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Harmonic conjugates of parabolic Bergman functions

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Abstract.

The parabolic Bergman space is the Banach space of solutions of some parabolic equations on the upper half space which have finite L^p norms. We introduce and study $L^{(\alpha)}$-harmonic conjugates of parabolic Bergman functions, and give a sufficient condition for a parabolic Bergman space to have unique $L^{(\alpha)}$-harmonic conjugates.

1. Introduction

Recently, Nishio, Shimomura, and Suzuki [4] have introduced parabolic Bergman spaces on the upper half-space and proved many interesting properties of these spaces. Parabolic Bergman spaces contain harmonic Bergman spaces studied by Ramey and Yi [6]. In this paper, we introduce and study $L^{(\alpha)}$-harmonic conjugates of parabolic Bergman functions, which are a generalized notion of usual harmonic conjugates of harmonic Bergman functions.

We describe the definition of parabolic Bergman spaces. Let H be the upper half-space of the $(n+1)$-dimensional Euclidean space $\mathbb{R}^{n+1}$, that is, $H = \{(x,t) \in \mathbb{R}^{n+1} \ ; \ x \in \mathbb{R}^n, \ t > 0\}$. For $1 \leq p < \infty$, the Lebesgue space $L^p(H, dV)$ is defined to be the Banach space of Lebesgue measurable functions on H with

$$\| u \|_p = \left(\int_H |u(x,t)|^p dV(x,t) \right)^{1/p} < \infty,$$

where dV is the Lebesgue volume measure on H. For $0 < \alpha \leq 1$, We define $L^{(\alpha)}$-harmonic functions on H. For $0 < \alpha < 1$, $(-\Delta)^\alpha$ is the

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convolution operator defined by

$$((-\Delta)^{\alpha}\varphi)(x,t) = -C_{n,\alpha}\lim_{\delta\downarrow 0}\int_{|y-x|>\delta}(\varphi(y,t)-\varphi(x,t))|y-x|^{-n-2\alpha}dy$$

for all $\varphi \in C_0^{\infty}(H)$, where $C_{n,\alpha} = -4^{\alpha}\pi^{-n/2}\Gamma((n+2\alpha)/2)/\Gamma(-\alpha) > 0$, and Δ is the Laplace operator with respect to x. For $0 < \alpha \le 1$, a parabolic operator $L^{(\alpha)}$ is defined by $L^{(\alpha)} = \frac{\partial}{\partial t} + (-\Delta)^{\alpha}$. (We note that when $\alpha = 1$, $L^{(1)}$ is the heat operator.) A continuous function u on H is said to be $L^{(\alpha)}$-harmonic if $L^{(\alpha)}u = 0$ in the sense of distributions, that is, $u\cdot\tilde{L}^{(\alpha)}\varphi \in L^1(H, dV)$ and $\int u\cdot\tilde{L}^{(\alpha)}\varphi dV = 0$ for all $\varphi \in C_0^{\infty}(H)$, where $\tilde{L}^{(\alpha)} = -\frac{\partial}{\partial t} + (-\Delta)^{\alpha}$ is the adjoint operator of $L^{(\alpha)}$. For $1 \le p < \infty$ and $0 < \alpha \le 1$, the parabolic Bergman space b_{α}^{p} is the set of all $L^{(\alpha)}$-harmonic functions on H which belong to $L^p(H, dV)$, and it is a Banach space with the L^p norm. It is known that $b_{\alpha}^{p} \subset C^{\infty}(H)$ (see Theorem 5.4 of [4]), and when $\alpha = 1/2$, $b_{1/2}^{p}$ coincides with harmonic Bergman spaces of Ramey and Yi (see Corollary 4.4 of [4]).

We introduce the definition of $L^{(\alpha)}$-harmonic conjugates of parabolic Bergman functions. For a function u on H such that $\partial u/\partial x_j$ and $\partial u/\partial t$ exist at every $(x,t) = (x_1, \ldots, x_n, t) \in H$, we write $\partial_{x_j}u = \partial u/\partial x_j$ and $\partial_t u = \partial u/\partial t$, respectively.

DEFINITION 1.1. *For a function $u \in b_{\alpha}^{p}$, the functions $v_1, \ldots, v_n$ are called $L^{(\alpha)}$-harmonic conjugates of u if $v_1, \ldots, v_n$ satisfy the following conditions:*

(1) $v_1, \ldots, v_n$ are $L^{(\alpha)}$-harmonic on H,

(2) $\partial_{x_j}v_k = \partial_{x_k}v_j$ and $\partial_{x_j}u = \partial_t v_j$ $(1 \le j, k \le n)$.

Usually, given a harmonic function u on H, the functions $v_1, \ldots, v_n$ on H are called harmonic conjugates of u if $(v_1, \ldots, v_n, u) = \nabla f$ for some harmonic function f on H. As mentioned above, $b_{1/2}^{p}$ coincide with harmonic Bergman spaces, and it is easy to see that when $\alpha = 1/2$ the conditions (1) and (2) of Definition 1.1 are equivalent to the definition of usual harmonic conjugates of harmonic Bergman functions. Hence, $L^{(\alpha)}$-harmonic conjugates are generalization of harmonic conjugates.

Many authors have studied and proved interesting and important results concerning properties of harmonic conjugates, (for instance, see Chapter III of [2]). One of the fundamental problems of harmonic conjugates is the boundedness of the conjugation operator. It is known that when $\alpha = 1/2$ there are unique harmonic conjugates $v_1, \ldots, v_n$ of a function $u \in b_{1/2}^{p}$ such that $v_j \in b_{1/2}^{p}$ (see Theorem 6.1 of [6]), and thus the conjugation operator is bounded on the harmonic Bergman spaces for all $1 \le p < \infty$. In this paper, we prove the following result (see Theorem 4.1): Let $0 < \alpha \le 1$ and $1 \le p < \infty$. If $\lambda = p(\frac{1}{2\alpha} - 1) > -1$ and $u \in b^p_\alpha$, then there exist unique $L^{(\alpha)}$-harmonic conjugates $v_1, \dots, v_n$ of u such that $v_j \in b^p_\alpha(\lambda)$, where $b^p_\alpha(\lambda)$ is the weighted parabolic Bergman spaces (see section 3 for the definition). Hence, we obtain the conjugation operator from b^p_α into $b^p_\alpha(\lambda)$ is bounded whenever $\lambda = p(\frac{1}{2\alpha} - 1) > -1$.

Throughout this paper, C will denote a positive constant whose value is not necessarily the same at each occurrence; it may vary even within a line.

2. Existence of $L^{(\alpha)}$-harmonic conjugates

When $\alpha = 1/2$, there are unique harmonic conjugates $v_1, \dots, v_n$ of a function $u \in b^p_{1/2}$ such that $v_j \in b^p_{1/2}$ (see Theorem 6.1 of [6]). In this section, we show that there exist $L^{(\alpha)}$-harmonic conjugates $v_1, \dots, v_n$ of a function $u \in b^p_\alpha$ such that $t^{\frac{1}{2\alpha}-1} v_j \in L^p(H, dV)$ whenever $p(\frac{1}{2\alpha} - 1) > -1$.

A fundamental solution of the parabolic operator $L^{(\alpha)}$ plays an important role for studying parabolic Bergman spaces. We define the fundamental solution of $L^{(\alpha)}$. For $x \in \mathbb{R}^n$, let

$$W^{(\alpha)}(x,t) = \begin{cases} \dfrac{1}{(2\pi)^n} \displaystyle\int_{\mathbb{R}^n} \exp(-t|\xi|^{2\alpha} + i\, x \cdot \xi)\, d\xi & t > 0 \\ 0 & t \le 0, \end{cases} \tag{2.1}$$

where $x \cdot \xi$ denotes the inner product on $\mathbb{R}^n$ and $|\xi| = (\xi \cdot \xi)^{1/2}$. The function $W^{(\alpha)}$ is the fundamental solution of $L^{(\alpha)}$ and $L^{(\alpha)}$-harmonic on H. We describe some properties of $W^{(\alpha)}$. We note that $W^{(\alpha)}(x,t) \ge 0$ and

$$\int_{\mathbb{R}^n} W^{(\alpha)}(x-y,s)dy = 1 \tag{2.2}$$

for all $x \in \mathbb{R}^n$ and $s > 0$. If $u \in b^p_\alpha$, then u satisfies the Huygens property, that is,

$$u(x,t) = \int_{\mathbb{R}^n} u(x-y, t-s) W^{(\alpha)}(y,s)dy \tag{2.3}$$

holds for all $x \in \mathbb{R}^n$ and $0 < s < t < \infty$ (see Theorem 4.1 of [4]). By (2.1), the fundamental solution $W^{(\alpha)}$ is in $C^\infty(H)$. Let $k \in \mathbb{N}_0$ and $\beta = (\beta_1, \cdots, \beta_n) \in \mathbb{N}_0^n$ be a multi-index, where $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Then, we define $\partial_x^\beta \partial_t^k = \partial_{x_1}^{\beta_1} \cdots \partial_{x_n}^{\beta_n} \partial_t^k = \partial^{|\beta|+k} / \partial x_1^{\beta_1} \cdots \partial x_n^{\beta_n} \partial t^k$. Clearly, we have

$$\partial_x^\beta \partial_t^k W^{(\alpha)}(x-y, t+s) = (-1)^{|\beta|} \partial_y^\beta \partial_s^k W^{(\alpha)}(x-y, t+s) \tag{2.4}$$

for all $(x,t),(y,s) \in H$. The following estimate is (1) of Proposition 1 of [5] : there exists a constant $C > 0$ such that

$$(2.5) \qquad |\partial_x^\beta \partial_t^k W^{(\alpha)}(x,t)| \leq \frac{Ct^{-k+1}}{(t+|x|^{2\alpha})^{\frac{n+|\beta|}{2\alpha}+1}} .$$

The following lemma is an immediate consequence of Theorem 1 of [5].

LEMMA 2.1. *Let $0 < \alpha \leq 1$, $1 \leq q < \infty$, $\theta \in \mathbb{R}$, $\beta \in \mathbb{N}_0^n$ be a multi-index, and $k \in \mathbb{N}$. If $\left(\frac{n+|\beta|}{2\alpha}+k\right)q - \left(\frac{n}{2\alpha}+1\right) > \theta > -1$, then there exists a constant $C > 0$ such that*

$$\int_H t^\theta |\partial_x^\beta \partial_t^k W^{(\alpha)}(x-y,t+s)|^q dV(x,t) \leq C s^{\frac{n}{2\alpha}+1-(\frac{n+|\beta|}{2\alpha}+k)q+\theta}$$

for all$(y,s) \in H$.

Let $c_k = \frac{(-2)^k}{k!}$. The following lemma is Theorem 6.7 of [4].

LEMMA 2.2. *Let $0 < \alpha \leq 1$ and $1 \leq p < \infty$. If $u \in b_\alpha^p$ and $(y,s) \in H$, then*

$$u(y,s) = -2c_{m+j} \int_H \partial_t^m u(x,t)\ t^{m+j} \partial_t^{j+1} W^{(\alpha)}(x-y,t+s) dV(x,t)$$

for all $m,\ j \in \mathbb{N}_0$.

PROPOSITION 2.3. *Let $0 < \alpha \leq 1$ and $1 \leq p < \infty$. If $\lambda = p(\frac{1}{2\alpha} - 1) > -1 - \frac{n}{2\alpha}$ and $u \in b_\alpha^p$, then there exist $L^{(\alpha)}$-harmonic conjugates $v_1, \ldots, v_n$ of u.*

PROOF. For each $1 \leq j \leq n$, let v_j be a function on H defined by

$$(2.6) \qquad v_j(y,s) = 2c_1 \int_H u(x,t)\ t\ \partial_{x_j} \partial_t W^{(\alpha)}(x-y,t+s) dV(x,t).$$

Since $p(\frac{1}{2\alpha} - 1) > -1 - \frac{n}{2\alpha}$, Lemma 2.1 implies that

$$t \partial_{x_j} \partial_t W^{(\alpha)}(\,\cdot - y,\ \cdot + s) \in L^q(H, dV),$$

where q is the exponent conjugate to p. Hence, the function v_j is well defined for all $(y,s) \in H$ when $p(\frac{1}{2\alpha} - 1) > -1 - \frac{n}{2\alpha}$. We show that

$v_1, \ldots, v_n$ are the $L^{(\alpha)}$-harmonic conjugates of u. Since $W^{(\alpha)}$ is $L^{(\alpha)}$-harmonic, so is v_j. Moreover, since (2.4) and Lemma 2.1 imply that

$$t\partial_{y_k}\partial_{x_j}\partial_t W^{(\alpha)}(\,\cdot - y,\ \cdot + s) \in L^q(H, dV)$$

for all $1 < q \leq \infty$, we can differentiate through the integral (2.6) with respect to y_k. Therefore we obtain $\partial_{y_k} v_j = \partial_{y_j} v_k$. Similarly, Lemmas 2.1 and 2.2 imply that $\partial_s v_j = \partial_{y_j} u$. □

REMARK 2.4. We note that when $0 < \alpha \leq \frac{1}{2}$, the assumption $\lambda = p(\frac{1}{2\alpha} - 1) > -1 - \frac{n}{2\alpha}$ of Proposition 2.3 always holds for all $1 \leq p < \infty$.

We consider an integrability condition of the function v_j which is defined in (2.6).

THEOREM 2.5. *Let $0 < \alpha \leq 1$ and $1 \leq p < \infty$. If $\lambda = p(\frac{1}{2\alpha} - 1) > -1$, then there exists a constant $C > 0$ such that*

$$\| t^{\frac{1}{2\alpha}-1} v_j \|_p \leq C \| u \|_p$$

for all $u \in b^p_\alpha$ and $1 \leq j \leq n$, where v_j is defined in (2.6).

PROOF. Let $c = \frac{1}{2\alpha} - 1$. We suppose that $p = 1$ (we note that when $p = 1$, $\lambda > -1$ for all $0 < \alpha \leq 1$). Then, (2.6) and the Fubini theorem imply that there exists a constant $C > 0$ such that

$$\begin{aligned} &\int_H |s^c v_j(y,s)| dV(y,s) \\ \leq\ & C \int_H |u(x,t)|\ t \int_H s^c |\partial_{x_j}\partial_t W^{(\alpha)}(x-y, t+s)| dV(y,s) dV(x,t). \end{aligned}$$

Therefore, Lemma 2.1 implies that $\| t^{\frac{1}{2\alpha}-1} v_j \|_1 \leq C \| u \|_1$.

Suppose that $p > 1$, and let q be the exponent conjugate to p. Then, the Hölder inequality shows that there exists a constant $C > 0$ such that

$$\begin{aligned} &|v_j(y,s)| \\ \leq\ & C \int_H |u(x,t)|\ t^{\frac{1}{pq}+\frac{1}{p}}\ t^{-\frac{1}{pq}+\frac{1}{q}} \\ &\times |\partial_{x_j}\partial_t W^{(\alpha)}(x-y, t+s)|^{\frac{1}{p}+\frac{1}{q}} dV(x,t) \\ \leq\ & C \left(\int_H |u(x,t)|^p t^{\frac{1}{q}+1} |\partial_{x_j}\partial_t W^{(\alpha)}(x-y,t+s)| dV(x,t) \right)^{\frac{1}{p}} \\ &\times \left(\int_H t^{-\frac{1}{p}+1} |\partial_{x_j}\partial_t W^{(\alpha)}(x-y,t+s)| dV(x,t) \right)^{\frac{1}{q}}. \end{aligned}$$

Since $\lambda = p(\frac{1}{2\alpha} - 1) > -1$, Lemma 2.1 implies that

$$\int_H t^{-\frac{1}{p}+1} |\partial_{x_j} \partial_t W^{(\alpha)}(x-y, t+s)| dV(x,t) \leq C s^{-(\frac{1}{2\alpha}+\frac{1}{p}-1)}.$$

Thus, by the Fubini theorem we have

$$\begin{aligned} &\int_H |s^c v_j(y,s)|^p dV(y,s) \\ \leq\ & C \int_H |u(x,t)|^p t^{\frac{1}{q}+1} \\ &\times \int_H s^{cp-(\frac{1}{2\alpha}+\frac{1}{p}-1)\frac{p}{q}} |\partial_{x_j} \partial_t W^{(\alpha)}(x-y, t+s)| dV(y,s) dV(x,t). \end{aligned}$$

Lemma 2.1 also implies that

$$\int_H s^{cp-(\frac{1}{2\alpha}+\frac{1}{p}-1)\frac{p}{q}} |\partial_{x_j} \partial_t W^{(\alpha)}(x-y, t+s)| dV(y,s) \leq C t^{-(\frac{1}{q}+1)}.$$

Therefore, we obtain $\| t^{\frac{1}{2\alpha}-1} v_j \|_p \leq C \| u \|_p$. □

3. Weighted parabolic Bergman spaces

In Proposition 2.3 and Theorem 2.5, we prove that the function v_j which is defined in (2.6) is $L^{(\alpha)}$-harmonic and in $L^p(H, t^\lambda dV)$, where $\lambda = p(\frac{1}{2\alpha} - 1)$. In order to study the $L^{(\alpha)}$-harmonic conjugates, we define weighted parabolic Bergman spaces. For any $\lambda > -1$, the weighted parabolic Bergman space $b^p_\alpha(\lambda)$ is the set of all $L^{(\alpha)}$-harmonic functions on H which belong to $L^p(H, t^\lambda dV)$. We note that any function $u \in L^p(H, t^\lambda dV)$ satisfies $u \cdot \tilde{L}^{(\alpha)} \varphi \in L^1(H, dV)$ for all $\varphi \in C_0^\infty(H)$. In fact, it is known that $u \cdot \tilde{L}^{(\alpha)} \varphi \in L^1(H, dV)$ for all $\varphi \in C_0^\infty(H)$ if and only if

$$\int_{t_1}^{t_2} \int_{\mathbb{R}^n} |u(x,t)| (1+|x|)^{-n-2\alpha} dV(x,t) < \infty$$

for all $t_2 > t_1 > 0$ (see Remark 2.2 of [4]). If $u \in L^p(H, t^\lambda dV)$ for some $1 \leq p < \infty$ and $\lambda > -1$, then elementary calculations show that $\int_{t_1}^{t_2} \int_{\mathbb{R}^n} |u(x,t)|(1+|x|)^{-n-2\alpha} dV(x,t) < \infty$ for all $t_2 > t_1 > 0$. Hence, $u \in L^p(H, t^\lambda dV)$ satisfies the integrability condition in the definition of $L^{(\alpha)}$-harmonic functions.

We give some properties of the weighted parabolic Bergman spaces. When $\lambda = 0$, the following lemma is Theorem 4.1 of [4]. We claim that $u \in b^p_\alpha(\lambda)$ also satisfies the Huygens property.

LEMMA 3.1. *Let $0 < \alpha \leq 1$, $1 \leq p < \infty$ and $\lambda > -1$. If $u \in b^p_\alpha(\lambda)$, then*

$$u(x,t) = \int_{\mathbb{R}^n} u(x-y, t-s) W^{(\alpha)}(y,s) dy$$

for all $x \in \mathbb{R}^n$ and $0 < s < t < \infty$.

PROOF. In the proof of Theorem 4.1 of [4], the Huygens property for $u \in b^p_\alpha$ derives from an $L^{(\alpha)}$-harmonicity of u and a local integrability of a function $U(t) = \int_{\mathbb{R}^n} |u(x,t)|^p dx$ on $(0,\infty)$. If $u \in b^p_\alpha(\lambda)$, then it is easy to check that the function $U(t)$ is also locally integrable on $(0,\infty)$. Therefore, u satisfies the Huygens property. □

REMARK 3.2. It was known that for $u \in b^p_\alpha$ the function $U(t) = \int_{\mathbb{R}^n} |u(x,t)|^p dx$ is decreasing on $(0,\infty)$ (see Lemma 5.6 of [4]). By Lemma 3.1 and the Minkowski inequality, for any $\lambda > -1$ the same result holds for $u \in b^p_\alpha(\lambda)$.

When $\lambda = 0$, the following lemma is Proposition 5.2 of [4].

LEMMA 3.3. *Let $0 < \alpha \leq 1$, $1 \leq p < \infty$ and $\lambda > -1$. Then there exists a constant $C > 0$ such that*

$$|u(x,t)| \leq C t^{-(\frac{n}{2\alpha}+\lambda+1)\frac{1}{p}} \left(\int_H |u(y,s)|^p s^\lambda dV(y,s) \right)^{\frac{1}{p}}$$

for all $(x,t) \in H$ and $u \in b^p_\alpha(\lambda)$.

PROOF. Since the proof of Lemma 3.3 is analogous to that of Proposition 5.2 of [4], we describe the outline of the proof. For fixed $0 < a_1 < a_2 < 1$, Lemma 3.1 implies that

$$u(x,t) = \frac{1}{(a_2-a_1)t} \int_{a_1 t}^{a_2 t} \int_{\mathbb{R}^n} u(y, t-s) W^{(\alpha)}(x-y, s) dy ds.$$

Then, using the Jensen inequality and (2.5), we have

$$\begin{aligned} &|u(x,t)| \\ \leq\ & Ct^{-(\frac{n}{2\alpha}+1)\frac{1}{p}}\left(\int_{a_1 t}^{a_2 t}\int_{\mathbb{R}^n}|u(y,t-s)|^p dyds\right)^{\frac{1}{p}} \\ =\ & Ct^{-(\frac{n}{2\alpha}+1)\frac{1}{p}}\left(\int_{a_1 t}^{a_2 t}(t-s)^{-\lambda}(t-s)^{\lambda}\int_{\mathbb{R}^n}|u(y,t-s)|^p dyds\right)^{\frac{1}{p}} \\ \leq\ & Ct^{-(\frac{n}{2\alpha}+\lambda+1)\frac{1}{p}}\left(\int_{a_1 t}^{a_2 t}(t-s)^{\lambda}\int_{\mathbb{R}^n}|u(y,t-s)|^p dyds\right)^{\frac{1}{p}}, \end{aligned}$$

because $(1-a_2)t < t-s < (1-a_1)t$ whenever $a_1 t < s < a_2 t$. Hence, we obtain

$$|u(x,t)| \leq Ct^{-(\frac{n}{2\alpha}+\lambda+1)\frac{1}{p}}\left(\int_0^\infty s^\lambda\int_{\mathbb{R}^n}|u(y,s)|^p dyds\right)^{\frac{1}{p}}.$$

□

By Lemma 3.1, $u \in b_\alpha^p(\lambda)$ is in $C^\infty(H)$. Thus, as in the proof of Lemma 3.3, we have the following lemma, which is Theorem 5.4 of [4] when $\lambda = 0$.

LEMMA 3.4. *Let $0 < \alpha \leq 1$, $1 \leq p < \infty$ and $\lambda > -1$. If $\beta \in \mathbb{N}_0^n$ is a multi-index and $k \in \mathbb{N}_0$, then there exists a constant $C > 0$ such that*

$$|\partial_x^\beta \partial_t^k u(x,t)| \leq Ct^{-(\frac{|\beta|}{2\alpha}+k)-(\frac{n}{2\alpha}+\lambda+1)\frac{1}{p}}\left(\int_H |u(y,s)|^p s^\lambda dV(y,s)\right)^{\frac{1}{p}}$$

for all $(x,t) \in H$ and $u \in b_\alpha^p(\lambda)$.

For $\delta > 0$ and a function u on H, we write $u_\delta(x,t) = u(x,t+\delta)$. We note that if $u \in b_\alpha^p(\lambda)$ then $u_\delta \in b_\alpha^p(\lambda)$ for all $\delta > 0$. In fact, if $u \in b_\alpha^p(\lambda)$, then

$$\begin{aligned} &\int_1^\infty t^\lambda \int_{\mathbb{R}^n}|u(x,t+\delta)|^p dxdt \\ \leq\ & C\int_1^\infty (t+\delta)^\lambda\int_{\mathbb{R}^n}|u(x,t+\delta)|^p dxdt \\ \leq\ & C\int_H |u(x,t)|^p t^\lambda dV. \end{aligned}$$

Moreover, Remark 3.2 implies that

$$\int_0^1 t^\lambda \int_{\mathbb{R}^n} |u(x,t+\delta)|^p dxdt \le U(\delta) \int_0^1 t^\lambda dt < \infty.$$

Hence, we have $u_\delta \in b^p_\alpha(\lambda)$.

When $\lambda = 0$, the following lemma is Lemma 6.6 of [4].

LEMMA 3.5. *Let $0 < \alpha \le 1$, $1 \le p < \infty$ and $\lambda > -1$. If $u \in b^p_\alpha(\lambda)$ and $(y,s) \in H$, then*

(3.1)

$$u_\delta(y,s) = -2c_{m+j} \int_H \partial_t^m u_\delta(x,t)\ t^{m+j} \partial_t^{j+1} W^{(\alpha)}(x-y,t+s) dV(x,t)$$

for all $m,\ j \in \mathbb{N}_0$ and $\delta > 0$.

PROOF. The proof of Lemma 3.5 is analogous to that of Lemma 6.6 of [4]. We only show that the integral (3.1) is well defined. By Lemma 3.4, there exist constants $C > 0$ and $0 < \varepsilon < 1$ such that

$$|\partial_t^m u_\delta(x,t)| \le C(t+\delta)^{-m-(\frac{n}{2\alpha}+\lambda+1)\frac{1}{p}} \le Ct^{-m-\varepsilon}\ \delta^{\varepsilon-(\frac{n}{2\alpha}+\lambda+1)\frac{1}{p}}.$$

Therefore, we have

$$|\partial_t^m u_\delta(x,t) t^{m+j} \partial_t^{j+1} W^{(\alpha)}(x-y,t+s)| \le Ct^{j-\varepsilon} |\partial_t^{j+1} W^{(\alpha)}(x-y,t+s)|.$$

Hence, Lemma 2.1 implies that $\partial_t^m u_\delta(x,t) t^{m+j} \partial_t^{j+1} W^{(\alpha)}(x-y,t+s) \in L^1(H, dV)$. □

THEOREM 3.6. *Let $0 < \alpha \le 1$, $1 \le p < \infty$, and $\lambda > -1$. If $\gamma > -1$ and non-negative integers ℓ, m satisfy*

$$\gamma + (\ell - m)p > -1, \tag{3.2}$$

then there exists a constant $C > 0$ such that

$$\int_H t^{\gamma+(\ell-m)p} |\partial_t^\ell u_\delta|^p dV \le C \int_H t^\gamma |\partial_t^m u_\delta|^p dV \tag{3.3}$$

for all $u \in b^p_\alpha(\lambda)$ and $\delta > 0$.

PROOF. Suppose that $p > 1$, and let q be the exponent conjugate to p. By (3.2), we can choose a constant $\eta > 0$ such that

$$\gamma + (\ell - m)p - \frac{p}{q}\eta > -1 \tag{3.4}$$

Moreover, let j be a non-negative integer such that

$$-\eta + \ell + j > -1 \tag{3.5}$$

and

$$\ell + j > \gamma + (\ell - m)p - \frac{p}{q}\eta. \tag{3.6}$$

Since, as in the proof of Lemma 3.5, there exist constants $C > 0$ and $0 < \varepsilon < 1$ such that

$$\begin{aligned} & |\partial_t^m u_\delta(x,t) t^{m+j} \partial_t^{\ell+j+1} W^{(\alpha)}(x-y, t+s)| \\ \leq \; & C t^{j-\varepsilon} |\partial_t^{\ell+j+1} W^{(\alpha)}(x-y,t+s)|, \end{aligned}$$

Lemma 2.1 implies that

$$\partial_t^m u_\delta(x,t) t^{m+j} \partial_t^{\ell+j+1} W^{(\alpha)}(x-y,t+s) \in L^1(H, dV).$$

Therefore, by Lemma 3.5 we have

(3.7)

$$\partial_s^\ell u_\delta(y,s) = -2c_{m+j} \int_H \partial_t^m u_\delta(x,t)\, t^{m+j} \partial_t^{\ell+j+1} W^{(\alpha)}(x-y, t+s) dV(x,t).$$

As in the proof of Theorem 2.5, the Hölder inequality implies that there exists a constant $C > 0$ such that

$$\begin{aligned} & |\partial_s^\ell u_\delta(y,s)|^p \\ \leq \; & C \left(\int_H t^{-\eta+\ell+j} |\partial_t^{\ell+j+1} W^{(\alpha)}(x-y,t+s)| dV(x,t) \right)^{\frac{p}{q}} \\ \times \; & \int_H |\partial_t^m u_\delta(x,t)|^p t^{\frac{p(\eta+m-\ell)}{q}+m+j} |\partial_t^{\ell+j+1} W^{(\alpha)}(x-y,t+s)| dV(x,t). \end{aligned}$$

By (3.5), Lemma 2.1 and the Fubini theorem imply that

$$\begin{aligned} & \int_H s^{\gamma+(\ell-m)p} |\partial_s^\ell u_\delta(y,s)|^p dV(y,s) \\ \leq \; & C \int_H s^{\gamma+(\ell-m)p-\frac{p}{q}\eta} \int_H |\partial_t^m u_\delta(x,t)|^p t^{\frac{p(\eta+m-\ell)}{q}+m+j} \\ & \times |\partial_t^{\ell+j+1} W^{(\alpha)}(x-y,t+s)| dV(x,t) dV(y,s) \\ = \; & C \int_H |\partial_t^m u_\delta(x,t)|^p t^{\frac{p(\eta+m-\ell)}{q}+m+j} \\ & \times \int_H s^{\gamma+(\ell-m)p-\frac{p}{q}\eta} |\partial_t^{\ell+j+1} W^{(\alpha)}(x-y,t+s)| dV(y,s) dV(x,t). \end{aligned}$$

By (3.4) and (3.6), Lemma 2.1 also implies that

$$\begin{aligned} & \int_H s^{\gamma+(\ell-m)p-\frac{p}{q}\eta} |\partial_t^{\ell+j+1} W^{(\alpha)}(x-y,t+s)| dV(y,s) \\ \leq \; & C t^{\gamma+(\ell-m)p-\frac{p}{q}\eta-(\ell+j)}. \end{aligned}$$

Hence, we obtain

$$\int_H s^{\gamma+(\ell-m)p}|\partial_s^\ell u_\delta(y,s)|^p dV(y,s) \le C\int_H t^\gamma |\partial_t^m u_\delta(x,t)|^p dV(x,t).$$

We suppose that $p=1$. Then, using (3.7) and the Fubini theorem, we have

$$\begin{aligned} &\int_H s^{\gamma+\ell-m}|\partial_s^\ell u_\delta(y,s)|dV(y,s) \\ \le\ & C\int_H |\partial_t^m u_\delta(x,t)|t^{m+j} \\ &\times \int_H s^{\gamma+\ell-m}|\partial_t^{\ell+j+1}W^{(\alpha)}(x-y,t+s)|dV(y,s)dV(x,t). \end{aligned}$$

Since we can choose a non-negative integer j such that $\gamma-m-j<0$, Lemma 2.1 implies that

$$\int_H s^{\gamma+\ell-m}|\partial_t^{\ell+j+1}W^{(\alpha)}(x-y,t+s)|dV(y,s) \le Ct^{\gamma-m-j}.$$

Hence, we have the theorem. □

For a function $u\in L^p(H,t^\lambda dV)$, define $\| u \|_{p,\lambda} = (\int_H |u|^p t^\lambda dV)^{1/p}$. We have the following inequalities.

COROLLARY 3.7. *Let $0<\alpha\le 1$, $1\le p<\infty$, and $\lambda>-1$. Then, there exists a constant $C>0$ such that*

$$C^{-1}\| u_\delta \|_{p,\lambda} \le \| t^\ell \partial_t^\ell u_\delta \|_{p,\lambda} \le C\| u_\delta \|_{p,\lambda} \tag{3.8}$$

for all $u\in b_\alpha^p(\lambda)$, $\delta>0$, and $\ell\in\mathbb{N}_0$.

4. Uniqueness of $L^{(\alpha)}$-harmonic conjugates

In this section, we show that $L^{(\alpha)}$-harmonic conjugates of $u\in b_\alpha^p$ are unique whenever $\lambda=p(\frac{1}{2\alpha}-1)>-1$.

THEOREM 4.1. *Let $0<\alpha\le 1$ and $1\le p<\infty$. If $\lambda=p(\frac{1}{2\alpha}-1)>-1$ and $u\in b_\alpha^p$, then there exist unique $L^{(\alpha)}$-harmonic conjugates $v_1,\dots,v_n$ of u on H such that $v_j\in b_\alpha^p(\lambda)$.*

PROOF. By Proposition 2.3 and Theorem 2.5, it suffices to prove the uniqueness of $L^{(\alpha)}$-harmonic conjugates of $u\in b_\alpha^p$ that belong to $b_\alpha^p(\lambda)$.

Suppose that $u_1, \ldots, u_n$ are also $L^{(\alpha)}$-harmonic conjugates of u such that $u_j \in b^p_\alpha(\lambda)$. Take arbitrary $\delta > 0$. Then by Corollary 3.7, there exists a constant $C > 0$ such that

$$\| t^{\frac{1}{2\alpha}-1}(v_j - u_j)_\delta \|_p \leq C \| t^{\frac{1}{2\alpha}} \partial_t (v_j - u_j)_\delta \|_p . \tag{4.1}$$

By the hypothesis and the definition of $L^{(\alpha)}$-harmonic conjugates, we have

$$\partial_t (v_j - u_j)_\delta = \partial_{x_j} u_\delta - \partial_{x_j} u_\delta \equiv 0.$$

Therefore, (4.1) and the continuity of $v_j - u_j$ imply that $v_j(x, t+\delta) = u_j(x, t+\delta)$ for all $(x,t) \in H$. Since $\delta > 0$ is arbitrary, we obtain $v_j = u_j$ as desired. □

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