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Hyperbolic Riemann surfaces without unbounded positive harmonic functions

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Abstract.

Let R be an open Riemann surface with Green's functions. It is proved that there exist no unbounded positive harmonic functions on R if and only if the minimal Martin boundary of R consists of finitely many points with positive harmonic measure.

§1. Introduction

Denote by O_G the class of open Riemann surfaces R such that there exist no Green's functions on R. We say that an open Riemann surface R is *parabolic* (resp. *hyperbolic*) if R belongs (resp. does not belong) to O_G.

For an open Riemann surface R, we denote by $HP(R)$ (resp. $HB(R)$) the class of *positive* (resp. *bounded*) harmonic functions on R. It is well-known that if R is parabolic, then $HP(R)$ and $HB(R)$ consist of constant functions (cf. [5]).

Hereafter, we consider only hyperbolic Riemann surfaces R. Let $\Delta = \Delta^R$ and $\Delta_1 = \Delta_1^R$ the *Martin boundary* of R and the *minimal Martin boundary* of R, respectively. The purpose of this paper is to prove the following.

Theorem. *Suppose that R is hyperbolic. Then the followings are equivalent:*

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(i) *there exist no unbounded positive harmonic functions on* R, *i.e.* $HP(R) \subset HB(R)$,

(ii) *the minimal Martin boundary* Δ_1^R *of* R *consists of finitely many points with positive harmonic measure.*

The above theorem combined with the Martin representation theorem yields the following.

COROLLARY. *Suppose that* R *is hyperbolic and there exist no unbounded positive harmonic functions on* R. *Then the linear space* $HB(R)$ *of bounded harmonic functions on* R *is of finite dimension.*

Denote by $\omega_z(\cdot)$ the *harmonic measure* on Δ^R with respect to $z \in R$. We also denote by $k_\zeta(z)$ $((\zeta, z) \in (R \cup \Delta^R) \times R)$ the *Martin kernel* on R with pole at ζ. The following proposition, which is easily proved, plays fundamental role in the proof of the above theorem.

PROPOSITION. *Let* ζ *belong to* Δ_1^R. *Then the Martin kernel* $k_\zeta(\cdot)$ *with pole at* ζ *is bounded on* R *if and only if the harmonic measure* $\omega.(\{\zeta\})$ *of the singleton* $\{\zeta\}$ *is positive.*

§2. Proof of Theorem

Let $k_\zeta(\cdot)$ be the Martin kernel on R with pole at ζ such that $k_\zeta(a) = 1$ for a fixed point $a \in R$. Consider the *canonical measure* χ of the harmonic function 1 in the Martin representation theorem, that is

$$1 = \int_{\Delta_1^R} k_\xi(z) d\chi(\xi). \tag{2.1}$$

As a relation between χ and harmonic measure ω_z, the following is known (c.f. [1, Satz 13.4]):

$$d\omega_z(\xi) = k_\xi(z) d\chi(\xi). \tag{2.2}$$

We first give the proof of Proposition in the introduction.

Proof of Proposition. We assume that the Martin kernel $k_\zeta(z)$ with pole at $\zeta \in \Delta_1^R$ is bounded on R. Take a positive constant M such that $k_\zeta(z) \leq M$ on R. Then, by the Martin representation theorem, we deduce that

$$\int_{\Delta_1^R} k_\xi(z) d\delta_\zeta(\xi) = k_\zeta(z) \leq M = \int_{\Delta_1^R} M k_\xi(z) d\chi(\xi),$$

where δ_ζ is the Dirac measure on Δ_1^R supported at ζ. Hence, by virtue of the fact that the mapping of HP functions to their canonical measures are lattice isomorphic (cf. [1,Forgesatz 13.1]), we see that $\delta_\zeta \le M\chi$ or $(1/M)\delta_\zeta \le \chi$ on Δ_1^R. From this and (2.2) it follows that

$$0 < \frac{k_\zeta(z)}{M} = k_\zeta(z)\frac{\delta_\zeta(\{\zeta\})}{M} \le k_\zeta(z)\chi(\{\zeta\}) = \omega_z(\{\zeta\}),$$

thus we have proved the '*only if part*'.

We next assume that $\omega_z(\{\zeta\}) > 0$. Then, by (2.2), we have

$$0 < \omega_z(\{\zeta\}) = k_\zeta(z)\chi(\{\zeta\}). \tag{2.3}$$

Hence $c := \chi(\{\zeta\})$ is a positive constant. On the other hand, $\omega_z(\{\zeta\}) \le 1$ on R. Therefore, in view of (2.3), we see that $k_\zeta(z) \le c^{-1}$ on R. Thus we have proved the 'if part'.

Applying Proposition proved above, we next give the proof of Theorem in the introduction.

Proof of Theorem. Since the implication (ii) $\Rightarrow$ (i) easily follows from Proposition and the Martin representation theorem, we only have to show the implication (i) $\Rightarrow$ (ii).

Suppose that (ii) is not the case although we are assuming that $HP(R) \subset HB(R)$. Then it easily follows from Proposition that Δ_1^R does not contain a point ζ with $\omega.(\{\zeta\}) = 0$. Therefore Δ_1^R consists of countably infinitely many points ζ_n $(n \in \mathbb{N})$ with $\omega.(\{\zeta_n\}) > 0$ and moreover each Martin kernel k_{ζ_n} is bounded on R. Put $M_n := \sup_{z\in R} k_{\zeta_n}(z)$. Then we deduce that

$$\int_{\Delta_1^R} k_\xi(z)d\left(\frac{1}{M_n}\right)\delta_{\zeta_n}(\xi) = \frac{k_{\zeta_n}(z)}{M_n} \le 1 = \int_{\Delta_1^R} k_\xi(z)d\chi(\xi),$$

where δ_{ζ_n} is the Dirac measure at ζ_n and χ is the measure in (2.1). Hence, by means of lattice isomorphic determination of canonical measures, we see that $(1/M_n)\delta_{\zeta_n} \le \chi$ for every $n \in \mathbb{N}$. Since the supports $\text{supp}(\delta_{\zeta_n})$ of $\{\delta_{\zeta_n}\}$ are mutually disjoint, this implies that $\sum_{n=1}^\infty (1/M_n)\delta_{\zeta_n} \le \chi$. Therefore we conclude that

$$\sum_{n=1}^\infty \frac{k_{\zeta_n}(z)}{M_n} = \int_{\Delta_1^R} k_\xi(z)d\left(\sum_{n=1}^\infty \frac{\delta_{\zeta_n}(\xi)}{M_n}\right) \le \int_{\Delta_1^R} k_\xi(z)d\chi(\xi) = 1.$$

Since $k_{\zeta_n}(a) = 1$, this yields that $\sum_{n=1}^\infty \dfrac{1}{M_n} \le 1$ and hence

$$\lim_{n\to\infty} M_n = +\infty. \tag{2.4}$$

In view of (2.4), we can choose a subsequence $\{M_{n_i}\}$ of $\{M_n\}$ such that

$$\sum_{i=1}^{\infty} \frac{1}{\sqrt{M_{n_i}}} < +\infty. \tag{2.5}$$

Put

$$h(z) := \sum_{i=1}^{\infty} \frac{1}{\sqrt{M_{n_i}}} k_{\zeta_{n_i}}(z).$$

By (2.5) and the Harnack principle, $h(z)$ is convergent and a positive harmonic function on R since $k_\zeta(a) = 1$ for every ζ. On the other hand, by the definition, $h(z) \geq \frac{1}{\sqrt{M_{n_i}}} k_{\zeta_{n_i}}(z)$ on R and therefore

$$\sup_{z\in R} h(z) \geq \frac{1}{\sqrt{M_{n_i}}} \sup_{z\in R} k_{\zeta n_i}(z) = \frac{1}{\sqrt{M_{n_i}}} M_{n_i} = \sqrt{M_{n_i}}.$$

Hence, by means of (2.4), we see that $\sup_{z\in R} h(z) = +\infty$ or $h \notin HB(R)$. This contradicts our primary assumption $HP(R) \subset HB(R)$.

The proof is herewith complete.

§3. Examples

In this section we will give examples of open Riemann surfaces R satisfying the condition $HP(R) \subset HB(R)$ in Theorem. We can moreover require for Δ_1^R to consist of p points of positive harmonic measure for an arbitrarily given integer $1 \leq p < \infty$ in advance.

Let O_{HP} be the class of open Riemann surfaces on which there exists no nonconstant positive harmonic functions. Recall the class O_G of open Riemann surfaces on which there exist no Green's functions. Then it holds that $O_G \subset O_{HP}$ (cf. e.g. [5]). Moreover the inclusion $O_G \subset O_{HP}$ is strict, that is, there exists an open Riemann surface T belonging to $O_{HP} \setminus O_G$ (cf. [6], [5]). Since $HP(T)$ consists of only constant functions, the Martin boundary Δ^T of T and hence the minimal Martin boundary Δ_1^T of T also consists of a single point ζ_0 and the Martin kernel k_{ζ_0} on T with pole at ζ_0 is equal to the constant function 1.

Consider a p-sheeted $(1 \leq p < \infty)$ *unlimited* (possibly branched) covering surface $\tilde{T}$ of T with its projection map π. Here we say that $\tilde{T}$ is unlimited if the following condition is satisfied: for any arc C in T with a as its initial point and any point $\tilde{a}$ over a, i.e. $\pi(\tilde{a}) = a$, there exists an arc $\tilde{C}$ in $\tilde{T}$ with $\tilde{a}$ as its initial point such that $\pi(\tilde{C}) = C$. By our preceeding result (cf. [2]), the minimal Martin boundary $\Delta_1^{\tilde{T}}$ of

$\tilde{T}$ consists of at most p points. Moreover, there exists $\tilde{T}$ such that $\Delta_1^{\tilde{T}}$ consists of exactly p points. Put $\Delta_1^{\tilde{T}} = \{\tilde{\zeta}_1, \cdots, \tilde{\zeta}_q\}$ $(1 \leq q \leq p)$ and denote by $\tilde{k}_{\tilde{\zeta}_i}$ the Martin kernel on $\tilde{T}$ with pole at $\tilde{\zeta}_i$. As a relation between $\tilde{k}_{\tilde{\zeta}_i}$ and the Martin kernel k_{ζ_0} on T, it holds that

$$\sum_{\tilde{z} \in \pi^{-1}(z)} \tilde{k}_{\tilde{\zeta}_i}(\tilde{z}) \leq c_i k_{\zeta_0}(z),$$

where c_i is a positive constant (cf. [3]). Hence $\tilde{k}_{\tilde{\zeta}_i}$ is bounded on $\tilde{T}$ for every i $(1 \leq i \leq q)$ since $k_{\zeta_0}^T = 1$. Consequently, by virtue of the Martin representation theorem, we see that $HP(\tilde{T}) \subset HB(\tilde{T})$.

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