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A subharmonic Hardy class and Bloch pullback operator norms

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Abstract.

We estimate the operator norm of the composition operators mapping Bloch space boundedly into Hardy spaces, *BMOA* space, Lipschitz spaces and mean Lipschitz spaces respectively.

§1. Introduction

This is to give a brief survey of a resent result on Bloch pullback operators, whose detailed proof will appear at [5]. Our purpose here is two-fold. One is to obtain hyperbolic version of Littlewood-Paley g-function equivalence, the other is to estimate the operator norm of Bloch-pullback operators. At first glance these two topics seem to be quite apart, but they are very closely related.

Let D be the unit disc of the complex plane and $S = \partial D$. Let H^p, $0 < p < \infty$, denote the classical Hardy space defined to consist of f holomorphic in D for which

$$\|f\|_{H^p} = \lim_{r \to 1} \left(\int_S |f(r\zeta)|^p \, d\sigma(\zeta) \right)^{1/p} < \infty,$$

where $d\sigma$ is the rotation invariant Lebesgue probability measure (Haar measure) on S.

For a holomorphic function f in D, the g-function of Littlewood-Paley defined as

$$g_f(\zeta) = \left(\int_0^1 (1-r)|f'(r\zeta)|^2 \, dr \right)^{1/2}, \quad \zeta \in S,$$

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satisfies the following beautiful and powerful relation

$$\|g_f\|_{L^p} \approx \|f - f(0)\|_{H^p} \tag{1.1}$$

(see [1] or [8], also see [16] for $1 < p < \infty$). Here and throughout, $L^p = L^p(S)$.

In parallel with H^p, there defined is ϱH^p consisting of holomorphic self map ϕ of D for which

$$\|\phi\|_{\varrho H^p} = \lim_{r\to 1}\left(\int_S \varrho\left(\phi(r\zeta), 0\right)^p \, d\sigma(\zeta)\right)^{1/p} < \infty,$$

where ϱ is the hyperbolic distance on D :

$$\varrho(z,w) = \frac{1}{2}\log\frac{1+|\varphi_z(w)|}{1-|\varphi_z(w)|}, \quad \varphi_z(w) = \frac{z-w}{1-\bar{z}w}.$$

We set $\lambda(z) = \log\frac{1}{1-|z|}$, $z \in D$. Note that if ϕ is a holomorphic self map of D, then $\lambda \circ \phi$ is subharmonic in D and radial limit $\phi^*(\zeta) = \lim_{r\to 1}\phi(r\zeta)$ exists almost every $\zeta \in S$, so $\phi \in \varrho H^p$ if and only if $\lambda \circ \phi^* \in L^p(S)$. Throughout, $dA(z)$ denotes the Lebesgue area measure of D normalized to be $A(D) = 1$.

Along with [6, 10] for previous results on pullback theory, we refer to [3, 16] for Hardy space theory and [2, 15] for composition operator theory.

§2. Hyperbolic g-function

Our first subject is the Littlewood-Paley type g-function that characterizes the membership of ϱH^p. See [4] and [6] for related previous works. We define, as in [4],

$$\varrho g_\phi(\zeta) = \int_0^1 (1-r)\left(\frac{|\phi'(r\zeta)|}{1-|\phi(r\zeta)|^2}\right)^2 dr, \quad \zeta \in S.$$

As our first result, we have the following hyperbolic analogue of (1.1).

Theorem 2.1. *Let $0 < p < \infty$. Then*

$$\|\varrho g_\phi\|_{L^p} \approx \|\lambda \circ \phi^*\|_{L^p} \tag{2.1}$$

for all holomorphic self map ϕ of D with $\phi(0) = 0$.

When $p = 1$, (2.1) follows immediately from the following.

Lemma 2.2. *Let ϕ be a holomorphic self map of D and $0 < p < \infty$. Then*

$$\int_D \log \frac{1}{|z|} \Delta(\lambda \circ \phi)^p(z)\, dA(z) \approx \|\lambda \circ \phi^*\|_{L^p}^p - (\lambda \circ \phi(0))^p .$$

For the proof of Theorem 2.1, we need several more techniques. We skip them and refer to [5].

§3. Norm of the Bloch-pullback operators

We next pass to our second subject, the Bloch pullback. It is known that there is a Bloch function having radial limits at no points of S, while functions of H^p should have radial limits almost everywhere on S. This observation give rise to the problem of characterizing holomorphic self maps ϕ of D for which $f \circ \phi \in H^p$ for every Bloch function f. It is so called "Bloch - H^p pullback problem" and the Bloch-pullback operator (induced by a holomorphic self map ϕ of D) means the composition operator C_ϕ defined on the Bloch space $\mathcal{B}$ by $C_\phi f = f \circ \phi$. H^p is a Banach space with norm $\|f\|_{H^p}$ when $1 \le p < \infty$, while it is a Frechet space with the compatible metric $\|f\|_{H^p}^p$ when $0 < p < 1$. The following characterization of the Bloch-H^p pullback operator shows a connection between Hardy space and hyperbolic Hardy class.

Theorem A [4, 6]. *Let $0 < p < \infty$ and ϕ be a holomorphic self map of D. Then C_ϕ maps $\mathcal{B}$ boundedly into H^p if and only if $\phi \in \varrho H^{p/2}$.*

As an application of Theorem 2.1, we moreover have the following theorem. Here, $\mathcal{B}^0$ denotes the subspace of $\mathcal{B}$ consisting of $f \in \mathcal{B}$ with $f(0) = 0$.

Theorem 3.1. *Let $0 < p < \infty$ and ϕ be a holomorphic self map of D with $\phi(0) = 0$. If we set $\|C_\phi\| = \sup\{\|C_\phi f\|_{H^p} : f \in \mathcal{B}^0, \|f\|_{\mathcal{B}} \le 1\}$ then it satisfies*

$$\|C_\phi\| \approx \|\lambda \circ \phi^*\|_{L^{p/2}}^{1/2}.$$

The assumption that $\phi(0) = 0$ is not essential restriction in the sense that if C_ϕ is bounded (or compact) then so is C_ψ with $\psi = \varphi_{\phi(0)} \circ \phi$. Note also that $C_\phi : \mathcal{B} \to Y$ is bounded if and only if $C_\phi : \mathcal{B}^0 \to Y$ is bounded.

As a limiting space of H^p, a similar problem might be asked for $BMOA$. $BMOA$, the space of holomorphic functions of bounded mean oscillation, consists of holomorphic f in D for which

$$\|f\|_{BMOA} = \sup_{a\in D}\left\{\lim_{r\to 1}\int_S |f\circ\varphi_a(r\zeta) - f(a)|^2\,d\sigma(\zeta)\right\}^{1/2} < \infty.$$

In parallel with $BMOA$, there defined is $\varrho BMOA$ consisting of holomorphic self map ϕ of D for which

$$\|\phi\|_{\varrho BMOA} = \sup_{a\in D}\lim_{r\to 1}\int_S \varrho\left(\phi\circ\varphi_a(r\zeta),\phi(a)\right)\,d\sigma(\zeta) < \infty.$$

The classes $\varrho BMOA$ as well as ϱH^p were defined and studied mainly as a hyperbolic counterpart of the corresponding Euclidean classes by S. Yamashita [11, 12, 13], and later studied by several authors in connection with the composition operators.

Theorem B [7]. *Let ϕ be a holomorphic self map of D. Then C_ϕ maps $\mathcal{B}$ boundedly into $BMOA$ if and only if $\phi\in\varrho BMOA$.*

Noting that the Möbius invariance of ϱ implies $\varrho\left(\phi\circ\varphi_a(z),\phi(a)\right)=\varrho\left(\varphi_{\phi(a)}\circ\phi\circ\varphi_a(z),0\right)$, it follows that $\phi\in\varrho BMOA$ if and only if

$$\sup_{a\in D}\|\lambda\circ(\varphi_{\phi(a)}\circ\phi\circ\varphi_a)^*\|_{L^1} < \infty.$$

Since $\log|1-\bar{\phi}(a)\phi\circ\varphi_a|$ is harmonic in D,

$$\|\lambda\circ(\varphi_{\phi(a)}\circ\phi\circ\varphi_a)^*\|_{L^1} = \|\lambda\circ(\phi\circ\varphi_a)^* - \lambda\circ\phi(a)\|_{L^1}$$

[7, (3.7)], so that the next theorem gives Theorem B. Here, as the norm of $BMOA$ we take $|f(0)|+\|f\|_{BMOA}$, which makes $BMOA$ a Banach space.

Theorem 3.2. *Let ϕ be a holomorphic self map of D with $\phi(0)=0$. Then the operator norm of C_ϕ from $\mathcal{B}^0$ boundedly into BMOA satisfies*

$$\|C_\phi\| \approx \sup_{a\in D}\|\lambda\circ(\phi\circ\varphi_a)^* - \lambda\circ\phi(a)\|_{L^1}^{1/2}.$$

$VMOA$, the space of holomorphic functions of vanishing mean oscillation, consists of holomorphic f in D for which

$$\lim_{|a|\to 1}\lim_{r\to 1}\int_S |f\circ\varphi_a(r\zeta) - f(a)|^2\,d\sigma(\zeta) = 0.$$

In parallel to $VMOA$, $\varrho VMOA$ is defined to consist of holomorphic self map ϕ of D for which

$$\lim_{|a|\to 1}\lim_{r\to 1}\int_S \varrho\,(\phi\circ\varphi_a(r\zeta),\phi(a))\;d\sigma(\zeta) \;=\; 0.$$

We have

Corollary C [9]. *Let ϕ be a holomorphic self map of D. Then C_ϕ maps $\mathcal{B}$ boundedly into $VMOA$ if and only if $\phi\in\varrho VMOA$.*

See [9] for previous study on $\varrho VMOA$.

§4. More on Bloch-pullback operator norm

We give some more examples of Banach space Y and resolve Bloch-Y pullback problem by further evaluating the operator norm of $C_\phi:\mathcal{B}\to Y$.

Let $\mathcal{D}$ denote the space of holomorphic functions f in D satisfying

$$\|f\|_{\mathcal{D}} := \left(\int_D |f'(z)|^2\;dA(z)\right)^{1/2} < \infty.$$

Then $\mathcal{D}$ is a Banach space with the norm $|f(0)|+\|f\|_{\mathcal{D}}$. Similarly, we let $\varrho\mathcal{D}$ denote the space of holomorphic self map ϕ of D satisfying

$$\|\phi\|_{\varrho\mathcal{D}} := \left(\int_D \frac{|\phi'(z)|^2}{(1-|\phi(z)|^2)^2}\;dA(z)\right)^{1/2} < \infty.$$

Then we have

Theorem 4.1. *Let ϕ be a holomorphic self map of D. Then C_ϕ maps $\mathcal{B}$ boundedly into $\mathcal{D}$ if and only if $\phi\in\varrho\mathcal{D}$. Moreover, if $\phi(0)=0$ then the operator norm of C_ϕ from $\mathcal{B}^0$ boundedly into $\mathcal{D}$ satisfies*

$$\|C_\phi\| \;\approx\; \|\phi\|_{\varrho\mathcal{D}}.$$

H^∞, consisting of bounded holomorphic functions, is a Banach space with the norm $\|f\|_{H^\infty}=\sup_{z\in D}|f(z)|$, while ϱH^∞ is defined to consist of holomorphic ϕ of D for which $|\phi|<c$ for some $c<1$.

Theorem 4.2. *If ϕ be a holomorphic self map of D, then $C_\phi:\mathcal{B}^0\to H^\infty$ is bounded if and only if $\phi\in\varrho H^\infty$. If $\phi(0)=0$, then the operator norm of C_ϕ from $\mathcal{B}^0$ boundedly into H^∞ satisfies*

$$\|C_\phi\| \;=\; \sup_{z\in D}\rho\circ\phi(z),$$

where ρ is defined by $\rho(w)=\rho(0,w)=\frac{1}{2}\log\frac{1+|w|}{1-|w|}$, $w\in D$.

Beyond H^∞, there are function spaces having smooth boundary conditions. We are going to mention about holomorphic Lipschitz spaces. For $0 < \alpha \leq 1$, we say, by definition, that $f \in Lip_\alpha$ if f is holomorphic in D, $f \in C(\bar{D})$, and satisfies the Lipschitz condition:

$$\|f\|_{Lip_\alpha} := \sup\left\{\frac{|f(z) - f(w)|}{|z - w|^\alpha} : z, w \in D, z \neq w\right\} < \infty.$$

Lip_α is a Banach space equipped with the norm $|f(0)| + \|f\|_{Lip_\alpha}$. Several different (but essentially same) notions for Lip_α are used in the literature. We followed that of [2].

Corresponding to this, there is hyperbolic Lipschitz class of Yamashita [14]. We say, by definition, that $\phi \in \varrho Lip_\alpha$ if ϕ is a holomorphic self map of D, $\phi \in C(\bar{D})$, and satisfies the hyperbolic Lipschitz condition:

$$\|\phi\|_{\varrho Lip_\alpha} := \sup\left\{\frac{\varrho(\phi(z), \phi(w))}{|z - w|^\alpha} : z, w \in D, z \neq w\right\} < \infty.$$

We have

Theorem 4.3. *Let $0 < \alpha \leq 1$ and ϕ be a holomorphic self map of D. Then $C_\phi : \mathcal{B} \to Lip_\alpha$ is bounded if and only if $\phi \in \varrho Lip_\alpha$. Further if $\phi(0) = 0$, then the operator norm of C_ϕ from $\mathcal{B}^0$ boundedly into Lip_α satisfies*

$$\|C_\phi\| = \|\phi\|_{\varrho Lip_\alpha}.$$

For $1 \leq p < \infty$ and $0 < \alpha < 1$, we say, by definition, that $f \in Lip_\alpha^p$ if $f \in H^p$ and satisfies the mean Lipschitz condition:

$$\|f\|_{Lip_\alpha^p} := \sup\left\{\frac{1}{t^\alpha}\left(\int_S |f(\eta\zeta) - f(\zeta)|^p d\sigma(\zeta)\right)^{\frac{1}{p}} : 0 < |1 - \eta| \leq t\right\} < \infty.$$

Lip_α^p is a Banach space equipped with the norm $\|\cdot\|_{H^p} + \|\cdot\|_{Lip_\alpha^p}$.

Corresponding to this, there is hyperbolic mean Lipschitz class of Yamashita [14]. We say, by definition, that $\phi \in \varrho Lip_\alpha^p$ if ϕ is a holomorphic self map of D, $\varrho(\phi^*) \in L^p(S)$, and ϕ satisfies the hyperbolic mean Lipschitz condition:

$$\|\phi\|_{\varrho Lip_\alpha^p} := \sup\left\{\frac{1}{t^\alpha}\left(\int_S \varrho\left(\phi(\eta\zeta), \phi(\zeta)\right)^p d\sigma(\zeta)\right)^{\frac{1}{p}} : 0 < |1 - \eta| \leq t\right\} < \infty.$$

We have

Theorem 4.4. *Let $1 \leq p < \infty$ and $0 < \alpha < 1$. Let ϕ be a holomorphic self map of D. Then $C_\phi : \mathcal{B} \to Lip^p_\alpha$ is bounded if and only if $\phi \in \varrho Lip^p_\alpha$. Furthermore, if $\phi(0) = 0$, then operator norm of C_ϕ from $\mathcal{B}^0$ boundedly into Lip^p_α satisfies*

$$\|C_\phi\| \approx \|\lambda \circ \phi^*\|^{1/2}_{L^{p/2}} + \|\phi\|_{\varrho Lip^p_\alpha}.$$

References

[1] P. Ahern and J. Bruna, Maximal and area integral characterization of Hardy-Sobolev spaces in the unit ball of $\mathbb{C}^n$, Revista Mathematica Iberoamericana, **4** (1988), 123–153.

[2] Carl C. Cowen and Barbara D. Maccluer, Composition operators on spaces of analytic functions, CRC Press, New York, 1994.

[3] P. L. Duren, Theory of H^p spaces, Academic Press New York, 1970.

[4] E. G. Kwon, Composition of Blochs with bounded analytic functions, Proc. Amer. Math. Soc., **124** (1996), 1473–1480.

[5] E. G. Kwon, Hyperbolic g-function and Bloch pullback operators, J. Math. Anal. Appl., to appear.

[6] E. G. Kwon, Hyperbolic mean growth of bounded holomorphic functions in the ball, Trans. Amer. Math. Soc., **355** (2003), 1269–1294.

[7] E. G. Kwon, On analytic functions of Bergman BMO in the ball, Canad. Math. Bull., **42** (1999), 97–103.

[8] Steven G. Krantz and Song-Ying Li, Area integral characterizations of functions in Hardy spaces on domains in $\mathbb{C}^n$, Complex Variables, **32** (1997), 373-399.

[9] S. Makhmutov and M. Tjani, Composition operators on some Möbius invariant Banach spaces, Bull. Austral. Math. Soc., **62** (2000), 1–9.

[10] W. Ramey and D. Ullrich, Bounded mean oscillations of Bloch pullbacks, Math. Ann., **291** (1991), 591–606.

[11] S. Yamashita, Functions with H^p hyperbolic derivative, Math. Scand., **13** (1983), 238–244.

[12] S. Yamashita, Holomorphic functions of hyperbolically bounded mean oscillation, Bollettino U.M.I.(6), l5-B (1986), 983–1000.

[13] S. Yamashita, Hyperbolic Hardy class and hyperbolically Dirichlet finite functions , Hokkaido Math. J., Special Issue **10** (1981), 709–722.

[14] S. Yamashita, Smoothness of the boundary values of functions bounded and holomorphic in the disk, Trans. Amer. Math. Soc., **272** (1982), 539–544.

[15] K. Zhu, Operator theory in function spaces, Marcel Dekker, New York, 1990.

[16] A. Zygmund, Trigonometric series, Cambridge Univ. Press, London, 1959.

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