal{I}_W(nH)),\ rk\, H^1(X, \mathcal{I}_W(nH))$$

are bounded by Cn^2 for some constant C.

Combining Corollary 2 with Riemann-Roch and Barth's Theorem 3, we get the following corollary:

Corollary 3. *The spaces $H^0(X, K_X(nH) \otimes \mathcal{I}_W)$ and $H^0(X, \mathcal{I}_W(nH))$ have dimension bounded by cn^2 for some constant c.*

We shall use the following consequence of the uniform position principle of Harris:

Lemma 10. *Let A be generic and let $W' \subset W$ be a subset of $W = Sing\, S_A$. Then if*

$$H^0(X, K_X(nH) \otimes \mathcal{I}_{W'}) \neq H^0(X, K_X(nH) \otimes \mathcal{I}_W),$$

W' imposes card W' *independent conditions to $H^0(X, K_X(nH))$. Similarly, if*

$$H^0(X, \mathcal{O}_X(nH) \otimes \mathcal{I}_{W'}) \neq H^0(X, \mathcal{O}_X(nH) \otimes \mathcal{I}_W),$$

W' imposes card W' *independent conditions to $H^0(X, \mathcal{O}_X(nH))$.*

Proof. Indeed, we represented in the previous proof the set W as the projection in X of a 0-dimensional subscheme $\widetilde{W}$ of $G \times X$, defined as the zero set of a generic transverse section of the vector bundle $\mathcal{F} \boxtimes \mathcal{O}_X(H)$ on $G \times X$. One verifies that the uniform position principle [8] applies to $\widetilde{W}$, and this allows to conclude that all subsets of W of given cardinality impose the same number of independent conditions to $H^0(X, K_X(nH))$ or $H^0(X, \mathcal{O}_X(nH))$. This number is then obviously equal to

$$Min\, (\text{card}\, W',\, a)$$

where $a = rk\, (rest:\ H^0(X, K_X(nH)) \rightarrow H^0(W, K_X(nH)_{|W})))$, resp.

$$a = rk\, (rest:\ H^0(X, \mathcal{O}_X(nH)) \rightarrow H^0(W, \mathcal{O}_W(nH)))$$

in the second case.

■

From now on, we will treat separately the uniruled and the K-trivial cases.

The uniruled case. We may assume (X, H) satisfies the inequality $H^2 K_X < 0$ of Lemma 4. We want to study the set Z of (2.3) for a generic surface $S \in \mid nH \mid$, and more precisely the irreducible components Z' of Z which are of dimension $\geq dim\, \mathbb{P}(V')$.

Degenerating S to S_A and applying Lemma 7, we find that the specialization Z'_s of Z' is contained in

$$Z_0 := \{(v, v') \in \mathbb{P}(V_A) \times \mathbb{P}(V'_A),\ vv'_{|W} = 0\},$$

where

$$V_A = H^0(S_A, K_{S_A}),\ V'_A = H^0(S_A, \mathcal{O}_{S_A}(nH)).$$

Lemma 11. *Z'_s is contained in the union*

$$\begin{aligned} &\mathbb{P}H^0(S_A, K_{S_A} \otimes \mathcal{I}_W) \times \mathbb{P}(V'_A) \\ &\cup \mathbb{P}(V_A) \times \mathbb{P}H^0(S_A, \mathcal{O}_{S_A}(nH) \otimes \mathcal{I}_W). \end{aligned} \tag{2.7}$$

Proof. We observe that Z_0 is a union of irreducible components indexed by subsets $W' \subset W$, with complementary set $W'' := W \setminus W'$:

$$Z_0 = \cup_{W' \subset W} Z_{W'},$$

$$Z_{W'} := \mathbb{P}H^0(S_A, K_{S_A} \otimes \mathcal{I}_{W'}) \times \mathbb{P}H^0(S_A, \mathcal{O}_{S_A}(nH) \otimes \mathcal{I}_{W''}).$$

We use now Lemma 10: it says that if both conditions

$$H^0(X, K_X(nH) \otimes \mathcal{I}_{W'}) \neq H^0(X, K_X(nH) \otimes \mathcal{I}_W),$$

$$H^0(X, \mathcal{O}_X(nH) \otimes \mathcal{I}_{W''}) \neq H^0(X, \mathcal{O}_X(nH) \otimes \mathcal{I}_W)$$

hold, then W' imposes $\mathrm{card}\, W'$ independent conditions to the linear system $H^0(X, K_X(nH))$ and W'' imposes $\mathrm{card}\, W''$ independent conditions to $H^0(X, \mathcal{O}_X(nH))$. Thus the codimension of $Z_{W'}$ in $\mathbb{P}(V_A) \times \mathbb{P}(V'_A)$ is equal to $\mathrm{card}\, W' + \mathrm{card}\, W'' = \mathrm{card}\, W$. But $\mathrm{card}\, W$ is equal to $\frac{n(n^2-1)}{6} H^3$ by Theorem 3, while the dimension of $V_A = H^0(S_A, K_{S_A}) \cong H^0(X, K_X(nH))$ is equal to

$$\frac{1}{6} n^3 H^3 + \frac{1}{4} n^2 K_X H^2 + \text{affine linear term in } n$$

by Riemann-Roch.

As $K_XH^2 < 0$, we conclude that for n large enough, if W' is as above, we have

$$codim\, Z' < dim\, \mathbb{P}(V_A),$$

and thus

$$dim\, Z_{W'} < dim\, \mathbb{P}(V'_A).$$

Thus, for large n, the only components of Z_0 which may have dimension $\geq dim\, \mathbb{P}(V'_A)$ are the two components $\mathbb{P}H^0(S_A, K_{S_A} \otimes \mathcal{I}_W) \times \mathbb{P}(V'_A)$ and $\mathbb{P}(V_A) \times \mathbb{P}H^0(S_A, \mathcal{O}_{S_A}(nH) \otimes \mathcal{I}_W)$. ∎

Corollary 4. *Assume S is generic and $Z' \subset \mathbb{P}(V) \times \mathbb{P}(V')$ is an irreducible component of Z which has dimension $\geq dim\, \mathbb{P}(V')$. Then either*
i) $dim\, pr_1(Z') \leq cn^2$ or
ii) $dim\, pr_2(Z') \leq cn^2$,
where c is the constant of Corollary 3.

Proof. By Lemma 11, the specialization Z'_s of Z' is contained in the union (2.7). As we have by Corollary 3

$$dim\, \mathbb{P}H^0(X, K_X(nH) \otimes \mathcal{I}_W) < cn^2,\ dim\, \mathbb{P}H^0(X, \mathcal{I}_W(nH)) < cn^2,$$

this implies that the cycle Z'_s satisfies:

$$h_1^{nc^2} h_2^{nc^2} [Z'_s] = 0 \text{ in } H^*(\mathbb{P}(V_A) \times \mathbb{P}(V'_A), \mathbb{Z}),$$

where

$$h_1 := pr_1^* c_1(\mathcal{O}_{\mathbb{P}(V_A)}(1)),\ h_2 := pr_2^* c_1(\mathcal{O}_{\mathbb{P}(V'_A)}(1)),$$

and $[Z'_s]$ is the cohomology class of the cycle Z'_s.

It follows that we also have

$$h_1^{nc^2} h_2^{nc^2} [Z'] = 0 \text{ in } H^*(\mathbb{P}(V) \times \mathbb{P}(V'), \mathbb{Z}). \tag{2.8}$$

We claim that this implies that i) or ii) holds. Indeed, as Z' is irreducible, there are well defined generic ranks k_1, k_2 of the projection $pr_{1|Z'}$, $pr_{2|Z'}$ respectively, which are also the generic ranks of the pull-back of the $(1,1)$-forms $pr_1^*\omega_1$, $pr_2^*\omega_2$ to Z', where ω_i are the Fubini-Study $(1,1)$-forms on $\mathbb{P}(V)$, $\mathbb{P}(V')$. As the form

$$pr_1^*\omega_1^{nc^2} \wedge pr_1^*\omega_2^{nc^2}$$

is semi-positive on Z', the condition (2.8) implies that everywhere on Z, we have

$$pr_1^*\omega_1^{nc^2} \wedge pr_2^*\omega_2^{nc^2} = 0.$$

As $dim\, Z' \geq 2cn^2$ and (pr_1, pr_2) is an immersion on the smooth locus of Z', this implies easily that either $k_1 = rk\, pr_1$ or $k_2 = rk\, pr_2$ has to be $< cn^2$, that is i) or ii). ∎

Corollary 5. *With the same assumptions as in the previous corollary, if $(v, v') \in Z'$, one has either*
i) $rk\, \mu_v : V' \to H^1(S, \Omega_S) < cn^2$, or
ii) $rk\, \mu_{v'} : V \to H^1(S, \Omega_S) < cn^2$,
where i) and ii) refer to the two cases of Corollary 4 and

$$\mu_v(\cdot) = \mu(v \otimes \cdot),\ \mu_{v'}(\cdot) = \mu(\cdot \otimes v').$$

Proof. Indeed, assume case i) of Corollary 4 holds. As $dim\, Z' \geq dim\, \mathbb{P}(V')$, the generic fibre of $pr_1 : Z' \to \mathbb{P}(V')$ has dimension $> dim\, \mathbb{P}(V') - cn^2$. But the generic fibre is, by definition of Z, equal to $\mathbb{P}(Ker\, \mu_v)$. Thus $rank\, \mu_v < cn^2$.

In case ii), we can do the same reasoning, as we have

$$dim\, Z' \geq dim\, \mathbb{P}(V') \geq dim\, \mathbb{P}(V).$$

∎

The proof that such a Z' does not exist, and thus, the proof of Proposition 2 in the uniruled case, concludes now with the following two Lemmas :

Lemma 12. *Let $S \in \mid nH \mid$ be generic, with n large enough. Let c be any positive constant. Then there exists a constant A such that the sets*

$$\Gamma = \{v \in \mathbb{P}(V),\ rk\, \mu_v < cn^2\}, \tag{2.9}$$

$$\Gamma' = \{v' \in \mathbb{P}(V'),\ rk\, \mu_{v'} < cn^2\}, \tag{2.10}$$

both have dimension bounded by A.

Lemma 13. *Let A be any positive constant. Let $S \in \mid nH \mid$ be generic, with n large enough (depending on A). Then the set*

$$B = \{v \in V,\ rk\, \mu_v < A\}$$

reduces to 0.

Indeed, we know by Corollary 5 that our set Z' should satisfy either $pr_1(Z') \subset \Gamma$ (case i) or $pr_2(Z') \subset \Gamma'$ (case ii). Thus by Lemma 12, one concludes that in case i), $dim\, pr_1(Z') \leq A$ and in case ii), $dim\, pr_2(Z') \leq A$, where A does not depend on n.

In case ii), it follows that $dim\, Z \leq dim\, \mathbb{P}(V) + A$ and as we have $dim\, \mathbb{P}(V) + A < dim\, \mathbb{P}(V')$ by Lemma 5, this gives a contradiction.

In case i), it follows, arguing as in the proof of Corollary 5, that for $(v, v') \in Z'$, one has $rk\, \mu_v < A$. This is impossible unless Z' is empty by Lemma 13. Thus, assuming Lemmas 12 and 13, Proposition 2 is proved for uniruled threefolds with $H^2(X, \mathcal{O}_X) = 0$. ∎

Proof of Lemma 12. Our first step is to reduce the statement to the case where S is a surface in $\mathbb{P}^3$. This is done as follows: we choose once for all a morphism

$$f : X \to \mathbb{P}^3$$

given by 4 sections of H, so that $f^*\mathcal{O}_{\mathbb{P}^3}(1) = H$. We shall prove the result for surfaces of the form $S = f^{-1}(\Sigma)$, where Σ is a generic smooth surface of degree n in $\mathbb{P}^3$. Let $f_S : S \to \Sigma$ be the restriction of f to S. We have trace maps

$$f_{S*} : H^1(S, \Omega_S(sH)) \to H^1(\Sigma, \Omega_\Sigma(s)),$$

$$f_{S*} : H^0(S, K_S(sH)) \to H^0(\Sigma, K_\Sigma(s))$$

for all integers s. We note now that the map μ admits obvious twists that we shall also denote by μ:

$$\mu : H^0(S, K_S(lH)) \otimes H^0(S, \mathcal{O}_S(nH)) \to H^1(S, \Omega_S(lH)).$$

Furthermore, we have similarly defined bilinear maps μ^Σ:

$$\mu^\Sigma : H^0(\Sigma, K_\Sigma(l)) \otimes H^0(\Sigma, \mathcal{O}_\Sigma(n)) \to H^1(\Sigma, \Omega_\Sigma(l)).$$

All the maps μ can be defined using the maps

$$\delta : H^0(S, K_S(lH)) \hookrightarrow H^1(S, \Omega_S((-n + l)H)),$$

induced by the exact sequence (which is itself a twist of the normal exact sequence)

$$0 \to \Omega_S(-nH) \to \Omega^2_{X|S} \to K_S \to 0,$$

twisted by lH, and then the product map

$$H^1(S, \Omega_S((-n + l)H)) \otimes H^0(S, \mathcal{O}_S(nH)) \to H^1(S, \Omega_S(lH)).$$

The same is true for the maps μ_Σ.

As there is a commutative diagram of normal exact sequences

$$\begin{array}{ccccccccc} 0 & \to & T_S & \to & T_{X|S} & \to & \mathcal{O}_S(nH) & \to & 0 \\ & & f_{S*}\downarrow & & f_*\downarrow & & \| & & \\ 0 & \to & f^*T_\Sigma & \to & f^*T_{\mathbb{P}^3|S} & \to & \mathcal{O}_S(nH) & \to & 0 \end{array},$$

where the bottom line is the normal bundle sequence of Σ pulled-back to S, it follows that for $v \in H^0(S, K_S(l))$ and $\eta \in H^0(\Sigma, \mathcal{O}_\Sigma(n))$, we have:

$$f_{S*}(\mu_v(f_S^*\eta)) = \mu^\Sigma_{f_{S*}(v)}(\eta), \tag{2.11}$$

Equation (2.11) implies that

$$rk\,(\mu^\Sigma_{f_{S*}(v)} : H^0(\Sigma, \mathcal{O}_\Sigma(n)) \to H^1(\Sigma, \Omega_\Sigma(l)))$$

$$\leq rk\,(\mu_v : H^0(S, \mathcal{O}_S(n)) \to H^1(S, \Omega_S(l))).$$

Let us now prove the first case of Lemma 12, namely for the set Γ. The second proof is done similarly.

Starting from a sufficiently ample H, one finds that $H^0(X, \mathcal{O}_X(4H))$ restricts surjectively onto $H^0(X_u, \mathcal{O}_{X_u}(4H))$, for any $u \in \mathbb{P}^3$, where $X_u := f^{-1}(u)$.

We have the following Lemma:

Lemma 14. *The image Γ_Σ of the composed map*

$$\Gamma \times H^0(X, \mathcal{O}_X(4H)) \xrightarrow{\nu} H^0(S, K_S(4H)) \xrightarrow{f_{S*}} H^0(\Sigma, K_\Sigma(4)),$$

where ν is the product, has dimension at least equal to $\frac{1}{N} dim\,\Gamma$, where

$$N := rk\,H^0(X, \mathcal{O}_X(4H)).$$

Proof. Indeed, as the restriction map

$$H^0(X, \mathcal{O}_X(4H)) \to H^0(X_u, \mathcal{O}_{X_u}(4H))$$

is surjective, if e_i is a basis of $H^0(X, \mathcal{O}_X(4H))$, the map

$$\Gamma \to \Gamma_\Sigma^D, \ \gamma \mapsto f_{S*}(\gamma e_i),$$

is injective. Thus $dim\,\Gamma \leq N dim\,\Gamma_\Sigma$.

■

On the other hand, if $v \in \Gamma$, $\alpha \in H^0(X, \mathcal{O}_X(4H))$, we have

$$rk\, \mu_{\alpha v} \leq rk\, \mu_v$$

because $\mu_{\alpha v} = \alpha\mu_v$. Thus we conclude that the following hold:

$$dim\, \Gamma_\Sigma \geq \frac{1}{N} dim\, \Gamma,$$

$$rk\, \mu_w^\Sigma \leq rk\, \mu_v \leq cn^2,$$

for all $w \in \Gamma_\Sigma$.

As N does not depend on n, it suffices to show the result for generic Σ in $\mathbb{P}^3$ and for the product

$$\mu^\Sigma : H^0(\Sigma, K_\Sigma(4)) \times H^0(\Sigma, \mathcal{O}_\Sigma(n)) \to H^1(\Sigma, \Omega_\Sigma(4)).$$

This last product is well known (cf [19],6.1.3) to identify to the multiplication in the Jacobian ring of Σ:

$$\mu_\Sigma : H^0(\Sigma, \mathcal{O}_\Sigma(n)) \times H^0(\Sigma, \mathcal{O}_\Sigma(n)) \to R_\Sigma^{2n}.$$

Thus we have to show that for generic Σ, the set

$$\Gamma_\Sigma := \{v \in H^0(\Sigma, \mathcal{O}_\Sigma(n)),\, rk\, \mu_{\Sigma,v} \leq cn^2\}$$

has dimension bounded by a constant which is independent of n.

For this, we specialize to the case where Σ is the Fermat surface, that is, its defining equation is $\sigma = \sum_0^3 X_i^n$. The Jacobian ideal of Σ is then generated by the X_i^{n-1}, and there is thus a natural action of the torus $(\mathbb{C}^*)^4$ on the Jacobian ring R_Σ, by multiplication of the coordinates by a scalar. The subspace

$$\Gamma_\Sigma \subset R_\Sigma^{n-1}$$

is thus invariant under $(\mathbb{C}^*)^4$. Note that the fixed points of the induced action on $\mathbb{P}(R_\Sigma^{n-1})$ are the monomials, and are thus isolated. It follows that we have the inequality

$$dim\, \overline{\Gamma}_\Sigma \leq \text{ number of fixed points on } \overline{\Gamma}_\Sigma.$$

Thus we have to bound the number of monomials

$$X_I = X_0^{i_0} X_1^{i_1} X_2^{i_2} X_3^{i_3},\ i_0 + i_2 + i_3 + i_4 = n,$$

such that

$$rk\, X_I : R_\Sigma^n \to R_\Sigma^{2n-1} \leq cn^2.$$

But the kernel of the multiplication by X_I above is equal to the ideal

$$X_0^{n-i_0}S^{i_0} + \dots X_3^{n-i_3}S^{i_3},$$

where $S^l := H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(l))$, and thus has dimension $\leq \sum_k rk\, S^{i_k}$. Hence, if $rk\, X_I \leq cn^2$, we must have

$$\sum_k rk\, S^{i_k} \geq rk\, S^n - cn^2, \tag{2.12}$$

with $\sum_k i_k = n$. It is not hard to see that there exists an integer $l > 0$ such that, if n is large enough and (2.12) holds for I, n, one of the i_k's has to be $\geq n - l$. Thus the other i_j's have to be non greater than l. This shows immediately that the number of such monomials is bounded by a constant independent of n and concludes the proof of Lemma 12. ∎

Proof of Lemma 13. The key point is the following fact from [6].

Proposition 3. *Let X be any projective manifold and H be a very ample line bundle on X. Let A be a given constant, and for $n > A$, let $M \subset H^0(X, \mathcal{O}_X(nH))$ be a subspace of codimension $\leq A$. Then*

$$H^0(X, \mathcal{O}_X(H)) \cdot M \subset H^0(X, \mathcal{O}_X((n+1)H))$$

has codimension $\leq A$, with strict inequality if M has no base-point.

Assume $v \in V$ satisfies the condition that $rk\, \mu_v < A$. Let $M := Ker\, \mu_v \subset H^0(S, \mathcal{O}_S(nH))$. By Proposition 3, we conclude that if $n > A$, we have

$$H^0(S, \mathcal{O}_S(H)) \cdot M \subset H^0(S, \mathcal{O}_S((n+1)H))$$

has codimension $< A$. Next, we consider for each l the map

$$\mu_v^l : H^0(S, \mathcal{O}_S((n+l)H)) \to H^1(S, \Omega_S(l)),$$

obtained as the composite of the twisted Kodaira-Spencer map

$$H^0(S, \mathcal{O}_S((n+l)H)) \to H^1(S, T_S(l)),$$

and the contraction with v, using the contraction map

$$H^0(S, K_S) \otimes H^1(S, T_S(l)) \to H^1(S, \Omega_S(l)).$$

We note that the kernel M_l of the map μ_v^l contains

$$M_1 \cdot H^0(S, \mathcal{O}_S((l-1)H)).$$

On the other hand, M_1 also contains the image of the map

$$H^0(S, T_X(H)_{|S}) \to H^0(S, \mathcal{O}_S((n+1)H))$$

induced by the normal bundle sequence twisted by H. We may assume that H is ample enough so that $H^0(X, T_X(1))$ is generated by global sections, and then M_1 has no base-point. Proposition 3 thus implies that if $n > A$, the numbers *corank* M_l are strictly decreasing, starting from $l \geq 1$. Hence we conclude that

$$M_A = H^0(S, \mathcal{O}_S((n+A)H)).$$

As n is large and A is fixed, we may assume that

$$H^0(X, K_X((2n-A)H)) \otimes H^0(X, \mathcal{O}_X((n+A)H)) \to H^0(X, K_X(3nH))$$

is surjective, and that the same is true after restriction to S. Thus we conclude that

$$M_A \cdot H^0(S, K_X((2n-A)H)_{|S}) = H^0(S, K_X(3nH)_{|S}).$$

We use now the definition of M_A, and the compatibility of the twisted Kodaira-Spencer maps and the maps $\lrcorner v$ with multiplication. This implies that for any $P \in H^0(S, K_X((3n)H)_{|S})$, sending to

$$\overline{P} \in H^1(S, T_S(K_S(2nH))),$$

via the map induced by the twisted normal bundle sequence

$$0 \to T_S(K_S(2nH)) \to T_{X|S}(K_S(2nH)) \to K_X(3nH)_{|S} \to 0,$$

we have

$$\overline{P} \lrcorner v = 0 \text{ in } H^1(S, \Omega_S(K_S(nH))). \tag{2.13}$$

We have now a map

$$\delta : H^1(S, \Omega_S(K_S(nH))) \to H^2(S, K_S),$$

induced by the exact sequence

$$0 \to K_S \to \Omega_X(K_X(2nH))_{|S} \to \Omega_S(K_S(nH)) \to 0,$$

and one knows (cf [4]) that up to a multiplicative coefficient, one has

$$\delta(\overline{P} \lrcorner v) = < v, res_S(P) >, \tag{2.14}$$

where on the right, $<,>$ is Serre duality between $H^0(S, K_S)$ and $H^2(S, \mathcal{O}_S)$, and the Griffiths residue map

$$H^0(X, K_X(3nH)) \stackrel{res_S}{\to} H^2(S, \mathcal{O}_S) \tag{2.15}$$

is described in [19], 6.1.2. The key point for us is that, because in our case $H^3(X, \Omega_X) = 0$ and because n is large enough, the residue map (2.15) is surjective, and thus (2.13) together with (2.14) imply that, for all $\eta \in H^2(S, \mathcal{O}_S)$, one has

$$< \eta, v >= 0,$$

which implies that $v = 0$. ∎

The Calabi-Yau case. Here X has trivial canonical bundle and satisfies $H^2(X, \mathcal{O}_X) = 0$. We use in this case a variant of Lemma 6. As K_X is trivial, the spaces V and V' are equal, and the pairing $\mu : V \times V' \to H^1(S, \Omega_S)$ is symmetric. Thus, using Bertini, Lemma 6 can be refined as follows (cf [15]):

Lemma 15. *Let $\mu : V \otimes V' \to H^1(S, \Omega_S)$ be symmetric and $q : H^1(S, \Omega_S) \to S^2V^*$ be its dual. Then the generic quadric in $Im\, q$ is non-singular if the following condition holds. There is no subset $Z \subset \mathbb{P}(V)$ contained in the base-locus of $Im\, q$ and satisfying:*

$$rk\, \mu_v \leq dim\, Z, \forall v \in Z.$$

We have to verify that such a Z does not exist for generic $S \in \mid nH \mid$, n large enough. Degenerating S to S_A as before, the base-locus of $Im\, q$ specializes to a subspace of the base-locus of $Im\, q_A$. We now use Lemma 7, together with Corollary 3, to conclude that the base-locus of $Im\, q_A$ has dimension $\leq cn^2$, for some c independent of n.

Thus the base-locus of $Im\, q$ also has dimension bounded by cn^2, for generic S.

By definition of Z, it follows that for $v \in Z$ one has

$$rk\, (\mu_v : V \to H^1(S, \Omega_S)) \leq cn^2.$$

Using Lemma 12, it follows that $dim\, Z \leq A$ for some constant A independent of N. But then, for $v \in Z$, one has

$$rk\, (\mu_v : V \to H^1(S, \Omega_S)) \leq A$$

which implies that Z is empty by Lemma 13. This concludes the proof of Proposition 2 when X is a Calabi-Yau threefold. ∎

§3. The case where $H^2(X, \mathcal{O}_X) \neq 0$

In this section, we show how to adapt the previous proof to the case where X is uniruled with $H^2(X, \mathcal{O}_X) \neq 0$.

In this case, a smooth birational model of X admits a map $\phi : X' \dashrightarrow \Sigma$, with generic fibre isomorphic to $\mathbb{P}^1$, where Σ is a smooth surface. Note that ϕ_* sends $H^3(X, \Omega_X)$ isomorphically to $H^2(\Sigma, \mathcal{O}_\Sigma)$.

We may assume that X' carries a line bundle H such that

$$H^2 K_{X'} < 0,$$

because there is a smooth birational model of X on which such an H exists, and by blowing-up this X' to an $\widetilde{X}$ with exceptional relatively anti-ample divisor E, we may assume that ϕ becomes defined, while an $\widetilde{H}$ of the form $\tau^* H - \epsilon E$ with small ϵ will still satisfy the property $\widetilde{H}^2.K_{\widetilde{X}} < 0$.

In the sequel X, H, ϕ will satisfy the properties above. For S a smooth surface in $\mid nH \mid$, we have the Gysin maps:

$$\phi_* : H^1(S, \Omega_S) \to H^1(\Sigma, \Omega_\Sigma),\ \phi_* : H^2(S, \mathcal{O}_S) \to H^2(\Sigma, \mathcal{O}_\Sigma),$$

$$\phi_* : H^2(S, \mathbb{Z}) \to H^2(\Sigma, \mathbb{Z}).$$

We will denote by

$$H^1(S, \Omega_S)_\Sigma,\ H^2(S, \mathcal{O}_S)_\Sigma,\ H^2(S, \mathbb{Z})_\Sigma$$

the respective kernels of these maps. The proof will use the following variant of Proposition 1:

Proposition 4. *Assume there is a $S \in \mid nH \mid$, and a $\lambda \in H^1(S, \Omega_S)_\Sigma$ such that the natural map*

$$\mu_\lambda : H^0(S, \mathcal{O}_S(nH)) \to H^2(S, \mathcal{O}_S)_\Sigma$$

defined as in (1.1) is surjective. Then the Hodge conjecture is true for integral Hodge classes on X.

Proof. We consider a simply connected open set in $\mid nH \mid$ parameterizing smooth surfaces and containing the point $0 \in\mid nH \mid$ which is the parameter for S. We study the infinitesimal variation of Hodge structure on $H^2(S_t, \mathbb{Z})_\Sigma$ for $t \in B$.

By the same reasoning as in the proof of Proposition 1, the existence of λ satisfying the property above implies that at some point $\lambda \in H^{1,1}(S)_{\mathbb{R},\Sigma}$, the natural map

$$\psi : H^{1,1}_{\mathbb{R},\Sigma} \to H^2(S,\mathbb{R})_\Sigma$$

is a submersion. Here on the left hand side, we have the real vector bundle with fibre $H^{1,1}(S_t)_{\mathbb{R},\Sigma}$ at the point t, and on each fibre $H^{1,1}(S_t)_{\mathbb{R},\Sigma}$, ψ is the inclusion $H^{1,1}(S_t)_{\mathbb{R},\Sigma} \subset H^2(S_t,\mathbb{R})_\Sigma$, followed by the topological isomorphism $H^2(S_t,\mathbb{R})_\Sigma \cong H^2(S,\mathbb{R})_\Sigma$.

This implies that the image of ψ contains an open cone and we deduce from this as in the proof of Proposition 1 that $H^2(S,\mathbb{Z})_\Sigma$ is generated over $\mathbb{Z}$ by classes α which are algebraic on some nearby fiber S_t.

Consider now the inclusion $j : S \to X$. It induces a surjective Gysin map $j_* : H^2(S,\mathbb{Z}) \to H^4(X,\mathbb{Z})$ by Lefschetz hyperplane theorem. On the other hand, we have a commutative diagram of Gysin maps:

$$\begin{array}{ccc} H^2(S,\mathbb{Z}) & \stackrel{j_*}{\to} & H^4(X,\mathbb{Z}) \\ \phi_* \downarrow & & \phi_* \downarrow \\ H^2(\Sigma,\mathbb{Z}) & = & H^2(\Sigma,\mathbb{Z}) \end{array} .$$

From this and the previous conclusion, we deduce that the group

$$Ker\,(\phi_* : H^4(X,\mathbb{Z}) \to H^2(\Sigma,\mathbb{Z})) = j_*H^2(S,\mathbb{Z})_\Sigma$$

is generated by classes of algebraic cycles on X.

Proposition 4 is then a consequence of the following:

Lemma 16. *Let α be an integral Hodge class of degree 2 on Σ. Then there is an algebraic 1-cycle Z on X such that $\alpha = \phi_*([Z])$.*

Indeed, assuming this lemma, if α is an integral Hodge class on X of degree 4, $\phi_*\alpha$ is an integral Hodge class of degree 2 on Σ, hence is equal to $\phi_*([Z])$ for some Z. Hence $\alpha - [Z]$ belongs to $Ker\,\phi_*$ and thus it is algebraic as we already proved. This proves the Proposition. ∎

Proof of Lemma 16. We may assume by Lefschetz $(1,1)$ theorem and because Σ is algebraic, that α is the class of a curve $C \subset S$ which is in general position. Thus

$$\phi_C : X_C := \phi^{-1}(C) \to C$$

is a geometrically ruled surface, which admits a section $C' \subset X_C$ (see [3], or [7] for a more general statement).

But then the curve $C' \subset X$ satisfies $\phi_*[C'] = [C]$. ∎

By Proposition 4, the proof of Theorem 2 in case where X is uniruled and satisfies $H^2(X,\mathcal{O}_X) \neq 0$ will now be a consequence of the following proposition.

Proposition 5. *Let the pair (X, H) satisfy the inequality*

$$H^2 K_X < 0.$$

Then for n large enough, for S a generic surface in $\mid nH \mid$, there is a $\lambda \in H^1(S, \Omega_S)_\Sigma$ which satisfies the property that

$$\mu_\lambda : H^0(S, \mathcal{O}_S(nH)) \to H^2(S, \mathcal{O}_S)_\Sigma$$

is surjective.

The proof works exactly as the proof of Proposition 2 in the unir-uled case. The only thing to note is the fact that the analogue of Proposition 13 still holds in this case, with $V = H^0(S, K_S)_\Sigma$, $V' = H^0(S, \mathcal{O}_S(nH))$. This is indeed the only place where we used the assumption $H^2(X, \mathcal{O}_X) = 0$.

In this case, we have an isomorphism

$$\phi_* : H^3(\Omega_X) \cong H^2(\Sigma, \mathcal{O}_\Sigma),$$

so that for $S \subset X$ a smooth surface

$$H^2(S, \mathcal{O}_S)_\Sigma = Ker\,(j_* : H^2(S, \mathcal{O}_S) \to H^3(X, \Omega_X)),$$

where j is the inclusion of S into X.

But the theory of Griffiths residues shows that the last kernel is precisely generated by residues $res_S\omega$, $\omega \in H^0(X, K_X(3nH))$. Thus, the arguments of Lemma 13 will show in this case that if $v \in H^0(S, K_S)$ satisfies $rank\,\mu_v \leq A$, where A is a given constant, and $S \in \mid nH \mid$ with n large enough, then

$$v \in (Ker\, j_*)^\perp,$$

where $\perp$ refers to Serre duality between $H^0(S, K_S)$ and $H^2(S, \mathcal{O}_S)$. But as $Ker\,\phi_* = Ker\, j_*$, we have

$$(Ker\, j_*)^\perp = \phi^* H^0(\Sigma, K_\Sigma).$$

Thus if furthermore $v \in H^0(S, K_S)_\Sigma$, we must have $v = 0$ because

$$H^0(S, K_S)_\Sigma \cap \phi^* H^0(\Sigma, K_\Sigma) = 0.$$

■

§4. Appendix

We give for the convenience of the reader the proof of the vanishing Lemma 9. Recall that we want to prove the vanishing of the spaces:

(1) $H^i(G, \bigwedge^i \mathcal{F}^*)$, $n+2-i \geq 0$, $i \geq 1$
(2) $H^{i-1}(G, \bigwedge^i \mathcal{F}^*)$, $n+2-i = 0$.
(3) $H^{i-2}(G, \bigwedge^i \mathcal{F}^*)$, $n+2-i = 0$.
(4) $H^{i-3}(G, \bigwedge^i \mathcal{F}^*)$, $n+2-i \leq 0$.

We use first the dual of the exact sequence (2.6) to get a resolution of $\bigwedge^i \mathcal{F}^*$:

$$\ldots \to \bigwedge^{i-1}(K \otimes \mathcal{E}^*) \otimes \mathcal{L}^{-1} \to \bigwedge^{i}(K \otimes \mathcal{E})^* \to \bigwedge^{i} \mathcal{F}^* \to 0.$$

This induces a spectral sequence converging to

$$H^i(G, \bigwedge^{i} \mathcal{F}^*),\ H^{i-1}(G, \bigwedge^{i} \mathcal{F}^*),\ H^{i-2}(G, \bigwedge^{i} \mathcal{F}^*),\ H^{i-3}(G, \bigwedge^{i} \mathcal{F}^*),$$

whose E_1 terms are

Case 1 $\qquad H^{i+s}(G, \bigwedge^{i-s}(K \otimes \mathcal{E}^*) \otimes \mathcal{L}^{-s}),\ n+2 \geq i \geq 1,\ i \geq s \geq 0,$

Case 2 $\qquad H^{i+s-1}(G, \bigwedge^{i-s}(K \otimes \mathcal{E}^*) \otimes \mathcal{L}^{-s}),\ i = n+2,\ i \geq s \geq 0,$

Case 3 $\qquad H^{i+s-2}(G, \bigwedge^{i-s}(K \otimes \mathcal{E}^*) \otimes \mathcal{L}^{-s}),\ i = n+2 i \geq s \geq 0$

Case 4 $\qquad H^{i+s-3}(G, \bigwedge^{i-s}(K \otimes \mathcal{E}^*) \otimes \mathcal{L}^{-s}),\ n+2 \leq i,\ i \geq s \geq 0$

respectively.

Let $P \subset \mathbb{P}(K) \times G$ be the incidence scheme, so P is a $\mathbb{P}^1$-bundle over G. Let pr_i, $i = 1, 2$ denote the projections from P to $\mathbb{P}(K)$ and G respectively. Let $H := pr_1^*\mathcal{O}(1)$ and denote also by $\mathcal{L}$ the pull-back of $\mathcal{L}$ to P. Then $pr_2^*\mathcal{E}^*$ fits into an exact sequence:

$$0 \to H^{-1} \to pr_2^*\mathcal{E}^* \to H \otimes \mathcal{L}^{-1} \to 0.$$

Thus the bundle

$$pr_2^*(\bigwedge^{i-s}(K \otimes \mathcal{E}^*) \otimes \mathcal{L}^{-s})$$

admits a filtration whose successive quotients are line bundles of the form

$$H^{-\alpha} \otimes (H \otimes \mathcal{L}^{-1})^{\beta} \otimes \mathcal{L}^{-s} = H^{-\alpha+\beta} \otimes \mathcal{L}^{-\beta-s},$$

where $\alpha + \beta = i - s$, $\alpha \geq 0$, $\beta \geq 0$. As we are interested in

$$H^*(G, \bigwedge^{i-s}(K \otimes \mathcal{E}^*) \otimes \mathcal{L}^{-s}) = H^*(G, R^0 pr_{2*}(pr_2^*(\bigwedge^{i-s}(K \otimes \mathcal{E}^*) \otimes \mathcal{L}^{-s}))),$$

it suffices to study the cohomology groups

$$H^*(P, H^{-\alpha+\beta} \otimes \mathcal{L}^{-\beta-s}),$$

with $-\alpha + \beta \geq 0$. These groups are equal to the groups

$$H^*(G, S^{-\alpha+\beta}\mathcal{E} \otimes \mathcal{L}^{-\beta-s})$$

which are partially computed in [18]. The conclusion is the following:

Lemma 17. *a) These groups vanish for* $* \neq n-2,\ 2(n-2)$ *and for* $\beta + s \leq n-2$.

b) For $* = n-2$*, these groups vanish if* $-s-\alpha+1 < 0$.

c) For $* = 2(n-2)$*, these groups vanish if* $-s-\alpha \geq -n+1$.

Case 1. Here $* = i + s$, and the following inequalities hold:

$$\beta \geq \alpha \geq 0,\ \beta + s \geq n-1 \tag{4.16}$$

and furthermore

$$1 \leq i \leq n+2,\ \alpha + \beta = i - s.$$

According to Lemma 17, in order to get a non trivial cohomology group, we have only two possibilities:

a) $i + s = n-2$, $-s-\alpha+1 \geq 0$.

b) $i + s = 2(n-2)$, $-s-\alpha < -n+1$.

In case a), we have $\beta + s \geq n-1$ and $\alpha + \beta + 2s = i + s = n-2$, which is clearly a contradiction as $\alpha + s \geq 0$.

In case b), we have $\beta + s \geq n-1$, $\alpha + s \geq n$ and thus

$$2n-1 \leq \alpha + \beta + 2s = i + s = 2(n-2)$$

which is clearly a contradiction. ■

Case 2. Now $* = i + s - 1$ and $i = n+2$. We have again the inequalities (4.16) and furthermore

$$i = n+2,\ \alpha + \beta = i - s.$$

By Lemma 17, in order to get a non trivial cohomology group, we have only two possibilities:

a) $i+s-1=n-2$, $-s-\alpha+1\geq 0$.

b) $i+s-1=2(n-2)$, $s+\alpha\geq n$.

In case a), we have $i=n+2$ and $s\geq 0$, hence $i+s-1=n-2$ is impossible.

In case b), we have $i+s=2n-3$, while $s+\alpha\geq n$ and $\beta+s\geq n-1$ give $\alpha+\beta+2s=i+s\geq 2n-1$, contradiction.

■

Case 3. Now $*=i+s-2$ and $i=n+2$. We have again the inequalities (4.16) and furthermore $i=n+2$, $\alpha+\beta=i-s$. As before, in order to get a non trivial cohomology group, we have only two possibilities:

a) $i+s-2=n-2$, $-s-\alpha+1\geq 0$.

b) $i+s-2=2(n-2)$, $s+\alpha\geq n$.

In case a), we have $i=n+2$ and $s\geq 0$, hence $i+s-2=n-2$ is impossible.

In case b), we have $i+s=2n-2$, while $s+\alpha\geq n$ and $\beta+s\geq n-1$ give $\alpha+\beta+2s=i+s\geq 2n-1$, contradiction.

■

Case 4. Now $*=i+s-3$ and $i\geq n+2$. We have again the inequalities (4.16) and furthermore $i\geq n+2$, $\alpha+\beta=i-s$. As before, in order to get a non trivial cohomology group, we have only two possibilities:

a) $i+s-3=n-2$, $-s-\alpha+1\geq 0$.

b) $i+s-3=2(n-2)$, $s+\alpha\geq n$.

In case a), we have $i\geq n+2$ and $s\geq 0$ thus $i+s-3=n-2$ is impossible.

In case b), we have $i+s=2n-1$, while $s+\alpha\geq n$ and $\beta+s\geq n-1$ give $\alpha+\beta+2s=i+s\geq 2n-1$. Thus we must have the two equalities

$$s+\alpha=n,\ \ \beta+s=n-1.$$

This contradicts the fact that $\beta\geq\alpha$.

■

References

[1] M. Artin and D. Mumford, Some elementary examples of unirational varieties which are not rational, Proc. London Math. Soc., **25** (1972), 75–95.

[2] W. Barth, Counting singularities of quadratic forms on vector bundles, In: Vector bundles and Differential Equations, Proceedings, Nice, 1979, (ed. A. Hirschowitz), Progress in Mathematics, **7**, Birkhäuser, 1980.

[3] A. Beauville, Surfaces algébriques complexes, Astérisque, Société Mathématique de France, Paris, **54** (1978).

[4] J. Carlson and P. Griffiths, Infinitesimal variations of Hodge structure and the global Torelli theorem, In: Géométrie algébrique, Angers, 1980, (ed. A. Beauville), Sijthoff-Noordhoff, 51–76.

[5] H. Clemens and P. Griffiths, The intermediate Jacobian of the cubic Threefold, Ann. of Math., **95** (1972), 281–356.

[6] M. Green, Restriction of linear series to hyperplanes and some results of Macaulay and Gotzmann, In: Algebraic curves and projective geometry, Trento, 1988, Lecture Notes in Math., **1389**, Springer, Berlin, 1989.

[7] T. Graber, J. Harris and J. Starr, Families of rationally connected varieties, J. Amer. Math. Soc., **16** (2003), 57–67.

[8] J. Harris, Galois groups of enumerative problems, Duke Math. J., **46**, 685–724.

[9] V. Iskovskikh and Y. Manin, Three dimensional quartics and counterexamples to the Lüroth problem, Math. USSR Sbornik, 1971, 141–166.

[10] J. Kollár, Rational curves on algebraic varieties, Ergebnisse der Math. und ihrer Grenzgebiete, Springer-Verlag, 1996.

[11] J. Kollár, Lemma p. 134, In: Classification of irregular varieties, (eds. E. Ballico, F. Catanese and C. Ciliberto), Lecture Notes in Math., **1515**, Springer, 1990.

[12] Y. Miyaoka, On the Kodaira dimension of minimal threefolds, Math. Ann., **281** (1988), 325–332.

[13] Y. Miyaoka and T. Peternell, Geometry of higher dimensional algebraic varieties, DMV Seminar, **26**, Birkhäuser, 1991.

[14] C. Soulé and C. Voisin, Torsion cohomology classes and algebraic cycles on complex projective manifolds,Advances in Mathematics, Special volume in honor of Michael Artin, Part I, **198**, 107–127.

[15] C. Voisin, Densité du lieu de Noether-Lefschetz pour les sections hyperplanes des variétés de Calabi-Yau de dimension trois, International Journal of Mathematics, **3** (1992), 699–715.

[16] C. Voisin, Sur l'application d'Abel-Jacobi des variétés de Calabi-Yau de dimension trois, Annales de l'ENS, 4eme série, **27** (1994), 209–226.

[17] C. Voisin, The Griffiths group of a general Calabi-Yau threefold is not finitely generated,Duke Math. J., **102** (2000), 151–186.

[18] C. Voisin, Green's generic syzygy conjecture for curves of even genus lying on a $K3$ surface, J. Eur. Math. Soc., **4** (2002), 363–404.

[19] C. Voisin, Hodge Theory and Complex Algebraic Geometry II, Cambridge studies in advanced mathematics, **77**, Cambridge Univ. Press, 2003.

Institut de mathématiques de Jussieu
175 rue du Chevaleret
75013 Paris
France
E-mail address: voisin@math.jussieu.fr