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Some potential theoretic results on an infinite network

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Abstract.

The greatest harmonic minorant of a superharmonic function is determined as the limit of a sequence of solutions for discrete Dirichlet problems on finite subnetworks. Without using the Green kernel explicitly, a positive superharmonic function is decomposed uniquely as a sum of a potential and a harmonic function. The infimum of a left directed family of harmonic functions is shown to be either $-\infty$ or harmonic. As applications, we study the reduced functions and their properties. We show the existence of the Green kernel with the aid of our reduced function.

§1. Introduction

Let $N = \{X, Y, K, r\}$ be an infinite network which is connected and locally finite and has no self-loop. Here X is a countable set of nodes, Y a countable set of arcs, K a node-arc incidence function and r a strictly positive real function on Y.

We say that a network $N' = \{X', Y', K', r'\}$ is a subnetwork of N if X' and Y' are subsets of X and Y respectively, K' is the restriction of K onto $X' \times Y'$ and r' is the restriction of r onto Y'. For simplicity, we write $N' =< X', Y' >$ in case $N' = \{X', Y', K', r'\}$ is a subnetwork of N. We say that $N' =< X', Y' >$ is a finite subnetwork of N if X' or Y' is a finite set. For later use, we recall a notion of an exhaustion. We say that a sequence of finite subnetworks $\{N_n\}(N_n =< X_n, Y_n >)$ of N

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is an exhaustion of N if

$$Y(x) := \{y \in Y; K(x,y) \neq 0\} \quad \subset \quad Y_{n+1} \text{ for all } x \in X_n,$$
$$X = \bigcup_{n=1}^{\infty} X_n \quad \text{and} \quad Y = \bigcup_{n=1}^{\infty} Y_n$$

Notice that $X_n \subset X_{n+1}$ and $Y_n \subset Y_{n+1}$. For notations and terminologies we mainly follow [2] and [3]. Let $L(X)$ be the set of all real functions on X and $L^+(X)$ be the set of all non-negative functions on X. For $x \in X$, denote by W_x the neighboring nodes of x, i.e.,

$$W_x = \{z \in X; K(x,y)K(z,y) = -1 \text{ for some } y \in Y(x)\}.$$

For every $u \in L(X)$, the Laplacian $\Delta u \in L(X)$ is defined by

$$\Delta u(x) = -t(x)u(x) + \sum\nolimits_{z \in W_x} t(x,z)u(z),$$

where

$$\begin{aligned} t(x) &= \sum\nolimits_{y \in Y} r(y)^{-1}|K(x,y)| \\ t(x,z) &= \sum\nolimits_{y \in Y} r(y)^{-1}|K(x,y)K(z,y)| \text{ for } z \neq x. \end{aligned}$$

Notice that $t(x,z) = t(z,x)$ and $t(x,z) = 0$ for $z \in X \setminus (W_x \cup \{x\})$

$$t(x) = \sum\nolimits_{z \in W_x} t(x,z).$$

We say that a function $u \in L(X)$ is superharmonic on a set $A \subseteq X$ if $\Delta u(x) \leq 0$ for all $x \in A$. We say that u is subharmonic on A if $-u$ is superharmonic on A. If u is both superharmonic and subharmonic on A, we say that u is harmonic on A. The following minimum principle and maximum principle are well-known:

Lemma 1.1 (Minimum principle). *Let X' be a finite subset of X. If u is superharmonic on X' and $u(x) \geq 0$ on $X \setminus X'$, then $u(x) \geq 0$ on X'.*

Lemma 1.2 (Maximum principle). *Let X' be a finite subset of X. If u is subharmonic on X' and $u(x) \leq 0$ on $X \setminus X'$, then $u(x) \leq 0$ on X'.*

Lemma 1.3 (Harnack's principle). *Let $\{X_n\}$ be a sequence of subsets of X such that $X_n \subset X_{n+1}$ and $X = \cup_{n=1}^{\infty} X_n$ and let $\{u_n\}$ be a sequence of functions on X such that $u_n(x) \leq u_{n+1}(x)$ on X. If u_n is superharmonic on X_n for every n, then the pointwise limit of $\{u_n\}$ is equal to either ∞ or a real valued superharmonic function.*

For a finite subnetwork $N' =< X', Y' >$ of N, the harmonic green function of N' with pole at $a \in X'$ is the unique function u determined by

$$\Delta u(x) = -\varepsilon_a(x) \text{ on } X' \text{ and } u(x) = 0 \text{ on } X \setminus X',$$

where ε_a denotes the characteristic function of $\{a\}$. Denote by $g_a^{N'}$ the harmonic Green function of N' with pole at a. Notice that $g_a^{N'}(b) = g_b^{N'}(a) > 0$ for all $a, b \in X'$(cf. [1]). For $f \in L(X)$, the Green potential $G_{N'}f$ is defined by

$$G_{N'}f(x) = \sum_{z \in X'} g_z^{N'}(x) f(z).$$

§2. The greatest harmonic minorant

We begin with a discrete Dirichlet problem:

Lemma 2.1. [1] *Let $f \in L(X)$ and $N' =< X', Y' >$ be a finite subnetwork of N. There exists a unique function u' such that*

$$\Delta u'(x) = 0 \text{ on } X' \text{ and } u'(x) = f(x) \text{ on } X \setminus X'.$$

Proof. The uniqueness follows from the maximum and minimum principles. We see easily that $u' = f + G_{N'}(\Delta f)$ satisfies our requirements. □

Denote by $h_f^{N'}$ the unique function u' determined in Lemma 2.1.

Corollary 2.1. *Let $N' =< X', Y' >$ be a finite subnetwork of N. Then $h_{\alpha f + \beta g}^{N'} = \alpha h_f^{N'} + \beta h_g^{N'}$ for $f, g \in L(X)$ and real numbers α, β.*

By Lemmas 1.1 and 1.2, we obtain

Lemma 2.2. *Let $N' =< X', Y' >$ be a finite subnetwork of N.*

(1) *If u is superharmonic on X', then $h_u^{N'}(x) \leq u(x)$ on X.*

(2) *If u is subharmonic on X', then $h_u^{N'}(x) \geq u(x)$ on X.*

Corollary 2.2. *If u is harmonic on X', then $h_u^{N'} = u$.*

Lemma 2.3. *Let $N' =< X', Y' >$ be a finite subnetwork of N and $u_1, u_2 \in L(X)$. If $u_1(x) \leq u_2(x)$ on X, then $h_{u_1}^{N'}(x) \leq h_{u_2}^{N'}(x)$ on X.*

Proof. Let $v(x) = h_{u_2}^{N'}(x) - h_{u_1}^{N'}(x)$. Then v is harmonic on X' and $v(x) = u_2(x) - u_1(x) \geq 0$ on $X \setminus X'$. By the minimum principle,$v(x) \geq 0$ on X'. Hence $v(x) \geq 0$ on X. □

Lemma 2.4. *Let $N' =< X', Y' >$ be a finite subnetwork of N. If u is a superharmonic function on X, then $h_u^{N'}$ is superharmonic on X.*

Proof. By Lemma 2.2, $h_u^{N'}(x) \leq u(x)$ on X. It suffices to show that $h_u^{N'}(x)$ is superharmonic on $X \setminus X'$. For $x \in X \setminus X'$, we have $h_u^{N'}(x) = u(x)$ and

$$\begin{aligned} \Delta h_u^{N'}(x) &= -t(x)h_u^{N'}(x) + \sum\nolimits_{z \in W_x} t(x,z)h_u^{N'}(z) \\ &\leq -t(x)u(x) + \sum\nolimits_{z \in W_x} t(x,z)u(z) = \Delta u(x) \leq 0. \end{aligned}$$

Therefore u is superharmonic on X. □

Lemma 2.5. *Let $N_1 =< X_1, Y_1 >$ and $N_2 =< X_2, Y_2 >$ be finite subnetworks of N such that $Y(x) \subset Y_2$ for all $x \in X_1$. If u is superharmonic on X, then $h_u^{N_1}(x) \geq h_u^{N_2}(x)$ on X.*

Proof. Let $v(x) = h_u^{N_1}(x) - h_u^{N_2}(x)$. Then $v(x) = u(x) - h_u^{N_2}(x) \geq 0$ on $X \setminus X_1$ and $\Delta v(x) = 0$ on X_1. Therefore $v(x) \geq 0$ on X by the minimum principle. □

Theorem 2.1. *Let u be superharmonic on X and $\{N_n\}$ be an exhaustion of N and put*

$$\pi_u(x) = \lim_{n \to \infty} h_u^{N_n}(x) \text{ for each } x \in X.$$

Then either $\pi_u = -\infty$ or $\pi_u \in L(X)$ is harmonic on X.

Proof. Put $u_n = h_u^{N_n}$. Then $u_{n+1}(x) \leq u_n(x) \leq u(x)$ on X and u_n is harmonic on X_n. By Harnack's principle, we see that the limit v of the sequence $\{-u_n\}$ is equal to either ∞ or a real valued superharmonic function on X. In case $v = \infty$, we have $\pi_u = -\infty$. Assume that $v \neq \infty$. Then we see $\pi_u = -v \in L(X)$ and $\Delta \pi_u(x) \geq 0$ on X. Let $x \in X$. Since N is locally finite, there exists n_0 such that $W_x \cup \{x\} \subset X_n$ for all $n \geq n_0$. Since u_n is harmonic on X_n and $u_n(z) \to \pi_u(z)$ for all $z \in W_x \cup \{x\}$ as $n \to \infty$, we have

$$\begin{aligned} \Delta \pi_u(x) &= -t(x)\pi_u(x) + \sum\nolimits_{z \in W_x} t(x,z)\pi_u(z) \\ &= \lim_{n \to \infty} \{-t(x)u_n(x) + \sum\nolimits_{z \in W_x} t(x,z)u_n(z)\} \\ &= \lim_{n \to \infty} \Delta u_n(x) = 0. \end{aligned}$$

In case $\pi_u \in L(X)$, we call π_u the harmonic part of u. Notice that π_u does not depend on the choice of an exhaustion of N and that $\pi_u(x) \leq u(x)$ on X. □

Proposition 2.1. *Let u_1, u_2 be superharmonic functions on X. If there exists a subharmonic minorant v of* $\min(u_1, u_2)$, *then*

$$\pi_{\min(u_1,u_2)}(x) \leq \min(\pi_{u_1}(x), \pi_{u_2}(x)) \quad \text{on } X.$$

Proof. Let $u = \min(u_1, u_2)$ and $\{N_n\}(N_n =< X_n, Y_n >)$ be an exhaustion of N. Then u is superharmonic and

$$v(x) \leq h_u^{N_n}(x) \leq h_{u_k}^{N_n}(x) \text{ on } X \text{ for } k = 1, 2.$$

Therefore $\pi_u(x) \leq \pi_{u_k}(x)$ on X for $k = 1, 2$. □

Proposition 2.2. *Let u_1 and u_2 be superharmonic functions on X. If they have subharmonic minorants, then $\pi_{u_1+u_2} = \pi_{u_1} + \pi_{u_2}$.*

Proof. Let N_n be the same as above. We have by Corollary 2.1

$$h_{u_1+u_2}^{N_n} = h_{u_1}^{N_n} + h_{u_2}^{N_n}.$$

□

Corollary 2.3. *Let u be a superharmonic function on X with a subharmonic minorant and let ϕ be a harmonic function on X. Then $\pi_{u+\phi} = \pi_u + \phi$.*

Theorem 2.2. *Let u be superharmonic on X. If u has a subharmonic minorant v, i.e., v is subharmonic on X and $v(x) \leq u(x)$ on X, then $v(x) \leq \pi_u(x)$ on X. Moreover, π_u is the greatest harmonic minorant of u.*

Proof. Let $\{N_n\}(N_n =< X_n, Y_n >)$ be an exhaustion of N. Since v is subharmonic on X and $v(x) \leq u(x)$ on X, we have

$$v(x) \leq h_v^{N_n}(x) \leq h_u^{N_n}(x) \text{ on } X$$

by Lemmas 2.2 and 2.3. Thus we have $v(x) \leq \pi_u(x)$ on X. If s is a harmonic minorant of u, then we have $s(x) = h_s^{N_n}(x) \leq h_u^{N_n}(x)$ on X by Corollary 2.2 and Lemma 2.3, so that $s(x) \leq \pi_u(x)$ on X. □

There are many characterizations for an infinite network N to be of hyperbolic type. We say here that N is of hyperbolic type (or shortly, hyperbolic) if there exists a nonconstant positive superharmonic function on X. It is well-known that N is hyperbolic if and only if N has a Green function, i.e., the limit g_a of $\{g_a^{N_n}\}$ exists and satisfies the condition:$\Delta g_a(x) = -\varepsilon_a(x)$ on X.

Without using this Green kernel explicitly, we introduce

Definition 2.1. *We say that a positive superharmonic function* u *is a potential if the greatest harmonic minorant of* u *is zero, i.e.,* $\pi_u = 0$.

Needless to say, we have $\pi_u \in L(X)$ if $u \in L^+(X)$ is superharmonic on X.

Theorem 2.3. *Let* N *be hyperbolic.*

(1) *If* u *is a potential, then* λu $(\lambda > 0)$ *is also a potential.*
(2) *If* u_1 *and* u_2 *are potentials, then* $u_1 + u_2$ *is also a potential.*
(3) *If* u_1 *is a potential and* u_2 *is a positive superharmonic function, then* $\min(u_1, u_2)$ *is a potential.*

Proof. (2) and (3) follow from Propositions 2.1 and 2.2. For (1), it suffices to note that $\pi_{\lambda u} = \lambda \pi_u$. □

Theorem 2.4. *Let* N *be hyperbolic.*

(1) *Assume that* v *is superharmonic on* X *and* u *is a potential. If* $u + v \in L^+(X)$, *then* $v \in L^+(X)$.
(2) *If* u *is a potential and if* v *is a subharmonic minorant of* u, *then* $v \leq 0$.
(3) *Assume that* u *is a superharmonic function with a subharmonic minorant* v. *Then* u *can be expressed uniquely as the sum of a potential and a harmonic function.*

Proof. Since $u \geq -v$ and $-v$ is subharmonic, we have $0 = \pi_u(x) \geq \pi_{-v}(x) \geq -v(x)$ on X. Thus (1) follows. The second assertion follows from the relation: $v(x) \leq \pi_v(x) \leq \pi_u(x) = 0$ on X. Let us prove (3). Since u has a subharmonic minorant, we have $\pi_u \in L(X)$ is harmonic. We take $p = u - \pi_u$. Then $p \in L^+(X)$ and $\pi_p = 0$ by Corollary 2.3. Therefore p is a potential. Assume that there exist potentials p_1, p_2 and harmonic functions h_1, h_2 satisfying the relation: $u = p_1 + h_1 = p_2 + h_2$. We have

$$p_1(x) \geq p_1(x) - p_2(x) = h_2(x) - h_1(x)$$

for all $x \in X$. We see by the above observation (2) that $h_2(x) - h_1(x) \leq 0$ on X. We obtain similarly $h_1(x) - h_2(x) \leq 0$ on X, and hence $h_1(x) = h_2(x)$. This shows the uniqueness of our decomposition. □

§3. Sets of Superharmonic Functions

We say that a set Φ of functions on X is left directed if for every $u_1, u_2 \in \Phi$, there exists $u \in \Phi$ such that $u \leq \min(u_1, u_2)$. We define $\inf \Phi$ by

$$\inf \Phi(x) = \inf\{u(x); u \in \Phi\}.$$

For simplicity, we set $X(a) = W_a \cup \{a\}$ for $a \in X$.

Theorem 3.1. *If Φ is a left directed family of harmonic functions on X, then* $\inf \Phi$ *is either equal to $-\infty$ identically or harmonic on X.*

Proof. For simplicity, put $h = \inf \Phi$. It suffices to show that h is harmonic on X unless $h = -\infty$. Let a be any node such that $h(a) > -\infty$. Since $X(a)$ is a finite set, we can find a sequence $\{u_n\}$ in Φ such that $u_{n+1}(x) \leq u_n(x)$ on X and $u_n(x) \to h(x)$ as $n \to \infty$ for every $x \in X(a)$. Since $u_n(a) = \frac{\sum_{x \in W_a} t(x,a) u_n(x)}{t(a)}$, we have $h(a) = \frac{\sum_{x \in W_a} t(x,a) h(x)}{t(a)}$. Since $h(a) > -\infty$, we see that $h(x) > -\infty$ for all $x \in W(a)$ and h is harmonic at a. Taking $b \in W(a)$ and proceeding as before we get $h(x) > -\infty$ for all $x \in W(b)$ and h is harmonic at b. Since any point $z \in X$ is connected to a by a finite number of edges we get $h(z) > -\infty$ and h is harmonic at z. Hence we have h is harmonic on X. □

Similarly we can prove

Theorem 3.2. *If Φ is a left directed family of superharmonic functions on X and* $\inf \Phi \in L(X)$, *then* $\inf \Phi$ *is superharmonic on X.*

Let us use a discrete analogue of Poisson's integral. For $u \in L(X)$ and $a \in X$, we define the function $P_a u \in L(X)$ by

$$\begin{aligned} P_a u(x) &= u(x) \ \text{ if } x \neq a \\ P_a u(a) &= \sum\nolimits_{x \in X} [t(a,x)/t(a)] u(x). \end{aligned}$$

Lemma 3.1. *Assume that u is superharmonic on X. Then $P_a u(x) \leq u(x)$ on X and $P_a u$ is superharmonic on X and harmonic at a.*

Proof. Since u is superharmonic at a, $P_a u(a) \leq u(a)$, so that $P_a u(x) \leq u(x)$ on X. For $x \notin X(a)$, it is clear that $P_a u$ is superharmonic at x. For $x \in W_a$, we have

$$\begin{aligned} \Delta P_a u(x) &= -t(x) P_a u(x) + \sum\nolimits_{z \in W_x} t(z,x) P_a u(z) \\ &\leq -t(x) u(x) + \sum\nolimits_{z \in W_x} t(z,x) u(z) = \Delta u(x) \leq 0. \end{aligned}$$

For $x = a$, we have

$$\Delta P_a u(a) = -t(a) P_a u(a) + \sum\nolimits_{z \in W_a} t(z,a) u(z) = 0.$$

□

Theorem 3.3. *Let A be a subset of X and Φ be a left directed family of superharmonic functions on X. If* $\inf \Phi \in L(X)$ *and $P_a u \in \Phi$ for all $a \in A$ and $u \in \Phi$, then* $\inf \Phi$ *is harmonic on A.*

Proof. Let us put $h = \inf \Phi$. Then h is superharmonic on X by Theorem 3.2. Let $a \in A$. Then $P_a h(x) \leq h(x)$ by Lemma 3.1. By our assumption, we have $h(a) \leq P_a u(a)$ for all $u \in \Phi$. There exists a sequence $\{u_n\}$ in Φ such that $u_n(x) \to h(x)$ as $n \to \infty$ for all $x \in X(a)$. We see easily that $P_a u_n(a) \to P_a h(a)$ as $n \to \infty$, so that $h(a) \leq P_a h(a)$. Namely, $h(a) = P_a h(a)$, i.e., $\Delta h(a) = 0$. □

§4. Reduced Functions and their properties

In this section, we always assume that N is hyperbolic. Denote by $SH^+(N)$ the set of all non-negative superharmonic functions on X. For $f \in L^+(X)$, let us put $\mathcal{S}_f = \{u \in SH^+(N); u(x) \geq f(x) \text{ on } X\}$ and

$$R_f(x) = \inf\{u(x); u \in \mathcal{S}_f\}.$$

Theorem 4.1. *The function R_f is superharmonic on X and harmonic on the set $\{x \in X; f(x) = 0\}$.*

Proof. We show that $\mathcal{S}_f$ is left directed. Let $u_1, u_2 \in \mathcal{S}_f$ and $u_3(x) = \min\{u_1(x), u_2(x)\}$ for $x \in X$. Then $u_3 \in SH^+(N)$ and $u_3(x) \geq f(x)$ on X. Thus $u_3 \in \mathcal{S}_f$. Since $R_f(x) \geq f(x) \geq 0$ on X, we see by Theorem 3.2 that R_f is superharmonic on X. Let $A = \{x \in X; f(x) = 0\}$. For any $u \in \mathcal{S}_f$, we see by Lemma 3.1 that $P_a u$ is superharmonic and $P_a u(x) = u(x) \geq f(x)$ for $x \neq a$. If $a \in A$, then $P_a u(a) \geq 0 = f(a)$. Therefore $P_a u \in \mathcal{S}_f$ for all $u \in \mathcal{S}_f$ and $a \in A$. Our assertion follows from Theorem 3.3. □

Let $u \in L^+(X)$ and A be a subset of X. The function

$$R_u^A(x) = \inf\{v(x); v \in SH^+(N), v(x) \geq u(x) \text{ on } A\}$$

is called the reduced function (or balayage) of u on A.

Theorem 4.2. *R_u^A is superharmonic in X and harmonic in $X \setminus A$.*

Proof. Consider the function $f \in L^+(X)$ defined by $f(x) = u(x)$ for $x \in A$ and $f(x) = 0$ for $x \in X \setminus A$. Then $R_u^A = R_f$ and our assertion follows from Theorem 4.1. □

Lemma 4.1. *If N is hyperbolic, there exists a potential p such that $p(x) > 0$ on X.*

Proof. By our definition, there exists a non-constant positive superharmonic function v. Our assertion is clear if v is not harmonic by Theorem 2.4. Assume that v is harmonic on X. For $a \in X$, we consider the function $s_a \in L(X)$ defined by $s_a(x) = \min(v(x), v(a))$ for $x \in X$. Then $s_a \in SH^+(N)$, $s_a(x) \le s_a(a) = v(a)$ on X. If $\Delta s_a(a) = 0$, then

$$\sum_{x \in W_a} t(x,a)[s_a(a) - s_a(x)] = 0$$

implies that $s_a(x) = s_a(a)$ on $X(a)$, i.e., $v(x) \ge v(a)$ on $X(a)$. Since v is harmonic, we must have $v(x) = v(a)$ on $X(a)$. Taking $a_1 \in X(a), a \ne a_1$, we consider $s_{a_1} = \min(v, v(a_1))$. If $\Delta s_{a_1}(a_1) = 0$, we obtain $v(x) = v(a)$ on $X(a) \cup X(a_1)$. After repeating this procedure a finite number of times, we obtain $b \in X$ such that $s_b = \min(v, v(b))$ and $\Delta s_b(b) < 0$, since v is non-constant. □

Theorem 4.3. *For any $a \in X$, there exists a unique bounded potential $G_a(x)$ such that $\Delta G_a(x) = -\varepsilon_a(x)$.*

Proof. We see by Theorems 4.1 and 4.2 that $u_a(x) = R_{\varepsilon_a} = R_1^{\{a\}}$ is superharmonic on X and harmonic on $X \setminus \{a\}$. Since $1 \in \mathcal{S}_{\varepsilon_a}$, we have $0 \le u_a(x) \le 1$ on X. Since N is hyperbolic, there exists a potential $p > 0$ by Lemma 4.1. Notice that $v(x) = p(x)/p(a) \in \mathcal{S}_{\varepsilon_a}$ and v is also a potential. Thus $u_a(x) \le v(x)$ on X and u_a is a potential by Theorem 2.3. We show that $\Delta u_a(a) < 0$. Supposing the contrary, u_a is harmonic on X. Since u_a is a potential, we must have $u_a = 0$. On the other hand, we have $u_a(a) = 1$. This is a contradiction. Let us put $G_a(x) = -u_a(x)/\Delta u_a(a)$. Then G_a is a bounded potential and $\Delta G_a(x) = -\varepsilon_a(x)$ on X.

We prove the uniqueness of G_a. Assume that there exists a potential ϕ such that $\Delta \phi(x) = -\varepsilon_a(x)$ on X. Let $h = \phi - G_a$. Then $\Delta h(x) = \Delta \phi(x) - \Delta G_a(x) = 0$ on X. Hence h is harmonic on X and $\phi = G_a + h$. By the uniqueness of the Riesz decomposition (Theorem 2.4(3)), we conclude that $h = 0$. Therefore $\phi = G_a$. □

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