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On the sharpness of certain approach regions

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Abstract.

In this survey, we describe joint work in collaboration with A. Stokolos, O. Svensson and T. Weiss. We consider the following question: How sharp is the Stolz approach region condition for the almost everywhere convergence of bounded harmonic functions? The issue was first settled in the rotation invariant case in the unit disc by Littlewood in 1927 and later examined, under less stringent conditions, by Aikawa in 1991. We show that our results are, in a precise sense, sharp.

§1. How sharp are the Stolz approach regions?

In this survey, we describe joint work in collaboration with A. Stokolos, O. Svensson and T. Weiss. Proofs appear elsewhere [8].

1.1. The unit disc in the plane

Consider the space H^∞ of all bounded holomorphic functions in the unit disc $\mathbb{D}$ in $\mathbb{C}$. How sharp is the Stolz (nontangential) approach

$$\Gamma_\alpha(e^{i\theta}) = \left\{ z \in \mathbb{D} : |z - e^{i\theta}| < (1+\alpha)(1-|z|) \right\} \tag{1.1}$$

for the a. e. boundary convergence of H^∞ functions?

A family $\gamma = \{\gamma(\theta)\}_{\theta\in[0,2\pi)}$ of subsets of $\mathbb{D}$, called an *approach*, may have the following properties:

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c: each $\gamma(\theta)$ is a curve in $\mathbb{D}$ ending at $e^{i\theta}$;
tg: each $\gamma(\theta)$ ends tangentially at $e^{i\theta}$;
aecv: each $h \in \mathrm{H}^\infty$ converges a. e. along $\gamma(\theta)$ to its Stolz boundary values.

The STRONG SHARPNESS STATEMENT is the following claim.

(SSS) *There is no approach* γ *satisfying* (**c**)&(**tg**)&(**aecv**).

This claim is coherent with a principle — implicit in Fatou [10] — whose first rendition is found in Littlewood [20], who showed that there is no **rotation invariant** approach γ satisfying (**c**)&(**tg**)&(**aecv**). Another rendition of this principle (with stronger conclusions) has been given by Aikawa [1], who proved that, if (**u**) is the condition:

u: the curves $\{\gamma(\theta)\}_\theta$ are uniformly bi-Lipschitz equivalent;

then there is no approach γ satisfying **(u)** and (**c**)&(**tg**)&(**aecv**).

Our first result[1] is a theorem of Littlewood type where the tangential curve is allowed to vary its shape, and we do not require uniformity in the order of tangency. Moreover, we show that, in a precise sense, Theorem 1.1 is sharp.

Theorem 1.1 (A sharp Littlewood type theorem). *Let* $\gamma : [0, 2\pi) \to 2^{\mathbb{D}}$ *be such that*

(c⋆): *for each* $\theta \in [0, 2\pi)$*, the set* $\{e^{i\theta}\} \cup \gamma(\theta)$ *is connected;*
(tg): *for each* $\alpha > 0$ *and* $\theta \in [0, 2\pi)$ *there exists* $\delta > 0$ *such that if* $z \in \gamma(\theta) \cap \Gamma_\alpha(e^{i\theta})$ *then* $|z - e^{i\theta}| > \delta$*;*
(reg): *for each open subset* O *of* $\mathbb{D}$ *the set*

$$\{\theta \in [0, 2\pi) : \gamma(\theta) \cap O \neq \emptyset\}$$

is a measurable subset of $[0, 2\pi)$.

Then there exists $h \in \mathrm{H}^\infty$ *with the property that, for almost every* $\theta \in [0, 2\pi)$*, the limit of* $h(z)$ *as* $z \to e^{i\theta}$ *and* $z \in \gamma(\theta)$ *does not exist.*

• Condition (**c⋆**) is strictly weaker than (**c**) but it *cannot* be relaxed to the minimal condition one may ask for:

(apprch): $e^{i\theta}$ belongs to the closure of $\gamma(\theta)$ for all θ

since Nagel and Stein [21] showed that there is a rotation invariant approach γ satisfying (**apprch**) and (**tg**)&(**aecv**). This discovery disproved a conjecture of Rudin [24], prompted by his construction of a highly oscillating inner function in $\mathbb{D}$. Thus, (**c⋆**) identifies the property of *curves* relevant to a theorem of Littlewood type.

[1] A preliminary version of this result was announced in Di Biase et al [7].

• It is not easy to see **(reg)** fail. The images of radii by an inner function satisfy **(reg)**: this example prompted Rudin [24] to ask about the truth value of (SSS). Observe that **(reg)** is a qualitative condition, while **(u)** is quantitative. The former is perhaps more commonly met than the latter. Furthermore, the conditions are independent of each other.

• Since our hypothesis do not impose any smoothness, neither on $\gamma(\theta)$ nor on the domain, a version of our theorem can be formulated, and proved as well, for domains with rough boundary, such as NTA domains in $\mathbb{R}^n$; see Theorem 1.3 below.

• Is it possible to prove Theorem 1.1 without assuming **(reg)**? Several theorems in Analysis do fail if we omit some regularity conditions, while others (typically those involving null sets) remain valid without 'regularity' hypothesis[2]. This question brings us back to the truth value of (SSS), and we prove the following result.

Theorem 1.2. *It is neither possible to prove the Strong Sharpness Statement, nor to disprove it.*

The proof uses a combination of methods of modern logic (developed after 1929) and harmonic analysis, based upon an insight about the location of the link that makes the combination possible. See Theorem 2.1, Theorem 2.2 and Theorem 2.3.

1.2. Nontangentially accessible domains in $\mathbb{R}^n$

Let h^∞ be the space of bounded harmonic functions on a bounded domain $D \subset \mathbb{R}^n$. Assume that D is NTA — as defined by Jerison and Kenig [17]. How sharp is the so-called *corkscrew approach*

$$\Gamma_\alpha(w) \stackrel{\text{def}}{=} \{z \in D : |z - w| < (1+\alpha)\text{dist}(z, \partial D)\} \tag{1.2}$$

for the boundary convergence for h^∞ functions, a. e. relative to harmonic measure?

Observe that D may be *twisting* a. e. relative to harmonic measure. In this case, the 'corkscrew' approach (1.2) does not look like a sectorial angle at all.

[2]A *regularity hypothesis* in a theorem is one which is not (formally) necessary to give meaning to the conclusion of the theorem. *A priori* it is not clear which theorems belong to which group. Egorov's theorem on pointwise convergence belongs to the first; see Bourbaki [2], p. 198. One example in the second group can be found in Stein [25], p. 251.

Theorem 1.1 lends itself to the task of formulating[3] the appropriate sharpness statement for NTA domains, without any further restrictions on the domain.

Theorem 1.3. *If D is an NTA domain in $\mathbb{R}^n$ and $\gamma = \{\gamma(w)\}_{w\in\partial D}$ is a family of subsets of D such that*

(c⋆): *for each $w \in \partial D$, $\gamma(w) \cup \{w\}$ is connected;*
(tg): *for each $\alpha > 0$ and $w \in \partial D$ there exists $\delta > 0$ such that if $z \in \gamma(w) \cap \Gamma_\alpha(w)$ then $|z - w| > \delta$;*
(reg): *for each open subset O of D the set*

$$\{w \in \partial D : \gamma(w) \cap O \neq \emptyset\}$$

is a measurable subset of ∂D (i. e. its characteristic function is resolutive);

then there exists $h \in \mathrm{h}^\infty$ such that for almost every $w \in \partial D$, with respect to harmonic measure, the limit of $h(z)$ as $z \to w$ and $z \in \gamma(w)$ does not exist.

• A condition such as rotation invariance, in place of **(reg)**, would have no meaning, since in this context there is no group suitably acting, not even locally.

• Observe that **(c⋆)** *cannot* be relaxed to the weaker condition

$$w \text{ belongs to the closure of } \gamma(w)\,, \text{ for each } w \in \partial\mathbb{D}\,. \tag{1.3}$$

Indeed, the first-named author showed the existence, for NTA domains in $\mathbb{R}^n$, of an approach γ, satisfying (1.3) and **(tg)**, along which all h^∞ functions converge to their boundary values taken along (1.2), a. e. relative to harmonic measure[4].

§2. Overview of the proofs

The core of the problem belongs to harmonic analysis, so we restrict ourselves, without loss of generality, to the space h^∞ of bounded harmonic functions on $\mathbb{D}$.

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[3]In formulating (and proving) our Theorem 1.1 we also had this goal in mind.

[4]In Di Biase [5], the existence is showed by reducing the problem to the discrete setting of a (not-necessarily-homogeneous) tree, rather than on the action of a group on the space. In general, in this context, there is no group suitably acting on the space.

The boundary of $\mathbb{D}$, denoted by $\partial\mathbb{D}$, is naturally identified to the quotient group $\mathbb{R}/2\pi\mathbb{Z}$, from which it inherits the *Lebesgue measure* m; thus, $m(\partial\mathbb{D}) = 2\pi$.

If $h \in \mathrm{h}^\infty$, the *Fatou set of* h, denoted by $\mathcal{F}(h) \subset \partial\mathbb{D}$, is the set of points $w \in \partial\mathbb{D}$, such that *the limit of* $h(z)$ *as* $z \to w$ *and* $z \in \Gamma_\alpha(w)$ *exists for all* $\alpha > 0$; this limit is denoted $h_\flat(w)$. Now, $m(\mathcal{F}(h)) = 2\pi$ and $h_\flat \in \mathrm{L}^\infty(\partial\mathbb{D})$; see Fatou [10].

The *Poisson extension* $P : \mathrm{L}^\infty(\partial\mathbb{D}) \to \mathrm{h}^\infty$ recaptures h from $h_\flat$, since $h = P[h_\flat]$.

If γ is a subset of $\mathbb{D} \times \partial\mathbb{D}$ and $w \in \partial\mathbb{D}$, *the shape of* γ *at* w is the set

$$\gamma(w) \stackrel{\text{def}}{=} \{z \in \mathbb{D} : (z,w) \in \gamma\} \subset \mathbb{D} .$$

An *approach* is a subset γ of $\mathbb{D} \times \partial\mathbb{D}$ such that (**apprch**) holds for all θ. One may think of γ as a family $\{\gamma(\theta)\}_{\theta\in[0,2\pi)}$ of subsets of $\mathbb{D}$. If $h \in \mathrm{h}^\infty$ and γ is an approach, then define the following two subsets of $\partial\mathbb{D}$: $\mathrm{C}(h,\gamma)$ is the set

$$\{w \in \mathcal{F}(h);\ h(z) \text{ converges to } h_\flat(w) \text{ as } z \to w \text{ and } z \in \gamma(w)\}$$

and $\mathrm{D}(h,\gamma)$ is the subset

$$\{w \in \partial\mathbb{D};\ h(z) \text{ does not have any limit as } z \to w \text{ and } z \in \gamma(w)\} .$$

If γ is an approach and $u : \mathbb{D} \to \mathbb{R}$ a function on $\mathbb{D}$, the function on $\partial\mathbb{D}$ given by

$$\gamma^\star(u)(w) \stackrel{\text{def}}{=} \sup\{|u(z)| : z \in \gamma(w)\}$$

is called the *maximal function of* u *along* γ at $w \in \partial\mathbb{D}$.

Lemma 2.1. *The following properties of an approach* γ *are equivalent:*

(a) $\gamma^\star$ *maps all continuous functions (on* $\mathbb{D}$*) to measurable functions (on* $\partial\mathbb{D}$*);*

(b) *for every open* $Z \subset \mathbb{D}$*, the boundary subset*

$$\gamma^\downarrow(Z) \stackrel{\text{def}}{=} \{w \in \partial\mathbb{D} : Z \cap \gamma(w) \neq \emptyset\} \subset \partial\mathbb{D}$$

is a measurable subset of $\partial\mathbb{D}$.

The subset in (b) is called the *shadow projected by* Z *along* γ. The proof of Lemma 2.1 is left to the reader[5]. The approach γ is called: *regular* if it satisfies (a) or (b) in Lemma 2.1; *rotation invariant* if $(z,w) \in \gamma$

[5]This circle of ideas is based on the work of E. M. Stein. Cf. Fefferman and Stein [11].

implies $(e^{i\theta}z, e^{i\theta}w) \in \gamma$ for all θ, z, w. A rotation invariant approach is regular.

2.1. The Independence Theorem

2.1.1. *Preliminary Remarks* Modern logic gives us tools that show that some statements can be neither proved nor disproved. The basic idea is familiar: if different models (or 'concrete' representations) of some axioms exhibit different properties, then these properties do not follow from those axioms. For example, the existence of a single, 'concrete' non commutative group shows that commutativity can not be derived from the group axioms. Similarly, the existence of different models of geometry shows that Euclid's Fifth Postulate does not follow from the others. Since the currently adopted system of axioms for Mathematics is ZFC[6], to prove a theorem amounts to deduce the statement from ZFC. A *model* of ZFC stands to ZFC as, say, a 'concrete' group stands to the axioms of groups. If ZFC is consistent, then it has several, different models. Gödel showed, in his *completeness theorem*, that a statement can be deduced from ZFC if and only if it holds in every model of ZFC; in particular, if it holds in some models but not in others, then it follows that it *can be neither proved nor disproved.* The tangential boundary behaviour of h^∞ functions is radically different in different models of ZFC[7].

2.1.2. *The Independence Result*

Theorem 2.1. *There is a model of ZFC in which there exists an approach γ satisfying* (**c**) *and* (**tg**) *and such that $C(h, \gamma)$ has measure equal to 2π for every $h \in \mathrm{h}^\infty$.*

Theorem 2.2. *There is a model of ZFC in which for every approach satisfying* (**c**$\star$) *and* (**tg**) *there exists $h \in \mathrm{h}^\infty$ such that $D(h, \gamma)$ has outer measure equal to 2π.*

Theorem 2.1 and Theorem 2.2, together with Gödel's completeness theorem, imply Theorem 1.2.

2.1.3. *A Consequence of ZFC* The following result shows that Theorem 2.2 cannot be improved[8]. Observe that while Theorem 2.1 only holds in some models of ZFC but not in others (and therefore, by Gödel's

[6]Acronym for Zermelo, Fraenkel and the Axiom of Choice. See Cohen [4], Drake [9], Jech [16], Kunen [19].

[7]Since an approach is a fairly arbitrary subset of $\mathbb{D} \times \partial\,\mathbb{D}$, *in retrospect* this result can be rationalized, but other examples in Analysis show that this rationalization is not *a priori* infallible.

[8]Theorem 2.3 in itself does not say whether (SSS) can be proved or not.

completeness theorem, the corresponding statement can not be deduced from ZFC) the following theorem can be deduced from ZFC and therefore it holds in any model of set theory (see the discussion in 2.1.1).

Theorem 2.3 (A consequence of ZFC). *There exists an approach γ satisfying* (**c**) *and* (**tg**) *such that for each $h \in \mathrm{h}^\infty$, the set $C(h,\gamma)$ has outer measure equal to 2π.*

Remark 2.1. We quote a remark made by Gödel in [12] about the Continuum Hypothesis, or Cantor's conjecture.

> Only someone who [...] denies that the concepts and axioms of classical set theory have any meaning (or any well-defined meaning) could be satisfied with such a solution, not someone who believes them to describe some well-determined reality. For in this reality Cantor's conjecture must be either true or false, and its undecidability from the axioms as knows today can only mean that these axioms do not contain a complete description of this reality; and such a belief is by no means chimerical, since it is possible to points out ways in which a decision of the question, even if it is undecidable from the axioms in their present form, might nevertheless be obtained.

It seems to us that Gödel's remark applies equally well to (SSS), for those who share the Platonist viewpoint of Gödel.

§3. How un-Stolz are the sharp approach regions in $\mathbb{C}^n$?

The theory of the boundary behaviour (from the viewpoint of the almost everywhere convergence) of holomorphic functions in the Hardy spaces, defined on a bounded pseudoconvex domain $\mathbb{D}$ with smooth boundary in $\mathbb{C}^n$, has been so far been sufficiently understood in a few cases only: the unit ball in $\mathbb{C}^n$ (Koranyi [18]; Hakim and Sibony [13]; Hirata [14]); finite type domains in $\mathbb{C}^2$ (Nagel et al [22]); convex finite type in $\mathbb{C}^n$ (Di Biase and Fischer [6]). The task is to give a precise (possibly intrinsic) description of the sharp approach, together with a proof of its sharpness, as well as a local Fatou theorem, coupled with the study of the area function, the maximal function along the sharp approach, and the L^p estimates relating these operators to each other, as well as a Calderón-Stein theorem, and so forth.

In the few cases that are sufficiently understood, a family of balls in the boundary (having certain covering and doubling properties) plays an important role in the theory; see Hörmander [15], Nagel et al [23], Stein [26]. However, in general, this structure seems to be missing; see Chirka[3] (whose results appear to have a conditional nature, i.e. conditional upon the occurrence of certain covering and doubling properties of certain boundary balls, that are rather difficult to verify).

In the few cases that are sufficiently understood, two features have been observed. The first one is that the sharp approach has a shape whose section, taken along a complex tangential direction, depends on the direction itself; [18]. For example, if $\mathbb{D}$ is the unit ball in $\mathbb{C}^n$, the shape of the sharp approach at a boundary point w can be described as the locus in the domain of the following inequality:

$$\frac{\operatorname{dist}(z,\partial\mathbb{D})}{\operatorname{dist}(z,w+T^c_w(\partial\mathbb{D}))}\geq C>0$$

where $T^c_w(\partial\mathbb{D})$ is the complex tangent space at w. The second feature is that the shape of the approach does change near weakly pseudoconvex points and yields sharper estimates for the associated maximal operator; see Nagel et al [22], [23] for the case $n=2$ and [6] for convex finite type domains in $\mathbb{C}^n$.

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