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The L^p resolvents for elliptic systems of divergence form

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Abstract.

We consider elliptic systems of divegence form in $\mathbb{R}^n$ under the limited smoothness assumptions on the coefficients. We construct L^p resolvents with evaluation of their operator norms, and derive the Gaussian bounds for heat kernels and estimates for resolvent kernels. These results extend those for single operators.

§1. Introduction

In [5] we considered a single elliptic operator of order $2m$ in divergence form, which is defined in $\mathbb{R}^n$ and has non-smooth coefficients, in the framework of L^p Sobolev spaces and constructed the resolvents. In [6, 7] we extended this result to an operator defined in a general domain with the Dirichlet boundary condition. Furthermore, in [7] we showed that the heat kernels and the resolvent kernels are differentiable (we exclude the diagonal set for the resolvent kernels) up to order $m-1+\sigma$ for any $\sigma \in (0,1)$ and evaluated their derivatives. These results correspond to the results by Tanabe [8] for single operators of non-divergence form.

The purpose of this paper is to extend the above results to elliptic systems defined in $\mathbb{R}^n$.

Let $x = (x_1, \dots, x_n)$ be a generic point in $\mathbb{R}^n$, $\alpha = (\alpha_1, \dots, \alpha_n)$ a multi-index with length $|\alpha| = \alpha_1 + \cdots + \alpha_n$, and

$$D^\alpha = D_1^{\alpha_1} \cdots D_n^{\alpha_n}, \qquad D_j = -\sqrt{-1}\frac{\partial}{\partial x_j} \quad (j = 1, \dots, n).$$

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Let $N \geq 1$ be an integer. We consider the elliptic operator in divergence form

$$(1.1) \qquad Au(x) = \sum_{|\alpha|\leq m,\, |\beta|\leq m} D^\alpha(a_{\alpha\beta}(x)D^\beta u(x))$$

in $\mathbb{R}^n$, where $a_{\alpha\beta}(x)$ is an $N \times N$ matrix $(a^{ij}_{\alpha\beta}(x))_{1\leq i\leq N,\, 1\leq j\leq N}$ and $u(x) = {}^t(u_1(x), \ldots, u_N(x))$. We allow the coefficients to be complex valued, whereas many literature such as [2, 4] deals with systems with real-valued coefficients. We denote by $a(x, \xi)$ the principal symbol of A:

$$a(x, \xi) = \sum_{|\alpha|=|\beta|=m} a_{\alpha\beta}(x)\xi^{\alpha+\beta}, \quad x \in \mathbb{R}^n,\ \xi \in \mathbb{R}^n.$$

Throughout this paper we assume the following.

(H1) All the coefficients $a^{ij}_{\alpha\beta}$ are measurable and bounded in $\mathbb{R}^n$.

(H2) The coefficients $a^{ij}_{\alpha\beta}$ with $|\alpha| = |\beta| = m$ are uniformly continuous in $\mathbb{R}^n$.

(H3) The operator A satisfies the Legendre-Hadamard condition, that is, there exists $\delta_A > 0$ such that

$$\operatorname{Re} {}^t\eta a(x, \xi)\eta \geq \delta_A|\xi|^{2m}|\eta|^2$$

for any $x \in \mathbb{R}^n$, $\xi \in \mathbb{R}^n$ and $\eta = {}^t(\eta_1, \ldots, \eta_N) \in \mathbb{R}^N$.

Let $1 \leq p \leq \infty$ and $\tau \in \mathbb{R}$. We denote by $L^p = L^p(\mathbb{R}^n)$ the space of p-integrable functions and define $H^{\tau,p}$ by

$$H^{\tau,p} = H^{\tau,p}(\mathbb{R}^n) = \{f \in \mathcal{S}'(\mathbb{R}^n) : \langle D\rangle^\tau f \in L^p(\mathbb{R}^n)\}$$

with norm $\|u\|_{H^{\tau,p}} = \|\langle D\rangle^\tau u\|_{L^p}$, where $\langle\xi\rangle = (1 + |\xi|^2)^{1/2}$.

For a Banach space X we define X^N to be the set of all $u = {}^t(u_1, \ldots, u_N)$ such that $u_j \in X$ for $1 \leq j \leq N$ with norm $\|u\|_{X^N} = \max_{1\leq j\leq N} \|u_j\|_X$, and $X^{N\times N}$ the set of all $N \times N$ matrices $a = (a_{ij})_{i,j}$ such that $a_{ij} \in X$ for $1 \leq i \leq N$, $1 \leq j \leq N$ with norm $\|a\|_{X^{N\times N}} = \max_{1\leq i\leq N, 1\leq j\leq N} \|a_{ij}\|_X$. When $X = \mathbb{R}$, we simply write $|a|$ for $\|a\|_{X^{N\times N}}$.

For an integer $k \geq 1$ it is sometimes convenient to write $f \in (H^{-k,p})^N$ as

$$(1.2) \qquad f = \sum_{|\alpha|\leq k} D^\alpha f_\alpha, \quad f_\alpha \in (L^p)^N$$

and note that the norm $\inf \sum_{|\alpha|\leq k} \|f_\alpha\|_{(L^p)^N}$ is equivalent to the norm $\|f\|_{(H^{-k,p})^N}$, where the infimum is taken over all the expressions in (1.2).

We mean by $T : X \to Y$ that T is a bounded linear operator from a Banach space X to a Banach space Y.

Let $1 < p < \infty$. Since $D^\alpha : (H^{\tau,p})^N \to (H^{\tau-|\alpha|,p})^N$ for $\tau \in \mathbb{R}$ and $a_{\alpha\beta} : (L^p)^N \to (L^p)^N$, we can regard A in (1.1) as a bounded operator from $(H^{m,p})^N$ to $(H^{-m,p})^N$. When we want to stress p, we write A as A_p. So we have

$$A = A_p = \sum_{|\alpha|,|\beta|\le m} D^\alpha a_{\alpha\beta} D^\beta : (H^{m,p})^N \to (H^{-m,p})^N.$$

We often use the following notations:

$$M_A = \max_{|\alpha|,\,|\beta|\le m} \|a_{\alpha\beta}\|_{(L^\infty)^{N\times N}}, \qquad \zeta_A = (n, m, N, \delta_A, M_A),$$

$$\omega_A(\varepsilon) = \max_{1\le i\le N,\,1\le j\le N}\ \max_{|\alpha|=|\beta|=m} \sup\{|a^{ij}_{\alpha\beta}(x) - a^{ij}_{\alpha\beta}(y)| : x, y \in \mathbb{R}^n,\ |x-y| \le \varepsilon\},$$

$$\Lambda(R,\theta) = \{\lambda \in \mathbb{C} : |\lambda| \ge R,\ \theta \le \arg\lambda \le 2\pi - \theta\}$$

for $\varepsilon > 0$, $R > 0$ and $0 < \theta < \pi$.

Let $\mu_j(x,\xi)$, $1 \le j \le N$ be all the eigenvalues of $a(x,\xi)$. By (H1) we have $|\mathrm{Im}\,\mu_j(x,\xi)| \le M_0|\xi|^{2m}$ with some constant M_0 depending only on n, m, N and M_A. On the other hand, (H3) implies $\mathrm{Re}\,\mu_j(x,\xi) \ge \delta_A|\xi|^{2m}$. Therefore we conclude that

$$-\kappa_A \le \arg\mu_j(x,\xi) \le \kappa_A, \tag{1.3}$$

where $\kappa_A = \arctan(M_0/\delta_A) \in (0, \pi/2)$. In [5, 6] we assumed $a(x,\xi) \ge \delta_A|\xi|^{2m}$ for a single operator, which is a stronger ellipticity condition than (H3). In this case we can take $\kappa_A = 0$.

§2. Main results

We are now ready to state the main theorems. The first theorem is concerned with the estimates of the type

$$\|(A_p - \lambda)^{-1}\|_{(H^{-i,p})^N \to (H^{j,p})^N} \le K|\lambda|^{-1+(i+j)/2m} \tag{2.1}$$

for $0 \le i \le m$ and $0 \le j \le m$ with some $K > 0$.

Theorem 2.1. *Let $p \in (1,\infty)$ and $\theta \in (\kappa_A, \pi/2)$. Then there exist $R = R(\theta, \zeta_A, \omega_A)$, $K_1 = K_1(p, \theta, \zeta_A)$ and $K_2 = K_2(\theta, \zeta_A)$ such that for $\lambda \in \Lambda(R,\theta)$ the resolvent $(A_p - \lambda)^{-1}$ exists and (2.1) holds for $0 \le i \le m$ and $0 \le j \le m$ with $K = K_1$, and for $0 \le i \le m-1$ and $0 \le j \le m-1$ with $K = K_2$.*

Moreover the resolvents are consistent in the sense that

$$(A_p - \lambda)^{-1} f = (A_q - \lambda)^{-1} f, \quad f \in (H^{-m,p})^N \cap (H^{-m,q})^N$$

when $\lambda \in \Lambda(R, \theta)$ *for any* $p, q \in (1, \infty)$.

For $p \in (1, \infty)$ we define the operator $A_{(p)}$ in $(L^p)^N$ by

$$D(A_{(p)}) = \{u \in (H^{m,p})^N : A_p u \in (L^p)^N\},$$
$$A_{(p)} u = A_p u \quad \text{for } u \in D(A_{(p)}).$$

It follows from Lemma 3.1 in Section 3 that $D(A_{(p)})$ is dense in $(H^{m,p})^N$, $A_{(p)}$ is a closed operator in $(L^p)^N$, and $(A_{(p)})^* = (A^*)_{(p^*)}$, where $p^* = p/(p-1)$ and A^* is the dual operator of A.

Let $h \in \mathbb{R}^n$. We define the difference operators Δ_h, $\Delta_h^{(1)}$ and $\Delta_h^{(2)}$ by $\Delta_h u(x) = u(x+h) - u(x)$, $\Delta_h^{(1)} F(x,y) = F(x+h, y) - F(x,y)$ and $\Delta_h^{(2)} F(x,y) = F(x, y+h) - F(x,y)$, respectively, for vector-valued functions u of $x \in \mathbb{R}^n$ and F of $(x,y) \in \mathbb{R}^n \times \mathbb{R}^n$. We set

$$\Delta = \{(x,x) : x \in \mathbb{R}^n\}.$$

For $t \in \mathbb{C} \setminus \{0\}$, $x \in \mathbb{R}^n$ and $C > 0$ we set

$$\Phi_m(t, x; C) = \exp\{-C(|x|^{2m}|t|^{-1})^{1/(2m-1)}\}.$$

Theorem 2.2. *Let* $p \in (1, \infty)$. *Then the operator* $-A_{(p)}$ *generates an analytic semigroup* $e^{-tA_{(p)}}$ *of angle* $\pi/2 - \kappa_A$ *with kernel* $U(t,x,y)$ *which is independent of* p *and satisfies the following estimates. For any* $\varepsilon \in (0, \pi/2 - \kappa_A)$ *and* $\sigma \in (0,1)$ *there exist* $C_1 = C_1(\varepsilon, \zeta_A)$, $C_2 = C_2(\varepsilon, \zeta_A)$, $C_3 = C_3(\varepsilon, \zeta_A, \omega_A)$, $C_1' = C_1'(\varepsilon, \sigma, \zeta_A)$, $C_2' = C_2'(\varepsilon, \sigma, \zeta_A)$ *and* $C_3' = C_3'(\varepsilon, \sigma, \zeta_A, \omega_A)$ *such that for* $|\alpha| < m$, $|\beta| < m$ *and* $|\arg t| \le \pi/2 - \kappa_A - \varepsilon$ *we have*

$$(2.2) \qquad |\partial_x^\alpha \partial_y^\beta U(t,x,y)| \le C_1 |t|^{-(n+|\alpha|+|\beta|)/2m} \Phi_m(t, x-y; C_2) e^{C_3|t|}$$

for $(x,y) \in \mathbb{R}^n \times \mathbb{R}^n$, *and*

$$(2.3) \qquad |\Delta_h^{(i)} \partial_x^\alpha \partial_y^\beta U(t,x,y)| \le C_1' |t|^{-(n+|\alpha|+|\beta|+\sigma)/2m} \Phi_m(t, x-y; C_2') e^{C_3'|t|} |h|^\sigma$$

for $i \in \{1, 2\}$, $h \in \mathbb{R}^n$ *and* $(x,y) \in \mathbb{R}^n \times \mathbb{R}^n$ *with* $2|h| \le |x-y|$.

Theorem 2.2 extends the result for $p = 2$ obtained by Auscher and Qafsaoui [1], who used the method of Morrey-Campanato spaces.

For $x \in \mathbb{R}^n$, $\lambda \in \mathbb{C}$, $\tau > 0$ and $C > 0$ we set

$$\Psi_m^\tau(x, \lambda; C) = \begin{cases} |\lambda|^{-1+\tau/2m} \exp(-C|\lambda|^{1/2m}|x|) & (\tau < 2m) \\ (1 + \log_+ |\lambda|^{1/2m}|x|) \exp(-C|\lambda|^{1/2m}|x|) & (\tau = 2m) \\ |x|^{2m-\tau} \exp(-C|\lambda|^{1/2m}|x|) & (\tau > 2m), \end{cases}$$

where $\log_+ s = \max\{0, \log s\}$ for $s > 0$.

Theorem 2.3. *Let $p \in (1, \infty)$ and $\theta \in (\kappa_A, \pi/2)$. Then there exists $R = R(\theta, \zeta_A, \omega_A)$ such that for $\lambda \in \Lambda(R, \theta)$ the resolvent $(A_{(p)} - \lambda)^{-1}$ exists and it has a kernel $G_\lambda(x, y)$ which is independent of p and satisfies the following estimates. For any $\sigma \in (0, 1)$ there exist $C_1 = C_1(\theta, \zeta_A)$, $C_2 = C_2(\theta, \zeta_A)$, $C_1' = C_1'(\sigma, \theta, \zeta_A)$ and $C_2' = C_2'(\sigma, \theta, \zeta_A)$ such that for $|\alpha| < m$, $|\beta| < m$ and $\lambda \in \Lambda(R, \theta)$ we have*

$$|\partial_x^\alpha \partial_y^\beta G_\lambda(x, y)| \leq C_1 \Psi_m^{n+|\alpha|+|\beta|}(x - y, \lambda; C_2) \tag{2.4}$$

for $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n \setminus \Delta$, and

$$|\Delta_h^{(i)} \partial_x^\alpha \partial_y^\beta G_\lambda(x, y)| \leq C_1' \Psi_m^{n+|\alpha|+|\beta|+\sigma}(x - y, \lambda; C_2')|h|^\sigma \tag{2.5}$$

for $i \in \{1, 2\}$, $h \in \mathbb{R}^n$ and $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n \setminus \Delta$ with $2|h| \leq |x - y|$.

Moreover $\partial_x^\alpha \partial_y^\beta G_\lambda(x, y)$ is continuous on Δ if $n + |\alpha| + |\beta| < 2m$.

§3. Partial proof of Theorem 2.1

Since $T = (T_{ij})_{i,j} : X^N \to Y^N$ and $T_{ij} : X \to Y$ for $1 \leq i \leq N$, $1 \leq j \leq N$ are equivalent, most properties of T can be reduced to those of T_{ij}. This enables us to obtain the main results along the same line as in the case of single operators. We first derive Lemma 3.1 below, which is weaker than Theorem 2.1 for the constants R and K may depend on p. Then Lemma 3.1 leads to Thorem 2.2, from which Theorems 2.1 and 2.3 follow.

In the following we give only the outline of the proofs except Lemma 3.3 whose proof is a little complicated when $N \geq 2$. The details for the case of single operaters are found in [5, 6, 7].

Lemma 3.1. *Let $p \in (1, \infty)$ and $\theta \in (\kappa_A, \pi/2)$. Then there exist $R_p = R(p, \theta, \zeta_A, \omega_A)$ and $K = K(p, \theta, \zeta_A)$ such that for $\lambda \in \Lambda(R_p, \theta)$ the resolvent $(A_p - \lambda)^{-1}$ exists and (2.1) holds for $0 \leq i \leq m$ and $0 \leq j \leq m$.*

Moreover the resolvents are consistent in the sense of Theorem 2.1.

The proof of Lemma 3.1 is given after some preparation.

Lemma 3.2 ([7]). *Let $\theta \in (\kappa_A, \pi/2)$. Then there exists a constant $C = C(\theta, \kappa_A) > 0$ such that*

$$|s - \lambda| \geq C(|s| + |\lambda|)$$

for $|\arg s| \leq \kappa_A$ and $\theta \leq \arg \lambda \leq 2\pi - \theta$.

Lemma 3.3. *Let $p \in (1, \infty)$, $\theta \in (\kappa_A, \pi/2)$ and fix $x_0 \in \mathbb{R}^n$. Then for $\lambda \in \Lambda(1, \theta)$ the operator $a(x_0, D) - \lambda : (H^{m,p})^N \to (H^{-m,p})^N$ has an inverse and there exists $K = K(p, \theta, \zeta_A)$ such that*

$$\|(a(x_0, D) - \lambda)^{-1}\|_{(H^{-i,p})^N \to (H^{j,p})^N} \leq K|\lambda|^{-1+(i+j)/2m} \tag{3.1}$$

for $0 \leq i \leq m$ and $0 \leq j \leq m$.

Proof. Set $b_\lambda(\xi) = (b_{\lambda ij}(\xi))_{i,j} = (a(x_0, \xi) - \lambda)^{-1}$. Then

$$b_{\lambda ij}(\xi) = (\det(a(x_0, \xi) - \lambda))^{-1} c_{\lambda ji}(\xi),$$

where $c_{\lambda ij}(\xi)$ is (i, j)-cofactor of the matrix $a(x_0, \xi) - \lambda$. By (H1), (1.3), Lemma 3.2 and $\operatorname{Re} \mu_j(x, \xi) \geq \delta_A |\xi|^{2m}$ we have

$$\begin{aligned} |c_{\lambda ij}(\xi)| &\leq C(|\xi|^{2m} + |\lambda|)^{N-1}, \\ |\det(a(x_0, \xi) - \lambda)| &= |\lambda - \mu_1(x_0, \xi)| \cdots |\lambda - \mu_N(x_0, \xi)| \\ &\geq C(|\xi|^{2m} + |\lambda|)^N. \end{aligned}$$

Since $\partial_\xi^\alpha b_\lambda(\xi)$ is written in the form

$$\sum_{\alpha^1 + \cdots + \alpha^k = \alpha} C_{\alpha\alpha^1 \ldots \alpha^k} b_\lambda(\xi) \cdot \partial_\xi^{\alpha^1} a(x_0, \xi) \cdots b_\lambda(\xi) \cdot \partial_\xi^{\alpha^k} a(x_0, \xi) \cdot b_\lambda(\xi)$$

with $1 \leq |\alpha^j| \leq 2m$ $(j = 1, \ldots, k)$, we have

$$\begin{aligned} |\partial_\xi^\alpha b_\lambda(\xi)| &\leq C \sum |\xi|^{2m - |\alpha^1|} \cdots |\xi|^{2m - |\alpha^k|} (|\xi|^{2m} + |\lambda|)^{-k-1} \\ &\leq C(|\xi|^{2m} + |\lambda|)^{-1 - |\alpha|/2m}. \end{aligned}$$

So we get

$$|\xi|^{|\gamma|} \left| \partial_\xi^\gamma \{\xi^{\alpha+\beta} b_\lambda(\xi)\} \right| \leq C|\lambda|^{-1 + (|\alpha| + |\beta|)/2m}$$

for $|\alpha| \leq m$, $|\beta| \leq m$ and $|\gamma| \leq [n/2] + 1$. Finally, by applying Mihlin's multiplier theorem to the operator $D^\alpha b_\lambda(D) D^\beta$ we get the lemma. □

Proof of Lemma 3.1. For $\varepsilon \in (0,1)$ we take a family of functions $\{\eta_{s\varepsilon}(x)\}_{s\in\mathbb{Z}^n}$ in $C_0^\infty(\mathbb{R}^n)$ such that

$$\sum_{s\in\mathbb{Z}^n} \eta_{s\varepsilon}(x)^2 = 1, \qquad \operatorname{supp}\eta_{s\varepsilon} \subset \{x\in\mathbb{R}^n : |x-\varepsilon s| < \varepsilon\},$$
$$|D^\alpha \eta_{s\varepsilon}(x)| \le C_{n,m}\varepsilon^{-|\alpha|} \quad \text{for } |\alpha| \le 2m,$$
$$\#\{s\in\mathbb{Z}^n : \eta_{s\varepsilon}(x) \ne 0\} \le 2^n \text{ for any } x\in\mathbb{R}^n.$$

We define a parametrix for $A-\lambda$ by

$$P_\lambda = \sum_{s\in\mathbb{Z}^n} \eta_{s\varepsilon} P_{s\lambda}\eta_{s\varepsilon}, \qquad P_{s\lambda} = (a(\varepsilon s, D) - \lambda)^{-1}.$$

Using the Leibniz formula, we have

$$(A-\lambda)P_\lambda = I + R_\lambda, \qquad R_\lambda = J_1 + J_2 + J_3 + J_4,$$

where I denotes the identity and

$$J_1 = \sum_{|\alpha|+|\beta|<2m} D^\alpha a_{\alpha\beta} D^\beta P_\lambda,$$
$$J_2 = \sum_{|\alpha|=|\gamma|=m}\sum_{\beta<\gamma} C_{1\gamma\beta} D^\alpha a_{\alpha\gamma}\Big(\sum_s \eta_{s\varepsilon}^{(\gamma-\beta)} D^\beta P_{s\lambda}\eta_{s\varepsilon}\Big),$$
$$J_3 = \sum_{|\alpha|=|\beta|=m} D^\alpha\Big(\sum_s (a_{\alpha\beta} - a_{\alpha\beta}(\varepsilon s))\eta_{s\varepsilon} D^\beta P_{s\lambda}\eta_{s\varepsilon}\Big),$$
$$J_4 = \sum_{|\gamma|=|\beta|=m}\sum_{\alpha<\gamma} C_{0\gamma\alpha} D^\alpha\Big(\sum_s a_{\gamma\beta}(\varepsilon s)\eta_{s\varepsilon}^{(\gamma-\alpha)} D^\beta P_{s\lambda}\eta_{s\varepsilon}\Big)$$

with some constants $C_{0\gamma\alpha}$ and $C_{1\gamma\beta}$. Careful calculation yields

$$\|P_\lambda R_\lambda^k\|_{(H^{-i,p})^N \to (H^{j,p})^N} \le K_0 K_1^k (\omega_A(\sqrt{n}\varepsilon) + \varepsilon^{-1}|\lambda|^{-1/2m})^k |\lambda|^{-1+(i+j)/2m}$$

for $0 \le i \le m$, $0 \le j \le m$ and $\lambda \in \Lambda(\varepsilon^{-2m}, \theta)$. So if we take $\varepsilon \in (0,1)$ and $R > 0$ so that

$$K_1\omega_A(\sqrt{n}\varepsilon) \le 4^{-1}, \qquad K_1\varepsilon^{-1}R^{-1/2m} \le 4^{-1}, \quad R \ge \varepsilon^{-2m},$$

then for $\lambda \in \Lambda(R,\theta)$ the series $\sum_{k=0}^\infty (-1)^k P_\lambda R_\lambda^k$ converges as an operator $(H^{-m,p})^N \to (H^{m,p})^N$ and it is a right inverse of $A-\lambda$. The duality aurgument shows that the right inverse is exactly $(A-\lambda)^{-1}$.

We also get the consistency of resolvents, since $(A-\lambda)^{-1}$ consists of three kinds of operators such as D^α, Fourier multipliers $(a(x_0, D) - \lambda)^{-1}$, and multiplication operators by functions in $(L^\infty)^N$, which are consistent in the sense of Theorem 2.1. □

§4. Proof of Theorem 2.2

Based on Lemma 3.1, we can prove Theorem 2.2. It is seen from (2.1) that $-A_{(p)}$ generates an analytic semigroup of angle $\pi/2 - \kappa_A$.

As for the heat kernel estimate, we shall first consider the case of $p = 2$. By the Sobolev embedding theorem and Lemma 3.1 we have range $(A_{(p)} - \lambda)^{-1} \subset (L^p)^N \cap (L^q)^N$ for p, q with $1 < p < q$, $p^{-1} - q^{-1} < m/n$ and

$$\|(A_{(p)} - \lambda)^{-1}\|_{(L^p)^N \to (L^q)^N} \leq C|\lambda|^{-1+(n/2m)(1/p-1/q)}$$

for $\lambda \in \Lambda(R_p, \theta) \cap \Lambda(R_q, \theta)$ with $\theta \in (\kappa_A, \pi/2)$.

Given $\varepsilon \in (0, 2^{-1}(\pi/2 - \kappa_A))$ and $\sigma \in (0,1)$, we take a sequence $\{p_j\}_{j=1}^k$ satisfying

$$2 = p_k < p_{k-1} < \cdots < p_1 = \max\{\tfrac{n}{1-\sigma}, 2\}, \qquad p_j^{-1} - p_{j-1}^{-1} < m/n,$$

and assume that $\lambda \in \Lambda(\kappa_A + \varepsilon, R)$ and $|\arg t| < \pi/2 - \kappa_A - 2\varepsilon$ with $R = \max\{R_{p_1}, \ldots, R_{p_k}\}$, where each R_{p_j}, $j = 1, \ldots, k$ is the constant defined for $p = p_j$ and $\theta = \kappa_A + \varepsilon$ in Lemma 3.1. Then we have

$$(A_{(2)} - \lambda)^{-k} = (A_{(p_1)} - \lambda)^{-1} \cdots (A_{(p_k)} - \lambda)^{-1}$$

and therefore

$$\|(A_{(2)} - \lambda)^{-k}\|_{(L^2)^N \to (L^\infty)^N} \leq K'|\lambda|^{-k+n/4m}.$$

This combined with the formula

$$\begin{aligned} e^{-tA_{(2)}} &= \frac{1}{2\pi\sqrt{-1}} \int_\Gamma e^{-t\lambda}(A_{(2)} - \lambda)^{-1}\, d\lambda \\ &= \frac{(k-1)!}{2\pi\sqrt{-1}} t^{1-k} \int_\Gamma e^{-t\lambda}(A_{(2)} - \lambda)^{-k}\, d\lambda, \end{aligned}$$

where Γ is a path in $\Lambda(\kappa_A + \varepsilon, R)$, gives

$$\|e^{-tA_{(2)}}\|_{(L^2)^N \to (L^\infty)^N} \leq C|t|^{-n/4m} e^{(\sin(\kappa_A+\varepsilon))^{-1}R|t|}.$$

Applying the kernel theorem to $e^{-2tA_{(2)}} = e^{-tA_{(2)}}(e^{-tA^*_{(2)}})^*$, we obtain

$$|U(2t, x, y)| \leq C^2|t|^{-n/2m} e^{2(\sin(\kappa_A+\varepsilon))^{-1}R|t|}. \tag{4.1}$$

The Gaussian estimate can be derived by Davies' method of exponential perturbation (cf [3]). To this end we set $A_\phi = e^{-\phi}Ae^{\phi}$, where $\phi(x) = \phi(x; \eta, R_0)$ is a C^∞ function of x with parameters $\eta \in \mathbb{R}^n$ and $R_0 > 0$

satisfying $\phi(x) = e^{x\eta}$ for $|x| \le R_0$ and $\partial^\alpha \phi \in L^\infty(\mathbb{R}^n)$ for $|\alpha| \le m$. Then the heat kernel $U_\phi(t,x,y)$ for A_ϕ satisfies the estimate similar to (4.1). So the relation

$$U(t,x,y) = e^{(x-y)\eta} U_\phi(t,x,y)$$

for $|x| \le R_0$ and $|y| \le R_0$ yields the Gaussian bounds.

We can get the estimates for the derivatives of $U(t,x,y)$ and their Hölder norms by using the fact $(A_{(p_1)} - \lambda)^{-1} : (L^{p_1})^N \to (B^{m-1+\sigma})^N$ in the above argument, where $B^{m-1+\sigma}$ denotes the Hölder space of order $m-1+\sigma$.

Finally we shall consider the case of $p \ne 2$. The Gaussian bounds yield $\sup_y \|U(t,\cdot,y)\|_{(L^1)^{N\times N}} < \infty$ and $\sup_x \|U(t,x,\cdot)\|_{(L^1)^{N\times N}} < \infty$. So the integral operator with kernel $U(t,x,y)$ is a bounded operator in $(L^p)^N$. Hence the consistency of resolvents shows that $e^{-tA_{(p)}}$ has the same integral kernel as $e^{-tA_{(2)}}$.

§5. Proof of Theorem 2.3

By Theorem 2.2 we have $\|e^{-tA_{(p)}}\|_{(L^p)^N \to (L^p)^N} \le Ce^{R|t|}$ with some C and R, and therefore

$$(A_{(p)} - \lambda)^{-1} = \int_0^\infty e^{t\lambda} e^{-tA_{(p)}}\, dt, \quad \lambda < -R.$$

Let $\theta \in (0, 2^{-1}(\pi/2 - \kappa_A))$. Deforming the integral path and using analytic continuation, we get the formulae such as

$$(A_{(p)} - \lambda)^{-1} = \int_{L_\theta} e^{t\lambda} e^{-tA_{(p)}}\, dt \tag{5.1}$$

for λ with $|\lambda| > (\sin\theta)^{-1} R$ and $\kappa_A + 2\theta \le \arg\lambda \le \pi$, where L_θ is the half line which runs from 0 to $\infty e^{\sqrt{-1}(\pi/2-\kappa_A-\theta)}$. Then the estimate for the resolvent kernel $G_\lambda(x,y)$ follows from (5.1) and the Gaussian bounds.

§6. Proof of Theorem 2.1

Let $p \in (1,\infty)$ and $\theta \in (\kappa_A, \pi/2)$. By Theorem 2.2 and (5.1) we have $\Lambda(R_0,\theta) \subset \rho(A_{(p)})$, the resolvent set of $A_{(p)}$, with some $R_0 = R_0(\theta, \zeta_A, \omega_A)$. On the other hand, by Lemma 3.1 we have $\Lambda(R_1,\theta) \subset \rho(A_p)$ with some $R_1 = R_1(p,\theta,\zeta_A,\omega_A)$. Based on these inclusions and the resolvent equation, we can take the constant R independent of p in Theorem 2.1. Furthermore, by using (5.1) and the Gaussian bounds we can also take the constant K_2 independent of p in Theorem 2.1.

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