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Wiener criterion for Cheeger p-harmonic functions on metric spaces

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Abstract.

We show that for Cheeger p-harmonic functions on doubling metric measure spaces supporting a Poincaré inequality, the Wiener criterion is necessary and sufficient for regularity of boundary points.

§1. Introduction

The well-known Wiener criterion in $\mathbf{R}^n$ states that a boundary point $x \in \partial\Omega$ is regular for p-harmonic functions (i.e. every solution of the Dirichlet problem with continuous boundary data is continuous at x) if and only if

$$\int_0^1 \left(\frac{\operatorname{Cap}_p(B(x,t) \setminus \Omega, B(x,2t))}{t^{n-p}} \right)^{1/(p-1)} \frac{dt}{t} = \infty,$$

where Cap_p is the p-capacity on $\mathbf{R}^n$. For $p = 2$, this was proved by Wiener [30]. For $1 < p < \infty$, the sufficiency part of the Wiener criterion is due to Maz′ya [25] and has been extended to more general equations in Gariepy–Ziemer [10], Heinonen–Kilpeläinen–Martio [12] and Danielli [8]. The necessity part for $1 < p < \infty$ was proved by Kilpeläinen–Malý [19] and extended to weighted equations by Mikkonen [26]. For subelliptic operators, the Wiener criterion was proved in Trudinger–Wang [29].

In the last decade, there has been a lot of development in the theory of p-harmonic functions on doubling metric measure spaces supporting a Poincaré inequality. The Dirichlet problem for such p-harmonic

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functions has been solved for rather general boundary data (including Sobolev and continuous functions) in e.g. Cheeger [7], Shanmugalingam [27] and [28], Kinnunen–Martio [22] and Björn–Björn-Shanmugalingam [2] and [3].

In Björn–MacManus–Shanmugalingam [6], the sufficiency part of the Wiener criterion was proved in linearly locally connected spaces. The proof in [6] applies both to Cheeger p-harmonic functions and to p-harmonic functions defined using the upper gradient. In this note, we show that for Cheeger p-harmonic functions the assumption of linear local connectedness can be omitted. Moreover, for Cheeger p-harmonic functions, the Wiener condition is also necessary, i.e. we have the following result.

Theorem 1.1. *Let X be a complete metric measure space with a doubling measure μ supporting a p-Poincaré inequality. Let $\Omega \subset X$ be open and bounded. Then the point $x \in \partial\Omega$ is Cheeger p-regular if and only if for some $\delta > 0$,*

$$\int_0^\delta \left(\frac{\operatorname{Cap}_p(B(x,t) \setminus \Omega, B(x,2t))}{t^{-p}\mu(B(x,t))} \right)^{1/(p-1)} \frac{dt}{t} = \infty. \tag{1.1}$$

Much of the theory of p-harmonic functions on metric spaces has been done for p-harmonic functions defined using the upper gradient. All those proofs go through for Cheeger p-harmonic functions as well (just replacing g_u by $|Du|$ throughout). On the other hand, certain results and methods which apply to Cheeger p-harmonic functions cannot be used for p-harmonic functions defined using the upper gradients. The proof of Theorem 1.1 is one such example: it uses Wolff potential estimates for supersolutions, as in Kilpeläinen–Malý [19]. For other examples, see e.g. Björn–MacManus–Shanmugalingam [6] or Björn–Björn–Shanmugalingam [2].

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§2. Preliminaries

We assume throughout the paper that $X = (X, d, \mu)$ is a complete metric space endowed with a metric d and a positive complete Borel measure μ such that $0 < \mu(B) < \infty$ for all balls $B \subset X$ (we make the convention that balls are nonempty and open). We also assume that the measure μ is *doubling*, i.e. that there exists a constant $C > 0$ such that for all balls $B = B(x,r) := \{y \in X : d(x,y) < r\}$ in X,

$$\mu(2B) \le C\mu(B),$$

where $\lambda B = B(x, \lambda r)$. Note that some authors assume that X is proper (i.e. that closed bounded sets are compact) rather than complete, but, since μ is doubling, X is complete if and only if X is proper.

Throughout the paper, $1 < p < \infty$ is fixed. In [13], Heinonen and Koskela introduced upper gradients as a substitute for the modulus of the usual gradient. The advantage of this new notion is that it can easily be used in metric spaces.

Definition 2.1. *A nonnegative Borel function g on X is an* upper gradient *of an extended real-valued function f on X if for all nonconstant rectifiable curves $\gamma : [0, l_\gamma] \to X$, parameterized by arc length ds,*

$$|f(\gamma(0)) - f(\gamma(l_\gamma))| \le \int_\gamma g\, ds \tag{2.1}$$

whenever both $f(\gamma(0))$ and $f(\gamma(l_\gamma))$ are finite, and $\int_\gamma g\, ds = \infty$ otherwise. If g is a nonnegative measurable function on X such that (2.1) holds for p-almost every curve, (i.e. it fails only for a curve family with zero p-modulus, see Definition 2.1 in Shanmugalingam [27]*), then g is a* p-weak upper gradient *of f.*

We further assume that X supports a *weak p-Poincaré inequality*, i.e. there exist constants $C > 0$ and $\lambda \ge 1$ such that for all balls $B \subset X$, all measurable functions f on X and all upper gradients g of f,

$$⨍_B |f - f_B|\, d\mu \le C(\operatorname{diam} B)\left(⨍_{\lambda B} g^p\, d\mu\right)^{1/p}, \tag{2.2}$$

where $f_B := ⨍_B f\, d\mu = \mu(B)^{-1} \int_B f\, d\mu$.

By Keith–Zhong [17] it follows that X supports a weak q-Poincaré inequality for some $q \in [1,p)$, which was earlier a standard assumption. As X is complete, it suffices to require that (2.2) holds for all compactly supported Lipschitz functions, see Heinonen–Koskela [14] or Keith [15], Theorem 2. There are many spaces satisfying these assumptions, such as Riemannian manifolds with nonnegative Ricci curvature and the Heisenberg groups. For a list of examples see e.g. Björn [5], and for more detailed descriptions see Heinonen–Koskela [13] or the monograph Hajłasz–Koskela [11]. The following Sobolev type spaces were introduced in Shanmugalingam [27].

Definition 2.2. *For $u \in L^p(X)$, let*

$$\|u\|_{N^{1,p}(X)} = \left(\int_X |u|^p \, d\mu + \inf_g \int_X g^p \, d\mu \right)^{1/p},$$

where the infimum is taken over all upper gradients of u. The Newtonian space *on X is the quotient space*

$$N^{1,p}(X) = \{u : \|u\|_{N^{1,p}(X)} < \infty\}/\sim,$$

where $u \sim v$ if and only if $\|u - v\|_{N^{1,p}(X)} = 0$.

Every $u \in N^{1,p}(X)$ has a unique *minimal p-weak upper gradient* $g_u \in L^p(X)$ in the sense that for every p-weak upper gradient g of u, $g_u \le g$ μ-a.e., see Corollary 3.7 in Shanmugalingam [28]. Theorem 6.1 in Cheeger [7] shows that for Lipschitz f,

$$g_f(x) = \limsup_{y \to x} \frac{|f(y) - f(x)|}{d(x,y)}.$$

Cheeger [7] uses a different definition of Sobolev spaces which leads to the same space, see Theorem 4.10 in [27]. Cheeger's definition yields the notion of partial derivatives in the following theorem, see Theorem 4.38 in [7].

Theorem 2.3. *Let X be a metric measure space equipped with a doubling Borel regular measure μ. Assume that X admits a weak p-Poincaré inequality for some $1 < p < \infty$.*

Then there exists $N \in \mathbf{N}$ and a countable collection (U_α, X^α) of measurable sets U_α and Lipschitz "coordinate" functions $X^\alpha : X \to \mathbf{R}^{k(\alpha)}$, $1 \le k(\alpha) \le N$, such that $\mu\bigl(X \setminus \bigcup_\alpha U_\alpha\bigr) = 0$ and for every Lipschitz $f : X \to \mathbf{R}$ there exist unique bounded vector-valued functions $d^\alpha f : U_\alpha \to \mathbf{R}^{k(\alpha)}$ such that for μ-a.e. $x \in U_\alpha$,

$$\lim_{r \to 0+} \sup_{y \in B(x,r)} \frac{|f(y) - f(x) - \langle d^\alpha f(x), X^\alpha(y) - X^\alpha(x) \rangle|}{r} = 0,$$

where $\langle \cdot \, , \cdot \rangle$ denotes the usual inner product in $\mathbf{R}^{k(\alpha)}$.

Cheeger shows that for μ-a.e. $x \in U_\alpha$, there is an inner product norm $|\cdot|_x$ on $\mathbf{R}^{k(\alpha)}$ such that for all Lipschitz f,

$$g_f(x)/C \le |d^\alpha f(x)|_x \le C g_f(x), \tag{2.3}$$

where C is independent of f and x, see p. 460 in [7]. We can assume that the sets U_α are pairwise disjoint and let $Df(x) = d^\alpha f(x)$ for $x \in U_\alpha$.

We shall in the following omit the subscript x in the norms $|\cdot|_x$ and use the notation

$$|Df| = |Df(x)| := |d^\alpha f(x)|_x. \tag{2.4}$$

Thus, (2.3) can be written as

$$g_f/C \le |Df| \le C g_f \quad \mu\text{-a.e. in } X. \tag{2.5}$$

The differential mapping $D : f \mapsto Df$ is linear and satisfies the Leibniz and chain rules. Also, $Df = 0$ μ-a.e. on every set where f is constant. See Cheeger [7] for these properties.

By Theorem 4.47 in [7] and Theorem 4.10 in Shanmugalingarm [27], Lipschitz functions are dense in $N^{1,p}(X)$. Using Theorem 10 in Franchi–Hajlasz–Koskela [9] or Keith [16], the "gradient" Du extends uniquely to the whole $N^{1,p}(X)$ and it satisfies (2.5) for every $u \in N^{1,p}(X)$.

Definition 2.4. *The p-capacity of a set $E \subset X$ is the number*

$$C_p(E) := \inf_u \|u\|^p_{N^{1,p}},$$

where the infimum is taken over all $u \in N^{1,p}(X)$ such that $u \ge 1$ on E.

For various properties as well as equivalent definitions of the p-capacity we refer to Kilpeläinen–Kinnunen–Martio [18] and Kinnunen–Martio [20], [21]. The p-capacity is the correct gauge for distinguishing between two Newtonian functions. If $u \in N^{1,p}(X)$, then $u \sim v$ if and only if $u = v$ outside a set of p-capacity zero. Moreover, Corollary 3.3 in Shanmugalingam [27] shows that if $u, v \in N^{1,p}(X)$ and $u = v$ μ-a.e., then $u \sim v$.

To be able to compare the boundary values of Newtonian functions we need a Newtonian space with zero boundary values. Let

$$N_0^{1,p}(\Omega) = \{f|_\Omega : f \in N^{1,p}(X) \text{ and } f = 0 \text{ in } X \setminus \Omega\}.$$

Throughout the paper, $\Omega \subset X$ will be a nonempty bounded open set in X such that $C_p(X \setminus \Omega) > 0$. (If X is unbounded then the condition $C_p(X \setminus \Omega) > 0$ is of course immediately fulfilled.)

§3. p-harmonic functions and regularity

There are two ways of generalizing p-harmonic functions to metric spaces, one based on the scalar-valued upper gradient g_u and the other using the vector-valued Cheeger gradient Du. In this paper, we are concerned with Cheeger p-harmonic functions given by the following definition.

Definition 3.1. *A function* $u \in N^{1,p}_{\mathrm{loc}}(\Omega)$ *is* Cheeger p-harmonic *in* Ω *if it is continuous and for all Lipschitz functions* φ *with compact support in* Ω,

$$\int_\Omega |Du|^p \, d\mu \le \int_\Omega |Du + D\varphi|^p \, d\mu, \tag{3.1}$$

or equivalently,

$$\int_\Omega |Du|^{p-2} Du \cdot D\varphi \, d\mu = 0,$$

where $\cdot$ *denotes the inner product giving rise to the norm* $|\cdot|$ *from (2.4) (note that it depends on* x*).*

As mentioned in the introduction, all properties which have been proved for p-harmonic functions defined using the upper gradient, also hold for Cheeger p-harmonic functions and will be used here without further notice. By Kinnunen–Shanmugalingam [24], every function satisfying (3.1) has a locally Hölder continuous representative which satisfies the Harnack inequality and the maximum principle. It is this representative that we call Cheeger p-harmonic.

The Dirichlet problem for Cheeger p-harmonic functions and rather general boundary data was solved using the Perron method in Björn–Björn-Shanmugalingam [3]. The construction is based on Cheeger p-superharmonic functions. The *upper Perron solution* for $f : \partial\Omega \to \mathbf{R}$ is

$$\overline{P}f(x) := \inf_u u(x), \quad x \in \Omega,$$

where the infimum is taken over all Cheeger p-superharmonic functions u on Ω bounded below such that

$$\liminf_{\Omega \ni y \to x} u(y) \ge f(x) \quad \text{for all } x \in \partial\Omega.$$

The lower Perron solution is defined by $\underline{P}f = -\overline{P}(-f)$, and if both solutions coincide, we let $Pf := \overline{P}f = \underline{P}f$ and f is called *resolutive.* Note that we always have $\underline{P}f \le \overline{P}f$, by Theorem 7.2 in Kinnunen–Martio [22]. The following comparison principle holds: If $f_1 \le f_2$ on $\partial\Omega$, then $Pf_1 \le Pf_2$ in Ω.

The following theorem is proved in [3], Theorems 5.1 and 6.1.

Theorem 3.2. *Let* $f \in C(\partial\Omega)$ *or* $f \in N^{1,p}(X)$*. Then* f *is resolutive. Moreover, if* $f \in N^{1,p}(X)$*, then* $Pf - f \in N^{1,p}_0(\Omega)$*.*

By Theorem 7.7 in Kinnunen–Martio [22], every Cheeger p-superharmonic function is a pointwise limit of an increasing sequence of p-supersolutions. A function $u \in N^{1,p}_{\mathrm{loc}}(\Omega)$ is a *p-supersolution* in Ω if for all nonnegative Lipschitz functions φ with compact support in Ω,

$$\int_\Omega |Du|^{p-2} Du \cdot D\varphi \, d\mu \geq 0.$$

We also have the following simple lemma.

Lemma 3.3. *Assume that* $f : \partial\Omega \to \overline{\mathbf{R}}$ *is resolutive. Let* $\Omega' \subset \Omega$ *be open and define* $h : \partial\Omega' \to \overline{\mathbf{R}}$ *by*

$$h(x) = \begin{cases} f(x), & \text{if } x \in \partial\Omega \cap \partial\Omega', \\ Pf(x), & \text{if } x \in \Omega \cap \partial\Omega'. \end{cases}$$

Then h *is resolutive with respect to* Ω' *and the Perron solution for* h *in* Ω' *is* $P_{\Omega'} h = Pf|_{\Omega'}$.

Proof. Let u be a Cheeger p-superharmonic function admissible in the definition of $\overline{P}f = Pf$. Then it is easily verified (using the lower semicontinuity of u) that $\lim_{\Omega' \ni y \to x} u(y) \geq h(x)$ for all $x \in \partial\Omega'$. Hence u is admissible in the definition of the upper Perron solution $\overline{P}_{\Omega'} h$ for h in Ω' and taking infimum over all such u shows that $\overline{P}_{\Omega'} h \leq Pf$ in Ω'. Applying the same argument to $-f$, we obtain

$$\underline{P}_{\Omega'} h = -\overline{P}_{\Omega'}(-h) \geq -P(-f) = Pf \geq \overline{P}_{\Omega'} h \geq \underline{P}_{\Omega'} h.$$

□

Definition 3.4. *A point* $x \in \partial\Omega$ *is* Cheeger p-regular *if*

$$\lim_{\Omega \ni y \to x} Pf(y) = f(x) \quad \text{for all } f \in C(\partial\Omega).$$

In Björn–Björn [1], regular boundary points have been characterized by means of barriers. Theorems 4.2 and 6.1 in [1] also give other equivalent characterizations of regularity. In particular, Theorem 6.1(f) in [1] shows that regularity is a local property:

Theorem 3.5. *Let* $x \in \partial\Omega$ *and* $\delta > 0$. *Then* x *is Cheeger* p*-regular with respect to* Ω *if and only if it is Cheeger* p*-regular with respect to* $\Omega \cap B(x, \delta)$.

§4. Proof of Theorem 1.1: sufficiency

We start by defining the relative capacity which appears in the Wiener criterion.

Definition 4.1. *Let $B \subset X$ be a ball and $E \subset B$. The* relative capacity *of E with respect to B is*

$$\operatorname{Cap}_p(E,B) = \inf_u \int_B |Du|^p \, d\mu,$$

where the infimum is taken over all $u \in N_0^{1,p}(B)$ such that $u \geq 1$ on E.

Lemma 3.3 in Björn [4] (combined with (2.5)) shows that the capacities Cap_p and C_p are in many situations equivalent and have the same zero sets. Moreover, $\operatorname{Cap}_p(B, 2B)$ is comparable to $r^{-p}\mu(B)$.

Unless otherwise stated, the letter C denotes various positive constants whose exact values are unimportant and may vary with each usage. The constant C is allowed to depend on the fixed parameters associated with the geometry of the space X.

Definition 4.2. *Let B be a ball and $K \subset B$ be compact. The* Cheeger p-potential *for K with respect to B is the Cheeger p-harmonic function in $B \setminus K$ with boundary data 1 on ∂K and 0 on ∂B. We extend the Cheeger p-potential u by 1 on K to have $u \in N_0^{1,p}(B)$.*

Lemma 3.2 in Björn–MacManus–Shanmugalingam [6] shows that the Cheeger p-potential u is a p-supersolution in B. Hence, by Proposition 3.5 in [6], there is a unique regular Radon measure $\nu \in N_0^{1,p}(B)^*$ such that

$$\int_B |Du|^{p-2} Du \cdot D\varphi \, d\mu = \int_B \varphi \, d\nu \quad \text{for all } \varphi \in N_0^{1,p}(B). \tag{4.1}$$

The sufficiency part of Theorem 1.1 will follow from the following lemma. It was proved in [6], Lemma 5.7, for p-harmonic functions defined using the upper gradient under the additional assumption that X is linearly locally connected. Here we show it without this assumption, but only for Cheeger p-harmonic functions. Estimates of this type appeared first in Maz′ya [25], where they were used to prove the sufficiency part of the Wiener criterion for nonlinear elliptic equations.

Lemma 4.3. *Let $B = B(x,r)$ and $K \subset \overline{B}$ be compact. Let u be the Cheeger p-potential for K with respect to $4B$. Then for $0 < \rho \leq r$ and $y \in B(x,\rho)$,*

$$1 - u(y) \leq \exp\biggl(-C \int_\rho^r \biggl(\frac{\operatorname{Cap}_p(B(x,t) \cap K, B(x,2t))}{t^{-p}\mu(B(x,t))} \biggr)^{1/(p-1)} \frac{dt}{t} \biggr).$$

Lemma 4.3 follows from the following lemma by iteration and the comparison principle in the same way as Lemma 5.7 in [6].

Lemma 4.4. *Let B, K and u be as in Lemma 4.3. Then*

$$\inf_B u \geq C\biggl(\frac{\operatorname{Cap}_p(K,4B)}{r^{-p}\mu(B)}\biggr)^{1/(p-1)}.$$

Proof. Let ν be the Radon measure given by (4.1). By Lemma 3.10 in [6], we have $\operatorname{supp}\nu \subset K$ and $\nu(K) = \operatorname{Cap}_p(K,4B)$. Lemma 4.8 in [6] then yields

$$\inf_B u \geq \inf_{2B} u + C\biggl(\frac{\nu(B)}{r^{-p}\mu(B)}\biggr)^{1/(p-1)} \geq C\biggl(\frac{\operatorname{Cap}_p(K,4B)}{r^{-p}\mu(B)}\biggr)^{1/(p-1)}.$$

□

The following corollary is proved in a similar way as Theorem 6.18 in Heinonen–Kilpeläinen–Martio [12]. See also Maz′ya [25].

Corollary 4.5. *Let $f : \partial\Omega \to \mathbf{R}$ be bounded and resolutive, and $x \in \partial\Omega$. Then for all sufficiently small $0 < \rho \leq r$,*

$$\sup_{\Omega\cap B(x,\rho)} (Pf - f(x)) \leq \sup_{\partial\Omega\cap B(x,4r)} (f - f(x))$$
$$+ \sup_{\partial\Omega}(f - f(x)) \exp\biggl(-C\int_\rho^r \biggl(\frac{\operatorname{Cap}_p(B(x,t)\setminus\Omega, B(x,2t))}{t^{-p}\mu(B(x,t))}\biggr)^{1/(p-1)}\frac{dt}{t}\biggr).$$

Proof. Let $B = B(x,r)$, $m = \sup_{\partial\Omega\cap 4B} f$ and $M = \sup_{\partial\Omega} f$. Note that by the maximum principle, $Pf \leq M$ in Ω. We can assume that $f(x) = 0$. Let u be the Cheeger p-potential for $K = \overline{B}\setminus\Omega$ in $4B$. Let h be as in Lemma 3.3 with $\Omega' := \Omega\cap 4B$. Then it is easily verified that $h \leq m + M(1-u)$ on $\partial\Omega'$. Lemma 3.3 and the comparison principle show that

$$Pf = P_{\Omega'}h \leq P_{\Omega'}(m + M(1-u)) = m + M(1-u) \quad \text{on } \Omega'$$

and Lemma 4.3 finishes the proof. □

To conclude the proof of the sufficiency part of Theorem 1.1, let $f \in C(\partial\Omega)$ and $\varepsilon > 0$ be arbitrary. There exists $r > 0$ such that $\sup_{\partial\Omega\cap B(x,4r)} |f - f(x)| \leq \varepsilon$. Condition (1.1) and Corollary 4.5 then imply that for sufficiently small ρ we have

$$\sup_{\Omega\cap B(x,\rho)} |Pf - f(x)| \leq 2\varepsilon.$$

Thus, Pf is continuous at x and as $f \in C(\partial\Omega)$ was arbitrary, x is Cheeger p-regular.

§5. Proof of Theorem 1.1: necessity

To obtain the necessity part of Theorem 1.1, we first formulate an estimate for p-supersolutions by means of Wolff potentials. It is similar to Theorem 1.6 in Kilpeläinen–Malý [19] and Corollary 4.11 in [6].

Lemma 5.1. *Let u be a nonnegative p-supersolution in $5B$, where $B = B(x,r)$. Let ν be the Radon measure given by* (4.1). *Then*

$$\lim_{\rho\to 0} \operatorname*{ess\,inf}_{B(x,\rho)} u \le C\biggl(\operatorname*{ess\,inf}_{3B} u + \int_0^r \biggl(\frac{\nu(B(x,t))}{t^{-p}\mu(B(x,t))}\biggr)^{1/(p-1)} \frac{dt}{t}\biggr).$$

Proof. It can be shown as in the proof of Theorem 3.13 in Mikkonen [26] that the above estimate holds with $\operatorname{ess\,inf}_{3B} u$ replaced by $\bigl(\fint_{\frac{1}{2}B} u^\gamma \, d\mu\bigr)^{1/\gamma}$ for all $\gamma > p-1$ (and C depending on γ). Theorem 4.3 in Kinnunen–Martio [23] shows that for γ close to $p-1$,

$$\biggl(\fint_{\frac{1}{2}B} u^\gamma \, d\mu\biggr)^{1/\gamma} \le C \operatorname*{ess\,inf}_{3B} u,$$

which concludes the proof. □

Corollary 5.2. *Let $u \in N_0^{1,p}(5B)$ be the Cheeger p-potential for a compact $K \subset \overline{B}$ in $5B$, where $B = B(x,r)$. Then*

$$\liminf_{y\to x} u(y) \le C \int_0^{2r} \biggl(\frac{\operatorname{Cap}_p(B(x,t)\cap K, B(x,2t))}{t^{-p}\mu(B(x,t))}\biggr)^{1/(p-1)} \frac{dt}{t}.$$

Proof. Let ν be the Radon measure given by (4.1). For $0 < t \le r$, let ν_t be the restriction of ν to $B(x,t)$ and $u_t \in N_0^{1,p}(5B)$ be the p-supersolution in $5B$ associated with ν_t as in (4.1), see Proposition 3.9 in Björn–MacManus–Shanmugalingam [6]. It satisfies

$$\int_{5B} |Du_t|^{p-2} Du_t \cdot D\varphi \, d\mu = \int_{5B} \varphi \, d\nu_t \quad \text{for all } \varphi \in N_0^{1,p}(5B). \tag{5.1}$$

Inserting $\varphi = (u_t - u)_+$ as a test function in both (4.1) and (5.1), a simple comparison yields $D(u_t - u)_+ = 0$ μ-a.e. in $5B$ (see e.g. Lemma 2.8 in [26]). Hence $u_t \le u \le 1$ in $5B$ and Lemma 3.10 in [6] implies

$$\nu_t(B(x,t)) \le \operatorname{Cap}_p(K \cap \overline{B}(x,t), 5B) \le \operatorname{Cap}_p(K\cap B(x,2t), B(x,4t)). \tag{5.2}$$

Let $a = \inf_{3B} u$. Then $a > 0$ by the maximum principle, and Lemma 5.4 in [6] shows that

$$\operatorname{Cap}_p(3B, 5B) \le \operatorname{Cap}_p(\{x : u \ge a\}, 5B) \le C a^{1-p} \operatorname{Cap}_p(K, 5B).$$

It follows that

$$a \le C\left(\frac{\operatorname{Cap}_p(K,5B)}{r^{-p}\mu(B)}\right)^{1/(p-1)}$$

$$\le C\int_r^{2r}\left(\frac{\operatorname{Cap}_p(K\cap B(x,t),B(x,2t))}{t^{-p}\mu(B(x,t))}\right)^{1/(p-1)}\frac{dt}{t}. \tag{5.3}$$

Inserting (5.2) and (5.3) into Lemma 5.1 finishes the proof of the corollary. □

To conclude the proof of the necessity part of Theorem 1.1, we apply Corollary 5.2 to $K = \overline{B}(x,r)\setminus\Omega$. Let u_r be the corresponding Cheeger p-potential with respect to $B(x,5r)$. If the integral in Theorem 1.1 converges, we can use Corollary 5.2 to find $r > 0$ sufficiently small so that

$$\liminf_{y\to x} u_r(y) < 1.$$

As u_r is the solution of the Dirichlet problem in $B(x,5r)\setminus K$ with the continuous boundary data 1 on K and 0 on $\partial B(x,5r)$, we see that x is not Cheeger p-regular for the open set $B(x,5r)\setminus K$. Theorem 3.5 then shows that x is not Cheeger p-regular for Ω either.

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